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ON THE CONVERGENCE THEORY OF TRUST–REGION–BASED ALGORITHMS FOR EQUALITY–CONSTRAINED OPTIMIZATION

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Abstract. In a recent paper, Dennis, El–Alem, and Maciel proved global convergence to a stationary point for a general trust–region–based algorithm for equality–constrained optimization. This general algorithm is based on appropriate choices of trust–region subproblems and seems particularly suitable for large problems.

This paper shows global convergence to a point satisfying the second–order necessary optimality conditions for the same general trust–region–based algorithm. The results given here can be seen as a generalization of the convergence results for trust–regions methods for unconstrained optimization obtained by Moré and Sorensen. The behavior of the trust radius and the local rate of convergence are analyzed. Some interesting facts concerning the trust–region subproblem for the linearized constraints, the quasi–normal component of the step, and the hard case are presented.

It is shown how these results can be applied to a class of discretized optimal control problems.

Key words. Equality–constrained optimization, trust regions, SQP methods, second–order necessary optimality conditions, local rate of convergence, hard case

AMS subject classification. 49M37, 90C30

1. Introduction.

Trust–region algorithms have been proved to be efficient and robust techniques to solve unconstrained optimization problems. An excellent survey in this area is Moré [22]. Other classical references for convergence results are Carter [3], Moré and Sorensen [23], Powell [26], and Shultz, Schnabel, and Byrd [29]. The standard techniques to handle the trust–region subproblems are the dogleg algorithm (Powell [25]), conjugate gradients (Steihaug [32] and Toint [33]), and Newton–like methods for the computation of locally constrained optimal steps (Gay [15], Moré and Sorensen [23], and Sorensen [30]). See also the book of Dennis and Schnabel [9]. Recent new algorithms to compute a locally constrained optimal step (in other words a step that satisfies a fraction of optimal decrease on the trust–region subproblem) that are very promising for large problems have been proposed by Rendl and Wolkowicz [28] and Sorensen [31].

Since the mid eighties a significant effort has been made to address the equality–constrained optimization problem. References are Celis, Dennis, and Tapia [4], Vardi [34] (see also El–Hallabi [14]), Byrd, Schnabel, and Shultz [2], Powell and Yuan [27], and El–Alem [13]. The fundamental questions associated with the application of trust–region algorithms to equality–constrained optimization are the decomposition of the step, the choice of the trust–region subproblems, and the choice of the merit function. During the first stages of the research conducted in this area it was not clear how to answer these questions properly. However, if we examine carefully the most recent

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references (Byrd and Omojokon [24], Dennis, El–Alem, and Maciel [7], El–Alem [12], [13], and Lalee, Nocedal, and Plantenga [21]) we can observe the same decomposition of the step (in its normal, or quasi–normal, and tangential components) and the same trust–region subproblems (the trust–region subproblem for the linearized constraints and the trust–region subproblem for the Lagrangian reduced to the tangent subspace). This is explained in great detail in Section 2 of this paper. As in the unconstrained case, the conditions that each component has to satisfy and the way they are computed might of course differ from algorithm to algorithm. We can see also in these most recent references that the merit function used is either the ℓ_2 penalty function without constraint term squared (cases of [21], [24]) or the augmented Lagrangian function (in [7], [12], [13]).

Consider now the equality–constrained optimization (ECO) problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & C(x) = 0, \end{array} \tag{1.1}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $c_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, \ldots, m$, $C(x) = \Big(c_1(x) \cdots c_m(x)\Big)^T$, and $m < n$. The functions f and c_i, $i = 1, \ldots, m$, are assumed to be at least twice continuously differentiable in the domain of interest.

In [7], Dennis, El–Alem, and Maciel have considered a general trust–region–based algorithm for the solution of the ECO problem (1.1). This general algorithm is very much like the algorithm proposed by Byrd and Omojokon [24][1]. As mentioned before, each step s is decomposed as $s^{\mathsf{n}} + s^{\mathsf{t}}$, where s^{n} is the quasi–normal component of the step, associated with trust–region subproblem for the linearized constraints and s^{t} is the tangential component, associated with the trust–region subproblem for the Lagrangian reduced to the tangent subspace. Each component of the step is only required to satisfy a fraction of Cauchy decrease on the corresponding trust–region subproblem. Another key feature of this general algorithm is the choice of the augmented Lagrangian as a merit function and the use of the El–Alem's scheme [11] to update the penalty parameter. Under appropriate assumptions, it can be shown that there exists a subsequence of iterates driving to zero the norm of the residual of the constraints and the norm of the gradient of the Lagrangian reduced to the tangent subspace (see [7][Section 8]). It is important to remark that this global convergence result is obtained under very mild conditions on the components of the step, on the multipliers estimates, and on the Hessian approximations. Thus, the Dennis, El–Alem, and Maciel [7] result is similar to the global result given by Powell [26] for unconstrained optimization.

One of the purposes of this paper is to show global convergence to a point satisfying the second–order necessary optimality conditions for this class of algorithms. Our result is similar to the results established by Moré and Sorensen [23], [30] for trust–region algorithms for unconstrained optimization. We accomplish this here by imposing a fraction of optimal decrease on the tangential component s^{t} of the step, by using exact second–order information, and by imposing conditions on the quasi–normal component s^{n} and on the Lagrange multipliers.

[1] The Thesis [24] was directed by Professor R. H. Byrd. The trust–region algorithm proposed here is usually referred as the Byrd and Omojokon algorithm.

In [2], Byrd, Schnabel, and Shultz have proposed a trust–region algorithm for equality–constrained optimization and established global convergence to a point satisfying the second–order necessary optimality conditions. However this algorithm does not belong to the class of trust–region algorithms considered here and their result is obtained with the use of the (exact) normal component and the least–squares multipliers update which we do not require in this paper. Other differences are that they use the ℓ_1 penalty function as the merit function and the analysis is carried out by using an orthogonal null–space basis. In recent papers, Coleman and Yuan [6] and El–Alem [12] have proposed trust–region algorithms for which they prove global convergence to points satisfying first–order and second–order necessary optimality conditions. Their algorithms use the (exact) normal component, an orthogonal null–space basis, and the least–squares multipliers update.

The conditions we need to impose to assure that a limit point of the sequence of iterates satisfies the second–order necessary optimality conditions are

$$\nabla_x \ell(x_k, \lambda_k)^T s_k^{\mathsf{n}} = \mathcal{O}(\delta_k \|C(x_k)\|) \text{ and } \|\Delta\lambda_k\| = \|\lambda_{k+1} - \lambda_k\| = \mathcal{O}(\delta_k),$$

where $\ell(x, \lambda) = f(x) + \lambda^T C(x)$, s_k^{n} is the quasi–normal component of the step s_k, and δ_k is the trust–region radius. In the case where $\|C(x_k)\|$ is small compared with δ_k, the first condition implies that any increase of the quadratic model of the Lagrangian from x_k to $x_k + s_k^{\mathsf{n}}$ is $\mathcal{O}(\delta_k^2)$. To see why this is relevant recall that a fraction of optimal decrease is being imposed on the tangential component s_k^{t} yielding a decrease of $\mathcal{O}(\delta_k^2)$ on the quadratic model. The second condition is needed for the same reasons because $\Delta\lambda_k$ also appears in the definition of predicted decrease. We show that both conditions are satisfied when either (i) the (exact) normal component and the least–squares multipliers are used; or (ii) the most reasonable choices of quasi–normal component and multipliers are made for a class of discretized optimal control problems. The former result is in agreement with the result obtained by El–Alem [12].

Gill, Murray, and Wright [17] and El–Alem [10] considered in their analyses that $\nabla_x \ell(x_k, \lambda_k)$ is $\mathcal{O}(\|s_k\|)$. In the latter work this assumption is used to prove local convergence results, and in the former to establish properties of an augmented Lagrangian merit function. We point out that this assumption implies that $\nabla_x \ell(x_k, \lambda_k)^T s_k^{\mathsf{n}}$ is $\mathcal{O}(\delta_k \|C(x_k)\|)$ since s_k is $\mathcal{O}(\delta_k)$ and we assume that s_k^{n} is $\mathcal{O}(\|C(x_k)\|)$.

We also prove that if the algorithm converges to a point where the reduced Hessian is positive definite, then the penalty parameter ρ_k is uniformly bounded and the trust–region radius δ_k is uniformly bounded away from zero, a desired property of a trust–region algorithm. In this case, particular choices of the multipliers and of the components s^{n} and s^{t} lead us to a q–quadratic rate of convergence in x.

A detailed treatment of the global convergence theory is given in Vicente [35].

The structure of the trust–region subproblem for the linearized constraints can be exploited to obtain some interesting results. We introduce a quasi–normal component that satisfies a fraction of optimal decrease on the trust–region subproblem for the linearized constraints. We show that the (exact) normal component shares this property. We also prove that if the algorithm is well behaved (for instance if the trust radius is uniformly bounded away from zero), then this subproblem has a natural tendency to fall into the so–called hard case.

We review the notation used in this paper. The Lagrangian function associated with the ECO problem (1.1) is defined by $\ell(x, \lambda) = f(x) + \lambda^T C(x)$, where

$\lambda \in \mathbb{R}^m$ is the Lagrange multiplier vector. The matrix $\nabla C(x)$ is given by $\nabla C(x) = \Big(\nabla c_1(x) \cdots \nabla c_m(x)\Big)$, where $\nabla c_i(x)$ represents the gradient of the function $c_i(x)$. Let $\nabla^2 f(x)$ and $\nabla^2 c_i(x)$ be the Hessian matrices of $f(x)$ and $c_i(x)$ respectively. We use subscripted indices to represent the evaluation of a function at a particular point of the sequences $\{x_k\}$ and $\{\lambda_k\}$. For instance, f_k represents $f(x_k)$ and ℓ_k is the same as $\ell(x_k, \lambda_k)$. The vector and matrix norms used are the ℓ_2 norms, and I_l represents the identity matrix of order l. Finally, $\lambda_1(A)$ denotes the smallest eigenvalue of the symmetric matrix A.

The structure of this paper is as follows. In Section 2, we introduce the trust–region subproblems and outline the general trust–region algorithm and the general assumptions. In Section 3, we present the global convergence theory. A class of discretized optimal control problems is introduced in Section 4 as a justification for the general form of our algorithms and theory. In Sections 5 and 6, we analyze respectively the behavior of the trust radius and the local rates of convergence. The trust–region subproblem for the linearized constraints is studied in Section 7. We end this paper with some summary conclusions.

2. Algorithm and general assumptions. The trust–region algorithm analyzed by Dennis, El–Alem, and Maciel [7] for the solution of the ECO problem (1.1), consists of computing, at each iteration k, a step s_k decomposed as $s_k = s_k^{\mathsf{n}} + s_k^{\mathsf{t}}$, where the components s_k^{n} and s_k^{t} are required to satisfy given conditions. If the step s_k is accepted, the algorithm continues by setting x_{k+1} to $x_k + s_k$. If the step is rejected then $x_{k+1} = x_k$.

2.1. Decomposition of the step. Suppose that $\|C_k\| \neq 0$. The component s_k^{n} is called the quasi–normal (or quasi–vertical) component of s_k and is required to satisfy a fraction of Cauchy decrease on the trust–region subproblem for the linearized constraints defined by

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|\nabla C_k^T s^{\mathsf{n}} + C_k\|^2 \\ \text{subject to} & \|s^{\mathsf{n}}\| \le r\delta_k, \end{array}$$

where $r \in (0,1)$ and δ_k is the trust radius. In other words, s_k^{n} has to satisfy

$$\|C_k\|^2 - \|\nabla C_k^T s_k^{\mathsf{n}} + C_k\|^2 \ge \sigma^{\mathsf{n}} \left(\|C_k\|^2 - \|\nabla C_k^T\, c_k^{\mathsf{n}} + C_k\|^2\right), \tag{2.1}$$

where $\sigma^{\mathsf{n}} > 0$ and c_k^{n} is the so–called Cauchy point for this trust–region subproblem, i.e. c_k^{n} is the optimal solution of

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|\nabla C_k^T c^{\mathsf{n}} + C_k\|^2 \\ \text{subject to} & c^{\mathsf{n}} \in span\{-\nabla C_k C_k\}, \\ & \|c^{\mathsf{n}}\| \le r\delta_k, \end{array}$$

and therefore

$$c_k^{\mathsf{n}} = \begin{cases} -\frac{\|\nabla C_k C_k\|^2}{\|\nabla C_k^T \nabla C_k C_k\|^2}\nabla C_k C_k & \text{if } \frac{\|\nabla C_k C_k\|^3}{\|\nabla C_k^T \nabla C_k C_k\|^2} \le r\delta_k, \\ -\frac{r\delta_k}{\|\nabla C_k C_k\|}\nabla C_k C_k & \text{otherwise.} \end{cases}$$

The component s_k^{n} also is required to satisfy the condition

$$\|s_k^{\mathsf{n}}\| \le \kappa_1 \|C_k\|, \tag{2.2}$$

where κ_1 is a positive constant independent of the iterate k of the algorithm. This condition is saying that close to feasibility the quasi–normal component has to be small.

In this paper, we require the quasi–normal component s_k^{n} also to satisfy

$$\nabla_x \ell_k^T s_k^{\mathsf{n}} \le \kappa_2 \|C_k\| \delta_k, \tag{2.3}$$

where κ_2 is a positive constant independent of the iterates. The important consequence of this condition is that if $\|C_k\|$ is small compared with δ_k, then any increase of the quadratic model of the Lagrangian along the quasi–normal component s_k^{n} is of $\mathcal{O}(\delta_k^2)$.

The two choices of s_k^{n} given in Sections 4.1 and 4.2 satisfy conditions (2.1), (2.2), and (2.3). Other choices have been suggested in [7], [20].

The component s_k^{t} is the tangential (or horizontal) component, and it must satisfy

$$\nabla C_k^T s_k^{\mathsf{t}} = 0,$$

i.e. it must lie in the null space $\mathcal{N}(\nabla C_k^T)$ of ∇C_k^T. Let W_k be an $n \times (n-m)$ matrix whose columns form a basis for $\mathcal{N}(\nabla C_k^T)$. Let also

$$q_k(s) = \ell_k + \nabla_x {\ell_k}^T s + \frac{1}{2} s^T H_k s$$

be a quadratic model of ℓ at (x_k, λ_k), where H_k is an approximation to $\nabla_{xx}^2 \ell(x_k, \lambda_k)$.

Since for any $s^{\mathsf{t}} \in \mathcal{N}(\nabla C_k^T)$, there exists a $\bar{s}^{\mathsf{t}} \in \mathbb{R}^{n-m}$ such that $s^{\mathsf{t}} = W_k \bar{s}^{\mathsf{t}}$, we consider also $\bar{q}_k^{\mathsf{t}}(\bar{s}^{\mathsf{t}})$ which is given by

$$\bar{q}_k^{\mathsf{t}}(\bar{s}^{\mathsf{t}}) = q_k(s_k^{\mathsf{n}} + W_k \bar{s}^{\mathsf{t}}) = q_k(s_k^{\mathsf{n}}) + \bar{g}_k^T \bar{s}^{\mathsf{t}} + \frac{1}{2} (\bar{s}^{\mathsf{t}})^T \bar{H}_k (\bar{s}^{\mathsf{t}})$$

with $\bar{H}_k = W_k^T H_k W_k$, $\bar{g}_k = W_k^T \nabla q_k(s_k^{\mathsf{n}})$ and $q_k(s_k^{\mathsf{n}}) = \ell_k + \nabla_x {\ell_k}^T s_k^{\mathsf{n}} + \frac{1}{2}(s_k^{\mathsf{n}})^T H_k (s_k^{\mathsf{n}})$.

If $\|\bar{g}_k\| \ne 0$, $\bar{s}_k^{\mathsf{t}}$ is required to satisfy a fraction of Cauchy decrease for the trust–region subproblem

$$\begin{array}{ll} \text{minimize} & \bar{q}_k^{\mathsf{t}}(\bar{s}^{\mathsf{t}}) \\ \text{subject to} & \|s_k^{\mathsf{n}} + W_k \bar{s}^{\mathsf{t}}\| \le \delta_k. \end{array}$$

Note that this is not a standard trust–region subproblem because s_k^{n} might not be orthogonal to $\mathcal{N}(\nabla C_k^T)$ and hence $\bar{s}^{\mathsf{t}} = 0$ might not be the center of the trust region. The steepest–descent direction at $\bar{s}^{\mathsf{t}} = 0$ associated with $\bar{q}_k^{\mathsf{t}}(\bar{s}^{\mathsf{t}})$ in the ℓ_2 norm is $-\bar{g}_k$. If we take into account the scaling matrix W_k, then the steepest–descent direction in the $\|W_k \cdot\|$ norm is given by $-(W_k^T W_k)^{-1} \bar{g}_k$. We consider the steepest–descent direction $-\bar{g}_k$ for $\bar{q}_k^{\mathsf{t}}(\bar{s}^{\mathsf{t}})$ on $\{\bar{s}^{\mathsf{t}} \in \mathbb{R}^{n-m} : \|s_k^{\mathsf{n}} + W_k \bar{s}^{\mathsf{t}}\| \le \delta_k\}$ and require $\bar{s}_k^{\mathsf{t}}$ to satisfy

$$q_k(s_k^{\mathsf{n}}) - q_k(s_k^{\mathsf{n}} + W_k \bar{s}_k^{\mathsf{t}}) \ge \bar{\sigma}^{\mathsf{t}} \left(q_k(s_k^{\mathsf{n}}) - q_k(s_k^{\mathsf{n}} + W_k \bar{c}_k^{\mathsf{t}}) \right), \tag{2.4}$$

where $\bar{\sigma}^{\mathsf{t}} > 0$, and $\bar{c}_k^{\mathsf{t}}$ is the Cauchy point for the ℓ_2 norm given by

$$\bar{c}_k^{\mathsf{t}} = \begin{cases} -\frac{\|\bar{g}_k\|^2}{\bar{g}_k^T \bar{H}_k \bar{g}_k} \bar{g}_k & \text{if } \frac{\|\bar{g}_k\|^2 \|W_k \bar{g}_k\|}{\bar{g}_k^T \bar{H}_k \bar{g}_k} \le \bar{\delta}_k \text{ and } \bar{g}_k^T \bar{H}_k \bar{g}_k > 0, \\ -\frac{\bar{\delta}_k}{\|W_k \bar{g}_k\|} \bar{g}_k & \text{otherwise,} \end{cases}$$

with $\bar{\delta}_k = \| -\tau_{max} W_k \bar{g}_k\|$ and

$$\tau_{max} = argmax\{\tau : \ \|s_k^{\mathsf{n}} - \tau W_k \bar{g}_k\| \leq \delta_k\}.$$

This is equivalent to saying that τ_{max} is the maximum step length along $s_k^{\mathsf{n}} - \tau W_k \bar{g}_k$ allowed inside the trust region defined by δ_k. It is easy to verify that

$$\bar{\delta}_k \in \Big((1-r)\delta_k, (1+r)\delta_k \Big).$$

The results given in this paper hold also if $\bar{c}_k^{\mathsf{t}}$ is defined along $-(W_k^T W_k)^{-1}\bar{g}_k$ provided the sequence $\{\|(W_k^T W_k)^{-1}\|\}$ is bounded. They are valid also if the coupled trust–region constraint $\|s_k^{\mathsf{n}} + W_k \bar{s}^{\mathsf{t}}\| \leq \delta_k$ is decoupled as $\|\bar{s}^{\mathsf{t}}\| \leq \delta_k$. In this latter case the parameter r defining the quasi–normal component s_k^{n} can have any positive value.

A step $\bar{s}_k^{\mathsf{t}}$ that satisfies this requirement can be computed by using Powell's dogleg algorithm [25] or by the conjugate–gradient algorithm adapted for trust regions by Steihaug [32] and Toint [33] (see also [7], [8], [21]).

In order to establish global convergence to a point satisfying the second–order necessary optimality conditions, we need $\bar{s}_k^{\mathsf{t}}$ to satisfy a fraction of optimal decrease on the following trust–region subproblem

$$\begin{aligned} &\text{minimize} \quad \bar{q}_k^{\mathsf{t}}(\bar{s}^{\mathsf{t}}) \\ &\text{subject to} \quad \|W_k \bar{s}^{\mathsf{t}}\| \leq \tilde{\delta}_k, \end{aligned} \tag{2.5}$$

where

$$\tilde{\delta}_k = \begin{cases} \bar{\delta}_k & \text{if } \|\bar{g}_k\| \neq 0 \\ (1-r)\delta_k & \text{otherwise.} \end{cases}$$

In other words, we require $\bar{s}_k^{\mathsf{t}}$ to satisfy the following conditions:

$$\begin{aligned} &\bar{q}_k^{\mathsf{t}}(0) - \bar{q}_k^{\mathsf{t}}(\bar{s}_k^{\mathsf{t}}) \geq \beta_1^{\mathsf{t}} \left(\bar{q}_k^{\mathsf{t}}(0) - \bar{q}_k^{\mathsf{t}}(\bar{s}_k^{*}) \right), \\ &\|W_k \bar{s}_k^{\mathsf{t}}\| \leq \beta_2^{\mathsf{t}} \tilde{\delta}_k, \end{aligned} \tag{2.6}$$

where β_1^{t}, $\beta_2^{\mathsf{t}} > 0$, and $\bar{s}_k^{*}$ is the optimal solution of the trust–region subproblem (2.5). This can be accomplished by applying the GQTPAR routine of Moré and Sorensen [23] or by using the algorithms recently proposed by Rendl and Wolkowicz [28] and Sorensen [31].

If $\bar{s}_k^{\mathsf{t}}$ satisfies a fraction of optimal decrease on the trust–region subproblem (2.5), then

$$\|s_k\| \leq \|s_k^{\mathsf{n}}\| + \|W_k \bar{s}_k^{\mathsf{t}}\| \leq r\delta_k + \beta_2^{\mathsf{t}} \tilde{\delta}_k \leq (r + \beta_2^{\mathsf{t}}(1+r))\delta_k.$$

If $\bar{s}_k^{\mathsf{t}}$ is only required to satisfy a fraction of Cauchy decrease, then $\|s_k\| = \|s_k^{\mathsf{n}} + W_k \bar{s}_k^{\mathsf{t}}\| \leq \delta_k$. We can combine both cases and write

$$\|s_k\| = \|s_k^{\mathsf{n}} + W_k \bar{s}_k^{\mathsf{t}}\| \leq \kappa_0 \delta_k, \tag{2.7}$$

where $\kappa_0 = \max\{r + \beta_2^{\mathsf{t}}(1+r), 1\}$.

It is also important to note that the definition of $\tilde{\delta}_k$ assures that the fraction of optimal decrease (2.6) implies the fraction of Cauchy decrease (2.4) provided $\beta_2^{\mathsf{t}} \geq 1$.

2.2. General trust–region algorithm. We introduce now the merit function and the corresponding actual and predicted decreases. The merit function used is the augmented Lagrangian

$$L(x,\lambda;\rho) = f(x) + \lambda^T C(x) + \rho C(x)^T C(x),$$

where ρ is the penalty parameter. The actual decrease $ared(s_k;\rho_k)$ at the iteration k is given by

$$ared(s_k;\rho_k) = L(x_k,\lambda_k;\rho_k) - L(x_{k+1},\lambda_{k+1};\rho_k).$$

The predicted decrease (see [7]) is the following:

$$pred(s_k;\rho_k) = L(x_k,\lambda_k;\rho_k) - \left(q_k(s_k) + \Delta\lambda_k^T(\nabla C_k^T s_k + C_k) + \rho_k\|\nabla C_k^T s_k + C_k\|^2\right).$$

To update the penalty parameter ρ_k we use the scheme proposed by El–Alem [11].

The Lagrange multipliers λ_k are required to satisfy

$$\|\Delta\lambda_k\| = \|\lambda_{k+1} - \lambda_k\| \le \kappa_3 \delta_k, \tag{2.8}$$

where κ_3 is a positive constant independent of k. This condition is not necessary for global convergence to a stationary point.

The general trust–region algorithm is given below.

Algorithm 2.1 (General trust–region algorithm).

1 Choose x_0, δ_0, λ_0, H_0, and W_0. Set $\rho_{-1} \ge 1$. Choose α_1, η_1, δ_{min}, δ_{max}, $\bar\rho$, and r such that $0 < \alpha_1$, $\eta_1 < 1$, $0 < \delta_{min} \le \delta_{max}$, $\bar\rho > 0$, and $r \in (0,1)$.

2 For $k = 0,1,2,\ldots$ do

2.1 If $\|C_k\| + \|W_k^T \nabla_x \ell_k\| + \gamma_k = 0$, where γ_k is given in (2.10), stop the algorithm and use x_k as a solution for the ECO problem (1.1).

2.2 Set $s_k^{\mathsf{n}} = s_k^{\mathsf{t}} = 0$.
If $\|C_k\| \ne 0$ then compute s_k^{n} satisfying (2.1), (2.2), (2.3), and $\|s_k^{\mathsf{n}}\| \le r\delta_k$.
If $\|W_k^T \nabla_x \ell_k\| + \gamma_k \ne 0$ then compute $\bar s_k^{\mathsf{t}}$ satisfying (2.6).
Set $s_k = s_k^{\mathsf{n}} + s_k^{\mathsf{t}} = s_k^{\mathsf{n}} + W_k \bar s_k^{\mathsf{t}}$.

2.3 Compute λ_{k+1} satisfying (2.8).

2.4 Compute $pred(s_k;\rho_{k-1})$:

$$q_k(0) - q_k(s_k) - \Delta\lambda_k^T(\nabla C_k^T s_k + C_k) + \rho_{k-1}\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right).$$

If $pred(s_k;\rho_{k-1}) \ge \frac{\rho_{k-1}}{2}\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right)$ then set $\rho_k = \rho_{k-1}$.
Otherwise set

$$\rho_k = 2\left(\frac{q_k(s_k) - q_k(0) + \Delta\lambda_k^T(\nabla C_k^T s_k + C_k)}{\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2}\right) + \bar\rho.$$

2.5 If $\frac{ared(s_k;\rho_k)}{pred(s_k;\rho_k)} < \eta_1$, set $\delta_{k+1} = \alpha_1\|s_k\|$ and reject s_k.
Otherwise accept s_k and choose δ_{k+1} such that

$$\max\{\delta_{min}, \delta_k\} \le \delta_{k+1} \le \delta_{max}.$$

2.6 If s_k was rejected set $x_{k+1} = x_k$ and $\lambda_{k+1} = \lambda_k$. Otherwise set $x_{k+1} = x_k + s_k$ and $\lambda_{k+1} = \lambda_k + \Delta\lambda_k$.

It is important to understand that the role of δ_{min} is just to reset δ_k after a step s_k has been accepted. During the course of finding such a step the trust radius can be decreased below δ_{min}. To our knowledge Zhang, Kim, and Lasdon [37] were the first to suggest this modification. We remark that the rules to update the trust radius in the previous algorithm can be much more complicated but those given suffice to prove convergence results and to understand the trust–region mechanism.

As a direct consequence of the way the penalty parameter is updated, we have the following result.

LEMMA 2.1. *The sequence* $\{\rho_k\}$ *satisfies*

$$\rho_k \geq \rho_{k-1} \geq 1 \quad and$$

$$pred(s_k;\rho_k) \geq \frac{\rho_k}{2}\Big(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\Big). \tag{2.9}$$

In order to establish global convergence results, we use the general assumptions given in [7]. These are Assumptions A.1–A.4. However for global convergence to a point that satisfies the second–order necessary optimality conditions, we also need Assumption A.5. We assume that for all iterations k, x_k and $x_k + s_k$ are in Ω, where Ω is an open subset of $\mathbb{R}^n$.

General assumptions

A.1 The functions f, c_i, $i = 1,\ldots,m$, are twice continuously differentiable in Ω.

A.2 The gradient matrix $\nabla C(x)$ has full column rank for all $x \in \Omega$.

A.3 The functions f, ∇f, $\nabla^2 f$, C, ∇C, $\nabla^2 c_i$, $i = 1,\ldots,m$, are bounded in Ω. The matrix $(\nabla C(x)^T \nabla C(x))^{-1}$ is uniformly bounded in Ω.

A.4 The sequences $\{W_k\}$, $\{H_k\}$, and $\{\lambda_k\}$ are bounded.

A.5 The Hessian approximation H_k is exact, i.e. $H_k = \nabla^2_{xx}\ell_k$, and $\nabla^2 f$ and $\nabla^2 c_i$, $i = 1,\ldots,m$, are Lipschitz continuous in Ω.

Assumptions A.3 and A.4 are equivalent to the existence of positive constants $\nu_0,\ldots,\nu_9$ such that $|f(x)| \leq \nu_0$, $\|\nabla f(x)\| \leq \nu_1$, $\|\nabla^2 f(x)\| \leq \nu_2$, $\|C(x)\| \leq \nu_3$, $\|\nabla C(x)\| \leq \nu_4$, $\|(\nabla C(x)^T \nabla C(x))^{-1}\| \leq \nu_5$, $\|\nabla^2 c_i(x)\| \leq \nu_6$, $i = 1,\ldots,m$, for all $x \in \Omega$, and $\|W_k\| \leq \nu_7$, $\|H_k\| \leq \nu_8$, and $\|\lambda_k\| \leq \nu_9$ for all k.

2.3. Predicted decrease along the tangential component. Consider again the trust–region subproblem (2.5). We can use the general assumptions and the structure of this subproblem to obtain a lower bound on the predicted decrease $q_k(s_k^{\mathsf{n}}) - q_k(s_k^{\mathsf{n}} + s_k^{\mathsf{t}})$ along the tangential component of the step.

It follows from the Karush–Kuhn–Tucker conditions that there exists a $\gamma_k \geq 0$ such that

$$\begin{aligned} &\bar{H}_k + \gamma_k W_k^T W_k \text{ is positive semi–definite,} \\ &\Big(\bar{H}_k + \gamma_k W_k^T W_k\Big)\bar{s}_k^* = -\bar{g}_k, \text{ and} \\ &\gamma_k\Big(\tilde{\delta}_k - \|W_k \bar{s}_k^*\|\Big) = 0. \end{aligned} \tag{2.10}$$

(It turns out that these conditions are also sufficient for $\bar{s}_k^*$ to solve the trust–region subproblem (2.5), see Gay [15] and Sorensen [30].) As a consequence of this we can write

$$\bar{q}_k^{\mathsf{t}}(0) - \bar{q}_k^{\mathsf{t}}(\bar{s}_k^*) = \frac{1}{2}\left(\|R_k \bar{s}_k^*\|^2 + \gamma_k \tilde{\delta}_k^2\right) \geq \frac{1}{2}\gamma_k \tilde{\delta}_k^2,$$

where $\bar{H}_k + \gamma_k W_k^T W_k = R_k^T R_k$. Hence we have

$$q_k(s_k^{\mathsf{n}}) - q_k(s_k^{\mathsf{n}} + s_k^{\mathsf{t}}) = \bar{q}_k^{\mathsf{t}}(0) - \bar{q}_k^{\mathsf{t}}(\bar{s}_k^{\mathsf{t}}) \geq \beta_1^{\mathsf{t}}\left(\bar{q}_k^{\mathsf{t}}(0) - \bar{q}_k^{\mathsf{t}}(\bar{s}_k^{*})\right) \geq \tfrac{1}{2}\beta_1^{\mathsf{t}}(1-r)^2\gamma_k\delta_k^2. \tag{2.11}$$

3. Global convergence. Dennis, El–Alem, and Maciel [7] have proved under Assumptions A.1–A.4 and conditions (2.1), (2.2), and (2.4) that

$$\liminf_{k\to+\infty}\left(\|W_k^T\nabla_x\ell_k\| + \|C_k\|\right) = 0. \tag{3.1}$$

In this section we assume that $\bar{s}_k^{\mathsf{t}}$ satisfies the fraction of optimal decrease (2.6) on the trust–region subproblem (2.5), as well as conditions (2.3), (2.8), and A.5 on s_k^{n}, λ_k, and H_k respectively, and show that (3.1) can be extended to

$$\liminf k \to +\infty \left(\|W_k^T\nabla_x\ell_k\| + \|C_k\| + \gamma_k\right) = 0. \tag{3.2}$$

The proof of (3.2) although simpler has the same structure as the proof given in [7].

We prove the result by contradiction, under the supposition that

$$\|W_k^T\nabla_x\ell_k\| + \|C_k\| + \gamma_k > \epsilon_{tol} \tag{3.3}$$

for all k. We start by analyzing the fraction of Cauchy and optimal decrease conditions.

LEMMA 3.1. *Let the general assumptions hold. Then*

$$\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2 \geq \kappa_4\|C_k\|\min\{\kappa_5\|C_k\|, r\delta_k\} \tag{3.4}$$

and

$$q_k(s_k^{\mathsf{n}}) - q_k(s_k) \geq \kappa_6\|\bar{g}_k\|\min\{\kappa_7\|\bar{g}_k\|, \kappa_8\delta_k\}, \tag{3.5}$$

and, moreover, since $\bar{s}_k^{\mathsf{t}}$ satisfies a fraction of optimal decrease for the trust–region subproblem (2.5),

$$q_k(s_k^{\mathsf{n}}) - q_k(s_k) \geq \kappa_9\gamma_k\delta_k^2, \tag{3.6}$$

where $\kappa_4, \ldots, \kappa_9$ are positive constants independent of the iterate k.

Proof. The conditions (3.4) and (3.5) are an application of Powell's result (see [26, Theorem 4], [22, Lemma 4.8]) followed by the general assumptions. The condition (3.6) is a restatement of (2.11) with $\kappa_9 = \frac{1}{2}\beta_1^{\mathsf{t}}(1-r)^2$. □

The following inequality is needed in the forthcoming lemmas.

LEMMA 3.2. *If the general assumptions hold, then*

$$q_k(0) - q_k(s_k^{\mathsf{n}}) - \Delta\lambda_k^T(\nabla C_k^T s_k + C_k) \geq -\kappa_{10}\|C_k\|\delta_k, \tag{3.7}$$

where κ_{10} is a positive constant independent of k.

Proof. The term $q_k(0) - q_k(s_k^{\mathsf{n}})$ can be bounded using (2.2), (2.3), and Assumption A.4, in the following way:

$$\begin{aligned} q_k(0) - q_k(s_k^{\mathsf{n}}) &= -\nabla_x\ell_k^T s_k^{\mathsf{n}} - \tfrac{1}{2}(s_k^{\mathsf{n}})^T H_k(s_k^{\mathsf{n}}) \\ &\geq -\kappa_2\|C_k\|\delta_k - \tfrac{1}{2}\|H_k\|\,\|s_k^{\mathsf{n}}\|^2 \\ &\geq -\kappa_2\|C_k\|\delta_k - \tfrac{1}{2}\nu_8 r\kappa_1\|C_k\|\delta_k. \end{aligned}$$

On the other hand, it follows from (2.8) and $\|\nabla C_k^T s_k + C_k\| \leq \|C_k\|$ that

$$-\Delta\lambda_k^T(\nabla C_k^T s_k + C_k) \geq -\kappa_3\|C_k\|\delta_k$$

If we combine these two bounds we get (3.7) with $\kappa_{10} = \kappa_2 + \frac{1}{2}\nu_8 r\kappa_1 + \kappa_3$. □

The following technical lemma gives us upper bounds on the difference between the actual decrease and the predicted decrease. The proof follows similar arguments as the proof of Lemma 6.3 in [11].

Lemma 3.3. *Let the general assumptions hold. There exist positive constants* $\bar{\kappa}_1,\ldots,\bar{\kappa}_7$ *independent of* k*, such that*

$$(3.8)\qquad \begin{aligned} |ared(s_k;\rho_k) - pred(s_k;\rho_k)| \;\leq\;& \bar{\kappa}_1\|s_k\|^3 + \bar{\kappa}_2\|\Delta\lambda_k\|\,\|s_k\|^2 \;+ \\ & \rho_k\Big(\bar{\kappa}_3\|s_k\|^3 + \bar{\kappa}_4\|C_k\|\,\|s_k\|^2\Big) \end{aligned}$$

and

$$(3.9)\qquad \begin{aligned} |ared(s_k;\rho_k) - pred(s_k;\rho_k)| \;\leq\;& \bar{\kappa}_5\|\Delta\lambda_k\|\,\|s_k\|^2 \;+ \\ & \rho_k\Big(\bar{\kappa}_6\|s_k\|^3 + \bar{\kappa}_7\|C_k\|\,\|s_k\|^2\Big). \end{aligned}$$

Proof. If we add and subtract $\ell(x_{k+1},\lambda_k)$ to $ared(s_k;\rho_k) - pred(s_k;\rho_k)$ and expand $\ell(\cdot,\lambda_k)$ around x_k we get

$$\begin{aligned} ared(s_k;\rho_k) - pred(s_k;\rho_k) \;=\;& \tfrac{1}{2}s_k^T\Big(H_k - \nabla_{xx}^2\ell(x_k + \pi_k^1 s_k,\lambda_k)\Big)s_k \\ & + \Delta\lambda_k^T(-C_{k+1} + C_k + \nabla C_k^T s_k) \\ & - \rho_k\Big(\|C_{k+1}\|^2 - \|\nabla C_k^T s_k + C_k\|^2\Big) \end{aligned}$$

for some $\pi_k^1 \in (0,1)$. Again using the Taylor expansion we can write

$$\begin{aligned} ared(s_k;\rho_k) - pred(s_k;\rho_k) \;=\;& \tfrac{1}{2}s_k^T\Big(H_k - \nabla_{xx}^2\ell(x_k + \pi_k^1 s_k,\lambda_k)\Big)s_k \\ & - \tfrac{1}{2}\textstyle\sum_{i=1}^m(\Delta\lambda_k)_i s_k^T\nabla^2 c_i(x_k + \pi_k^2 s_k)s_k \\ & - \rho_k\Big(\textstyle\sum_{i=1}^m c_i(x_k + \pi_k^3 s_k)(s_k)^T\nabla^2 c_i(x_k + \pi_k^3 s_k)(s_k) \\ & + (s_k)^T\nabla C(x_k + \pi_k^3 s_k)\nabla C(x_k + \pi_k^3 s_k)^T(s_k) \\ & - (s_k)^T\nabla C(x_k)\nabla C(x_k)^T(s_k)\Big)\,, \end{aligned}$$

where $\pi_k^2,\ \pi_k^3 \in (0,1)$. Now we expand $c_i(x_k + \pi_k^3 s_k)$ around $c_i(x_k)$. This and the general assumptions give us the estimate (3.8) for some positive constants $\bar{\kappa}_1,\ldots,\bar{\kappa}_4$.

The inequality (3.9) follows from (3.8) and $\rho_k \geq 1$. □

The following three lemmas bound the predicted decrease. They correspond respectively to Lemmas 7.6, 7.7, and 7.8 given in [7].

Lemma 3.4. *Let the general assumptions hold. Then the predicted decrease in the merit function satisfies*

$$(3.10)\qquad \begin{aligned} pred(s_k;\rho) \;\geq\;& \kappa_6\|\bar{g}_k\|\min\{\kappa_7\|\bar{g}_k\|,\kappa_8\delta_k\} - \kappa_{10}\|C_k\|\delta_k \\ & +\rho\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|\right)^2, \end{aligned}$$

and also

$$(3.11)\quad pred(s_k;\rho) \;\geq\; \kappa_9\gamma_k\delta_k^2 - \kappa_{10}\|C_k\|\delta_k + \;\rho\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|\right)^2,$$

for any $\rho > 0$*.*

Proof. The two conditions (3.10) and (3.11) follow from a direct application of (3.7) and from the two different lower bounds (3.5) and (3.6) on $q_k(s_k^{\mathrm{n}}) - q_k(s_k)$. ∎

LEMMA 3.5. *Let the general assumptions hold, and assume that* $\|W_k^T\nabla_x\ell_k\| + \|C_k\| + \gamma_k > \epsilon_{tol}$*. If* $\|C_k\| \leq \alpha\delta_k$*, where* α *satisfies*

$$(3.12)\quad \alpha \;\leq\; \min\left\{\frac{\epsilon_{tol}}{3\delta_{max}}, \frac{\epsilon_{tol}}{6\nu_7\nu_8\kappa_1\delta_{max}}, \frac{\kappa_6\epsilon_{tol}}{12\kappa_{10}\delta_{max}}\min\left\{\frac{\kappa_7\epsilon_{tol}}{6\delta_{max}}, \kappa_8\right\}, \frac{\kappa_9\epsilon_{tol}}{6\kappa_{10}}\right\},$$

then the predicted decrease in the merit function satisfies either

$$(3.13)\; pred(s_k;\rho) \;\geq\; \frac{\kappa_6}{2}\|\bar{g}_k\|\min\{\kappa_7\|\bar{g}_k\|, \kappa_8\delta_k\} + \rho\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right)$$

or

$$(3.14)\qquad pred(s_k;\rho) \;\geq\; \tfrac{\kappa_9}{2}\gamma_k\delta_k^2 + \rho\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right),$$

for any $\rho > 0$*.*

Proof. From $\|W_k^T\nabla_x\ell_k\| + \|C_k\| + \gamma_k > \epsilon_{tol}$ and the first bound on α given by (3.12), we get

$$\|W_k^T\nabla_x\ell_k\| + \gamma_k > \frac{2}{3}\epsilon_{tol}.$$

Thus either $\|W_k^T\nabla_x\ell_k\| > \frac{1}{3}\epsilon_{tol}$ or $\gamma_k > \frac{1}{3}\epsilon_{tol}$. Let us first assume that $\|W_k^T\nabla_x\ell_k\| > \frac{1}{3}\epsilon_{tol}$. Then it follows from the second bound on α given by (3.12) that

$$\begin{aligned}
\|\bar{g}_k\| &= \|W_k^T\nabla_x\ell_k + W_k^T H_k s_k^{\mathrm{n}}\| \\
&\geq \|W_k^T\nabla_x\ell_k\| - \|W_k^T H_k s_k^{\mathrm{n}}\| \\
&\geq \tfrac{1}{3}\epsilon_{tol} - \nu_7\nu_8\kappa_1\|C_k\| \\
&\geq \tfrac{1}{6}\epsilon_{tol}.
\end{aligned}$$

Using this, (3.10), $\delta_k \leq \delta_{max}$, and the third bound on α given by (3.12), we obtain

$$\begin{aligned}
pred(s_k;\rho) &\geq \tfrac{\kappa_6}{2}\|\bar{g}_k\|\min\{\kappa_7\|\bar{g}_k\|, \kappa_8\delta_k\} + \tfrac{\kappa_6\epsilon_{tol}}{12}\min\{\tfrac{\kappa_7\epsilon_{tol}}{6}, \kappa_8\delta_k\} \\
&\quad - \kappa_{10}\delta_{max}\|C_k\| + \rho\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right) \\
&\geq \tfrac{\kappa_6}{2}\|\bar{g}_k\|\min\{\kappa_7\|\bar{g}_k\|, \kappa_8\delta_k\} + \rho\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right).
\end{aligned}$$

Now suppose that $\gamma_k > \frac{1}{3}\epsilon_{tol}$. To establish (3.14), we combine (3.11) and the last bound on α given by (3.12) and get

$$\begin{aligned}
pred(s_k;\rho) &\geq \tfrac{\kappa_9}{2}\gamma_k\delta_k^2 + \left(\tfrac{\kappa_9}{6}\epsilon_{tol}\delta_k - \kappa_{10}\|C_k\|\right)\delta_k + \rho\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right) \\
&\geq \tfrac{\kappa_9}{2}\gamma_k\delta_k^2 + \rho\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right).
\end{aligned}$$

∎

We can set ρ to ρ_{k-1} in Lemma 3.5 and conclude that, if $\|W_k^T \nabla_x \ell_k\| + \|C_k\| + \gamma_k > \epsilon_{tol}$ and $\|C_k\| \leq \alpha\delta_k$, then the penalty parameter at the current iterate does not need to be increased. See Step 2.4 of Algorithm 2.1.

The proof of the next lemma follows the argument given in the proof of Lemma 3.5 to show that either $\|\bar{g}_k\| > \frac{1}{6}\epsilon_{tol}$ or $\gamma_k > \frac{1}{3}\epsilon_{tol}$ holds.

LEMMA 3.6. *Let the general assumptions hold, and assume that* $\|W_k^T \nabla_x \ell_k\| + \|C_k\| + \gamma_k > \epsilon_{tol}$. *If* $\|C_k\| \leq \alpha\delta_k$, *where* α *satisfies (3.12), then there exists a constant* $\kappa_{11} > 0$ *such that*

$$pred(s_k;\rho_k) \geq \kappa_{11}\delta_k^2. \tag{3.15}$$

Proof. By Lemma 3.5 we know that either (3.13) or (3.14) holds. Now we set $\rho = \rho_k$. In the first case we use $\|\bar{g}_k\| > \frac{1}{6}\epsilon_{tol}$ and get

$$\begin{aligned} pred(s_k;\rho_k) &\geq \tfrac{\kappa_6\epsilon_{tol}}{12}\min\{\tfrac{\kappa_7\epsilon_{tol}}{6}, \kappa_8\delta_k\} \\ &\geq \tfrac{\kappa_6\epsilon_{tol}}{12}\min\{\tfrac{\kappa_7\epsilon_{tol}}{6\delta_{max}}, \kappa_8\}\delta_k \\ &\geq \tfrac{\kappa_6\epsilon_{tol}}{12\delta_{max}}\min\{\tfrac{\kappa_7\epsilon_{tol}}{6\delta_{max}}, \kappa_8\}\delta_k^2. \end{aligned}$$

In the second case we use $\gamma_k > \frac{1}{3}\epsilon_{tol}$, to obtain

$$pred(s_k;\rho_k) \geq \frac{\kappa_9\epsilon_{tol}}{6}\delta_k^2.$$

Hence (3.15) holds with

$$\kappa_{11} = \min\left\{\frac{\kappa_6\epsilon_{tol}}{12\delta_{max}}\min\left\{\frac{\kappa_7\epsilon_{tol}}{6\delta_{max}}, \kappa_8\right\}, \frac{\kappa_9\epsilon_{tol}}{6}\right\}.$$

□

Next we prove under the supposition (3.3), that the penalty parameter ρ_k is bounded.

LEMMA 3.7. *Let the general assumptions hold. If* $\|W_k^T \nabla_x \ell_k\| + \|C_k\| + \gamma_k > \epsilon_{tol}$ *for all* k, *then*

$$\rho_k \leq \rho_*,$$

where ρ_* *does not depend on* k, *and thus* $\{\rho_k\}$ *and* $\{L_k\}$ *are bounded sequences.*

Proof. If ρ_k is increased at iteration k, then it is updated according to the rule

$$\rho_k = 2\left(\frac{q_k(s_k) - q_k(0) + \Delta\lambda_k^T(\nabla C_k^T s_k + C_k)}{\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2}\right) + \bar{\rho}.$$

We can write

$$\begin{aligned} \tfrac{\rho_k}{2}\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right) &= \nabla_x \ell(x_k,\lambda_k)^T s_k^{\mathsf{n}} + \tfrac{1}{2}(s_k^{\mathsf{n}})^T H_k (s_k^{\mathsf{n}}) \\ &\quad -(q_k(s_k^{\mathsf{n}}) - q_k(s_k)) + \Delta\lambda_k^T(\nabla C_k^T s_k + C_k) \\ &\quad +\tfrac{\bar{\rho}}{2}\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right). \end{aligned}$$

By applying (3.4) to the left hand side and (3.5) and (3.7) to the right hand side, we obtain

$$\begin{aligned}\tfrac{\rho_k}{2}\kappa_4\|C_k\|\min\{\kappa_5\|C_k\|, r\delta_k\} &\le \kappa_{10}\delta_k\|C_k\| + \tfrac{\bar{\rho}}{2}\Big(-2(\nabla C_k C_k)^T s_k - \|\nabla C_k^T s_k\|^2\Big)\\ &\le (\kappa_{10} + \bar{\rho}\kappa_0\nu_4)\delta_k\|C_k\|.\end{aligned}$$

If ρ_k is increased at iteration k, then from Lemma 3.5 we certainly know that $\|C_k\| > \alpha\delta_k$, where α satisfies (3.12). Now we use this fact to establish that

$$\left(\frac{\kappa_4}{2}\min\{\kappa_5\alpha, r\}\right)\rho_k \le \kappa_{10} + \bar{\rho}\kappa_0\nu_4.$$

We have proved that $\{\rho_k\}$ is bounded. From this and the general assumptions we conclude that $\{L_k\}$ is also bounded. ∎

We can prove also under the supposition (3.3), that the trust radius is bounded away from zero.

LEMMA 3.8. *Let the general assumptions hold. If* $\|W_k^T\nabla_x\ell_k\| + \|C_k\| + \gamma_k > \epsilon_{tol}$ *for all* k, *then*

$$\delta_k \ge \delta_* > 0,$$

where δ_* *does not depend on* k.

Proof. If s_{k-1} was an acceptable step, then $\delta_k \ge \delta_{min}$. If not then $\delta_k = \alpha_1\|s_{k-1}\|$, and we consider the cases $\|C_{k-1}\| \le \alpha\delta_{k-1}$ and $\|C_{k-1}\| > \alpha\delta_{k-1}$, where α satisfies (3.12). In both cases we use the fact

$$1 - \eta_1 \le \left|\frac{ared(s_{k-1};\rho_{k-1})}{pred(s_{k-1};\rho_{k-1})} - 1\right|.$$

Case I. $\|C_{k-1}\| \le \alpha\delta_{k-1}$. From Lemma 3.6, inequality (3.15) holds for $k = k-1$. Thus we can use $\|s_{k-1}\| \le \kappa_0\delta_{k-1}$, (2.8) and (3.9) with $k = k-1$, to obtain

$$\left|\frac{ared(s_{k-1};\rho_{k-1})}{pred(s_{k-1};\rho_{k-1})} - 1\right| \le \frac{(\bar{\kappa}_5\kappa_0\kappa_3\delta_{k-1}^2 + \rho_*\bar{\kappa}_6\kappa_0^2\delta_{k-1}^2 + \rho_*\bar{\kappa}_7\alpha\kappa_0\delta_{k-1}^2)\|s_{k-1}\|}{\kappa_{11}\delta_{k-1}^2}.$$

Thus $\delta_k = \alpha_1\|s_{k-1}\| \ge \frac{\alpha_1(1-\eta_1)\kappa_{11}}{\bar{\kappa}_5\kappa_0\kappa_3 + \rho_*\bar{\kappa}_6\kappa_0^2 + \rho_*\bar{\kappa}_7\alpha\kappa_0} \equiv \kappa_{12}$.

Case II. $\|C_{k-1}\| > \alpha\delta_{k-1}$. In this case from (2.9) and (3.4) with $k = k-1$ we get

$$\begin{aligned}pred(s_{k-1};\rho_{k-1}) &\ge \tfrac{\rho_{k-1}}{2}\kappa_4\|C_{k-1}\|\min\{\kappa_5\|C_{k-1}\|, r\delta_{k-1}\}\\ &\ge \rho_{k-1}\kappa_{13}\delta_{k-1}\|C_{k-1}\|\\ &\ge \rho_{k-1}\alpha\kappa_{13}\delta_{k-1}^2,\end{aligned}$$

where $\kappa_{13} = \frac{\kappa_4}{2}\min\{\kappa_5\alpha, r\}$. Again we use $\rho_{k-1} \ge 1$, (2.8) and (3.9) with $k = k-1$, and this time the last two lower bounds on $pred(s_{k-1};\rho_{k-1})$, and write

$$\begin{aligned}\left|\tfrac{ared(s_{k-1};\rho_{k-1})}{pred(s_{k-1};\rho_{k-1})} - 1\right| &\le \frac{\rho_{k-1}(\bar{\kappa}_5\kappa_0\kappa_3 + \bar{\kappa}_6\kappa_0^2)\delta_{k-1}^2\|s_{k-1}\|}{\rho_{k-1}\alpha\kappa_{13}\delta_{k-1}^2} + \frac{\rho_{k-1}\bar{\kappa}_7\kappa_0\delta_{k-1}\|C_{k-1}\|\,\|s_{k-1}\|}{\rho_{k-1}\kappa_{13}\delta_{k-1}\|C_{k-1}\|}\\ &\le \left(\frac{\bar{\kappa}_5\kappa_0\kappa_3 + \bar{\kappa}_6\kappa_0^2 + \bar{\kappa}_7\alpha\kappa_0}{\alpha\kappa_{13}}\right)\|s_{k-1}\|.\end{aligned}$$

Hence $\delta_k = \alpha_1\|s_{k-1}\| \ge \frac{\alpha_1(1-\eta_1)\alpha\kappa_{13}}{\bar{\kappa}_5\kappa_0\kappa_3 + \bar{\kappa}_6\kappa_0^2 + \bar{\kappa}_7\alpha\kappa_0} \equiv \kappa_{14}$.

The result follows by setting $\delta_* = \min\{\delta_{min}, \kappa_{12}, \kappa_{14}\}$. □

The next result is needed also for the forthcoming Theorem 3.1.

LEMMA 3.9. *Let the general assumptions hold. If* $\|W_k^T \nabla_x \ell_k\| + \|C_k\| + \gamma_k > \epsilon_{tol}$ *for all* k*, then an acceptable step is always found in finitely many trial steps.*

Proof. Let us prove the assertion by contradiction. Assume that for a given $\bar{k}$, $x_k = x_{\bar{k}}$ for all $k \geq \bar{k}$. This means that $\lim_{k\to+\infty} \delta_k = 0$ and all steps are rejected after iteration $\bar{k}$. See Steps 2.5 and 2.6 of Algorithm 2.1. We can consider the cases $\|C_k\| \leq \alpha\delta_k$ and $\|C_k\| > \alpha\delta_k$, where α satisfies (3.12), and appeal to arguments similar to those used in Lemma 3.8 to conclude that in any case

$$\left|\frac{ared(s_k;\rho_k)}{pred(s_k;\rho_k)} - 1\right| \leq \kappa_{15}\delta_k, \quad k \geq \bar{k}\,,$$

where κ_{15} is a positive constant independent of the iterates. Since we are assuming that $\lim_{k\to+\infty} \delta_k = 0$, we have $\lim_{k\to+\infty} \frac{ared(s_k;\rho_k)}{pred(s_k;\rho_k)} = 1$ and this contradicts the rules that update the trust radius. See Step 2.5 of Algorithm 2.1. □

Now we finally can state our first asymptotic result.

THEOREM 3.1. *Under the general assumptions, the sequence of iterates* $\{x_k\}$ *generated by the Algorithm 2.1 satisfies*

$$\liminf_{k\to+\infty} \left(\|W_k^T \nabla_x \ell_k\| + \|C_k\| + \gamma_k\right) = 0. \tag{3.16}$$

Proof. Suppose that there exists an $\epsilon_{tol} > 0$ such that $\|W_k^T \nabla_x \ell_k\| + \|C_k\| + \gamma_k > \epsilon_{tol}$ for all k.

At each iteration k either $\|C_k\| \leq \alpha\delta_k$ or $\|C_k\| > \alpha\delta_k$, where α satisfies (3.12). In the first case we appeal to Lemmas 3.6 and 3.8 and obtain

$$pred(s_k;\rho_k) \geq \kappa_{11}\delta_*^2.$$

If $\|C_k\| > \alpha\delta_k$, we have from $\rho_k \geq 1$, (2.9), (3.4), and Lemma 3.8, that

$$pred(s_k;\rho_k) \geq \frac{\kappa_4}{2}\alpha \min\{\kappa_5\alpha, r\}\delta_*^2.$$

Hence there exists a positive constant κ_{16} not depending on k such that $pred(s_k;\rho_k) \geq \kappa_{16}$. From Lemma 3.9, we can ignore the rejected steps and work only with successful iterates. So, without loss of generality, we have

$$L_k - L_{k+1} = ared(s_k;\rho_k) \geq \eta_1 pred(s_k;\rho_k) \geq \eta_1\kappa_{16}.$$

Now, if we let k go to infinity, this contradicts the boundedness of $\{L_k\}$. □

From this result we can state our global convergence result: existence of a limit point of the sequence of iterates generated by the algorithm satisfying the second–order necessary optimality conditions. This result generalizes those obtained for unconstrained optimization by Sorensen [30] and Moré and Sorensen [23].

THEOREM 3.2. *Let the general assumptions hold. Assume that* $W(x)$ *and* $\lambda(x)$ *are continuous functions and* $\lambda_k = \lambda(x_k)$ *for all* k*.*

If $\{x_k\}$ *is a bounded sequence generated by Algorithm 2.1, then there exists a limit point* x_* *such that*

- $C(x_*) = 0$,

- $W(x_*)^T \nabla f(x_*) = 0$, *and*
- $\nabla^2_{xx} \ell(x_*, \lambda(x_*))$ *is positive semi–definite on* $\mathcal{N}(\nabla C(x_*)^T)$.

Moreover, if $\lambda(x_*)$ *is such that* $\nabla_x \ell(x_*, \lambda(x_*)) = 0$ *then* x_* *satisfies the second–order necessary optimality conditions.*

Proof. Let $\{k_i\}$ be the index subsequence considered in (3.16). Since $\{x_{k_i}\}$ is bounded, it has a subsequence $\{x_{k_j}\}$ that converges to a point x_* and for which

$$\lim_{j \to +\infty} \left(\|W_{k_j}^T \nabla_x \ell_{k_j}\| + \|C_{k_j}\| + \gamma_{k_j} \right) = 0. \tag{3.17}$$

Now from this and the continuity of $C(x)$, we get $C(x_*) = 0$. Then we use the continuity of $W(x)$ and $\nabla f(x)$ to obtain

$$W(x_*)^T \nabla f(x_*) = 0.$$

Since $\lambda_1(\cdot)$ is a continuous function, we can use (2.10), $\lim_{j \to +\infty} \gamma_{k_j} = 0$, the continuity of $W(x)$, $\lambda(x)$, and of the second derivatives of $f(x)$ and $c_i(x)$, $i = 1, \ldots, m$, to obtain

$$\lambda_1 \left(W(x_*)^T \nabla^2_{xx} \ell(x_*, \lambda(x_*)) W(x_*) \right) \geq 0.$$

This shows that $\nabla^2_{xx} \ell(x_*, \lambda(x_*))$ is positive semi–definite on $\mathcal{N}(\nabla C(x_*)^T)$. □

The continuity of an orthogonal null space basis has been discussed in [1], [5], [16]. A class of nonorthogonal null space basis is described in Section 4.1.

The equation $\nabla_x \ell(x_*, \lambda(x_*)) = 0$ is satisfied for consistent updates of the Lagrange multipliers like the least–squares update (4.7) or the adjoint update (4.3).

4. Examples.

4.1. A class of discretized optimal control problems. We now introduce an important class of ECO problems where we can find convenient matrices W_k, quasi–normal components s_k^{n}, and multipliers λ_k satisfying all the requirements needed for our analysis. The numerical solution of many discretized optimal control problems involves solving the ECO problem

$$\begin{array}{ll} \text{minimize} & f(y, u) \\ \text{subject to} & C(y, u) = 0, \end{array} \tag{4.1}$$

where $y \in \mathbb{R}^m$, $u \in \mathbb{R}^{n-m}$ and $x = \begin{pmatrix} y \\ u \end{pmatrix}$ (see [8], [19], [20]). The variables in y are the state variables and the variables in u are the control variables. Other applications include parameter identification, inverse, and flow problems and design optimization. In many situations there are bounds on the control variables, but this is not considered here. Another interesting aspect of these problems is that $\nabla C(x)^T$ can be partitioned as

$$\nabla C(x)^T = \begin{pmatrix} C_y(x) & C_u(x) \end{pmatrix},$$

where $C_y(x)$ is a square matrix of order m.

In this class of problems the following assumption traditionally is made:

(4.2) The partial Jacobian $C_y(x)$ is nonsingular and its inverse is uniformly bounded in Ω.

As a consequence of this, the columns of

$$W(x) = \begin{pmatrix} -C_y(x)^{-1}C_u(x) \\ I_{n-m} \end{pmatrix}$$

form a basis for the null space of $\nabla C(x)^T$.

The usual choice for λ_k in these problems is the so–called adjoint multipliers

$$\lambda_k = -C_y(x_k)^{-T}\nabla_y f(x_k). \tag{4.3}$$

It follows directly from the continuity of $\nabla C(x)$ and the uniformly boundedness of $C_y(x)^{-1}$ that $W(x)$ varies continuously with x. Furthermore $\lambda(x) = -C_y(x)^{-T}\nabla_y f(x)$ is a continuous function of x with bounded derivatives.

Using the structure of the problem we can define the quasi–normal component s_k^{n} (see references [8], [19], [20]) as

$$s_k^{\mathsf{n}} = \begin{pmatrix} -\varsigma_k C_y(x_k)^{-1}C_k \\ 0 \end{pmatrix}, \tag{4.4}$$

where

$$\varsigma_k = \begin{cases} 1 & \text{if } \|C_y(x_k)^{-1}C_k\| \le r\delta_k, \\ \frac{r\delta_k}{\|C_y(x_k)^{-1}C_k\|} & \text{otherwise.} \end{cases}$$

As we will see in Section 7, the quasi–normal component (4.4) satisfies a fraction of optimal decrease and hence a fraction of Cauchy decrease on the trust–region subproblem for the linearized constraints.

Other choices for quasi–normal components are given in [20]. All these quasi–normal components are of the form

$$s_k^{\mathsf{n}} = \begin{pmatrix} (s_k^{\mathsf{n}})_y \\ 0 \end{pmatrix}. \tag{4.5}$$

Lemma 4.1. *If s_k^{n} verifies (4.5) and λ_k is given by (4.3), then conditions (2.3) and (2.8) are satisfied.*

Proof. From (4.3) and (4.5) we can see that

$$\nabla_x \ell_k^T s_k^{\mathsf{n}} = \begin{pmatrix} 0 \\ \nabla_u f(x_k) + C_u(x_k)^T\lambda_k \end{pmatrix}^T \begin{pmatrix} (s_k^{\mathsf{n}})_y \\ 0 \end{pmatrix} = 0$$

and condition (2.3) is trivially satisfied. Condition (2.8) follows from the existence of bounded derivatives for $\lambda(x) = -C_y(x)^{-T}\nabla_y f(x)$ in Ω. ∎

4.2. The normal component and the least–squares multipliers. Consider again the general ECO problem (1.1). If the component s_k^{n} of the step s_k is orthogonal to the null space of ∇C_k^T, then it is a multiple of $\nabla C_k(\nabla C_k^T\nabla C_k)^{-1}C_k$. If we also require that s_k^{n} lies inside the trust region of radius $r\delta_k$, then it is given by

$$s_k^{\mathsf{n}} = \begin{cases} -\nabla C_k(\nabla C_k^T\nabla C_k)^{-1}C_k & \text{if } \|\nabla C_k(\nabla C_k^T\nabla C_k)^{-1}C_k\| \le r\delta_k, \\ -\xi_k\nabla C_k(\nabla C_k^T\nabla C_k)^{-1}C_k, & \text{otherwise,} \end{cases} \tag{4.6}$$

where $\xi_k = \frac{r\delta_k}{\|\nabla C_k(\nabla C_k^T \nabla C_k)^{-1} C_k\|}$. If the quasi–normal component s_k^{n} of the step is given by (4.6), then it is called normal. As we will see in the Section 7, the normal component (4.6) satisfies a fraction of optimal decrease and hence a fraction of Cauchy decrease on the trust–region subproblem for the linearized constraints.

LEMMA 4.2. *The quasi–normal component (4.6) and the least–squares update*

$$\lambda_k = -(\nabla C_k^T \nabla C_k)^{-1} \nabla C_k^T \nabla f_k \tag{4.7}$$

satisfy conditions (2.3) and (2.8).

Proof. It can be easily confirmed that $\nabla_x \ell_k^T s_k^{\mathsf{n}} = 0$. The condition (2.8) holds since $\lambda(x) = -(\nabla C(x)^T \nabla C(x))^{-1} \nabla C(x)^T \nabla f(x)$ has bounded derivatives in Ω. □

5. The behavior of the trust radius. In Sections 5 and 6 we no longer need to consider that the tangential component $\bar{s}_k^{\mathsf{t}}$ satisfies a fraction of optimal decrease on the trust–region subproblem (2.5). It suffices to assume the fraction of Cauchy decrease condition (2.4). We assume that the component s_k^{n} satisfies conditions (2.1) and (2.2).

We need to strengthen conditions (2.3) and (2.8) in the following way:

$$\nabla_x \ell_k^T s_k^{\mathsf{n}} \le \kappa_2' \|C_k\| \, \|s_k\|, \tag{5.1}$$

$$\|\Delta\lambda_k\| = \|\lambda_{k+1} - \lambda_k\| \le \kappa_3' \|s_k\|, \tag{5.2}$$

$$\|s_k^{\mathsf{n}}\| \le \kappa_4' \|s_k\|, \tag{5.3}$$

where κ_2', κ_3', and κ_4' are positive constants independent of the iterates. The choices of s_k^{n} and λ_k suggested in Section 4 satisfy these requirements as well. See Lemmas 4.1 and 4.2 for the first two conditions. It is obvious that the normal component (4.6) satisfy (5.3). The quasi–normal component (4.4) also satisfies (5.3) (see [35][Lemma 5.6.1]).

The next theorems show that if $\lim_{k\to+\infty} x_k = x_*$ and $\nabla_{xx}^2 \ell(x_*, \lambda(x_*))$ is positive definite on $\mathcal{N}(\nabla C(x_*)^T)$, then the penalty parameter ρ_k is uniformly bounded and the trust radius δ_k is uniformly bounded away from zero.

THEOREM 5.1. *Let the general assumptions hold and $W(x)$ and $\lambda(x)$ be continuous. If $\{x_k\}$ converges to x_* and $\nabla_{xx}^2 \ell(x_*, \lambda(x_*))$ is positive definite on $\mathcal{N}(\nabla C(x_*)^T)$, then $\{\rho_k\}$ is a bounded sequence.*

Proof. First since $\nabla_{xx}^2 \ell(x_*, \lambda(x_*))$ is positive definite on $\mathcal{N}(\nabla C(x_*)^T)$ and $\nabla^2 f(x)$, $\nabla^2 c_i(x)$, $i = 1, \ldots, m$, $W(x)$, and $\lambda(x)$ are continuous functions of x, there exists a neighborhood $\mathcal{N}(x_*)$ of x_* and a $\bar{\gamma} > 0$ such that for any x in $\mathcal{N}(x_*)$,

$$\lambda_1 \left(W(x)^T \nabla_{xx}^2 \ell(x, \lambda(x)) W(x) \right) \ge \bar{\gamma}.$$

Since $\bar{q}_k^{\mathsf{t}}(\bar{s}_k^{\mathsf{t}}) - \bar{q}_k^{\mathsf{t}}(0) \le 0$ we can write

$$\frac{1}{2} (\bar{s}_k^{\mathsf{t}})^T \bar{H}_k (\bar{s}_k^{\mathsf{t}}) \le -(\bar{s}_k^{\mathsf{t}})^T \bar{g}_k \le \|\bar{s}_k^{\mathsf{t}}\| \, \|\bar{g}_k^{\mathsf{t}}\|.$$

Thus for all k such that $x_k \in \mathcal{N}(x_*)$ we have

$$\frac{1}{2} \bar{\gamma} \|\bar{s}_k^{\mathsf{t}}\|^2 \le \|\bar{s}_k^{\mathsf{t}}\| \, \|\bar{g}_k\|,$$

and this implies

$$\|s_k^{\mathsf{t}}\| \le \frac{2\nu_7}{\bar{\gamma}}\|\bar{g}_k\|. \tag{5.4}$$

Now by using (3.5) and (5.4), we have for all k such that $x_k \in \mathcal{N}(x_*)$, that

$$\begin{aligned} q_k(s_k^{\mathsf{n}}) - q_k(s_k) &\ge \kappa_6\|\bar{g}_k\| \min\{\kappa_7\|\bar{g}_k\|, \kappa_8\delta_k\} \\ &\ge \kappa_{17}\|s_k^{\mathsf{t}}\|^2, \end{aligned} \tag{5.5}$$

where $\kappa_{17} = \frac{\kappa_6\bar{\gamma}}{2\nu_7}\min\{\frac{\kappa_7\bar{\gamma}}{2\nu_7}, \frac{\kappa_8}{1+r}\}$.

Now let $\|C_k\| \le \alpha'\|s_k\|$ where the positive constant α' is defined later. Using similar arguments as in Lemma 3.2, it follows from (2.2), (5.1), (5.2), $\|C_k\| \le \alpha'\|s_k\|$, and Assumption A.4 that

$$q_k(0) - q_k(s_k^{\mathsf{n}}) - \Delta\lambda_k^T(\nabla C_k^T s_k + C_k) \ge -\kappa_{10}'\|C_k\|\,\|s_k\|, \tag{5.6}$$

where $\kappa_{10}' = \kappa_2' + \frac{1}{2}\nu_8\kappa_1^2\alpha' + \kappa_3'$.

From (2.2) and $\|C_k\| \le \alpha'\|s_k\|$ we also get

$$\begin{aligned} \|s_k\|^2 \le \left(\|s_k^{\mathsf{n}}\| + \|s_k^{\mathsf{t}}\|\right)^2 &\le 2\|s_k^{\mathsf{n}}\|^2 + 2\|s_k^{\mathsf{t}}\|^2 \\ &\le 2\alpha'\kappa_1^2\|C_k\|\,\|s_k\| + 2\|s_k^{\mathsf{t}}\|^2, \end{aligned}$$

which together with (5.5) and (5.6) implies

$$\begin{aligned} pred(s_k;\rho) \ge\ & \tfrac{1}{4}\kappa_{17}\|s_k\| + \left(\tfrac{1}{4}\kappa_{17}\|s_k\| - (\alpha'\kappa_1^2\kappa_{17} + \kappa_{10}')\|C_k\|\right)\|s_k\| \\ & + \rho\left(\|C_k\|^2 - \|\nabla C_k^T s_k + C_k\|^2\right), \end{aligned} \tag{5.7}$$

for all $\rho > 0$. We now need to impose the following condition on α':

$$\alpha' \le \frac{\kappa_{17}}{4\alpha'\kappa_1^2\kappa_{17} + 4\kappa_{10}'}. \tag{5.8}$$

Now we set $\rho = \rho_{k-1}$ in (5.7) and conclude that the penalty parameter does not need to be increased if $\|C_k\| \le \alpha'\|s_k\|$ (see Step 2.4 of Algorithm 2.1). Hence, if ρ_k is increased then $\|C_k\| > \alpha'\|s_k\|$ holds, and by using (5.1)–(5.3) we obtain:

$$q_k(0) - q_k(s_k^{\mathsf{n}}) - \Delta\lambda_k^T(\nabla C_k^T s_k + C_k) \ge -\kappa_{10}''\|C_k\|\,\|s_k\|, \tag{5.9}$$

with $\kappa_{10}'' = \kappa_2' + \frac{1}{2}\nu_8\kappa_1\kappa_4' + \kappa_3'$. Recall from the proof of Lemma 3.7 that if ρ_k is increased then

$$\frac{\rho_k}{2}\kappa_4\|C_k\| \min\left\{\kappa_5\|C_k\|, \frac{r}{\kappa_0}\|s_k\|\right\} \le (\kappa_{10}'' + \bar{\rho}\nu_4)\|s_k\|\,\|C_k\|,$$

which in turn implies

$$\left(\frac{\kappa_4}{2}\min\left\{\kappa_5\alpha', \frac{r}{\kappa_0}\right\}\right)\rho_k \le \kappa_{10}'' + \bar{\rho}\nu_4 \quad \Longleftrightarrow \quad \rho_k \le \rho_*'.$$

This completes the proof of the Theorem. □

THEOREM 5.2. *Let the general assumptions hold and $W(x)$ and $\lambda(x)$ be continuous. If $\{x_k\}$ converges to x_* and $\nabla^2_{xx}\ell(x_*, \lambda(x_*))$ is positive definite on $\mathcal{N}(\nabla C(x_*)^T)$, then δ_k is uniformly bounded away from zero and eventually all iterations are successful.*

Proof. The proof of the theorem is based on the boundedness of $\{\rho_k\}$. We consider the cases $\|C_k\| > \alpha'\|s_k\|$ and $\|C_k\| \le \alpha'\|s_k\|$, where α' satisfies (5.8).

If $\|C_k\| > \alpha'\|s_k\|$, then from (2.7), (2.9), and (3.4), we find that

$$pred(s_k; \rho_k) \ge \rho_k \frac{\kappa_4}{2}\|C_k\| \min\{\kappa_5\|C_k\|, r\delta_k\} \ge \rho_k\kappa_{18}\|s_k\|^2, \tag{5.10}$$

where $\kappa_{18} = \frac{\kappa_4\alpha'}{2}\min\{\kappa_5\alpha', \frac{r}{\kappa_0}\}$. In this case it follows from (3.9), (5.10), and $\rho_k \ge 1$ that

$$\left|\frac{ared(s_k;\rho_k)}{pred(s_k;\rho_k)} - 1\right| \le \left(\frac{\bar\kappa_5\kappa_3'}{\kappa_{18}} + \frac{\bar\kappa_6}{\kappa_{18}}\right)\|s_k\| + \frac{\bar\kappa_7}{\kappa_{18}}\|C_k\|. \tag{5.11}$$

Now, suppose that $\|C_k\| \le \alpha'\|s_k\|$. From (5.7) with $\rho = \rho_k$ we obtain

$$pred(s_k; \rho_k) \ge \frac{\kappa_{17}}{4}\|s_k\|^2.$$

Now we use (3.9) and $\rho_k \le \rho_*$ to get

$$\left|\frac{ared(s_k;\rho_k)}{pred(s_k;\rho_k)} - 1\right| \le \left(\frac{4\bar\kappa_5\kappa_3'}{\kappa_{17}} + \frac{4\bar\kappa_6\rho_*}{\kappa_{17}}\right)\|s_k\| + \frac{4\bar\kappa_7\rho_*}{\kappa_{17}}\|C_k\|. \tag{5.12}$$

It follows from Theorem 8.4 in [7] that

$$\liminf_{k\to+\infty}\left(\|W_k^T\nabla_x\ell_k\| + \|C_k\|\right) = 0.$$

From this result, the continuity of $C(x)$, and the convergence of $\{x_k\}$ we obtain $\lim_{k\to+\infty}\|C_k\| = 0$.

Finally from (5.11), (5.12), and the limits $\lim_{k\to+\infty} x_k = x_*$, $\lim_{k\to+\infty}\lambda_k = \lambda(x_*)$, and $\lim_{k\to+\infty}\|C_k\| = 0$, we finally get

$$\lim_{k\to+\infty}\left|\frac{ared(s_k;\rho_k)}{pred(s_k;\rho_k)}\right| = 1,$$

which by the rules for updating the trust radius in Step 2.5 of Algorithm 2.1 shows that δ_k is uniformly bounded away from zero. □

6. Local rate of convergence. In order to obtain q–quadratic local rates of convergence, we require the reduced tangential component $\bar s_k^{\mathsf t}$ to satisfy (2.4) and the following condition:

$$\text{if } \bar H_k \text{ is positive definite and } \|\bar H_k^{-1}\bar g_k\| \le \bar\delta_k \text{ then } \bar s_k^{\mathsf t} = -\bar H_k^{-1}\bar g_k. \tag{6.1}$$

6.1. Discretized optimal control formulation. Consider again problem (4.1) and assume that this problem has the structure described in Section 4.1. We can now use Theorem 5.2 to obtain a local rate of convergence.

THEOREM 6.1. *Suppose that the ECO problem is of the form (4.1). Let the general assumptions and assumption (4.2) hold and assume that $\{x_k\}$ converges to x_*. In addition to this, let $\bar s_k^{\mathsf t}$, $s_k^{\mathsf n}$, and λ_k be given by (6.1), (4.4) and (4.3).*

If $\nabla^2_{xx}\ell(x_*, \lambda_*)$ *is positive definite on* $\mathcal{N}(\nabla C(x_*)^T)$*, where*

$$\lambda_* = -C_y(x_*)^{-T}\nabla_y f(x_*),$$

then x_k *converges q–quadratically to* x_**.*

Proof. It can be shown by appealing to Theorem 8.4 in [7] that $\nabla_x \ell(x_*, \lambda_*) = 0$. It follows from Theorem 5.2 that δ_k is uniformly bounded away from zero. Thus there exists a positive integer $\bar{k}$ such that for all $k \geq \bar{k}$, $\bar{s}^{\mathsf{t}}_k = -\bar{H}_k^{-1}\bar{g}_k$ and $s^{\mathsf{n}}_k = \begin{pmatrix} -C_y(x_k)^{-1}C_k \\ 0 \end{pmatrix}$. Now the rate of convergence follows from [19]. ◻

6.2. Normal component and least–squares multipliers. Consider the general ECO problem (1.1) again and suppose that the quasi–normal component is the normal component (4.6) and λ_k is given by (4.7).

We can now use Theorem 5.2 to obtain the desired local rate of convergence. It is assumed that the orthogonal null–space basis is a continuous function of x.

THEOREM 6.2. *Let the general assumptions hold and assume that* $\{x_k\}$ *converges to* x_**. In addition to this, let* $\bar{s}^{\mathsf{t}}_k$*,* s^{n}_k*, and* λ_k *be given by (6.1), (4.6), and (4.7).*

If $\nabla^2_{xx}\ell(x_*, \lambda_*)$ *is positive definite on* $\mathcal{N}(\nabla C(x_*)^T)$*, where*

$$\lambda_* = -\left(\nabla C(x_*)^T \nabla C(x_*)\right)^{-1} \nabla C(x_*)^T \nabla f(x_*),$$

then x_k *converges q–quadratically to* x_**.*

Proof. It can be shown by appealing to Theorem 8.4 in [7] that $\nabla_x \ell(x_*, \lambda_*) = 0$. It follows from Theorem 5.2 that δ_k is uniformly bounded away from zero. Thus there exists a positive integer $\bar{k}$ such that for all $k \geq \bar{k}$, $\bar{s}^{\mathsf{t}}_k = -\bar{H}_k^{-1}\bar{g}_k$ and $s^{\mathsf{n}}_k = -\nabla C_k(\nabla C_k^T \nabla C_k)^{-1}C_k$. The q–quadratic rate of convergence follows from [18], [36]. ◻

7. The trust–region subproblem for the linearized constraints. In this section we investigate a few aspects of the trust–region subproblem for the linearized constraints

$$\text{(7.1)} \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}\|\nabla C_k^T s^{\mathsf{n}} + C_k\|^2 \\ \text{subject to} & \|s^{\mathsf{n}}\| \leq r\delta_k. \end{array}$$

First we prove that the normal component (4.6) and the quasi–normal component (4.4) always give a fraction of optimal decrease on this trust–region subproblem.

THEOREM 7.1. *Let the general assumptions hold. Then:*

(i) The normal component (4.6) satisfies a fraction of optimal decrease on the trust–region subproblem for the linearized constraints, i.e. there exists a positive constant β^{n}_1 *such that*

$$\text{(7.2)} \qquad \|C_k\|^2 - \|\nabla C_k^T s^{\mathsf{n}}_k + C_k\|^2 \geq \beta^{\mathsf{n}}_1 \left(\|C_k\|^2 - \|\nabla C_k^T s^*_k + C_k\|^2\right),$$

where s^*_k *is the optimal solution of (7.1).*

(ii) In addition, assume assumption (4.2). The quasi–normal component (4.4) satisfies the fraction of optimal decrease (7.2).

Proof. (i) If $\|\nabla C_k(\nabla C_k^T \nabla C_k)^{-1} C_k\| \le r\delta_k$, then s_k^{n} solves (7.1), and the result holds for any positive value of β_1^{n} in $(0,1]$. If this is not the case, then

$$\|C_k\|^2 - \|\nabla C_k^T s_k^{\mathsf{n}} + C_k\|^2 = \xi_k(2-\xi_k)\|C_k\|^2 \ge \xi_k\|C_k\|^2 \ge \frac{r\delta_k}{\nu_4\nu_5}\|C_k\|, \tag{7.3}$$

since $\|\nabla C_k(\nabla C_k^T \nabla C_k)^{-1} C_k\| \le \nu_4\nu_5\|C_k\|$ and $\xi_k \le 1$.

We also have

$$\begin{aligned}
\|C_k\|^2 - \|\nabla C_k^T s_k^* + C_k\|^2 &= -2(\nabla C_k C_k)^T s_k^* - (s_k^*)^T(\nabla C_k \nabla C_k^T)(s_k^*) \\
&\le 2\nu_4\|C_k\|\,\|s_k^*\| + \nu_4^2\|s_k^*\|^2 \\
&\le 2\nu_4 r\delta_k\|C_k\| + \nu_4^2 r\delta_k\|s_k^*\| \\
&\le (2\nu_4 r + \nu_4^3\nu_5 r)\delta_k\|C_k\|,
\end{aligned}$$

since $\|\nabla C_k(\nabla C_k^T \nabla C_k)^{-1}\|\|C_k\| > r\delta_k \ge \|s_k^*\|$. Combining this last inequality with (7.3) we get

$$\|C_k\|^2 - \|\nabla C_k^T s_k^{\mathsf{n}} + C_k\|^2 \ge \frac{1}{\nu_4^2\nu_5(2+\nu_4^2\nu_5)}\left(\|C_k\|^2 - \|\nabla C_k^T s_k^* + C_k\|^2\right),$$

and this completes the proof of (i).

(ii) If $\|C_y(x_k)^{-T} C_k\| \le r\delta_k$ then s_k^{n} solves (7.1), and (7.2) holds for any positive value of β_1^{n}. If this is not the case, we have

$$\begin{aligned}
\|C_k\|^2 - \|\nabla C_k^T s_k^{\mathsf{n}} + C_k\|^2 &= \|C_k\|^2 - \left\| -\varsigma_k \nabla C_k^T \begin{pmatrix} C_y(x_k)^{-1} C_k \\ 0 \end{pmatrix} + C_k\right\|^2 \\
&= \varsigma_k(2-\varsigma_k)\|C_k\|^2 \\
&\ge \frac{r\delta_k}{\nu_{10}}\|C_k\|,
\end{aligned} \tag{7.4}$$

where ν_{10} is the uniform bound on $\|C_y(x_k)^{-1}\|$. Now the rest of the proof follows as in (i). ∎

As a consequence of this theorem, we have immediately that the normal component (4.6) and the quasi–normal component (4.4) give a fraction of Cauchy decrease on the trust–region subproblem for the linearized constraints.

To compute a step s_k^{n} that gives a fraction of optimal decrease on the trust–region subproblem for the linearized constraints we can also use the techniques proposed in [23], [28], [31].

In the next theorem we show that the trust–region subproblem (7.1), due to its particular structure, tends to fall in the hard case in the latest stages of the algorithm. This result is relevant in our opinion since the algorithms proposed in [23], [28], [31] deal with the hard case.

The trust–region subproblem (7.1) can be rewritten as

$$\begin{aligned}
\text{minimize} \quad & \tfrac{1}{2}C_k^T C_k + (\nabla C_k C_k)^T s^{\mathsf{n}} + \tfrac{1}{2}(s^{\mathsf{n}})^T(\nabla C_k \nabla C_k^T)(s^{\mathsf{n}}) \\
\text{subject to} \quad & \|s^{\mathsf{n}}\| \le r\delta_k.
\end{aligned} \tag{7.5}$$

The matrix $\nabla C_k \nabla C_k^T$ is always positive semi–definite and, under the general assumptions, has rank m. Let $E_k(0)$ denote the eigenspace associated with the eigenvalue 0, i.e. $E_k(0) = \{v_k \in \mathbb{R}^n : \nabla C_k \nabla C_k^T v_k = 0\}$. The hard case is defined by the two following conditions:

(a) $(v_k)^T(\nabla C_k C_k) = 0$ for all v_k in $E_k(0)$ and

(b) $\|(\nabla C_k \nabla C_k^T + \mu I_n)^{-1} \nabla C_k C_k\| < r\delta_k$ for all $\mu > 0$.

THEOREM 7.2. *Under the general assumptions, if* $\lim_{k\to+\infty} \frac{\|C_k\|}{\delta_k} = 0$ *then there exists a* k_h *such that, for all* $k \geq k_h$*, the trust–region subproblem (7.5) falls in the hard case as defined above by (a) and (b).*

Proof. First we show that (a) holds at every iteration of the algorithm. If $v_k \in E_k(0)$,

$$\nabla C_k \nabla C_k^T v_k = 0.$$

Multiplying both sides by $(\nabla C_k^T \nabla C_k)^{-1} \nabla C_k^T$ gives us

$$\nabla C_k^T v_k = 0.$$

Thus $(v_k)^T(\nabla C_k C_k) = 0$ for all v_k in $E_k(0)$.

Now we prove that there exists a k_h such that (b) holds for every $k \geq k_h$. Since $g_k(\mu) = \|(\nabla C_k \nabla C_k^T + \mu I_n)^{-1} \nabla C_k C_k\|$ is a monotone strictly decreasing function of μ for $\mu > 0$,

$$\lim_{\mu \to 0^+} g_k(\mu) < r\delta_k$$

is equivalent to $g_k(\mu) < r\delta_k$, for all $\mu > 0$. Also, from the singular value decomposition of ∇C_k, we obtain

$$\lim_{\mu \to 0^+} g_k(\mu) = \| \lim_{\mu \to 0^+} (\nabla C_k \nabla C_k^T + \mu I_n)^{-1} \nabla C_k C_k\| = \|\nabla C_k (\nabla C_k^T \nabla C_k)^{-1} C_k\|.$$

Hence $g_k(\mu) < r\delta_k$ holds for all $\mu > 0$ if and only if $\|\nabla C_k (\nabla C_k^T \nabla C_k)^{-1} C_k\| < r\delta_k$.

Now since $\lim_{k\to+\infty} \frac{\|C_k\|}{\delta_k} = 0$, there exists a k_h such that $\|C_k\| < \frac{r}{\nu_4 \nu_5} \delta_k$ for all $k \geq k_h$. Thus $\|\nabla C_k (\nabla C_k^T \nabla C_k)^{-1} C_k\| \leq \nu_4 \nu_5 \|C_k\| < r\delta_k$, for all $k \geq k_h$, and this completes the proof of the theorem. □

We complete this section with the following corollary.

COROLLARY 7.1. *Under the general assumptions, if* $\lim_{k\to+\infty} \|C_k\| = 0$ *and the trust radius is uniformly bounded away from zero, then there exists a* k_h *such that, for all* $k \geq k_h$*, the trust–region subproblem (7.5) falls in the hard case as defined above by (a) and (b).*

Proof. If $\lim_{k\to+\infty} \|C_k\| = 0$ and the trust radius is uniformly bounded away from zero then $\lim_{k\to+\infty} \frac{\|C_k\|}{\delta_k} = 0$ and Theorem 7.2 can be applied. □

8. Concluding remarks. In Theorems 3.1 and 3.2 we have established global convergence to a point satisfying the second–order necessary optimality conditions for the general trust–region–based algorithm considered in this paper. In Theorem 5.2 we have proved that the trust radius is, under sufficient second–order optimality conditions, bounded away from zero. With the help of this result we analyzed local rates of convergence for different choices of steps and multipliers. We believe that these results complement the theory developed by Dennis, El–Alem, and Maciel in [7] that proves global convergence to a stationary point. We have also given a detailed analysis of the trust–region subproblem for the linearized constraints.

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