Normal matrices and their principal submatrices of co-order one

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Abstract

Let \( A \) be a normal matrix, \( v \) be any of its indices, \( A - v \) be the matrix obtained from \( A \) by deleting the \( v \)th row and column, and \( \lambda \) be an eigenvalue of \( A - v \). In our paper we construct the eigenspace of \( A \) associated with \( \lambda \) from that of \( A - v \). In particular, it is shown that if there is a (unique) Jordan block of size strictly greater than one in the part of the Jordan form of \( A - v \) corresponding to \( \lambda \), then the geometric multiplicity of \( \lambda \) decreases by one under the transition from \( A - v \) to \( A \) (in other words, the typical change of the spectral properties holds for \( \lambda \)). The results obtained are applied to circulant matrices. Moreover, in Appendix to our paper we consider almost regular tournament matrices as principal submatrices of co-order one of regular tournament matrices. In particular, it is observed that the Brualdi–Li tournament matrix \( B_{2n} \) of order \( 2n \) is permutationally similar to a principal submatrix of co-order one of the circulant matrix of order \( 2n + 1 \) with the first row \( 0, 1, \ldots, 1, 0, \ldots, 0 \). As a consequence of this fact, the weak Brualdi–Li conjecture is formulated for principal submatrices of co-order one of the adjacency matrices of Cayley tournaments.

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1. Introduction

The relations between the spectral properties of matrices and those of their principal submatrices are considered in many papers (see [1–15]). A particular attention is paid to the case when a principal submatrix is obtained by deleting some row and column from the original matrix $A$. In [16,17] such a submatrix is called a principal submatrix of co-order one and is denoted by $A - v$, where $v$ is the index of the deleted row and column. Using the spectral resolution of the identity for a self-adjoint operator and the fact that any principal submatrix of a Hermitian matrix $A$ is also Hermitian, it is not difficult to describe the change of the eigenspace corresponding to a given eigenvalue $\lambda$ under the transition from $A - v$ to $A$. In the present paper we study this problem for an arbitrary matrix (see Lemma 1) and consider the case of a normal matrix $A$ in detail (see Theorem 1). In [16] the change of the Jordan form under adding a generic column and row to an arbitrary matrix was determined. We show that the typical change takes place for an eigenvalue $\lambda$ of $A - v$ if $A$ is a normal matrix and there is a nontrivial Jordan block in the part of the Jordan form of $A - v$ associated with $\lambda$ (for instance, these conditions hold for the circulant matrix with the first row $0, 1, 0, \ldots, 0$). This result (see Proposition 1) is a direct consequence of the proof of Theorem 1. Another corollary of this proof (see Proposition 2 and Remark 3) is devoted to normal matrices which commute with sufficiently large groups of permutation matrices (for instance, any circulant matrix belongs to this class). In particular, we show that in some sense the spectral properties of their principal submatrices of co-order one are close to those of a normal matrix.

It is not difficult to prove that a regular tournament matrix $T$ is normal and every almost regular tournament matrix is a principal submatrix of co-order one of such $T$. Lately a lot of papers have been devoted to the spectral properties of almost regular tournaments. In particular, the Brualdi–Li problem posed in [18] was intensively considered. In Appendix to our paper we give a new representation for the Brualdi–Li matrix and formulate the weak Brualdi–Li conjecture based on this representation. Moreover, some classes of matrices which have the smallest Perron value among all almost regular tournament matrices of a given (even) order are obtained. The reader will also find different analogs of the Brualdi–Li conjecture for principal submatrices of co-order one of arbitrary irreducible regular matrices.

2. Construction of the eigenspace of $A$ for $\lambda$ from that of $A - v$: the general case

We start this section with some necessary spectral definitions and notation. Let $A$ be an arbitrary matrix of order $n$ and $V(A)$ be the index-set of $A$. For any $v \in V(A)$, denote by $A - v$ the matrix obtained from $A$ by deleting the $v$th column and row. It is clear that the order of $A - v$ is one less than that of $A$. In the sequel, we shall say that $A - v$ is a principal submatrix of co-order one.

Let $\eta_v$ and $\xi_v$ be the vectors obtained from the $v$th column and row of $A$, respectively, by deleting the diagonal entry $A(v, v)$. In our proofs we shall always assume that $v$ is the first index of $A$. In this case the matrix $A$ has the form

$$
\begin{pmatrix}
A(v, v) & \xi_v^T \\
\eta_v & A - v
\end{pmatrix}.
$$

By definition, the $v$-extension of a vector $\xi$ whose entries are indexed by the set $V(A - v)$ is the vector whose $v$th entry is equal to 0 and all the others coincide with the corresponding entries of $\xi$. It is not difficult to check that

$$
\begin{pmatrix}
A(v, v) & \xi_v^T \\
\eta_v & A - v
\end{pmatrix} \begin{pmatrix}
0 \\
0
\end{pmatrix} = \begin{pmatrix}
(\xi_v, \xi) \\
(A - v)\xi
\end{pmatrix},
$$

(1)
where \((\cdot, \cdot)\) is defined for any two vectors \(\xi, \eta \in \mathbb{C}^{n-1}\) in the following way: 
\[
(\xi, \eta) = \xi^\top \eta = \sum_w \xi(w)\overline{\eta}(w).
\]
In our paper we shall mainly use this scalar product. If \((\xi, \eta) = 0\), then we shall write \(\xi \perp \eta\) and say that \(\xi\) is orthogonal to \(\eta\). We shall also use the standard scalar product 
\[
(\xi, \eta)_\mathbb{C} = \sum_w \xi(w)\overline{\eta}(w).
\]
In particular, if \((\xi, \eta)_\mathbb{C} = 0\), then we shall say that \(\xi\) is \(\mathbb{C}\)-orthogonal to \(\eta\).

Let \(L, L_1\), and \(L_2\) be three linear spaces such that \(L_1 \cup L_2 \subseteq L\). By definition, \(L\) is the sum of \(L_1\) and \(L_2\) if any vector \(\xi\) of \(L\) can be represented in the form \(\xi = \xi_1 + \xi_2\) for some \(\xi_1 \in L_1\) and \(\xi_2 \in L_2\). Its sum \(L\) of \(L_1\) and \(L_2\) is direct if \(L_1 \cap L_2 = \{0\}\). In this case we shall write \(L = L_1 + L_2\). If any vector of \(L_1\) is \(\mathbb{C}\)-orthogonal to every vector of \(L_2\), then \(L\) is the \(\mathbb{C}\)-orthogonal sum of \(L_1\) and \(L_2\), which is denoted by \(L_1 \oplus L_2\).

Let \(I\) be the identity matrix. If \((A - \lambda I)\xi = 0\) for some \(\xi \neq 0\), then \(\lambda\) is an eigenvalue of \(A\) and \(\xi\) is an eigenvector of \(A\) for \(\lambda\). All such vectors taken together with 0 form the eigenspace \(E_A(\lambda)\) of \(A\) for \(\lambda\). Its dimension \(g_A(\lambda)\) is called the geometric multiplicity of the eigenvalue \(\lambda\).

It is not difficult to show using (1) that the \(v\)-extension of an eigenvector \(\xi\) of \(A - v\) for \(\lambda\) is an eigenvector of \(A\) for the same eigenvalue \(\lambda\) iff \((\xi, \xi_v) = 0\). In particular, this implies the inequality 
\[
g_{A-v}(\lambda) - 1 \leq g_A(\lambda) \leq g_{A-v}(\lambda) + 1.
\]

Let \((A - v)^\top\) be the transpose of the matrix \(A - v\). By Fredholm’s theorems, there exists a vector \(\eta\) such that 
\[
((A - v) - \lambda I)\eta = \eta_v \iff \eta_v \perp E_{(A-v)^\top}(\lambda).
\]
It is clear that the difference between any two solutions of 
\[(A - v) - \lambda I)\eta = \eta_v\] 
is an eigenvector of \(A - v\) for \(\lambda\). Let us denote the set of all solutions of this equation by 
\[
((A - v) - \lambda I)^{-1}\eta_v.
\]

**Lemma 1.** Let \(A\) be an arbitrary square matrix and \(v\) be any index of \(A\). Assume that \(\lambda\) is an eigenvalue of \(A - v\).

Then

\(1\) \(g_A(\lambda) = g_{A-v}(\lambda) - 1 \iff E_{A-v}(\lambda)\) is not orthogonal to \(\xi_v\) and \(E_{(A-v)^\top}(\lambda)\) is not orthogonal to \(\eta_v\) \(\iff E_A(\lambda)\) is a proper subspace of the \(v\)-extension of \(E_{A-v}(\lambda)\);

\(2\) \(g_A(\lambda) = g_{A-v}(\lambda) \iff \exists \xi_v \perp E_{A-v}(\lambda)\) or \(\exists \eta_v \perp E_{(A-v)^\top}(\lambda)\) or both of these conditions hold but 
\[-A(v, v) + (\xi_v, \eta^*) \neq -\lambda\] 
for any \(\eta^* \in ((A - v) - \lambda I)^{-1}\eta_v\) \(\iff E_A(\lambda)\) is the \(v\)-extension of \(E_{A-v}(\lambda)\) or \(E_{(A-v)^\top}(\lambda)\) is the \(v\)-extension of \(E_{A-v}(\lambda)\) or both of these conditions hold;

\(3\) \(g_A(\lambda) = g_{A-v}(\lambda) + 1 \iff \xi_v \perp E_{A-v}(\lambda), \ \eta_v \perp E_{(A-v)^\top}(\lambda)\) and 
\[-\lambda = -A(v, v) + (\xi_v, \eta^*)\] 
for any \(\eta^* \in ((A - v) - \lambda I)^{-1}\eta_v\) \(\iff\) for any \(\eta^* \in ((A - v) - \lambda I)^{-1}\eta_v, \ the\ following\ decomposition\ takes\ place:

\[
E_A(\lambda) = \begin{pmatrix} 0 \\ E_{A-v}(\lambda) \end{pmatrix} + \left(\begin{pmatrix} -1 \\ \eta^* v \end{pmatrix}\right).
\]

**Proof.** Let us consider the intersection \(E_{A-v}(\lambda) \cap \langle \xi_v \rangle^\perp\) of the eigenspace \(E_{A-v}(\lambda)\) with the hyperplane \(\langle \xi_v \rangle^\perp\) that is orthogonal to the linear hull \(\langle \xi_v \rangle\) of \(\xi_v\). It is clear that its \(v\)-extension belongs to \(E_A(\lambda)\) and its dimension is not less than \(g_{A-v}(\lambda) - 1\). Thus, if \(g_A(\lambda) = g_{A-v}(\lambda) - 1\), then \(E_A(\lambda)\) coincides with the \(v\)-extension of \(E_{A-v}(\lambda) \cap \langle \xi_v \rangle^\perp\) and there exists a vector in \(E_{A-v}(\lambda)\) that is not orthogonal to \(\xi_v\). It is clear that the matrix \(A^\perp\) can be represented in the form

\[
\begin{pmatrix} A(v, v) & \eta_v^\top \\ \xi_v & (A - v)^\top \end{pmatrix}.
\]

So, the vector \(\eta_v\) plays the same role for \(A^\perp\) as \(\xi_v\) plays for the original matrix \(A\). In particular, since the equalities \(g_A(\lambda) = g_{A-v}(\lambda) - 1\) and \(g_A^\top(\lambda) = g_{(A-v)^\top}(\lambda) - 1\) are equivalent
to each other, we can also state that there exists a vector in $E_{(A - v)\top}(\lambda)$ that is not orthogonal to $\eta_v$.

Suppose that $g_A(\lambda) = g_{A - v}(\lambda)$. If $\xi_v \perp E_{A - v}(\lambda)$, then $E_A(\lambda)$ coincides with the $v$-extension of $E_{A - v}(\lambda)$. In the opposite case there exists a vector $\eta^*$ such that the vector whose $v$th entry is equal to $-1$ and whose restriction to $V(A - v)$ coincides with $\eta^*$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$. From the equation

$$
\begin{pmatrix}
(A(v, v) & \xi_v^\top \\
\eta_v & A - v
\end{pmatrix}
\begin{pmatrix}
-1 \\
n_{v}
\end{pmatrix}
= \begin{pmatrix}
-(A(v, v) + (\xi_v, \eta^*)) \\
-\eta_v + (A - v)\eta^*
\end{pmatrix} = \lambda \begin{pmatrix}
-1 \\
n_{v}
\end{pmatrix}
$$

(2)

it follows that $((A - v) - \lambda I)\eta^* = \eta_v$. In particular, for any $\eta \in E_{(A - v)\top}(\lambda)$, we have

$$
(\eta, \eta_v) = (\eta, ((A - v) - \lambda I)\eta^*) = \left(\left((A - v)^\top - \lambda I\right)\eta, \eta^*\right) = 0.
$$

Thus, $\eta_v \perp E_{(A - v)\top}(\lambda)$ and therefore, $E_{A\top}(\lambda)$ coincides with the $v$-extension of $E_{(A - v)\top}(\lambda)$.

Assume now that $g_A(\lambda) = g_{A - v}(\lambda) + 1$. It is clear that for any eigenvector of $A$ for $\lambda$ whose $v$th entry is zero, its restriction to $V(A - v)$ belongs to $E_{A - v}(\lambda)$. This implies that if any eigenvector of $A$ for $\lambda$ has zero $v$th entry, then $E_A(\lambda)$ is a subspace of the $v$-extension of $E_{A - v}(\lambda)$. But this is impossible because of our condition on the geometric multiplicities. Thus, there exists a vector in $E_A(\lambda)$ whose $v$th entry is not equal to zero and there exist exactly $g_A(\lambda)$ linearly independent eigenvectors of $A$ for $\lambda$ whose $v$th entries are zero. The restrictions of these eigenvectors to $V(A - v)$ are eigenvectors of $A - v$ for $\lambda$ and therefore the linear hull of them coincides with $E_{A - v}(\lambda)$. This implies that the $v$-extension of $E_{A - v}(\lambda)$ is a subspace of $E_A(\lambda)$. In particular, $\xi_v \perp E_{A - v}(\lambda)$ and similarly, $\eta_v \perp E_{(A - v)\top}(\lambda)$.

From (2) it follows that the vector whose $v$th entry is equal to $-1$ and whose restriction to $V(A - v)$ coincides with $\eta^*$ is an eigenvector of $A$ associated with $\lambda$ iff $\eta^*$ is a solution of the system:

$$
\begin{cases}
-A(v, v) + (\xi_v, \eta^*) = -\lambda, \\
(A - v)\eta^* - \lambda\eta^* = \eta_v.
\end{cases}
$$

Since $\xi_v \perp E_{A - v}(\lambda)$, the vector $\eta^* + \xi$ is also a solution of this system for any $\xi \in E_{A - v}(\lambda)$. Thus, any solution of the equation $((A - v) - \lambda I)\eta^* = \eta_v$ can be taken as a solution of the above system. The lemma is proved. □

**Remark 1.** Lemma 1 also implies the observation (i) on page 1009 in [13]: If $A$ and $A - v$ have some eigenvalue $\lambda$ in common, then either $A$ or $A^\top$ admits a nonzero eigenvector with zero $v$th entry associated with $\lambda$.

By definition, if $(A - \lambda I)^m\xi = 0$ for some $m \in \mathbb{N}$ and $\xi \neq 0$, then $\xi$ is a **generalized eigenvector** corresponding to the eigenvalue $\lambda$. All such vectors taken together with 0 form the **generalized eigenspace** $L_A(\lambda)$ of $A$ for $\lambda$. Its dimension $n_A(\lambda)$ is called the **algebraic multiplicity** of the eigenvalue $\lambda$ of $A$ and coincides with the multiplicity of the complex number $\lambda$ as a root of the characteristic polynomial $\det(A - zI)$. It is clear that $E_A(\lambda)$ is a subspace of $L_A(\lambda)$ and therefore $n_A(\lambda) \geq g_A(\lambda)$. If $n_A(\lambda) = g_A(\lambda)$, then $\lambda$ is called a **semi-simple eigenvalue**. It is interesting to compare the condition $g_A(\lambda) = g_{A - v}(\lambda) + 1$ with the condition $n_A(\lambda) = n_{A - v}(\lambda) + 1$ when $\lambda$ is not a semi-simple eigenvalue of $A$ and $A - v$. Substituting $A - zI$ for $A$ in formula (0.8.5) [19], we obtain

$$
\det(A - zI) = \left(A(v, v) - z - (\xi_v, ((A - v) - zI)^{-1}\eta_v)\right) \det((A - v) - zI).
$$
From this identity it follows that the equality $n_A(\lambda) = n_{A-v}(\lambda) + 1$ takes place iff the function $z - A(v, v) + ((A - v) - z I)^{-1} \eta_v$ is analytic at the point $z = \lambda$ and this point is one of its zeros. Almost the same function really appears in the case of $g_A(\lambda) = g_{A-v}(\lambda) + 1$ (see statement (3) of Lemma 1). Thus, though the two conditions on the multiplicities are not equivalent to each other, the corresponding expressions are similar.

3. Construction of the eigenspace of $A$ for $\lambda$ from that of $A - v$: the case of a normal matrix

In this section, we consider normal matrices. There are many (equivalent) definitions of a normal matrix. The reader will find them in [20,21]. Throughout the paper, we use only the following: $A$ is normal iff $E_A^\top(\lambda) = \overline{E_A(\lambda)}$ for any eigenvalue $\lambda$ of $A$ (see condition (12) in [20]). Note that this equality implies that $\lambda$ is a semi-simple eigenvalue of $A$. Indeed, in the opposite case there exists a nonzero eigenvector $\xi$ of $A$ associated with $\lambda$ and a vector $\xi' \neq 0$. This contradiction shows that $\lambda$ must be a semi-simple eigenvalue of $A$.

In a similar way, one can show that $L_{(A-v)\top}(\lambda)^{-1}$ coincides with the direct sum of the generalized eigenspaces of $A - v$ associated with the eigenvalues of $A - v$ different from $\lambda$. It is also well known that the restriction $((A - v) - \lambda I)|_1$ of the operator $(A - v) - \lambda I$ to this direct sum is invertible. Denote its inverse by $((A - v) - \lambda I)^{-1}_1$. Then, for a normal matrix $A$, the statement of Lemma 1 takes the following form:

**Theorem 1.** Let $A$ be a normal matrix and $\lambda$ be an eigenvalue of $A - v$. Then

1. $g_A(\lambda) = g_{A-v}(\lambda) - 1 \iff E_A(\lambda)$ is a proper subspace of the $v$-extension of $E_{A-v}(\lambda)$;
2. $g_A(\lambda) = g_{A-v}(\lambda) \iff E_A(\lambda)$ is the $v$-extension of $E_{A-v}(\lambda)$;
3. $g_A(\lambda) = g_{A-v}(\lambda) + 1 \iff \eta_v \perp E_{(A-v)\top}(\lambda) = \overline{E_{A-v}(\lambda)}$ and

$$E_A(\lambda) = \begin{pmatrix} 0 \\ E_{A-v}(\lambda) \end{pmatrix} \oplus \begin{pmatrix} -1 \\ ((A - v) - \lambda I)^{-1}_1 \eta_v \end{pmatrix}.$$ 

**Proof.** Assume that $g_A(\lambda) = g_{A-v}(\lambda)$. Then, by statement (2) of Lemma 1, either $E_A(\lambda)$ is the $v$-extension of $E_{A-v}(\lambda)$ or $E_A^\top(\lambda)$ is the $v$-extension of $E_{(A-v)\top}(\lambda)$. Since $A$ is normal, $E_A^\top(\lambda) = \overline{E_A(\lambda)}$ and therefore in any case every eigenvector of $A$ has zero $v$th entry. Under the condition on the geometric multiplicities, this is possible if and only if $E_A(\lambda)$ coincides with the $v$-extension of $E_{A-v}(\lambda)$.

Consider now the case of $g_A(\lambda) = g_{A-v}(\lambda) + 1$. Let us take some element $\xi^*$ in $((A - v)^\top - \lambda I)^{-1}_1 \xi_v$ and some element $\eta^*$ in $((A - v) - \lambda I)^{-1}_1 \eta_v$. By statement (3) of Lemma 1 applied to both $A$ and $A^\top$, the equality $E_{A^\top}(\lambda) = \overline{E_A(\lambda)}$ can be rewritten in the form

$$\begin{pmatrix} E_{(A-v)^\top}(\lambda) \\ (\xi^*) \end{pmatrix} + \begin{pmatrix} -1 \\ E_{A-v}(\lambda) \end{pmatrix} + \begin{pmatrix} -1 \\ \eta^* \end{pmatrix}.$$ 

Let $\eta$ be any vector in $E_{(A-v)^\top}(\lambda)$. By (3), we have
\( \left( \begin{array}{c} 0 \\ \eta \end{array} \right) = \left( \begin{array}{c} 0 \\ \xi \end{array} \right) + \alpha \left( \begin{array}{c} -1 \\ \eta^v \end{array} \right) \)

for some \( \xi \in E_{A-v}(\lambda) \) and \( \alpha \in \mathbb{C} \). It is clear that \( \alpha = 0 \). In this case \( \eta = \bar{\xi} \) and therefore \( E_{(A-v)^\top} \subseteq E_{A-v}(\lambda) \). We recall that \( \dim E_{(A-v)^\top}(\lambda) = \dim E_{A-v}(\lambda) \). Thus, \( E_{(A-v)^\top}(\lambda) = E_{A-v}(\lambda) \). In particular, \( \lambda \) is a semi-simple eigenvalue of \( A - v \).

By statement (3) of Lemma 1, \( \eta_v \perp E_{(A-v)^\top}(\lambda) \) and therefore \( \eta_v \) belongs to the direct sum of the generalized eigenspaces of \( A - v \) associated with the eigenvalues of \( A - v \) different from \( \lambda \). In this case, the vector \( ((A - v) - \lambda I)^{-1} \eta_v \) is well defined.

Since \( \mathbb{C}^{n-1} = E_{A-v}(\lambda) \oplus E_{(A-v)^\top}(\lambda)^\perp \), this vector is \( \mathbb{C} \)-orthogonal to \( E_{A-v}(\lambda) \) and therefore the eigenvector of \( A \) whose \( v \)th entry is \( -1 \) and whose restriction to \( V(A - v) \) coincides with \( ((A - v) - \lambda I)^{-1} \eta_v \) is \( \mathbb{C} \)-orthogonal to the \( v \)-extension of \( E_{A-v}(\lambda) \). The theorem is proved. \( \square \)

The proof of Theorem 1 implies the following statement which will be very useful in the sequel.

**Corollary 1.** Let \( A \) be a normal matrix and \( \lambda \) be an eigenvalue of \( A - v \). Assume that \( g_A(\lambda) \geq g_{A-v}(\lambda) \). Then \( E_{(A-v)^\top}(\lambda) = E_{A-v}(\lambda) \).

It is easy to see that the matrix obtained from the \( n \times n \) permutation matrix

\[
C = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]  

by deleting the first column and row is the Jordan matrix of size \( n - 1 \). So, a principal submatrix of co-order one of a normal matrix can be non-diagonalizable. The proposition formulated below is devoted to this case and directly follows from Corollary 1.

**Proposition 1.** Let \( A \) be a normal matrix and \( \lambda \) be an eigenvalue of \( A - v \). Assume that the eigenvalue \( \lambda \) is not semi-simple. Then

\[
g_A(\lambda) = g_{A-v}(\lambda) - 1.
\]

By Lemma 3 [22], if the eigenvalue \( \lambda \) of the original matrix \( A \) is semi-simple, then the eigenvalue \( \lambda \) of \( A - v \) has at most one nontrivial Jordan block in the corresponding part of the Jordan form of \( A - v \). Let \( q \) be the size of the largest Jordan block associated with the eigenvalue \( \lambda \) of \( A - v \). Assume that \( q > 1 \) and consider any basis \( \xi_1, \xi_2, \ldots, \xi_{q-1}, \xi_q \) of the eigenspace \( E_{A-v}(\lambda) \) such that the eigenvector \( \xi_q \) belongs to a Jordan chain of length \( q \).

Suppose that \( (\xi_k, \xi_v) = 0 \). In this case the \( v \)-extension of \( \xi_k \) is an eigenvector of \( A \) for \( \lambda \). Since \( A \) is a normal matrix, we have \( E_{A^\top}(\lambda) = \overline{E_A(\lambda)} \). This means that the \( v \)-extension of \( \bar{\xi}_k \) is an eigenvector of \( A^\top \) for \( \lambda \) and therefore \( \bar{\xi}_k \) is an eigenvector of \( (A - v)^\top \) for \( \lambda \). Since \( q > 1 \), there exists a vector \( \xi_k^\prime \) such that \( ((A - v) - \lambda I)\xi_k^\prime = \xi_k \) and therefore

\[
(\xi_k, \bar{\xi}_k) = \left( ((A - v) - \lambda I)\xi_k^\prime, \xi_k \right) = (\xi_k^\prime, (A - v)^\top - \lambda I)\xi_k = 0.
\]

This contradiction to the evident inequality \( (\xi_k, \bar{\xi}_k) > 0 \) shows that \( (\xi_k, \xi_v) \neq 0 \). So, we can define

\[
\alpha_i = (\xi_i, \xi_v) / (\xi_k, \xi_v)
\]
for every $i = 1, \ldots, k - 1$. It is clear that the eigenvectors $\xi_1 - \alpha_1 \xi_k, \ldots, \xi_{k-1} - \alpha_{k-1} \xi_k$ are linearly independent and orthogonal to $\xi_v$. Thus, $E_A(\lambda)$ coincides with the $v$-extension of the linear subspace that spans these eigenvectors.

**Remark 2.** From Proposition 1 it follows that if $A$ is normal and some eigenvalue $\lambda$ of $A - v$ has a nontrivial Jordan block, then this Jordan block completely disappears from the part of the Jordan form associated with $\lambda$ under the transition from $A - v$ to $A$. In [16] it was shown that this change of the Jordan form is typical in the following sense: it holds under adding a generic column and row to an arbitrary matrix whose eigenvalue $\lambda$ has exactly one Jordan block of size strictly greater than one.

Let $A$ be an arbitrary matrix. It is natural to ask how close to a normal matrix the matrix $A$ is. Almost all known measures of nonnormality of matrices use the absolute values of the eigenvalues (see [23]). But from our point of view, it is also natural to consider “discrete” measures which concern the eigenspaces of $A$ and $A^\top$.

We say that $A$ is $k$-normal if $E_{A^\top}(\lambda) = E_A(\lambda)$ for any eigenvalue $\lambda$ with $g_A(\lambda) \geq k$. It is clear that $A$ is 1-normal iff $A$ is normal, and every simple matrix is 2-normal. By statements (i) and (iii) of Lemma 1 [24], every tournament matrix is 2-normal. The following statement yields another nontrivial example of a 2-normal matrix.

**Proposition 2.** Let $A$ be a circulant matrix. Then for any index $v$ of $A$, the submatrix $A - v$ is 2-normal.

**Proof.** By definition, $A$ admits the representation

$$A = \sum_{k=0}^{n-1} a_{k+1} C^k$$

for some numbers $a_1, \ldots, a_n$, where $C$ is the unitary matrix introduced above (see (4)). It is clear that the matrix $A$ is normal. Obviously, we can assume that $V(A) = V(C) = \{1, \ldots, n\}$. It is not difficult to check that $C^{k-1}$ is also a permutation matrix for any $k = 1, \ldots, n$, and the only 1 in the first row belongs to the $k$th column. Let $C^{k-1}_{lk}$ be the matrix obtained from $C^{k-1}$ by deleting the first row and $k$th column. It is easy to see that $C^{k-1}_{lk}$ is a permutation matrix. From the equality $AC^{k-1} = C^{k-1} A$ it follows that $(A - 1)C^{k-1}_{lk} = C^{k-1}_{lk}(A - k)$. In particular, this means that any two principal submatrices of co-order one of $A$ are permutationally similar to each other.

It is well known (see formula (1.2.13) in [19]) that

$$-\frac{\partial}{\partial z} \det(A - zI) = \sum_{v \in V(A)} \det((A - v) - zI).$$

Since in our case all the principal submatrices of co-order one have the same characteristic polynomial, the identity

$$\det((A - v) - zI) = -|V(A)|^{-1} \frac{\partial}{\partial z} \det(A - zI)$$

holds for any $v \in V(A)$. If $g_{A-v}(\lambda) \geq 2$, then $\lambda$ is also an eigenvalue of the original matrix $A$. From identity (5) it follows that $n_A(\lambda) = n_{A-v}(\lambda) + 1$ (see [22, Lemma 1]). By Proposition 1, this equality is possible if $\lambda$ is a semi-simple eigenvalue of $A - v$. Thus, $g_A(\lambda) = g_{A-v}(\lambda) + 1$. By Corollary 1, in this case we have $E_{(A-v)^\tau}(\lambda) = E_{A-v}(\lambda)$. The proposition is proved. □
Remark 3. By definition, a \((0, 1)\)-matrix \(A\) is the adjacency matrix of a vertex-transitive digraph, if for any two indices \(v\) and \(w\) of \(A\), there is a permutation matrix \(P\) such that \(AP = PA\) and \(P(v, w) = 1\). From the proof of Proposition 2 it follows that its statement also holds for the adjacency matrix \(A\) of a vertex-transitive digraph if \(A\) is normal. In particular, the adjacency matrix of a Cayley digraph on an abelian group satisfies these conditions (see [25]). Recall that any \((0, 1)\)-circulant matrix is the adjacency matrix of a Cayley digraph on some cyclic group.

Below, in Appendix to our paper, we shall show that principal submatrices of co-order one of circulant matrices naturally appear in different problems concerning tournaments with extremal spectral properties.

Appendix A. A look at almost regular tournament matrices as principal submatrices of co-order one of regular tournament matrices

By definition, a tournament matrix \(T\) is a \((0, 1)\)-matrix that satisfies the equation
\[
T + T^\top = J - I,
\]
where \(J\) and \(I\) are the all ones and identity matrices, respectively. In other words, tournament matrices are the adjacency matrices of tournaments, which form a well-studied class of digraphs. Denote by \(|T|\) the order of \(T\). If every row sum coincides with the corresponding column sum (and therefore is equal to \((|T| - 1)/2\)), then the tournament matrix \(T\) is regular (we note that in this case \(|T|\) is necessarily odd). In other words, \(T\) is regular if
\[
TJ = JT.
\]
From (A.1) and (A.2) it follows that a regular tournament matrix is normal. Moreover, its spectrum contains the Perron root \((|T| - 1)/2\), and its other eigenvalues lie on the line \(\text{Re } z = -1/2\) (see [26]).

By Levinger’s inequality (see [27]), for any tournament matrix \(T\) we have
\[
\rho(T) \leq \rho\left(\frac{T + T^\top}{2}\right) = \rho\left(\frac{J - I}{2}\right) = \frac{|T| - 1}{2},
\]
where \(\rho(T)\) is the spectral radius of \(T\). So, \((|T| - 1)/2\) is the maximum Perron root over all tournament matrices of odd order \(|T|\) and this maximal value is attained iff \(T\) is a regular tournament matrix (see [26]). For even \(|T|\), it is not known which tournament matrices \(T\) maximize the Perron root, but it is shown in [28] that for sufficiently large \(|T|\) the maximizers must be almost regular. By definition, a tournament matrix \(T\) of even order is almost regular if half of its row sums are \(|T|/2\) and the other half are \((|T| - 2)/2\). The following lemma whose proof is trivial yields an explicit connection between almost regular and regular tournament matrices.

Lemma A.1. Every almost regular tournament matrix is a principal submatrix of co-order one of a regular tournament matrix.

In [18] it was conjectured that the maximum of the Perron root over the class of tournament matrices of even order \(2n\) is attained for the Brualdi–Li matrix \(B_{2n}\), which has the form
\[
\begin{pmatrix}
U_n & U_n^\top \\
U_n^\top + I_n & U_n
\end{pmatrix},
\]
where \(I_n\) is the identity matrix of order \(n\) and \(U_n\) is the \(n \times n\) strictly upper triangular matrix with ones in all entries above the main diagonal. It is not difficult to check that
The Brualdi–Li matrix $B_{2n}$ is permutationally similar to a principal submatrix of co-order one of the circulant matrix $C_{2n+1}$.

It is not difficult to show that the numbers $e^{2\pi i k/(2n+1)}$, where $k = 1, \ldots, 2n + 1$, are the eigenvalues of the circulant matrix $C$ of order $2n + 1$ with the first row $0, 1, 0, \ldots, 0$ (see (4) in the previous section). This means that the spectrum of the circulant matrix $A = \sum_{k=0}^{2n} a_{k+1} C^k$ with the first row $a_1, \ldots, a_{2n+1}$ can be represented in the form

$$\text{Spec}(A) = \left\{ f \left( e^{2\pi i k/(2n+1)} \right) : k = 1, \ldots, 2n + 1 \right\},$$

where $f(z) = \sum_{k=0}^{2n} a_{k+1} z^k$. It is clear that $C_{2n+1} = \sum_{k=1}^{n} C^k$ and therefore $f(z) = \frac{z^{n+1} - 1}{z - 1} - 1$ in our case. So, for any $k = 1, \ldots, 2n$, the complex number

$$\frac{1}{1 + (-1)^k e^{\pi i k/(2n+1)}} - 1$$

is an eigenvalue of $C_{2n+1}$. It is clear that all these numbers considered together with the Perron root $n$ are distinct. Thus, all the eigenvalues of $C_{2n+1}$ are simple.

Let $v$ be any index of $C_{2n+1}$ and $C_{2n+1} - v$ be the matrix obtained from $C_{2n+1}$ by deleting the $v$th column and row. Suppose that the geometric multiplicity of some eigenvalue $\lambda$ of $C_{2n+1} - v$ is strictly greater than one. Then $\lambda$ is also an eigenvalue of $C_{2n+1}$ and therefore identity (5) in the previous section implies that

$$n_{C_{2n+1}}(\lambda) = n_{C_{2n+1} - v}(\lambda) + 1 \geq g_{C_{2n+1} - v}(\lambda) + 1 \geq 3.$$

But this inequality contradicts the fact that any eigenvalue of $C_{2n+1}$ is simple. Thus, the geometric multiplicity of every eigenvalue of $C_{2n+1} - v$ is equal to one. In other words, $C_{2n+1} - v$ is a simple matrix. In [29] a stronger result is proved. It states that all the eigenvalues of the Brualdi–Li matrix $B_{2n}$ are simple. Of course, this result would follow directly from the fact that $C_{2n+1} - v$ is a simple matrix if we could also show that the matrix $C_{2n+1} - v$ is diagonalizable. In connection with this simple observation, the following general question arises:
Which of circulant matrices have nondiagonalizable principal submatrices of co-order one?

Above we have given one example of such a matrix. It is the circulant matrix with the first row 0, 1, 0, . . ., 0 which has already been considered in Section 3. It is possible that there are no other irreducible circulant matrices whose principal submatrices of co-order one are nondiagonalizable but, at the moment, we cannot either prove or disprove this conjecture.

The above question can be also addressed to the adjacency matrices of Cayley digraphs on arbitrary abelian groups (recall that according to [25], they are normal). By definition, the Cayley digraph $D$ on a group $G$ with respect to a subset $S$ of its distinct elements different from the identity element $e$ is the directed graph with vertex-set $G$, and with $(g, h)$ an arc in $D$ if $hg^{-1} \in S$. In particular, the circulant matrix $C_{2n+1}$ is the adjacency matrix of the Cayley digraph on the cyclic group $\langle g \rangle$ of odd order $2n+1$ with respect to $S = \{g, \ldots, g^n\}$. The Cayley digraph $D$ on $G$ with respect to $S$ is a regular tournament iff $S \cap S^{-1} = \emptyset$ and $S \cup S^{-1} \cup e = G$. In this case we shall say that $D$ is a Cayley tournament. It is clear that the circulant matrix $C_{2n+1}$ is the adjacency matrix of a Cayley tournament. So, we can formulate the weak Brualdi–Li conjecture as the restriction of the original Brualdi–Li conjecture to principal submatrices of co-order one of Cayley tournament matrices:

$$\rho(C_{2n+1} - v) = \max\{\rho(T - v): T \text{ is a Cayley tournament matrix of order } 2n+1\}.$$ 

We also believe that the Cayley digraph on the cyclic group $\langle g \rangle$ of order $m$ with respect to $S = \{g, \ldots, g^k\}$ provides a solution to the analogous problem for the adjacency matrices of connected regular digraphs of degree $k$ on $m$ vertices.

For any tournament matrix $T$, let us introduce the quantity

$$\nu(T) = \frac{(T_1 | T|, T_1 | T|) - (|T| - 1)^2}{4}.$$ 

In [30] it was shown that the Perron root of $T$ belongs to the closed interval

$$\left[(|T| - 2)/4 + \sqrt{|T|^2 - 16\nu(T)/4}, (|T| - 1)/2\right].$$ 

If $T$ is an almost regular tournament matrix, then

$$\nu(T) = (|T| - 2)^2/8 + |T|^2/8 - (|T| - 1)^2/4 = 1/4.$$ 

So, we have the inequality

$$\rho(T) \geq (n - 1)/2 + \sqrt{(n^2 - 1)/4} \quad (A.4)$$

for every almost regular tournament matrix $T$ of order $2n$. From the results of [30] it follows that equality holds in (A.4) iff the matrix $T$ has exactly $|T| - 2$ eigenvalues with real part equal to $-1/2$. Below we shall give some important examples of such matrices.

For a tournament matrix $T$ of order $n$, we let

$$M_T = \begin{pmatrix} T & T^\top \\ T^\top + I & T \end{pmatrix}.$$ 

It is not difficult to check that $M_T$ is an almost regular tournament matrix of order $2n$. In [31] it was shown that if $T$ is a regular tournament matrix of odd order $n$, then $\det(zI - M_T) = \det(z^2I - (2z + 1)T)$ (see the proof of Theorem 4.1 therein). In particular, $z$ is an eigenvalue of $M_T$ if and only if $z^2 = (2z + 1)\lambda$ for some eigenvalue $\lambda$ of $T$. It is clear that the Perron
root \( \rho(M_T) \) is the biggest solution to the equation \( z^2 = (2z + 1)(n - 1)/2 \) and therefore equals \((n - 1)/2 + \sqrt{(n^2 - 1)/4} \). So, we have

**Proposition A.2.** For any regular tournament matrix \( T \) of order \( n \), the matrix \( M_T \) has the minimal possible Perron root among all almost regular tournament matrices of order \( 2n \).

Is it true that if \( n \) is an odd number, then every minimizer of order \( 2n \) has the form \( M_T \) for some regular tournament matrix \( T \)? In order to answer this question, let us consider Hadamard tournaments (see [32]). By definition, a tournament matrix \( H_{2n+1} \) of order \( 2n+1 \) is a Hadamard tournament matrix if

\[
H_{2n+1} H_{2n+1}^\top = (n + 1)/2I + (n - 1)/2J.
\]  
(A.5)

From equality (A.5) it follows that the number \( n \) is odd and therefore the order of every Hadamard tournament is congruent to \( 3 \) (mod \( 4 \)). The existence of a Hadamard tournament of any such order is a difficult unsolved problem since Hadamard tournament matrices of order \( 2n+1 \) are coexistent with skew Hadamard matrices of order \( 2n+2 \) (see [33]). It is well known (see [34]) that for any prime \( p \) congruent to \( 3 \) (mod \( 4 \)) and an odd positive integer \( m \), the Paley tournament of order \( pm \) is an example of a Hadamard tournament. Really it is believed that for any positive integer congruent to \( 3 \) (mod \( 4 \)), there exists a Hadamard tournament of the corresponding order. But up to now this statement is only a conjecture.

From equality (A.5) it follows that the sum of squared entries in every row of \( H_{2n+1} \) is equal to \( n \). Since \( H_{2n+1} \) is a \((0,1)\)-matrix, the sum of the entries themselves is also equal to \( n \). This means that \( n \) is the Perron root of \( H_{2n+1} \) and \( 1_{2n+1} \) is the associated eigenvector. In particular, every Hadamard tournament is regular and therefore its adjacency matrix is normal. Let \( \lambda \) be any eigenvalue of \( H_{2n+1} \) different from the Perron root \( n \) and \( \xi \) be the corresponding eigenvector. The equality \( H_{2n+1} + H_{2n+1}^\top = J - I \) allows us to rewrite (A.5) as

\[
H_{2n+1}^2 + H_{2n+1} + (n + 1)/2I - (n + 1)/2J = 0.
\]

Since \( J\xi = 0 \), the eigenvalue \( \lambda \) satisfies the equation

\[
\lambda^2 + \lambda + (n + 1)/2 = 0.
\]

Its roots are \(-1/2 + i\sqrt{2n + 1}/2\) and \(-1/2 - i\sqrt{2n + 1}/2\). So, the spectrum of \( H_{2n+1} \) can be represented in the following form (see [35]):

- multiplicity: \( n \)
- eigenvalues: \(-1/2 - i\sqrt{2n + 1}/2\) \quad \(-1/2 + i\sqrt{2n + 1}/2\) \quad \( n \)

Let \( v \) be any index of \( H_{2n+1} \) and \( H_{2n+1} - v \) be the matrix obtained from \( H_{2n+1} \) by deleting the \( v \)th row and column. Then the algebraic multiplicity of \(-1/2 + i\sqrt{2n + 1}/2\) and \(-1/2 - i\sqrt{2n + 1}/2\) as eigenvalues of \( H_{2n+1} - v \) is equal to \( n - 1 \) (in the opposite case the order of \( H_{2n+1} - v \) must be strictly greater than \( 2n \)). As we have noticed above, the existence of \( 2n - 2 \) eigenvalues with real part equal to \(-1/2\) already implies that the almost regular tournament matrix \( H_{2n+1} - v \) is a minimizer. Nevertheless, we continue our calculation.

Denote by \( x \) the Perron root of \( H_{2n+1} - v \). Let \( y \) be the other real eigenvalue. Then the spectrum of \( H_{2n+1} - v \) can be represented as follows:

- multiplicity: \( n - 1 \)
- eigenvalues: \(-1/2 - i\sqrt{2n + 1}/2\) \quad \(-1/2 + i\sqrt{2n + 1}/2\) \quad \( y \) \quad \( x \)

Let us determine the values of \( x \) and \( y \). Since \( \text{Tr}(H_{2n+1} - v)^2 = \text{Tr}(H_{2n+1} - v) = 0 \), we have \( x + y = n - 1 \) and \( x^2 + y^2 = n(n - 1) \). This implies that \( x \) satisfies the quadratic equation
\[ x^2 - (n-1)x - (n-1)/2 = 0 \] and therefore equals \((n-1)/2 + \sqrt{(n^2-1)/4}\). This means that the tournament matrix \(H_{2n+1} - v\) is also a minimizer.

**Proposition A.3.** A principal submatrix of co-order one of a Hadamard tournament matrix of order \(2n+1\) has the minimal possible Perron root among all almost regular tournament matrices of order \(2n\).

In [31] it was proved that if a regular tournament matrix \(T\) has \(k\) distinct eigenvalues, then \(M_T\) has \(2k\) distinct eigenvalues (see the statement (ii) of Theorem 4.1 therein). Every tournament matrix has at least three distinct eigenvalues. So, the spectrum of \(M_T\) contains at least six distinct eigenvalues. However, we have seen above that \(H_{2n+1} - v\) has exactly four distinct eigenvalues and therefore it cannot be represented as \(M_T\) for some regular tournament matrix \(T\).

**References**


