# Globally Convergent DC Trust-Region Methods

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#### Abstract

In this paper, we investigate the use of DC (Difference of Convex functions) models and algorithms in the application of trust-region methods to the solution of a class of nonlinear optimization problems where the constrained set is closed and convex (and, from a practical point of view, where projecting onto the feasible region is computationally affordable). We consider DC local models for the quadratic model of the objective function used to compute the trust-region step, and apply a primal-dual subgradient method to the solution of the corresponding trust-region subproblems.

One is able to prove that the resulting scheme is globally convergent for first-order stationary points. The theory requires the use of exact second-order derivatives but, in turn, the computation of the trust-region step asks only for one projection onto the feasible region (in comparison to the calculation of the generalized Cauchy point which may require more).

The numerical efficiency and robustness of the proposed new scheme when applied to bound-constrained problems is measured by comparing its performance against some of the current state-of-the-art nonlinear programming solvers on a vast collection of test problems.

Keywords: Trust-region methods, DC algorithm, global convergence, bound constraints.

# 1 Introduction

Consider the constrained nonlinear programming problem

$$\min f(x) \quad \text{subject to} \quad x \in C, \tag{1}$$

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where  $C \subseteq \mathbb{R}^n$  is a nonempty closed convex set and  $f : \mathbb{R}^n \to \mathbb{R}$  is a twice continuously differentiable function. We have in mind a constraint set C over which projections are computationally affordable (like a set defined by bounds on the variables or other simpler settings such as the one considered in [9]). However, the algorithms and theory proposed in this paper apply to any closed convex set C.

Trust-region methods are widely acknowledged to be among the most efficient and robust methods for solving nonlinear optimization problems (see [8, 26]). A trust-region step results from the approximate solution of the trust-region subproblem, where a quadratic model of fis minimized over a trust-region ball of pre-specified size, possibly intersected with the feasible region C in the constrained case. When constraints of the form  $x \in C$  are of polyhedral type, they can be naturally added to the trust-region subproblem (which would then consist of a quadratic program if the norm used in the trust-region ball is the  $\ell_{\infty}$  one). Most trust-region methods compute the trust-region step in a way that the decrease produced in the quadratic model is a fraction of what is obtained by the so-called generalized Cauchy point, computed by determining the gradient-projected path (see [8, Chapter 12]).

The purpose of this paper is to integrate the DC Algorithm (DCA) in a trust-region framework for the solution of problem (1). DCA is a primal-dual subgradient method designed for solving a general DC program, i.e., an optimization problem where one minimizes the difference of convex functions on the whole space. Note that minimizing a DC function over a convext set C can be restated as a general DC program by using the indicator function  $\chi_C$  of C. We apply DCA to the approximate solution of the trust-region subproblems, exploring specific DC decompositions of the quadratic models. The overall approach is shown to be globally convergent to first-order critical points (when the second-order information used in the quadratic model DC decompositions is exact). We will see that the theory requires only one DCA iteration to solve the trust-region subproblem, which amounts to only one projection onto the feasible region (and here we recall that the computation of the generalized Cauchy point may take more than one projection).

Our numerical experiments are focused entirely on the solution of bound-constrained problems. The numerical tests reported in this paper showed us that a few (cheap) DCA steps suffice to compute decently accurate trust-region steps, resulting in an efficient and reasonably robust algorithm. The minimization of a nonlinear function subject to bounds on the variables has been the subject of intense previous work, along many possible avenues. Major classes of algorithms for bound-constrained problems include the ones based on: active or  $\epsilon$ -active set methods (see, e.g., [1, 13, 32] and more recently [18] for a short review on active set methods); trust-region methods (see, e.g., [6, 7, 14, 22, 24]); interior-point methods (see, e.g., [5, 10, 19]); line-search projected gradient methods (see, e.g., [2] and the references therein; see also [3, 25, 35] for a limited memory BFGS method); and filter type methods (see [31]). The approach proposed and analyzed in this paper belongs to the trust-region class but also shares the flavor of projected gradient methods.

We organize our contribution in the following way. In Section 2 we provide some background on the DC Algorithm. Our DC trust-region method is introduced and analyzed in Section 3. The two following sections are devoted to present our numerical findings. First we provide in Section 4 practical details of the implementation of the DC trust-region method, as well as information on how the numerical experiments were done and compared. The numerical results are then presented and commented on in Section 5. Some final conclusions are reported in Section 6. The norms and inner products used in the paper are the Euclidian ones.

# 2 DC programming, algorithm, and models

Let us start by recalling some basic notions from Convex Analysis and Nonsmooth Calculus which will be needed afterwards (see [4, 29, 30]). In the sequel, the space  $\mathbb{R}^n$  is equipped with the Euclidean inner product  $\langle \cdot, \cdot \rangle$ . Let  $\Gamma_0(\mathbb{R}^n)$  be the 'convex cone' of all the lower semicontinuous proper (i.e., not identically equal to  $+\infty$ ) convex functions defined on  $\mathbb{R}^n$  and taking values in  $\mathbb{R} \cup \{+\infty\}$ .

For  $g \in \Gamma_0(\mathbb{R}^n)$ , the subdifferential  $\partial g(z)$  of g at a point z in its effective domain  $\{z \in \mathbb{R}^n : g(z) < +\infty\}$  is defined by

$$\partial g(z) = \{ w \in \mathbb{R}^n : \langle w, d \rangle \le g(z+d) - g(z), \ \forall d \in \mathbb{R}^n \}$$

(by convention  $\partial g(z) = \emptyset$  if z is not in the effective domain of g). The indicator function  $\chi_C$  of a nonempty set C is defined by  $\chi_C(z) = 0$  if  $z \in C$ ,  $+\infty$  otherwise. The normal cone N(C, z) of a nonempty, closed, convex set C at  $z \in C$  (the polar of the tangent cone) coincides with  $\partial \chi_C(z)$ , i.e.,

$$N(C,z) = \{ u \in \mathbb{R}^n : \langle u, w - z \rangle \le 0, \forall w \in C \} = \partial \chi_C(z).$$

For  $\varphi \in \Gamma_0(\mathbb{R}^n)$ , its conjugate  $\varphi^*$  is defined by

$$\varphi^*(w) = \sup\{\langle z, w \rangle - \varphi(z) : z \in \mathbb{R}^n\}$$

and it holds  $\varphi^* \in \Gamma_0(\mathbb{R}^n)$  and  $(\varphi^*)^* = \varphi$ . The latter relation provides the crucial characterization of  $\varphi \in \Gamma_0(\mathbb{R}^n)$  as a pointwise supremum of a collection of its affine minorants:

$$\varphi(z) = \sup\{\langle z, w \rangle - \varphi^*(w) : w \in \mathbb{R}^n\}.$$
(2)

A standard DC program is of the form (with the usual convention  $(+\infty) - (+\infty) = +\infty$ )

$$\inf\{f(z) := g(z) - h(z) : z \in \mathbb{R}^n\},\tag{3}$$

where  $g, h \in \Gamma_0(\mathbb{R}^n)$ . DC programming deals with the vector space  $DC(\mathbb{R}^n) = \Gamma_0(\mathbb{R}^n) - \Gamma_0(\mathbb{R}^n)$ . Such a function f is called a DC function, and g - h a DC decomposition of f, while the convex functions g and h are the DC components of f. Note that minimizing a DC function f = g - h on a nonempty closed convex C set can be recast into the standard form (3) by changing g to  $g + \chi_C$ .

Using (2), the DC duality [21, 27, 28] associates a primal DC program with its dual defined as

$$\inf\{h^*(w) - g^*(w) : w \in \mathbb{R}^n\},\$$

which is also a DC program with the same optimal value.

The DC Algorithm (DCA) is based on local optimality and DC duality, and has been introduced by Pham Dinh Tao in 1986 and extensively developed by Le Thi Hoai An and Pham Dinh Tao since 1994 (see [20, 21, 27, 28], and the references therein), being successfully applied to a number of classes of problems, including large-scale instances. DCA constructs two sequences  $\{z^l\}$  and  $\{w^l\}$  (of trial solutions of the primal and dual programs, respectively) which are improved at each iteration such that: (i) the sequences  $\{g(z^l) - h(z^l)\}$  and  $\{h^*(w^l) - g^*(w^l)\}$  are decreasing; (ii) their corresponding limit points  $z^{\infty}$  and  $w^{\infty}$  satisfy local optimality conditions, respectively for the primal and the dual. DCA is a descent method, without line search but globally convergent. Algorithm 2.1 (DC Algorithm (DCA)) Initialization Choose  $z^0 \in \mathbb{R}^n$ .

For l = 0, 1, ...

- 1. Compute  $w^l \in \partial h(z^l)$ .
- 2. Compute  $z^{l+1} \in \partial g^*(w^l)$ , i.e.,  $z^{l+1}$  is a solution of the convex program

 $\min\{g(z) - \langle z, w^l \rangle : z \in \mathbb{R}^n\}.$ 

If some stopping criterion is met, then stop, otherwise go to Step 1.

**Output** Return  $z^{l+1}$  and  $g(z^{l+1}) - h(z^{l+1})$  as the best known approximate solution and objective function value, respectively.

The type of algorithms for solving problem (1) of interest to us in this paper are based on the iterative minimization of quadratic models on the intersection of C with a trust region, and for this purpose we want to use DC programming. Note that when the set C is defined by bounds on the variables and we choose the  $\ell_{\infty}$ -norm for the trust region, the resulting trust-region subproblems will consist of minimizing a quadratic function subject to box constraints.

In the sequel, the closed ball with center  $x \in \mathbb{R}^n$  and radius  $\varepsilon > 0$  is denoted by  $B(x, \varepsilon)$ . Given  $x \in \mathbb{R}^n$ , we form a Taylor quadratic model of f around this point

 $\in \mathbb{R}$ , we form a Taylor quadratic model of f around this point

$$m(x+p,x) = f(x) + \langle \nabla f(x), p \rangle + \frac{1}{2} \langle p, \nabla^2 f(x)p \rangle.$$

Note that when  $\nabla^2 f$  is Lipschitz continuous with constant  $\kappa > 0$  on  $B(x, \Delta)$ , one has

$$|f(x+p) - m(x+p,x)| \leq \frac{\kappa}{6}\Delta^3, \tag{4}$$

for all  $p \in B(0, \Delta)$ .

The DC decomposition of m(x + p, x) of most interest to us is

$$m(x+p,x) = m_g(x+p,x) - m_h(x+p,x),$$

where

$$m_g(x+p,x) = \frac{\rho_x}{2} \|p\|^2 + \chi_D(p)$$
 and  $m_h(x+p,x) = \frac{\rho_x}{2} \|p\|^2 - m(x+p,x),$ 

 $\rho_x = \|\nabla^2 m(x+p,x)\| = \|\nabla^2 f(x)\|$ , and  $D = (C - \{x\}) \cap B(0,\Delta)$  is the intersection of C (shifted by x) with the trust region  $B(0,\Delta)$ .

### 3 The DC trust-region method

At the iteration k, a step  $p_k$  is computed by approximately solving the trust-region subproblem

min 
$$m(x_k + p, x_k)$$
 subject to  $p \in D_k = (C - \{x_k\}) \cap B(0, \Delta_k),$  (5)

using the DCA (Algorithm 2.1) and the DC decomposition

$$m(x_k + p, x_k) = \left(\frac{\rho_{x_k}}{2} \|p\|^2 + \chi_{D_k}(p)\right) - \left(\frac{\rho_{x_k}}{2} \|p\|^2 - m(x_k + p, x_k)\right),$$

with  $\rho_{x_k} = \|\nabla^2 f(x_k)\| + \epsilon$ , where  $\epsilon$  is a small positive quantity added to guarantee that  $\rho_{x_k}$  stays uniformly bounded away from zero.

The following algorithm summarizes our trust-region method using DCA for the trust-region subproblem minimization. The notation  $P_W(z) = \arg \min_{w \in W} ||w - z||$  denotes the projection of z onto a closed, convex set W. The algorithm is written without a stopping criterion to generate an infinite sequence of iterates for the subsequent analysis.

#### Algorithm 3.1 (DC trust-region algorithm)

#### Step 0 (initialization):

Choose an initial point  $x_0 \in C$  and an initial trust-region radius  $\Delta_0 > 0$ . Select a positive integer  $l_0$ . Choose constants  $\eta_1, \gamma_1, \gamma_2 \in (0, 1)$ . Start with k = 0 and set  $p_{-1} = 0$ .

### Step 1 (step calculation using DCA for subproblem):

Obtain  $p_k$ , with  $||p_k|| \leq \Delta_k$ , by using DCA to approximately solve the trust-region subproblem (5), as follows:

Set 
$$p_k^0 = P_{D_k}(p_{k-1})$$
.  
For  $l = 0, 1, \dots, l_0 - 1$ 

1.

2. Compute  $q_k^l = \rho_{x_k} p_k^l - \nabla m(x_k + p_k^l, x_k)$ .

3. Set 
$$p_k^{l+1} = P_{D_k}(q_k^l/\rho_{x_k})$$
.

Set 
$$p_k = p_k^{l_0}$$

Step 2 (acceptance of trial point): Compute  $f(x_k + p_k)$  and define

$$\tau_k = \frac{f(x_k) - f(x_k + p_k)}{m(x_k, x_k) - m(x_k + p_k, x_k)}.$$

If  $\tau_k \geq \eta_1$ , then  $x_{k+1} = x_k + p_k$ . Otherwise define  $x_{k+1} = x_k$ .

## Step 3 (trust-region radius update):

If  $\tau_k \geq \eta_1$  then  $\Delta_{k+1} \in [\Delta_k, +\infty)$ , otherwise  $\Delta_{k+1} \in [\gamma_1 \Delta_k, \gamma_2 \Delta_k]$ . Increment k by 1 and go to Step 1.

In fact, regarding Step 1 where DCA (Algorithm 2.1) is applied to solve the trust-region subproblem (5), we point out that  $q_k^l$  corresponds to  $w^l = \nabla h(z^l)$  in Algorithm 2.1 and that  $p_k^{l+1}$ , the solution of  $\min\{m_g(x_k + p) - \langle p, q_k^l \rangle : p \in \mathbb{R}^n\}$ , corresponds to  $z^{l+1}$  in Algorithm 2.1.

Note that the minimal effort per iteration in this algorithm (when  $l_0 = 1$ ) amounts to one projection, and that this compares to the computation of a generalized Cauchy point (see [8, Algorithm 12.2.2]) for trust-region methods when applied to general convex constrained problems which may take more than one projection.

Another point is that when  $p_k^0 = 0$  and  $l_0 = 1$ , a main step of Algorithm 3.1 resembles a DCA step applied to the original smooth problem of the form (1), since  $q_k^0 = \rho_{x_k} p_k^0 - \nabla m(x_k + p_k^0, x_k) = -\nabla f(x_k)$ . However the choice of  $\rho_{x_k}$  is local and it does not render a true global DC decomposition for the original problem.

We are now in a position to show global convergence to first-order stationary point.

**Theorem 3.1** Let  $\{x_k\}$  be a sequence generated by Algorithm 3.1 applied to a twice continuously differentiable function f for which  $\nabla^2 f$  is Lipschitz continuous on C. Then the sequence  $\{f(x_k)\}$  is decreasing and  $\lim_{k\to+\infty} ||x_{k+1}-x_k|| = 0$ . Moreover, every limit point  $x_{\infty}$  of the sequence  $\{x_k\}$  is a first-order critical point of problem (1), that is,  $0 \in \nabla f(x_{\infty}) + N(C, x_{\infty})$ , where  $N(C, x_{\infty})$  stands for the normal cone of the convex set C at the point  $x_{\infty}$ .

**Proof.** Consider first the convex quadratic function in  $p \in \mathbb{R}^n$  given by  $m_h(x_k + p, x_k) = \frac{\rho_{x_k}}{2} ||p||^2 - m(x_k + p, x_k)$ . Hence, for every k and  $l = 0, 1, \ldots, l_0 - 1$ , one has directly from the first-order characterization of convexity of  $m_h(x_k + p, x_k)$ 

$$\langle q_k^l, p_k^{l+1} - p_k^l \rangle \leq m_h(x_k + p_k^{l+1}, x_k) - m_h(x_k + p_k^l, x_k),$$
 (6)

where  $q_k^l = \nabla m_h(x_k + p_k^l, x_k) = \rho_{x_k} p_k^l - \nabla m(x_k + p_k^l, x_k)$ . On the other hand, since  $p_k^{l+1} = P_{D_k}(q_k^l/\rho_{x_k})$ , from the definition of projection one has

$$\rho_{x_k} \langle q_k^l / \rho_{x_k} - p_k^{l+1}, p_k^l - p_k^{l+1} \rangle \leq 0,$$

which is equivalent to

$$\langle q_k^l, p_k^l - p_k^{l+1} \rangle \leq \frac{\rho_{x_k}}{2} \|p_k^l\|^2 - \frac{\rho_{x_k}}{2} \|p_k^{l+1}\|^2 - \frac{\rho_{x_k}}{2} \|p_k^l - p_k^{l+1}\|^2.$$
 (7)

From inequalities (6) and (7), one then obtains

$$m(x_k + p_k^l, x_k) - m(x_k + p_k^{l+1}, x_k) \ge \frac{\rho_{x_k}}{2} \|p_k^l - p_k^{l+1}\|^2$$

and therefore

$$m(x_k, x_k) - m(x_k + p_k, x_k) = \sum_{l=0}^{l_0 - 1} [m(x_k + p_k^l, x_k) - m(x_k + p_k^{l+1}, x_k)] \\ \geq \frac{\rho_{x_k}}{2} \sum_{l=0}^{l_0 - 1} \|p_k^l - p_k^{l+1}\|^2 \geq \frac{\rho_{x_k}}{2l_0} \|p_k\|^2.$$
(8)

Now denote by  $\kappa$  the Lipschitz constant of  $\nabla^2 f$  on C. By the definition of  $\tau_k$  and by applying a Taylor expansion [11, Lemma 4.1.14] (in the numerator below), one has

$$|\tau_k - 1| = \left| \frac{m(x_k + p_k, x_k) - f(x_k + p_k)}{m(x_k, x_k) - m(x_k + p_k, x_k)} \right| \le \frac{(\kappa/6) \|p_k\|^3}{\rho_{x_k} \|p_k\|^2 / 2l_0} = \frac{\kappa l_0}{3\rho_{x_k}} \|p_k\|.$$

Thus, since  $\rho_{x_k} \ge \epsilon$  and  $||p_k|| \le \Delta_k$ , if

$$\Delta_k \leq \frac{3(1-\eta_1)\epsilon}{\kappa l_0},$$

then the iteration is successful. One can conclude that there is an infinity of successful iterations. Moreover, from the trust-region update of the algorithm, one has that

$$\Delta_k \geq \Delta_{\min} = \frac{3(1-\eta_1)\epsilon\gamma_1}{\kappa l_0}$$
 for all  $k$ .

Furthermore, ignoring the unsuccessful iterations where there is no displacement, one obtains

$$f(x_k) - f(x_{k+1}) = \tau_k \left[ m(x_k, x_k) - m(x_{k+1}, x_k) \right] \ge \frac{\rho_{x_k} \eta_1}{2l_0} \|x_k - x_{k+1}\|^2.$$
(9)

Consequently,  $f(x_k)$  is a monotonically decreasing sequence. Since f is bounded from below,  $f(x_k)$  converges. As a result one has that  $\lim_{k\to+\infty} ||x_{k+1} - x_k|| = 0$ .

Let  $x_{\infty}$  be a limit point of the sequence  $\{x_k\}$ , say,  $\lim_{i \to +\infty} x_{k_i} = x_{\infty}$  for some subsequence  $\{x_{k_i}\}$  of  $\{x_k\}$ . Since we proved above that there is an infinity of successful iterations, for all  $i = 1, 2, \ldots$  there exists an index  $j_i \ge 1$  such that

$$x_{k_i} = x_{k_i+1} = \cdots = x_{k_i+j_i-1} \neq x_{k_i+j_i}$$

One knows from (9) that  $\lim_{i\to+\infty} \|p_{k_i+j_i-1}^{l_0}\| = \|p_{k_i+j_i-1}\| = 0$ , and by using this and taking limits in (8), we obtain  $\lim_{i\to+\infty} \|p_{k_i+j_i-1}^1\| = 0$ . Since

$$\begin{aligned} x_{k_i+j_i-1} + p_{k_i+j_i-1}^1 \\ &= P_{C \cap B(x_{k_i+j_i-1}, \Delta_{k_i+j_i-1})} \left( x_{k_i+j_i-1} - \nabla f(x_{k_i+j_i-1}) / \rho_{x_{k_i+j_i-1}} - \nabla^2 f(x_{k_i}) p_{k_i+j_i-1}^1 / \rho_{x_{k_i}} \right) \end{aligned}$$

we have

$$\langle -\nabla f(x_{k_i})/\rho_{x_{k_i}} - \nabla^2 f(x_{k_i})p_{k_i+j_i-1}^1/\rho_{x_{k_i}} - p_{k_i+j_i-1}^1, x - x_{k_i} - p_{k_i+j_i-1}^1 \rangle \le 0$$

for all  $x \in C \cap B(x_{k_i}, \Delta_{k_i+j_i-1})$ . By taking the limits  $x_{k_i} \to x_{\infty}$ ,  $p_{k_i+j_i-1}^1 \to 0$ , and  $\rho_{x_{k_i}} \to \|\nabla^2 f(x_{\infty})\| + \epsilon$ , one obtains

$$\langle -\nabla f(x_{\infty})/(\|\nabla^2 f(x_{\infty})\| + \epsilon), x - x_{\infty} \rangle \leq 0$$

Recalling that  $\Delta_k \geq \Delta_{\min} > 0$  for all k, one obtains the desired conclusion  $-\nabla f(x_{\infty}) \in N(C, x_{\infty})$ .

Interestingly, it is possible to replace  $\tau_k$  by

$$\tau_k^{new} = \frac{2l_0(f(x_k) - f(x_k + p_k))}{\rho_{x_k} \|p_k\|^2}$$
(10)

and obtain a similar result.

**Corollary 3.1** Let  $\{x_k\}$  be a sequence generated by Algorithm 3.1, under the modification (10), applied to a twice continuously differentiable function f for which  $\nabla^2 f$  is Lipschitz continuous on C. Then the sequence  $\{f(x_k)\}$  is decreasing and  $\lim_{k\to+\infty} ||x_{k+1} - x_k|| = 0$ . Moreover, every limit point  $x_\infty$  of the sequence  $\{x_k\}$  is a first-order critical point of problem  $(\mathcal{P})$ , that is,  $0 \in \nabla f(x_\infty) + N(C, x_\infty)$ .

**Proof.** From (8) one obtains that  $\tau_k \ge \eta$  implies  $\tau_k^{new} \ge \eta$ . Thus, if an iteration is successful for Algorithm 3.1 so it is for the modified version of the algorithm. The rest of the proof is exactly as in the one of Theorem 3.1.

The search direction  $p_k$  could have also been computed by solving approximately the trustregion subproblem (5) using the DCA (Algorithm 2.1) and the DC decomposition

$$m(x_k + p, x_k) = \left( m(x_k + p, x_k) + \frac{\rho_{x_k}}{2} \|p\|^2 + \chi_{D_k}(p) \right) - \left( \frac{\rho_{x_k}}{2} \|p\|^2 \right),$$

with  $\rho_{x_k} = \max\{-\lambda_{\min}(\nabla^2 f(x_k)), 0\} + \epsilon$ , where  $\lambda_{\min}(\cdot)$  denotes the smallest eigenvalue of a matrix. The authors believe that it is possible to obtain the same convergent result for this

decomposition as the one described in Theorem 3.1. However, each internal iteration of DCA would have then required the solution of an auxiliary problem of the form

min 
$$m(x_k + p, x_k) + \frac{\rho_{x_k}}{2} ||p||^2 - \langle p, q_k^l \rangle$$
 subject to  $p \in D_k$ ,

which would have been more expensive when compared to what happens in Algorithm 3.1, where the bulk of the work per one internal iteration of DCA amounts to one projection onto  $D_k$ .

# 4 Implementation issues, test problems, and profiles

#### 4.1 Implementation issues

To provide an assessment of the proposed methodology we developed an implementation for Algorithm 3.1, called TRDC (Trust Region Difference of Convex). As already mentioned in the introduction, our implementation only addresses bound-constrained problems, i.e., problems of the form (1) where  $C = \{x \in \mathbb{R}^n : \ell \leq x \leq u\}$ , with  $\ell \in (\mathbb{R} \cup \{-\infty\})^n$  and  $u \in (\mathbb{R} \cup \{+\infty\})^n$ . To make projections onto  $(C - \{x_k\}) \cap B(0, \Delta_k)$  fast, see (5), we considered  $B(0, \Delta_k)$  defined using the  $\ell_{\infty}$ -norm. Since the solvers that we are using later for the numerical results are implemented in C++ (IPOPT) and Fortran (Lancelot B and TRON), and in order to provide a fair comparison, our implementation of TRDC was made in C.

While Algorithm 3.1 requests a positive integer  $l_0$  (the number of internal DCA iterations), performing more internal iterations than needed to solve the trust-region subproblem (5) will lead to inefficiency. Also, considering  $\rho_{x_k} = \|\nabla^2 f(x_k)\| + \epsilon$  may also lead to a high number of DCA internal iterations. Therefore, we consider an adaptive strategy for updating  $\rho_{x_k}$ , making it also dependent on the DCA internal loop counter l. Thus,  $\rho_{x_k}$  will be hereafter denoted by  $\rho_{x_k}^l$ . We start with a smaller value  $\rho_{x_k}^0$  (set to  $2^{-2} (\|\nabla^2 f(x_k)\| + \epsilon)$ ,  $\epsilon = 0.1$ , in our implementation), and multiply it by a factor of  $\rho_{factor} = 2$  in each inner iteration l. We then stop the DCA internal loop if  $m(x_k, x_k) - m(x_k + p_k^{l+1}, x_k) \ge C \|p_k^{l+1}\|^2$ , with  $C = 10^3$ . Such a stopping criterion is totally consistent with our theory, even when  $\rho_{x_k}^l$  is still below  $\|\nabla^2 f(x_k)\| + \epsilon$ , since we can see from the subsequent use of the lower bound (8) that all it is required is a reduction in the predicted decrease of the order of the square of the step (being irrelevant the size of the constant multiplying it). Finally, a maximum number of  $l_0 = 300$  DCA inner iterations is also considered.

As in TRON, which is also a bound-constrained type solver, we stop the external iterations, declaring success, whenever the absolute error in the objective function is small,  $|f(x_k) - f(x_k + p_k)| < \epsilon_{tol}$ , and the predicted reduction is small,  $m(x_k, x_k) - m(x_k + p_k, x_k) < \epsilon_{tol}$ , with  $\epsilon_{tol}$  set to  $10^{-12}$  as in TRON. A run of TRDC is stopped unsuccessfully if it exceeds a maximum number of external iterations (maxiter), a maximum of total internal DCA iterations (maxiterDCA), or a maximum of objective function evaluations (maxfeval), with maxiterDCA =  $10^7$ , maxfeval = 1000, and maxiter = 1000.

To improve numerical performance, and as in **IPOPT**, we considered instead a scaled objective function  $f^*$ , given by  $f^*(x) = \zeta f(x)$ , with

$$\zeta = \min\left(1, \frac{100}{\|\nabla f(x_0)\|}\right),\,$$

where  $x_0$  is the projection onto the feasible region of the user provided initial guess (e.g., given by CUTEr [15]). The scaling parameter  $\zeta$  is computed at the algorithm initialization and kept fixed for the remaining procedure. When  $x_0$  is not provided, we compute a feasible initial guess in the following componentwise fashion: the middle value of the bounds when both are finite, the finite bound when one of the bounds is finite, or 0 whenever the variable is free.

A final implementation issue is related to the update of the trust-region radius  $\Delta_k$ , described in **Step 3** of Algorithm 3.1. We provide the details of the updating scheme for  $\Delta_k$  in the following algorithm.

#### Trust-region radius update

• If  $\tau_k > \eta_3$ ,

- then increase the trust-region radius by setting  $\Delta_{k+1} = \min(\bar{\gamma}_3 \Delta_k, 1000)$ ,
- otherwise, if  $\tau_k < \eta_1$ 
  - \* then set  $\Delta_{k+1} = \bar{\gamma}_1 \Delta_k$
  - \* otherwise if  $\tau_k < \eta_2$ ,
    - · then set  $\Delta_{k+1} = \bar{\gamma}_2 \Delta_k$
    - · otherwise set  $\Delta_{k+1} = \Delta_k$ .

By taking  $0 < \eta_1 \le \eta_2 \le \eta_3 < 1$  and  $0 < \bar{\gamma}_1 \le \bar{\gamma}_2 < 1$ ,  $\bar{\gamma}_3 \ge 1$ , this scheme satisfies the conditions required in Step 3 of Algorithm 3.1. In practice, we started with  $\Delta_0 = 1$  and used  $\eta_1 = 10^{-3}$ ,  $\eta_2 = 0.25$ ,  $\eta_3 = 0.75$ ,  $\bar{\gamma}_1 = 0.5$ ,  $\bar{\gamma}_2 = 0.5$ , and  $\bar{\gamma}_3 = 2$ .

### 4.2 Test problems

In order to insure a proper comparison of the implemented solver with state-of-the-art optimization solvers, we decided to consider the CUTEr [15] test problems collection. From the complete test set there available, we selected all the unconstrained and bound-constrained problems, resulting in the 271 test problems reported in Table 1.

#### 4.3 Profiles

Using a large number of test problems demands for an aggregated way to show the numerical results. For a better visualization and brevity in the presentation of the numerical results, we are providing performance profiles obtained by using the procedure described in [12]. We consider also the modification made in [33] for the case where the metric used for performance does not always return a strictly positive value, as required in the original performance profiles. The major advantage of performance profiles is that they can be presented in one figure, by plotting, for the different solvers, a cumulative distribution function  $v(\pi)$  representing a performance ratio.

The performance ratio is defined by setting  $r_{p,s} = \frac{t_{p,s}}{\min\{t_{p,z}:z\in\mathcal{S}\}}$ ,  $p \in \mathcal{P}$ ,  $s \in \mathcal{S}$ , where  $\mathcal{P}$  is the test set,  $\mathcal{S}$  is the set of solvers, and  $t_{p,s}$  is a measure of performance of the application of solver s on test problem p. Then, one defines  $v_s(\pi) = \frac{1}{|\mathcal{P}|} \operatorname{size} \{p \in \mathcal{P} : r_{p,s} \leq \pi\}$ , where  $|\mathcal{P}|$  is the number of test problems. The value of  $v_s(1)$  is then the percentage of times that the solver swins over the remaining ones (or ties the best solver). If we are only interested in determining which solver is the best (in the sense that wins the most), we compare the values of  $v_s(1)$  for all the solvers. At the other end,  $v_s(\pi)$  for large values of  $\pi$  indicates the percentage of problems solved successfully by solver s, and thus serves as a measure of robustness.

EqPTUAR         1         KOEEHELB         3         PALMEREA         8         EXPLIN         1.200         SERVEND         5.000           AKIVA         2         MEYER3         3         PALMERED         8         EXPLIN2         1200         SCGBINE         5000           BERMEC         2         PFITILS         3         PALMEREC         8         EXPLIN2         1200         SCABINE         5000           CAMELG         2         PFITILS         3         PALMERCE         8         EXPLINA         2000         SRASINE         5000           CLIFF         WEDS         3         PALMERCE         8         RAYBENDS         2050         TESTQUAD         5000           CUBE         2         YFIT         3         PALMERCE         8         DIXMAANB         3000         TOUINTCSS         5000           DENSCHM         2         ALLINIT         4         VIERREAM         8         DIXMAANB         3000         CLILIT         5001           DENSCHM         2         HATTLDA         4         SPARANGUR         9         DIXMAANG         3000         CLILATE         5041           HATR         1         HIMMELBE         4	Problem	n	Problem	n	Problem	n	Problem	n	Problem	n
KITVA         2         PKIYER3         3         PALMERED         8         EXPQUAD         1200         SCHWUTT         5000           BEALE         2         PFITILS         3         PALMEREC         8         EXPQUAD         1200         SCGNINE         5000           BERNCC         2         PFITILS         3         PALMERCE         8         ELINVERSE         1999         SINQUAD         5000           CAMELG         2         PFITILS         3         PALMERCE         8         RAVEENDS         2050         TESTQUAD         5000           CUBE         2         PFITILS         3         PALMERCE         8         DIXMAAN         3000         TUNTGS         5000           DEMSCHNE         2         ALLINIT         4         SIBRERE         9         DIXMAAN         3000         TRIDIA         5003           DENSCHNF         2         BRUWNDEN         4         PALMEREE         9         DIXMAAN         3000         CLPLATEA         5041           EXPETIT         2         HATMEDB         4         SSCLPATH         10         DIXMAAN         3000         CLPLATEB         5041           HATMELB         2         NAUMERS	BQP1VAR	1	KOEBHELB	3	PALMER5A	8	EXPLIN	1200	SBRYBND	5000
EALE         2         PFITILS         3         PALMERGE         8         EXPQUAD         1.200         SCGSINE         5.000           BRKMCC         2         PFITILS         3         PALMERCE         8         LINVERSE         1.909         SINQUAD         5000           CAMEL6         2         PFITILS         3         PALMERCE         8         RAYBENDS         2060         SPARSINE         5000           CLIFF         2         VFIT         3         PALMERCE         8         DIXMAAN         3000         TQUARTIC         5000           DENSCHMA         2         VFIT         3         PALMERCE         8         DIXMAAN         3000         TQUARTIC         5000           DENSCHMC         2         ALLINIT         4         SSG8         DIXMAAN         3000         RCUNLS         5001           DENSCHMC         2         BROWNER         4         DIXMAAN         3000         RCUPLATEA         5041           EXPFIT         2         HATELDF         4         SCIPATH         10         DIXMAAN         3000         CLFLATEA         5041           HIMELBC         2         PALMERER         BIXMAANC         3000         SCIPAT	AKIVA	2	MEYER3	3	PALMER5D	8	EXPLIN2	1200	SCHMVETT	5000
PRRMCC         2         PFITZLS         3         PALMER6E         8         LINURESE         199         SINQUAD         5000           CAMELG         PFITTLS         3         PALMER7C         8         RATERINC         2000         SPASINE         5000           CLIFF         2         WEEDS         3         PALMER7C         8         RATERINC         2050         SROSENHR         5000           CLIFF         2         WEEDS         3         PALMER7C         8         RATERINC         2050         TESTQUAD         5000           CUEE         2         YFITU         3         PALMER8E         8         DIXMAANG         3000         TQUARTIC         5000           DENSCHN         2         ALLINIT         4         VIBREAM         8         DIXMAANG         3000         CLPLATEA         5041           DITL         2         HATFLDB         4         SPECAN         9         DIXMAANG         3000         CLPLATEC         5041           HAILBERTA         2         HAIMELB         4         SPECAN         9         DIXMAANG         3000         CLPLATEC         5041           HAILBERTA         4         SPECAN         9	BEALE	2	PFIT1LS	3	PALMER5E	8	EXPQUAD	1200	SCOSINE	5000
BROWNES         2         PFITISLS         3         PALMERCE         8         EDENSCH         2005         SROSENER         5000           CAMEL6         2         VETT         3         PALMERCE         8         RAYEENDS         2050         TESTUJUAD         5000           CUEF         2         VETT         3         PALMERSC         8         DIXMAANA         3000         TUNTGSS         5000           DENSCHNA         2         ALLINIT         4         S368         8         DIXMAANC         3000         TUNTA         5000           DENSCHNC         2         ALLINIT         4         S368         8         DIXMAANC         3000         CLPLATE         5041           DENSCHNC         2         HATFLDA         4         SPECAN         9         DIXMAANC         3000         CLPLATE         5041           HATRY         2         HATS         4         SPECAN         9         DIXMAANC         3000         SCC         5184           HIMMELBF         4         HISENT         10         DIXMAANC         3000         NDC         5184           HIMMELBF         2         PALMER14         GSBORNEB         11         DIXMA	BRKMCC	2	PFIT2LS	3	PALMER6C	8	LINVERSE	1999	SINQUAD	5000
CAMELE         2         PFITALS         3         PALMERTE         8         RAYBENDS         2050         SRGSENBR         5000           CLIFF         2         WEEDS         3         PALMERCE         8         RAYBENDS         2050         TESTQUAD         5000           CUBE         2         YFITU         3         PALMERCE         8         DIXMAANR         3000         TUINTOSS         5000           DENSCHNC         2         ALLINIT         4         ST86         8         DIXMAANC         3000         RCUDINTOSS         5001           DENSCHNC         2         ALLINITU         4         VIERBEAM         8         DIXMAANC         3000         RCULATE         5041           DITL         2         HATFLDB         4         SPECAN         9         DIXMAANG         3000         CLPLATEE         5041           HATELDE         4         HIMELET         10         DIXMAANG         3000         SCC         5184           HIMMELBE         2         KOWDSE         4         DSCIPATH         10         DIXMAANG         3000         NCRSIDNI         5476           HS3         2         PALMERA         4         DSTMAANK	BROWNBS	2	PFIT3LS	3	PALMER6E	8	EDENSCH	2000	SPARSINE	5000
CLIFF         2         VEEDS         3         PALMERSC         8         RAYBENDS         2050         TESTQUAD         5000           CUBE         2         YFIT         3         PALMERSC         8         DIXMAANA         3000         TUDARTIC         5000           DENSCHNB         2         ALLINIT         4         S268         8         DIXMAANC         3000         TCUDARTIC         5000           DENSCHNC         2         ALLINIT         4         VIERBEAM         8         DIXMAANC         3000         SCONDLIS         5002           DENSCHNC         2         ALLINIT         4         SPARSQUR         9         DIXMAANC         3000         SCONDLIS         5001           DENSCHNC         2         ALMPELD         4         SPARSQUR         9         DIXMAANC         3000         SCLPLATEE         5041           HATEND         1         DIXMAANT         3000         SCC         5184           HIMMELBG         2         PALMERTE         4         DIXMAANT         3000         SCC         5184           HIMMELBG         2         PALMERTE         4         DIXMAANT         3000         NESION         5366	CAMEL6	2	PFIT4LS	3	PALMER7C	8	RAYBENDL	2050	SROSENBR	5000
CUBE         2         YFIT         3         PALMERSE         8         DIXMAANB         3000         TOINTGSS         5000           DENSCHNA         2         YITU         3         PALMERSE         8         DIXMAANB         3000         TQUARTIC         5000           DENSCHNC         2         ALLINITU         4         VIERBEAM         8         DIXMAANE         3000         SCONDILS         5002           DENSCHNC         2         HATFLDA         4         SPARSQUR         9         DIXMAANF         3000         CLPLATEA         5041           LYNTL         2         HATFLDB         4         SPECAN         9         DIXMAANF         3000         CLPLATEE         5041           HILBERTT         2         HIMMELBF         4         HILBERTB         10         DIXMAANF         3000         SCC         5184           HIMMELBE         2         PALMER1B         4         WATSON         12         CHAIMANI         3000         TORSIONT         5476           HIMMELBE         2         PALMER2         4         DIXMAANK         15         WODS         4000         TORSIONT         5476           HS3         2         PALMER2B <td>CLIFF</td> <td>2</td> <td>WEEDS</td> <td>3</td> <td>PALMER7E</td> <td>8</td> <td>RAYBENDS</td> <td>2050</td> <td>TESTQUAD</td> <td>5000</td>	CLIFF	2	WEEDS	3	PALMER7E	8	RAYBENDS	2050	TESTQUAD	5000
DENSCHNA         2         YFITU         3         PALMERSE         8         DIXMAANE         3000         TQUARTIC         5000           DENSCHNE         2         ALLINIT         4         S368         8         DIXMAANE         3000         TRIDIA         5000           DENSCHNF         2         BROWNDEN         4         PALMERSE         9         DIXMAANE         3000         CLPLATER         5001           DJTL         2         HATFLDA         4         SPARSQUR         9         DIXMAANE         3000         CLPLATER         5041           HAIRY         2         HIMMELBF         4         HILBERTB         10         DIXMAANH         3000         CLPLATEC         5041           HIMMELBE         2         NALMERS         4         DSCIPATH         10         DIXMAANL         3000         MISURFO         5306           HIMMELBE         2         PALMER2E         4         HIXMAANK         5         WODN         5476           HS1         2         PALMER2B         4         HATFLDC         25         DRCAVILQ         4489         TORSIDNA         5476           HS3         2         PALMER4B         4         BQPCASIM	CUBE	2	YFIT	3	PALMER8C	8	DIXMAANA	3000	TOINTGSS	5000
DENSCHNE         2         ALLINIT         4         S368         8         DIXMAANC         3000         TIDIA         5000           DENSCHNC         2         ALLINITU         4         VIRBEAM         8         DIXMAAND         3000         SCINDILS         5000           DJTL         2         HATFLDA         4         SPARSQUR         9         DIXMAANF         3000         CLPLATEA         5041           EXPFTT         2         HATFLDA         4         SPARSQUR         9         DIXMAANF         3000         CLPLATEA         5041           HILMELF         4         HILEERTB         DIXMAANH         3000         DCC         5184           HIMMELBE         2         KOWOSB         4         OSCIPATH         10         DIXMAANI         3000         NDS         5366           HIMMELBH         2         PALMER1B         4         WATSON         12         CHAINNOU         0000         NORNDTOR         5476           HS3         2         PALMER2B         4         HATCLDC         25         DRCAV2LQ         4489         TORSIDN2         5476           HS4         2         PALMER3B         4         BQPGASIM         500	DENSCHNA	2	YFITU	3	PALMER8E	8	DIXMAANB	3000	TQUARTIC	5000
DENSCHINC         2         ALLINITU         4         VIBRBEAM         8         DIXMAAND         3000         SCONDILS         5002           DENSCHNF         2         BRGWNDEN         4         PALMERSE         9         DIXMAANE         3000         CLPLATEA         5041           EXPFIT         2         HATFLDA         4         SPARAGUR         9         DIXMAANE         3000         CLPLATEB         5041           HAIRY         2         HIMMELBF         4         HILBERTA         10         DIXMAANI         3000         CLPLATEB         5041           HILBERTA         2         HSS         4         HSI10         10         DIXMAANI         3000         SCC         5184           HIMMELB         2         KAURSE         4         DIXMAANI         15         WODS         4000         TORSIDNI         5476           HIMMELP1         2         PALMER2         4         HATFLC         25         DRCAVILQ         4489         TORSIDNI         5476           HS3         2         PALMER3         3         SPK         50         DRCAVILQ         4489         TORSIDNI         5476           HS4         2         PALMER4	DENSCHNB	2	ALLINIT	4	S368	8	DIXMAANC	3000	TRIDIA	5000
DENSCHNF         2         BROWNDEN         4         PALMER5B         9         DIXMAANE         3000         BRATUID         5003           DJTL         2         HATFLDA         4         SPARSQUR         9         DIXMAANE         3000         CLPLATEA         5041           EXPFIT         2         HATFLDB         4         SPECAN         9         DIXMAANF         3000         CLPLATEA         5041           HALRY         HIMMELBF         4         HILBERTB         10         DIXMAANT         3000         DSC         5184           HIMMELBE         2         KOWOSB         4         OSCIPATH         10         DIXMAANI         3000         SSC         5184           HIMMELBH         2         PALMER1B         4         WATSON         12         CHAINNOU         4000         NOBNDTOR         5476           HS1         2         PALMER2B         4         HATCLDC         25         DRCAVILQ         4489         TORSIDN2         5476           HS3         2         PALMER3B         4         BPCASIM         50         DRCAV3LQ         4489         TORSIDN3         5476           HS3         2         PALMER4B         4	DENSCHNC	2	ALLINTTU	4	VTBRBEAM	8	DTXMAAND	3000	SCOND1LS	5002
DJTL         2         HATFLDA         4         SPARSQUR         9         DIXMAANF         3000         CLPLATEA         5041           EXPFIT         2         HATFLDB         4         SPECAN         9         DIXMAANF         3000         CLPLATEB         5041           HATRY         2         HIMMELBF         4         HILBERTB         10         DIXMAANI         3000         CLPLATED         5041           HIMMELBR         2         HS38         4         HS110         10         DIXMAANI         3000         SEC         5184           HIMMELBR         2         PALMER1         4         OSCIPATH         10         DIXMAANI         3000         MINSURFO         5306           HIMMELP1         2         PALMER2         4         HATFLDC         25         DRCAVILQ         4489         TORSIONI         5476           HS3         2         PALMER3         4         BQFGASIM         50         SPMSRTLS         4999         TORSIONI         5476           HS3         2         PALMER4         4         BQFGASIM         50         BDQRTIC         5000         TORSIONI         5476           HS3         2         PALMER4	DENSCHNF	2	BROWNDEN	4	PALMER5B	9	DIXMAANE	3000	BRATU1D	5003
EXPFIT         2         HATFLDB         4         SPECAN         9         DIXMAANG         3000         CLPLATEB         5041           HAIRY         2         HIMMELBF         4         HILBERTB         10         DIXMAANG         3000         CLPLATEC         5041           HILBERTA         2         HS38         4         HS110         10         DIXMAANJ         3000         DCC         5184           HIMMELBG         2         RAUNOSB         4         OSCIPATH         10         DIXMAANJ         3000         MIXDURD         5306           HIMMELBH         2         PALMER1B         4         WATSON         12         CHAINWOU         4000         NOBNDTOR         5476           HIMMELP1         2         PALMER24         HATFLDC         25         DRCAVILQ         4489         TORSION2         5476           HS3         2         PALMER34         4         BQCASIM         50         DRCAVILQ         4489         TORSION5         5476           HS3         PALMER44         BQPGASIM         50         SPMSRTLS         4999         TORSION5         5476           HS4         2         PALMER44         CHRRINRDS         500	D.ITI.	2	HATFI.DA	4	SPARSOUR	9	DIXMAANE	3000	CL.PL.ATEA	5041
LAIRY         2         HIMMELBF         4         HILBERTB         10         DIXMAANH         3000         CLPLATEC         5041           HILBERTA         2         HS38         4         HS110         10         DIXMAANH         3000         CLPLATEC         5041           HIMMELBE         2         KOWOSB         4         OSCIPATH         10         DIXMAANJ         3000         SSC         5184           HIMMELBE         2         PALMER1E         4         WATSON         12         CHATNAND         4000         MOBIDTOR         5476           HIMMELP1         2         PALMER2B         4         MATFLOC         25         DRCAVILQ         4489         TORSION2         5476           HS3         2         PALMER2B         4         BQPGASIM         50         DRCAV2LQ         4489         TORSION5         5476           HS3         2         PALMER4B         4         BQPGASIM         50         SPMSRTLS         4999         TORSION5         5476           HS3         2         PALMER4B         4         CHANGSNS         50         BDQRTIC         5000         TORSION5         5476           HS4         2         PALMERA5 <td>EXPETT</td> <td>2</td> <td>HATFLDB</td> <td>4</td> <td>SPECAN</td> <td>9</td> <td>DIXMAANG</td> <td>3000</td> <td>CLPLATER</td> <td>5041</td>	EXPETT	2	HATFLDB	4	SPECAN	9	DIXMAANG	3000	CLPLATER	5041
MARK         2         HIMBER         1         HIMBER         1         DIAMINATION         0000         DEC         5184           HILBERTA         2         BASSA         4         BSID         10         DIXMANNI         3000         DDC         5184           HIMMELBB         2         PALMER14         4         OSCIPATH         10         DIXMANNI         3000         MINSURFO         5306           HIMMELBG         2         PALMER14         4         WATSON         12         CHATNWOO         4000         NOBNDTOR         5476           HIS         2         PALMER2B         4         HATFLDC         25         DRCAV2LQ         4489         TORSION3         5476           HS3         2         PALMER3         4         BQPGABIM         50         DRCAV2LQ         4489         TORSION5         5476           HS4         2         PALMER4         4         EQPCASIM         50         BDEXP         5000         TORSION5         5476           HUMPS         18455         TOINTOR         50         BDEXP         5000         TORSION5         5476           LOGHAITY         2         BIGGS6         COINTOR         50	HATRY	2	HIMMELBE	4	HILBERTR	10	DIXMAANH	3000	CLPLATEC	5041
ILIDICITIA         IDIC	HTLRERTA	2	HS38	4	HS110	10	DIXMAANT	3000	ODC	5184
INIMILED         2         DALMER1         4         OUSDATING         DIXMAANL         DOOD         MINSURFO         5306           HIMMELBH         2         PALMER14         4         WATSON         12         CHAINWOO         4000         NOBSIDIR         5476           HIMMELP1         2         PALMER2         4         DIXMAANK         15         WODDS         4000         NOBSIDIR         5476           HS1         2         PALMER2B         4         HATFLDC         25         DRCAV1LQ         4489         TORSION1         5476           HS3         2         PALMER3B         4         BQFGABIM         50         DRCAV3LQ         4489         TORSION5         5476           HS3         2         PALMER4B         4         BQFGABIM         50         DRCAV3LQ         4489         TORSION5         5476           HS4         2         PALMER4B         4         EQFGABIM         50         BDQRTIC         5000         TORSIONA         5476           HS5         2         PSPDOC         4         ERRINROS         50         BDQRTIC         5000         TORSIONA         5476           LOGHAIRY         BIGGSS         6         TO	HIMMEIRR	2	KUMUZB	4	ПОТТРАТН	10	DIXMAANI	3000	55C	5184
INIMELED         2         FALMENT         4         ODECNALD         11         DIAMAD         5000         MISONI         5000           HIMMELP1         2         PALMER1B         4         WATSON         12         CHAINWOO         4000         TORSIONI         5476           HS1         2         PALMER2B         4         HATFLDC         25         DRCAVILQ         4489         TORSIONI         5476           HS2         2         PALMER3         4         BPCABIN         50         DRCAVILQ         4489         TORSIONS         5476           HS3         2         PALMER3         4         BQPGABIN         50         DRCAVILQ         4489         TORSIONS         5476           HS4         2         PALMER4         BQPGASIN         50         SPMSRTLS         4999         TORSIONS         5476           HS5         2         PSPDOC         4         ERNINDS         500         TORSIONS         5476           JENSMP         2         OSBORHEA         5         TOINTGOR         50         BRUGNDT         5000         TORSIONS         5476           LOGHAIRY         2         BIGGS5         6         VAREICVL         50	HIMMEI BC	2	DAI MER 1	1	OSBORNER	11	DIXMAANI	3000	MINGUREO	5306
INTERLEDI       2       FALMERID       4       WATSON       12       GUARINGO       4000       TORSIONI       5476         HS1       2       PALMER2B       4       HATFLDC       25       DRCAVILQ       4489       TORSIONI       5476         HS2       2       PALMER3B       4       BQPGABIM       50       DRCAVILQ       4489       TORSIONI       5476         HS3       2       PALMER4B       4       BQPGABIM       50       DRCAVILQ       4489       TORSIONI       5476         HS3       0       2       PALMER4       4       BQPGASIM       50       SPMSRTLS       4999       TORSIONIS       5476         HS4       2       PALMER4B       4       CHNROSNB       50       BDQRTIC       5000       TORSIONIS       5476         HS5       2       PSPDC       4       ERRINROS       50       BDQRTIC       5000       TORSIONIS       5476         JENSMP       2       DISGOS       6       TOINTYDOR       50       BROSDN7D       5000       TORSIONIS       5476         LOGRAS       2       BIGGS       6       DECONVU       61       CHENHARK       5000       TORSIONIS	UTMMET DU	2	PALMERI DAIMEDID	4	USBURNEB	10	CUATNUOO	4000	MINSORFO	5300
HINDELF1       2       FALMER2B       4       DIAMARNK       13       WOUDS       4000       TORSION1       5476         HS2       2       PALMER2B       4       BPC       30       DRCAV12LQ       4489       TORSION2       5476         HS3       2       PALMER2B       4       BQPGAEIM       50       DRCAV12LQ       4489       TORSION5       5476         HS3MOD       2       PALMER4B       4       BQPGAEIM       50       DRCAV3LQ       4489       TORSION5       5476         HS4       2       PALMER4B       4       EQRGASIM       50       BRWERLS       4999       TORSION5       5476         HS5       2       PSPD0C       4       ERRINCOS       50       BDQRTIC       5000       TORSION5       5476         JENSMP       2       DIGGS3       6       TOINTOR       50       BOYDN7D       5000       TORSION5       5476         LOGRAIRY       2       BIGGS5       6       VAREIGVL       50       BRYBND       5000       TORSION5       5476         MEAHAT       2       BIGGS6       OECONVB       61       CHENHARK       5000       TORSION5       5476		2	PALMERID	4	WAISON DTYMAANV	15	LIDODG	4000	TOPSTONI	5470 E476
HS1         2         FALMERAB         4         INFLUC         23         DRCAVILQ         4489         TORSION3         5476           HS3         2         PALMERAB         4         BQPGABIM         50         DRCAVILQ         4489         TORSION4         5476           HS3         2         PALMERAB         4         BQPGABIM         50         DRCAVILQ         4489         TORSION4         5476           HS4         2         PALMERAB         4         CHNROSNE         50         BDEXP         5000         TORSION5         5476           HUMPS         2         HS45         5         TOINTGOR         50         BDEXP         5000         TORSIONE         5476           JENSMP         2         OSBGRNEA         5         TOINTOR         50         BROYDN7D         5000         TORSIONE         5476           LOGRAS         2         BIGGS6         6         DECONVU         61         CHAGGLY         5000         TORSIONE         5476           MARATOSB         2         BIGGS6         G         DECONVU         61         CHAGGLY         5000         TORSIONE         5476           MEALAT         HEARTGLS         6	UC1	2	PALMERZ	4	UIAMAANA UATELDO	10		4000	TORSIONI	5470
HS2       2       PALMER3B       4       SPR       30       DRCAV2LQ       4489       TURSION3       5476         HS3MOD       2       PALMER3B       4       BQPGASIM       50       DRCAV3LQ       4489       TURSION3       5476         HS4       2       PALMER4B       4       BQPGASIM       50       SPMSRTLS       4999       TURSION4       5476         HS5       2       PSPDOC       4       ERRINROS       50       BDQRTIC       5000       TORSIONA       5476         JENSMP       2       OSBORAEA       5       TOINTGOR       50       BDQRTIC       5000       TORSIONC       5476         LOGRAS       2       BIGGS3       6       TOINTQOR       50       BROYDN7D       5000       TORSIONC       5476         LOGRAS       2       BIGGS6       CARNETOVL       61       CHABHAK       5000       TORSIONE       5476         MARTOSE       2       PALMER1A       6       HYDC2OLS       99       DQRTIC       5000       RIDSENA       5625         S0308       2       PALMER4A       6       SENORS       100       FLETCBV2       5000       CURLY10       10000         <		2	PALMER2D	4	ATFLDC	20	DRCAVILQ	4409	TODGION2	5470
HS3         2         PALMERA         4         BQPGABIN         50         DRCAVSLQ         4499         TURSION4         5476           HS3M0D         2         PALMERAB         4         BQPGASIM         50         SPMSRTLS         4999         TURSION4         5476           HS4         2         PALMERAB         4         CHNROSNE         50         BDEXP         5000         TURSIONE         5476           HUMPS         2         HS45         5         TOINTGOR         50         BDQRTIC         5000         TURSIONE         5476           LOGRAIRY         2         BIGGS3         6         TOINTQOR         50         BRYEND         5000         TORSIONE         5476           LOGRAIS         2         BIGGS3         6         TOINTQOR         50         BRYEND         5000         TORSIONE         5476           MARTOSE         2         BIGGS5         6         VAREIGVL         50         BRYEND         5000         TORSIONE         5476           MARTOSE         2         BALMERAS         6         MECONVB         61         CRAGGLVY         5000         TORSIONE         5476           MARTOSE         2         BALMERAS <td>HS2</td> <td>2</td> <td>PALMER3</td> <td>4</td> <td>3PK</td> <td>30</td> <td>DRCAV2LQ</td> <td>4489</td> <td>TURSIUNS</td> <td>5476</td>	HS2	2	PALMER3	4	3PK	30	DRCAV2LQ	4489	TURSIUNS	5476
HS3MUD       2       PALMER44       4       EQPCASIM       50       SPNSKILS       4999       IOKSIUNS       5476         HS4       2       PALMER4B       4       CHNROSNB       50       ARWHEAD       5000       TORSIUNA       5476         HUMPS       2       HS45       5       TOINTGOR       50       BDQRTIC       5000       TORSIUNA       5476         LOGRAIRY       2       BIGGS3       6       TOINTGOR       50       BDQND7D       5000       TORSIUNA       5476         LOGROS       2       BIGGS5       6       VAREIGVL       50       BRYBND       5000       TORSIUNE       5476         MARATOSB       2       BIGGS6       O DECONVB       61       CHAGLVY       5000       TORSIUNE       5476         MDHOLE       2       HART6       6       DECONVU       61       CRAGGLVY       5000       FMINSRF2       5625         S308       2       PALMER1A       6       HYDC2LS       99       DQRTIC       5000       CURLY10       10000         SIM2BQP       2       PALMERA4       6       SENSORS       100       FLETCBV2       5000       CURLY10       10000	H53	2	PALMER3B	4	BUPGABIM	50	DRCAV3LU	4489	IURSIUN4	5476
HS42PALMER4B4CHNNUSNE50AKWHEADS000IDKSIDN65476HS52PSPD0C4ERRINROS50BDEXP5000TORSIDNA5476HUMPS2HS455TOINTGOR50BDQRTIC5000TORSIDNE5476LOGHAIRY2BIGGS36TOINTQOR50BROYDN7D5000TORSIDNE5476LOGRAS2BIGGS66VAREIGVL50BRYBND5000TORSIDNE5476MARATOSE2BIGGS66DECONVB61CHENHARK5000TORSIDNE5476MDHOLE2HART66DECONVU61CRAGGLVY5000TORSIDNE5476MEXHAT2HEARTGLS6MINSURF64DQDRTIC5000IMINSURF5625S3082PALMER1A6HYDC2OLS99DQRTIC5000CURLY1010000SIMEQP2PALMERAA6CHEBYQAD100ENGVAL15000CURLY2010000SINEQP2PALMERAA6SENSORS100FLETCBV35000CURLY1010000SINSER2PALMERAA6ARGLINA200FLETCBV35000CURLY2010000SINSER2PALMERAA6ARGLINA200FLETCBV35000JILBRNG110000SINSER2PALMERAA6BROWNAL200INDEF5000JILBRNG21	HS3MUD	2	PALMER4	4	BUPGASIM	50	SPMSRILS	4999	TURSIUNS	5476
HSS         2         PSPDUC         4         ERRINRUS         50         BDEXP         5000         TURSLUNA         5476           HUMPS         2         HS45         5         TOINTGOR         50         BDQRTIC         5000         TURSLUNA         5476           LOGHAIRY         2         BIGGS3         6         TOINTGOR         50         BROYDN7D         5000         TURSLUNC         5476           LOGROS         2         BIGGS6         6         DECONVB         61         CHENNARK         5000         TURSLUNC         5476           MARATOSB         2         BIGGS6         6         DECONVB         61         CHENNARK         5000         TURSLUNF         5625           MDHOLE         2         HART6         6         MINSURF         64         DQRTIC         5000         RUMSURF         5625           S308         2         PALMER1A         6         HNOCOLS         99         DQRTIC         5000         CURLY10         10000           SIMEQP         2         PALMER3A         6         MANCINO         100         FLETCBV3         5000         CURLY10         10000           SINEVAL         2         PALMER4A	HS4	2	PALMER4B	4	CHNRUSNB	50	ARWHEAD	5000	TURSIUN6	5476
HUMPS         2         H845         5         TUINTGUR         50         BDQRTIC         5000         TURSIUNB         5476           JENSMP         2         OSBORNEA         5         TOINTPSP         50         BIGGSB1         5000         TURSIUNE         5476           LOGHAIRY         2         BIGGS3         6         TUINTGUR         50         BROYDN7D         5000         TURSIUNE         5476           LOGROS         2         BIGGS5         6         VAREIGVL         50         BRYBND         5000         TURSIUNE         5476           MARATOSB         2         BIGGS6         6         DECONVE         61         CHENHARK         5000         TURSIUNE         5476           MDHOLE         2         HART6         6         DECONVE         61         CHENHARK         5000         TURSIUNE         5476           S008         2         PALMERAS         6         MINSURF         64         DQDRTIC         5000         INSURF         5625           S308         2         PALMERAA         6         CHEBYQAD         100         ENGVAL1         5000         CURLY10         10000           SIMEQP         PALMERAA         6 <td>HS5</td> <td>2</td> <td>PSPDOC</td> <td>4</td> <td>ERRINROS</td> <td>50</td> <td>BDEXP</td> <td>5000</td> <td>TORSIONA</td> <td>5476</td>	HS5	2	PSPDOC	4	ERRINROS	50	BDEXP	5000	TORSIONA	5476
JENSMP         2         OSBORNEA         5         TOINTPSP         50         BIGGSD         5000         TORSIONC         5476           LOGHAIRY         2         BIGGSS         6         TOINTQOR         50         BROYDN7D         5000         TORSIONC         5476           LOGROS         2         BIGGS5         6         VAREIGVL         50         BRYBND         5000         TORSIONE         5476           MARATOSB         2         BIGGS5         6         DECONVB         61         CHENHARK         5000         TORSIONE         5476           MCHOLE         2         HART6         6         DECONVU         61         CHAGGUY         5000         FMINSRF2         5625           MEXHAT         2         HEART6LS         6         MINSURF         64         DQDRTIC         5000         CURLNER         6218           SIMEQP         2         PALMERA         6         CHEBYQAD         100         FLETCBV2         5000         CURLY10         10000           SINEVAL         2         PALMERA         6         SENCRS         100         FLETCBV2         5000         CURLY20         10000           SINEVAL         2         PAL	HUMPS	2	HS45	5	TOINTGOR	50	BDQRTIC	5000	TORSIONB	5476
LOGHATRY         2         BIGGS3         6         TOINTQOR         50         BROYDN7D         5000         TORSTOND         5476           MARATOSB         2         BIGGS5         6         VAREIGVL         50         BRYBND         5000         TORSIONE         5476           MARATOSB         2         BIGGS6         6         DECONVB         61         CHENHARK         5000         TORSIONF         5476           MDHOLE         2         HART6         6         DECONVU         61         CRAGGLVY         5000         FMINSURF         5625           ROSENBR         2         PALMER1A         6         HYDC2OLS         99         DQRTIC         5000         CUINISURF         5625           S308         2         PALMER2A         6         CHEBYQAD         100         ENGVAL1         5000         CUINY10         10000           SIMBQP         2         PALMER3A         6         MANCINO         100         FLETCBV3         5000         CURLY10         10000           SIMBQP         2         PALMER4A         6         SENSORS         100         FLETCBV3         5000         DIXON3DQ         10000           SINSER         2	JENSMP	2	OSBORNEA	5	TOINTPSP	50	BIGGSB1	5000	TORSIONC	5476
LOGROS         2         BIGGS5         6         VAREIGVL         50         BRYEND         5000         TORSIONE         5476           MARATOSB         2         BIGGS6         6         DECONVB         61         CHENHARK         5000         TORSIONE         5476           MDHOLE         2         HART6         6         DECONVU         61         CRAGLVY         5000         FMINSRF2         5625           MEXHAT         2         HEART6LS         6         MINSURF         64         DQDRTIC         5000         NLMSURF         5625           S08         2         PALMER1A         6         HYDC2OLS         99         DQRTIC         5000         RCIBUA         6218           S1M2BQP         2         PALMER2A         6         CHEBYQAD         100         ENCVAL1         5000         CURLY10         10000           SIMSQP         2         PALMER3A         6         MARCINA         200         FLETCBV3         5000         CURLY20         10000           SINSER         2         PALMER5C         6         ARGLINC         200         GENHUMPS         5000         JILBRNG1         10000           SNAIL         2         PALMER7	LOGHAIRY	2	BIGGS3	6	TOINTQOR	50	BROYDN7D	5000	TORSIOND	5476
MARATOSE         2         BIGGS6         6         DECONVB         61         CHENHARK         5000         TÜRSIDNF         5476           MDHOLE         2         HART6         6         DECONVU         61         CRAGGLVY         5000         FMINSRF2         5625           MEXHAT         2         HEART6LS         6         MINSURF         64         DQDRTIC         5000         LMINSURF         5625           S08         2         PALMER1A         6         HYDC2OLS         99         DQRTIC         5000         RIDGENA         6218           S1M2BQP         2         PALMER2A         6         CHEBYQAD         100         FLETCBV2         5000         CURLY10         10000           SIMEQP         2         PALMER3A         6         MACINO         100         FLETCBV3         5000         CURLY10         10000           SINEVAL         2         PALMER5C         6         ARGLINC         200         FREUROTH         5000         CURLY20         10000           SINSER         2         PALMER7A         6         ARGLINC         200         ILARMTP5         5000         JILBRNG1         10000           SAALL         2 <td< td=""><td>LOGROS</td><td>2</td><td>BIGGS5</td><td>6</td><td>VAREIGVL</td><td>50</td><td>BRYBND</td><td>5000</td><td>TORSIONE</td><td>5476</td></td<>	LOGROS	2	BIGGS5	6	VAREIGVL	50	BRYBND	5000	TORSIONE	5476
MDHOLE         2         HART6         6         DECONVU         61         CRAGGLVY         5000         FMINSRF2         5625           MEXHAT         2         HEART6LS         6         MINSURF         64         DQDRTIC         5000         LMINSURF         5625           SO88         2         PALMER1A         6         HYDC2OLS         99         DQRTIC         5000         RUMSURF         5625           S308         2         PALMER1A         6         CHEBYQAD         100         ENGVAL1         5000         GRIDGENA         6218           SIM2BQP         2         PALMER3A         6         CHEBYQAD         100         FLETCBV2         5000         CURLY10         10000           SIMEQP         2         PALMER4A         6         SENSORS         100         FLETCBV3         5000         CURLY10         10000           SINEVAL         2         PALMER6A         6         ARGLINA         200         FREUROTH         5000         DIXON3DQ         10000           SINSER         2         PALMER7A         6         ARGLINA         200         ILIARWHD         5000         JNLBRNG1         10000           ZANGWIL2         PALMER7A	MARATOSB	2	BIGGS6	6	DECONVB	61	CHENHARK	5000	TORSIONF	5476
MEXHAT2HEARTGLS6MINSURF64DQDRTIC5000LMINSURF5625ROSENBR2PALMER1A6HYDC2OLS99DQRTIC5000NLMSURF5625S3082PALMER2A6CHEBYQAD100ENGVAL15000GRIDGENA6218SIM2BQP2PALMER3A6MANCINO100FLETCBV25000CUSINE10000SIMBQP2PALMER4A6SENSORS100FLETCBV35000CURLY1010000SINEVAL2PALMER5C6ARGLINA200FLETCBV35000CURLY2010000SISSER2PALMER6A6ARGLINC200GENHUMPS5000DIXON3DQ10000SNAIL2PALMER7A6ARGLINC200GENHUMPS5000JNLBRNG110000BARD3PALMER1D7PENALTY2200LIARWHD5000JNLBRNG110000BOX23AIRCRFTB8VARDIM200MCCORMCK5000JNLBRNG810000DENSCHND3MAXLIKA8GENROSE500NONCVXU25000NCVXBQP110000DENSCHNE3OSLBQP8PROBPENL500NONCVXU25000NCVXBQP110000DENSCHNE3SALMER1C8GENROSE500NONCVXU35000NCVXBQP110000DENSCHNE3PALMER1C8GER1000NONSCOMP<	MDHOLE	2	HART6	6	DECONVU	61	CRAGGLVY	5000	FMINSRF2	5625
ROSENBR2PALMER1A6HYDC2OLS99DQRTIC5000NLMSURF5625S3082PALMER2A6CHEBYQAD100ENGVAL15000GRIDGENA6218SIM2BQP2PALMER3A6MANCINO100FLETCBV25000COSINE10000SIMEQP2PALMER3A6MANCINO100FLETCBV35000CURLY1010000SINEVAL2PALMER6C6ARGLINA200FLETCHBV5000CURLY2010000SISSER2PALMER6A6ARGLINB200FREURDTH5000DIXON3DQ10000SNAIL2PALMER7A6ARGLINC200GENHUMPS5000DIXON3DQ10000ZANGWIL22PALMER7A6ARGLINC200INDEF5000JNLBRNG110000BARD3PALMER1D7PENALTY2200LIARWHD5000JNLBRNG210000BOX33HEART8LS8HADAMALS400MOREBV5000JNLBRNGB10000DENSCHND3MAXLIKA8GENROSE500NONCVXU25000NCVXBQP110000EG13PALMER1C8QR3DLS610NONDIA5000NCVXBQP210000DENSCHND3PALMER1C8EXTROSNB1000NCVXBQP310000GROWTHLS3PALMER1C8EXTROSNB1000NONCXBQP310000G	MEXHAT	2	HEART6LS	6	MINSURF	64	DQDRTIC	5000	LMINSURF	5625
S3082PALMER2A6CHEBYQAD100ENGVAL15000GRIDGENA6218SIM2EQP2PALMER3A6MANCINO100FLETCBV25000COSINE10000SIMEQP2PALMER3A6SENSORS100FLETCBV35000CURLY1010000SINEVAL2PALMER5C6ARGLINA200FLETCHBV5000CURLY2010000SINEVAL2PALMER5C6ARGLINB200FREURDTH5000CVRLQP110000SINEVAL2PALMER6A6ARGLINC200GENHUMPS5000DIXON3DQ10000SNAIL2PALMER7A6ARGLINC200GENHUMPS5000DIXON3DQ10000ZANGWIL22PALMER1D7PENALTY2200LIARWHD5000JNLBRNG110000BARD3PALMER1D7PENALTY2200LIARWHD5000JNLBRNG210000BOX23AIRCRFTB8VARDIM200MCCORMCK5000JNLBRNGB10000BOX33HEART3LS8HADAMALS400MOREBV5000JNLBRNG110000DENSCHND3MAXLIKA8GENROSE500NONVCVXU25000NCVXBQP110000DENSCHNE3OSLBQP8PROBPENL500NONVCVXUN5000NCVXBQP310000ENGVAL23PALMER1C8GR3DLS610NONDIA<	ROSENBR	2	PALMER1A	6	HYDC20LS	99	DQRTIC	5000	NLMSURF	5625
SIM2BQP       2       PALMER3A       6       MANCINO       100       FLETCBV2       5000       COSINE       10000         SIMBQP       2       PALMER4A       6       SENSORS       100       FLETCBV3       5000       CURLY10       10000         SINEVAL       2       PALMER5C       6       ARGLINA       200       FLETCBV       5000       CURLY20       10000         SINEVAL       2       PALMER5C       6       ARGLINA       200       FREUROTH       5000       CVXBQP1       10000         SINEVAL       2       PALMER6A       6       ARGLINC       200       GENHUMPS       5000       DIXON3DQ       10000         SNAIL       2       PALMER7A       6       ARGLINC       200       INDEF       5000       JNLBRNG1       10000         BARD       3       PALMER1D       7       PENALTY2       200       LIARWHD       5000       JNLBRNG2       10000         BOX2       3       AIRCRFTB       8       VARDIM       200       MCCORMCK       5000       JNLBRNG8       10000         DENSCHND       3       MAXLIKA       8       GENROSE       500       NONCVXU2       5000       NCVXBQP1	S308	2	PALMER2A	6	CHEBYQAD	100	ENGVAL1	5000	GRIDGENA	6218
SIMBQP         2         PALMER4A         6         SENSORS         100         FLETCBV3         5000         CURLY10         10000           SINEVAL         2         PALMER5C         6         ARGLINA         200         FLETCHBV         5000         CURLY20         10000           SINEVAL         2         PALMER5C         6         ARGLINA         200         FLETCHBV         5000         CURLY20         10000           SINEVAL         2         PALMER6A         6         ARGLINB         200         FREUROTH         5000         DIXON3DQ         10000           SNAIL         2         PALMER7A         6         ARGLINC         200         GENHUMPS         5000         JNLBRNG1         10000           BARD         3         PALMER1D         7         PENALTY2         200         LIARWHD         5000         JNLBRNG2         10000           BOX2         3         AIRCRFTB         8         VARDIM         200         MCCORMCK         5000         JNLBRNG2         10000           DENSCHND         3         MAXLIKA         8         GENROSE         500         NORCVXU2         5000         NCVXBQP1         10000           DENSCHNE         3<	SIM2BQP	2	PALMER3A	6	MANCINO	100	FLETCBV2	5000	COSINE	10000
SINEVAL         2         PALMER5C         6         ARGLINA         200         FLETCHBV         5000         CURLY20         10000           SISSER         2         PALMER6A         6         ARGLINB         200         FREUROTH         5000         CVXBQP1         10000           SNAIL         2         PALMER7A         6         ARGLINC         200         GENHUMPS         5000         DIXON3DQ         10000           ZANGWIL2         2         PALMER7A         6         BROWNAL         200         INDEF         5000         JNLBRNG1         10000           BARD         3         PALMER1D         7         PENALTY2         200         LIARWHD         5000         JNLBRNG2         10000           BOX2         3         AIRCRFTB         8         VARDIM         200         MCCORMCK         5000         JNLBRNG8         10000           BOX3         3         HEART8LS         8         HADAMALS         400         MOREBV         5000         JNLBRNG8         10000           DENSCHND         3         MAXLIKA         8         GENROSE         500         NONCVXUN         5000         NCVXBQP1         10000           EG1         3	SIMBQP	2	PALMER4A	6	SENSORS	100	FLETCBV3	5000	CURLY10	10000
SISSER         2         PALMER6A         6         ARGLINB         200         FREUROTH         5000         CVXBQP1         10000           SNAIL         2         PALMER7A         6         ARGLINC         200         GENHUMPS         5000         DIXON3DQ         10000           ZANGWIL2         2         PALMER8A         6         BROWNAL         200         INDEF         5000         JNLBRNG1         10000           BARD         3         PALMER1D         7         PENALTY2         200         LIARWHD         5000         JNLBRNG2         10000           BOX2         3         AIRCRFTB         8         VARDIM         200         MCCORMCK         5000         JNLBRNG8         10000           BOX3         3         HEART8LS         8         HADAMALS         400         MOREBV         5000         JNLBRNG8         10000           DENSCHND         3         MAXLIKA         8         GENROSE         500         NONCVXU2         5000         NCVXBQP1         10000           DENSCHNE         3         OSLBQP         8         PROBPENL         500         NONCVXUN         5000         NCVXBQP3         10000           EG1         3	SINEVAL	2	PALMER5C	6	ARGLINA	200	FLETCHBV	5000	CURLY20	10000
SNAIL2PALMER7A6ARGLINC200GENHUMPS5000DIXON3DQ10000ZANGWIL22PALMER8A6BROWNAL200INDEF5000JNLBRNG110000BARD3PALMER1D7PENALTY2200LIARWHD5000JNLBRNG210000BOX23AIRCRFTB8VARDIM200MCCORMCK5000JNLBRNGA10000BOX33HEART8LS8HADAMALS400MOREBV5000JNLBRNGB10000DENSCHND3MAXLIKA8GENROSE500NONCVXU25000NCVXBQP110000DENSCHNE3OSLBQP8PROBPENL500NONCVXU15000NCVXBQP210000EG13PALMER1C8QR3DLS610NONDIA5000OBSTCLAE10000GROWTHLS3PALMER2C8EXTROSNB1000NONSCOMP5000OBSTCLAL10000GULF3PALMER2E8FLETCHCR1000PENTDI5000OBSTCLBL10000HATFLDD3PALMER3C8SINEALI1000POWELLSG5000OBSTCLBL10000HATFLDFL3PALMER3E8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLI	SISSER	2	PALMER6A	6	ARGLINB	200	FREUROTH	5000	CVXBQP1	10000
ZANGWIL22PALMER8A6BROWNAL200INDEF5000JNLBRNG110000BARD3PALMER1D7PENALTY2200LIARWHD5000JNLBRNG210000BOX23AIRCRFTB8VARDIM200MCCORMCK5000JNLBRNGA10000BOX33HEART8LS8HADAMALS400MOREBV5000JNLBRNGB10000DENSCHND3MAXLIKA8GENROSE500NONCVXU25000NCVXBQP110000DENSCHNE3OSLBQP8PROBPENL500NONCVXU15000NCVXBQP210000EG13PALMER1C8QR3DLS610NONDIA5000OBSTCLAE10000ENGVAL23PALMER1E8EG21000NONSCOMP5000OBSTCLAE10000GROWTHLS3PALMER2C8EXTROSNB1000NONSCOMP5000OBSTCLAL10000GULF3PALMER3C8FLETCHCR1000PENTDI5000OBSTCLBL10000HATFLDD3PALMER3C8SINEALI1000POWELLSG5000OBSTCLBL10000HATFLDFL3PALMER3E8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLIN	SNAIL	2	PALMER7A	6	ARGLINC	200	GENHUMPS	5000	DIXON3DQ	10000
BARD3PALMER1D7PENALTY2200LIARWHD5000JNLBRNG210000BOX23AIRCRFTB8VARDIM200MCCORMCK5000JNLBRNGA10000BOX33HEART8LS8HADAMALS400MOREBV5000JNLBRNGB10000DENSCHND3MAXLIKA8GENROSE500NONCVXU25000NCVXBQP110000DENSCHNE3OSLBQP8PROBPENL500NONCVXUN5000NCVXBQP210000EG13PALMER1C8QR3DLS610NONDIA5000NCVXBQP310000ENGVAL23PALMER1E8EG21000NONSCOMP5000OBSTCLAE10000GROWTHLS3PALMER2C8EXTROSNB1000NONSCOMP5000OBSTCLAL10000GULF3PALMER2E8FLETCHCR1000PENTDI5000OBSTCLBL10000HATFLDD3PALMER3C8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER3E8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY2010000HS253	ZANGWIL2	2	PALMER8A	6	BROWNAL	200	INDEF	5000	JNLBRNG1	10000
BOX23AIRCRFTB8VARDIM200MCCORMCK5000JNLBRNGA10000BOX33HEART8LS8HADAMALS400MOREBV5000JNLBRNGB10000DENSCHND3MAXLIKA8GENROSE500NONCVXU25000NCVXBQP110000DENSCHNE3OSLBQP8PROBPENL500NONCVXUN5000NCVXBQP210000EG13PALMER1C8QR3DLS610NONDIA5000NCVXBQP310000ENGVAL23PALMER1E8EG21000NONDQUAR5000OBSTCLAE10000GROWTHLS3PALMER2C8EXTROSNB1000NONSCOMP5000OBSTCLAL10000GULF3PALMER3C8FLETCHCR1000PENTDI5000OBSTCLBL10000HATFLDD3PALMER3C8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER3E8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER4E8MSQRTALS1024QUARTC5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY2010000HS2533AAAAAAAAA	BARD	3	PALMER1D	7	PENALTY2	200	LIARWHD	5000	JNLBRNG2	10000
BOX33HEART8LS8HADAMALS400MOREBV5000JNLBRNGB10000DENSCHND3MAXLIKA8GENROSE500NONCVXU25000NCVXBQP110000DENSCHNE3OSLBQP8PROBPENL500NONCVXUN5000NCVXBQP210000EG13PALMER1C8QR3DLS610NONDIA5000NCVXBQP310000ENGVAL23PALMER1E8EG21000NONDQUAR5000OBSTCLAE10000GROWTHLS3PALMER2C8EXTROSNB1000NONSCOMP5000OBSTCLAL10000GULF3PALMER2E8FLETCHCR1000PENTDI5000OBSTCLBL10000HATFLDD3PALMER3C8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER3E8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER4E8MSQRTALS1024QUARTC5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY2010000	BOX2	3	AIRCRFTB	8	VARDIM	200	MCCORMCK	5000	JNLBRNGA	10000
DENSCHND3MAXLIKA8GENROSE500NONCVXU25000NCVXBQP110000DENSCHNE3OSLBQP8PROBPENL500NONCVXUN5000NCVXBQP210000EG13PALMER1C8QR3DLS610NONDUA5000NCVXBQP310000ENGVAL23PALMER1E8EG21000NONDQUAR5000OBSTCLAE10000GROWTHLS3PALMER2C8EXTROSNB1000NONSCOMP5000OBSTCLAL10000GULF3PALMER2E8FLETCHCR1000PENTDI5000OBSTCLBL10000HATFLDD3PALMER3C8PENALTY11000POWELLSG5000OBSTCLBU10000HATFLDFL3PALMER3E8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER4C8MSQRTALS1024QUARTC5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY2010000	BOX3	3	HEART8LS	8	HADAMALS	400	MOREBV	5000	JNLBRNGB	10000
DENSCHNE         3         OSLBQP         8         PROBPENL         500         NONCVXUN         5000         NCVXBQP2         10000           EG1         3         PALMER1C         8         QR3DLS         610         NONDUA         5000         NCVXBQP3         10000           ENGVAL2         3         PALMER1C         8         EG2         1000         NONDQUAR         5000         OBSTCLAE         10000           GROWTHLS         3         PALMER2C         8         EXTROSNB         1000         NONSCOMP         5000         OBSTCLAE         10000           GULF         3         PALMER2E         8         FLETCHCR         1000         PENTDI         5000         OBSTCLBL         10000           HATFLDD         3         PALMER3C         8         FLETCHCR         1000         POWELLSG         5000         OBSTCLBL         10000           HATFLDD         3         PALMER3C         8         SINEALI         1000         QRTQUAD         5000         OBSTCLBU         10000           HATFLDE         3         PALMER3E         8         SINEALI         1000         QRTQUAD         5000         OSURLY10         10000           HATFLDFL         <	DENSCHND	3	MAXLIKA	8	GENROSE	500	NONCVXU2	5000	NCVXBQP1	10000
EG13PALMER1C8QR3DLS610NONDIA5000NCVXBQP310000ENGVAL23PALMER1E8EG21000NONDQUAR5000OBSTCLAE10000GROWTHLS3PALMER2C8EXTROSNB1000NONSCOMP5000OBSTCLAL10000GULF3PALMER2E8FLETCHCR1000PENTDI5000OBSTCLBL10000HATFLDD3PALMER3C8PENALTY11000POWELLSG5000OBSTCLBU10000HATFLDE3PALMER3E8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER4C8MSQRTALS1024QUARTC5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY2010000HS2533AAAAAA	DENSCHNE	3	OSLBQP	8	PROBPENL	500	NONCVXUN	5000	NCVXBQP2	10000
ENGVAL23PALMER1E8EG21000NONDQUAR5000OBSTCLAE10000GROWTHLS3PALMER2C8EXTROSNB1000NONSCOMP5000OBSTCLAL10000GULF3PALMER2E8FLETCHCR1000PENTDI5000OBSTCLBL10000HATFLDD3PALMER3C8PENALTY11000POWELLSG5000OBSTCLBU10000HATFLDE3PALMER3E8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER4C8MSQRTALS1024QUARTC5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY2010000HS2533FALMER4E8SURATALS1024SUDLINSUDLINSUDLIN	EG1	3	PALMER1C	8	QR3DLS	610	NONDIA	5000	NCVXBQP3	10000
GROWTHLS         3         PALMER2C         8         EXTROSNB         1000         NONSCOMP         5000         OBSTCLAL         10000           GULF         3         PALMER2E         8         FLETCHCR         1000         PENTDI         5000         OBSTCLBL         10000           HATFLDD         3         PALMER3C         8         PENALTY1         1000         POWELLSG         5000         OBSTCLBU         10000           HATFLDE         3         PALMER3E         8         SINEALI         1000         QRTQUAD         5000         OBSTCLBU         10000           HATFLDFL         3         PALMER4C         8         MSQRTALS         1024         QUARTC         5000         SCURLY10         10000           HELIX         3         PALMER4E         8         MSQRTBLS         1024         QUDLIN         5000         SCURLY20         10000           HS25         3	ENGVAL2	3	PALMER1E	8	EG2	1000	NONDQUAR	5000	OBSTCLAE	10000
GULF         3         PALMER2E         8         FLETCHCR         1000         PENTDI         5000         OBSTCLBL         10000           HATFLDD         3         PALMER3C         8         PENALTY1         1000         POWELLSG         5000         OBSTCLBM         10000           HATFLDE         3         PALMER3E         8         SINEALI         1000         QRTQUAD         5000         OBSTCLBM         10000           HATFLDFL         3         PALMER4E         8         MSQRTALS         1024         QUARTC         5000         SCURLY10         10000           HELIX         3         PALMER4E         8         MSQRTBLS         1024         QUDLIN         5000         SCURLY20         10000           HS25         3	GROWTHLS	3	PALMER2C	8	EXTROSNB	1000	NONSCOMP	5000	OBSTCLAL	10000
HATFLDD3PALMER3C8PENALTY11000POWELLSG5000OBSTCLBM10000HATFLDE3PALMER3E8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER4C8MSQRTALS1024QUARTC5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY2010000HS253	GULF	3	PALMER2E	8	FLETCHCR	1000	PENTDI	5000	OBSTCLBL	10000
HATFLDE3PALMER3E8SINEALI1000QRTQUAD5000OBSTCLBU10000HATFLDFL3PALMER4C8MSQRTALS1024QUARTC5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY2010000HS253	HATFLDD	3	PALMER3C	8	PENALTY1	1000	POWELLSG	5000	OBSTCLBM	10000
HATFLDFL3PALMER4C8MSQRTALS1024QUARTC5000SCURLY1010000HELIX3PALMER4E8MSQRTBLS1024QUDLIN5000SCURLY2010000HS253	HATFLDE	3	PALMER3E	8	SINEALI	1000	QRTQUAD	5000	OBSTCLBU	10000
HELIX 3 PALMER4E 8 MSQRTBLS 1024 QUDLIN 5000 SCURLY20 10000 HS25 3	HATFLDFI.	3	PALMER4C	8	MSORTALS	1024	QUARTC	5000	SCURLY10	10000
HS25 3	HELIX	3	PALMER4E	8	MSORTBLS	1024	QUDLTN	5000	SCURLY20	10000
	HS25	3		-						

Table 1: CUTEr test problems used in the numerical results.  $10\,$ 

Clearly, when for a certain problem  $p \in \mathcal{P}$  one has  $\min\{t_{p,z} : z \in \mathcal{S}\} \leq 0$ , the value  $r_{p,s}$  becomes meaningless or undefined for all  $s \in \mathcal{S}$ . We considered two possibilities to overcome this problem in our numerical results. One is to simply exclude all problems where such a situation happens, reducing the number of test problems to be included and then using the original performance profiles [12]. The second one is to keep all problems and to choose a way to deal with problems where  $t_{p,s} \leq 0$  happens for at least one solver s. We considered  $r_{p,s} = t_{p,s} + 1 - \min\{t_{p,z} : z \in \mathcal{S}\}$  whenever  $\min\{t_{p,z} : z \in \mathcal{S}\} < 0.0001$  to overcome the possibility of  $r_{p,s}$  being meaningless or undefined (see [33] for further details).

# 5 Numerical results

Since our proposed method uses second order derivatives we decided to compare it against TRON [23, 24], IPOPT [34], and Lancelot B [16] (available under the GALAHAD library [17]), which represent well the state-of-the-art optimization solvers where second order derivatives are used. TRON was specially developed to address bound-constrained optimization problems, while IPOPT and Lancelot B can handle more general constrained optimization problems.

Since the computational effort made per iteration of IPOPT, Lancelot B, TRDC, and TRON is substantial different, we chose to compare the overall CPU time taken by the solvers. As it was said before, TRDC and TRON have similar stopping criteria, being the one for TRON slightly more advantageous for declaring success since it uses either the relative or the absolute error in function values. IPOPT and Lancelot B were run using a tolerance of  $10^{-5}$  (the default value for Lancelot B) in their stopping criteria: Lancelot B uses the norm of the projected gradient while IPOPT uses the maximum between a scaled norm of the gradient of the Lagrangian and the complementarity residual.

The numerical experiments were made in an Intel(R) Core(TM) Duo CPU computer, running at 2.66GHz, under a Linux operating system, using recent versions for all the solvers (TRON version 1.2, IPOPT version 3.10.1, and GALAHAD version dated of February 2011). The same exact CUTEr collection (version date CUTEr: Mon Jan 8 15:36:20 EST 2007) was used by the four solvers, and for each problem the same CUTEr initial point was considered. For each problem, a maximum running time of 3600 seconds was imposed to all solvers (i.e., a failure is declared for a solver on a problem if it is unable to provide an answer in less than 3600 seconds). The considered CPU time corresponds to the CPU time taken by the solver (excluding the CUTEr setup time) measured in seconds with two decimal places. Due to this limitation in measuring the solver CPU time, one can easily obtain  $t_{p,s} = 0$  for many 'easy' problems.

A first set of performance profiles is provided in Figures 1 and 2, where all the test problems were considered and then the performance profiles from [33] were used. For a matter of visibility around  $v_s(1)$  and along  $v_s(\pi)$  for large values of  $\pi$ , we considered two subfigures in each figure. These profiles were depicted trusting the exit flag produced by each solver in order to check if success was attained on solving a given problem. From these performance profiles one can conclude that the TRDC solver is competitive in both efficiency and robustness, when compared to the other solvers, specially for problems with less than 2000 variables.

A closer look at these numerical results reveals that for problems FLETCBV3, FLETCHBV, INDEF, QRTQUAD, SCURLY10, and SCURLY20 all the solvers were unable to converge. These problems seem to have an unbounded objective function and therefore were removed in the next round of profiles. Additionally, we list, on Table 2, the problems where a solver reported an unsuccessful



Figure 1: Performance profiles [33] for CPU time used by IPOPT, Lancelot B, TRDC, and TRON. Problems with  $n \leq 2000$ .



Figure 2: Performance profiles [33] for CPU time used by IPOPT, Lancelot B, TRDC, and TRON. All problems.

IPOPT	Lancelot B	TRDC		
OSCIPATH	ARGLINB	CURLY20		
PALMER5A	ARGLINC	EXPLIN		
	BIGGSB1	EXPLIN2		
	DJTL	EXPQUAD		
	HEART6LS	FREUROTH		
	LOGHAIRY	NCVXBQP1		
	OSCIPATH	NCVXBQP2		
	PALMER5A	NCVXBQP3		
	PALMER7A	PENALTY2		
		QUDLIN		
		RAYBENDL		
		RAYBENDS		
		SINQUAD		

Table 2: Test problems considered as successfully solved despite un unsuccessful exit flag.

exitflag, but the final objective function value is close to the one reported by the other solvers. For these problems we will now consider the run to be successful and use the corresponding CPU time. (To be more rigorous, suppose solver *B* solved problem *p* successfully. A solver *A* is considered to have solved problem *p* successfully, even when reporting an unsuccessful error flag, in the situations where either (i)  $f_{p,A} \leq f_{p,B}$  or (ii)  $\frac{|f_{p,A}-f_{p,B}|}{\max(1,\min(|f_{p,A}|,|f_{p,B}|))} \leq 10^{-3}$ , where  $f_{p,A}$  and  $f_{p,B}$  represent the final objective function values of the two solvers for problem *p*.) We point out that TRON always reports a successful exit flag (equal to zero), i.e., the only unsuccessful cases are due to exceeding the 3600 seconds running time limit.

The new performance profiles for the restricted test set are depicted in Figures 3 and 4. The relative position of each solver is similar, but these new profiles reassure the robustness of the TRDC solver, as it was able to solve more than 90% of the problems.

We have also built other performance profiles, namely for the cases where the considered problems had dimensions  $n \leq 10, 50, 100, 500, 1000, 3000, 5000$ . We also plotted the performance profiles for each type of constraints available in the problems: unconstrained problems ( $\ell = (-\infty)^n$  and  $u = (+\infty)^n$ ), bound-constrained problems with at least one finite bound and one infinite bound, and bound-constrained problems with  $\ell \in \mathbb{R}^n$  and  $u \in \mathbb{R}^n$ . Since we observed no major differences between these performance profiles and the ones reported before, we decided not to present them here for sake of brevity.

For a matter of completeness we also report our numerical findings using the original performance profiles [12]. In order to be able to plot these profiles we exclude from the test set all 'easy' problems for which  $\min\{t_{p,z} : z \in S\} = 0$  (i.e., all problems where at least one solver terminated using  $t_{p,s} = 0$ ). This technique is the same as the one used in [18], where numerical results were restricted to problems with a running time of the fastest solver exceeding .01 seconds. These new performance profiles (for the restricted test set previously described) are presented in Figures 5 and 6. The number of problems is reduced from 167 to 30 (when  $n \leq 2000$ ) and from 265 to 124 (all dimensions considered). While these new performance profiles are in accordance with the original ones in [12], the number of problems considered is considerably smaller and



Figure 3: Performance profiles [33] for CPU time used by IPOPT, Lancelot B, TRDC, and TRON. Restricted test set, problems with  $n \leq 2000$ .



Figure 4: Performance profiles [33] for CPU time used by IPOPT, Lancelot B, TRDC, and TRON. All problems in the restricted test set.



Figure 5: Performance profiles [12] for CPU time used by IPOPT, Lancelot B, TRDC, and TRON. Restricted test set, problems with  $n \leq 2000$  except the 'easy' ones.

the best solvers are likely to be in disadvantage since problems where a solver attains  $t_{p,s} = 0$  are removed from the analysis (disregardless of other solvers having or not  $t_{p,s} \gg 0$ ).

From the depicted profiles for  $n \leq 2000$ , one can observe that Lancelot B is the most efficient solver, while IPOPT, Lancelot B, and TRON are the most robust. TRDC presents a similar performance in terms of efficiency and attains a robustness of about 90%. When all the problems in the restricted test set are considered we observe a slight advantage for the TRDC solver.

# 6 Conclusions

A trust-region type method has been proposed, analyzed, and implemented, involving a new minimum requirement, from the solution of the trust-region subproblems, for achieving global convergence to first-order stationary points. Such a requirement, different from Cauchy or generalized Cauchy points, is related to the application of a first step of a primal-dual subgradient method (the DC algorithm), and it necessarily involves the knowledge of second-order derivatives, although it only requires one projection onto the feasible set. One is able to prove, in a relatively short and clean argument, that all limit points of the sequence of iterates are first-order critical. The numerical experiments reported show that the new approach is competitive with state-of-the-art solvers for problems with bounds on the variables.

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Figure 6: Performance profiles [12] for CPU time used by IPOPT, Lancelot B, TRDC, and TRON. All problems in the restricted test set, except the 'easy' ones.

# References

- D. P. Bertesekas. Projected Newton methods for optimization problems with simple constraints. SIAM J. Control Optim., 20:221–246, 1982.
- [2] E. G. Birgin, J. M. Martínez, and M. Raydan. Nonmonotone spectral projected gradient methods on convex sets. SIAM J. Optim., 10:1196–1211, 2000.
- [3] R.H. Byrd, P. Lu, J. Nocedal, and C. Zhu. A limited memory algorithm for bound constrained optimization. SIAM J. Sci. Comput., 16:1190–1208, 1995.
- [4] F. H. Clarke. Optimization and Nonsmooth Analysis. John Wiley & Sons, New York, New York, 1983. Reissued by SIAM, Philadelphia, 1990.
- [5] T. F. Coleman and Y. Li. An interior trust region approach for nonlinear minimization subject to bounds. SIAM J. Optim., 6:418–445, 1996.
- [6] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. Global convergence of a class of trust region algorithms for optimization with simple bounds. SIAM J. Numer. Anal., 25:433–460, 1988.
- [7] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. Correction to the paper on global convergence of a class of trust region algorithms for optimization with simple bounds. *SIAM J. Numer. Anal.*, 26:764–767, 1989.
- [8] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. *Trust-Region Methods*. MPS-SIAM Series on Optimization. SIAM, Philadelphia, 2000.

- [9] Y.-H. Dai. Fast algorithms for projection on an ellipsoid. SIAM J. Optim., 16:986–1006, 2006.
- [10] J. E. Dennis and L. N. Vicente. Trust-region interior-point algorithms for minimization problems with simple bounds. In H. Fisher, B. Riedmüller, and S. Schäffer, editors, *Applied Mathematics and Parallel Computing*, pages 97–107. Physica-Verlag, Springer-Verlag, Berlin, 1996. Festschrift for Klaus Ritter.
- [11] J. E. Dennis Jr. and R. B. Schnabel. Numerical Methods for Unconstrained Optimization and Nonlinear Equations. Prentice–Hall, Englewood Cliffs, (republished by SIAM, Philadelphia, in 1996, as Classics in Applied Mathematics, 16), 1983.
- [12] E. D. Dolan and J. J. Moré. Benchmarking optimization software with performance profiles. Math. Program., 91:201–213, 2002.
- [13] F. Facchinei, J. Júdice, and J. Soares. An active set Newton's algorithm for large-scale nonlinear programs with box constraints. SIAM J. Optim., 8:158–186, 1998.
- [14] A. Friedlander, J. M. Martínez, and S. A. Santos. A new trust region algorithm for bound constrained minimization. Appl. Math. Optim., 30:235–266, 1994.
- [15] N. I. M. Gould, D. Orban, and Ph. L. Toint. Contrained and unconstrainted test environement, revisited. http://cuter.rl.ac.uk/cuter-www.
- [16] N. I. M. Gould, D. Orban, and Ph. L. Toint. Results from a numerical evaluation of LANCELOT B. Internal Report 2002-1, Numerical Analysis Group, Rutherford Appleton Laboratory, Chilton, England, 2002.
- [17] N. I. M. Gould, D. Orban, and Ph. L. Toint. GALAHAD, a library of thread-safe Fortran 90 packages for large-scale nonlinear optimization. ACM Trans. Math. Software, 29:353–372, 2004.
- [18] W. W. Hager and H. Zhang. A new active set algorithm for box constrained optimization. SIAM J. Optim., 17:526–557, 2006.
- [19] M. Heinkenschloss, M. Ulbrich, and S. Ulbrich. Superlinear and quadratic convergence of affine-scaling interior-point Newton methods for problems with simple bounds without strict complementarity assumption. *Math. Program.*, 86:615–635, 1999.
- [20] H. A. Le Thi and T. Pham Dinh. Large scale molecular optimization from distance matrices by a D.C. optimization approach. SIAM J. Optim., 14:77–114, 2003.
- [21] H. A. Le Thi and T. Pham Dinh. The DC (difference of convex functions) programming and DCA revisited with DC models of real world nonconvex optimization problems. Ann. Oper. Res., 133:23–46, 2005.
- [22] M. Lescrenier. Convergence of trust region algorithms for optimization with bounds when strict complementarity does not hold. SIAM J. Numer. Anal., 28:476–495, 1991.
- [23] C.-J. Lin and J. J. Moré. TRON, a trust region Newton method for the solution of large bound-constrained optimization problems. http://www.mcs.anl.gov/~more/tron/.

- [24] C.-J. Lin and J. J. Moré. Newton's method for large bound-constrained optimization problems. SIAM J. Optim., 9:1100–1127, 1999.
- [25] J. L. Morales and J. Nocedal. Remark on "Algorithm 778. L-BFGS-B, Fortran subroutines for Large-Scale bound constrained optimization". ACM Trans. Math. Software, 38:7:1–7:4, 2011.
- [26] J. Nocedal and S. J. Wright. Numerical Optimization. Springer-Verlag, Berlin, second edition, 2006.
- [27] T. Pham Dinh and H. A. Le Thi. Convex analysis approach to D.C. programming: Theory, algorithms and applications. Acta Math. Vietnam., 22:289–355, 1997.
- [28] T. Pham Dinh and H. A. Le Thi. A D.C. optimization algorithm for solving the trust-region subproblem. SIAM J. Optim., 8:476–505, 1998.
- [29] R. T. Rockafellar. Convex Analysis. Princeton University Press, Princeton, 1970.
- [30] R. T. Rockafellar and R. J.-B. Wets. Variational Analysis. Springer, Berlin, 1997, third printing in 2009.
- [31] C. Sainvitu and Ph. L. Toint. A filter-trust-region method for simple-bound constrained optimization. Optim. Methods Softw., 22:835–848, 2007.
- [32] A. Schwartz and E. Polak. Family of projected descent methods for optimization problems with simple bounds. J. Optim. Theory Appl., 92:1–31, 1997.
- [33] A. I. F. Vaz and L. N. Vicente. A particle swarm pattern search method for bound constrained global optimization. J. Global Optim., 39:197–219, 2007.
- [34] A. Wächter and L. T. Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Math. Program.*, 106:25–57, 2006.
- [35] C. Zhu, R.H. Byrd, P. Lu, and J. Nocedal. Algorithm 778. L-BFGS-B, Fortran subroutines for Large-Scale bound constrained optimization. ACM Trans. Math. Software, 23:550–560, 1997.