# Optimization ... in 45 minutes

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Department of Mathematics
University of Coimbra

Slides written with the help of Rohollah Garmanjani (Nima)

# Presentation outline

1. Importance of Optimization (illustration by Linear Programming)
2. Application of Optimization (illustration by Classification)
3. Classes of Optimization Problems
4. Type of Mathematics used in Optimization (illustration by Integrality, Convexity, Non-smooth Calculus)

# Importance of Optimization

Optimization has many applications in different fields such as

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And many other applications in Physics, Chemistry, Geology ...

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Roughly 5-10% of Math. journals are in Optimization and related fields. Here are some of Optimization journals:

| | | | |
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| Mathematical Programming | Mathematical Programming Computation | SIAM Journal on Optimization | SIAM Journal on Control and Optimization |
| Mathematics of Operations Research | EURO Journal on Computational Optimization | Operations Research | INFORMS J. Computing |
| Computational Optimization and Applications | IIE Transactions | Journal of Combinatorial Optimization | Journal of Global Optimization |
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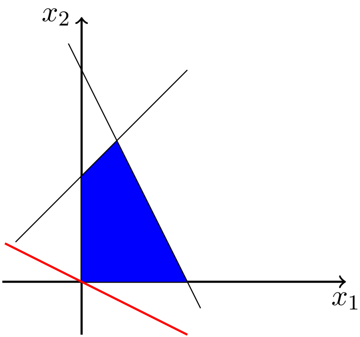
Optimization is broadly classified by AMS (under 90XX, 49XX, 65XX).

# Linear Programming (LP)

Optimizing a linear objective function subject to a number of linear equality or inequality constraints.

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + x_2 \le 2 \\ & 2x_1 + x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$



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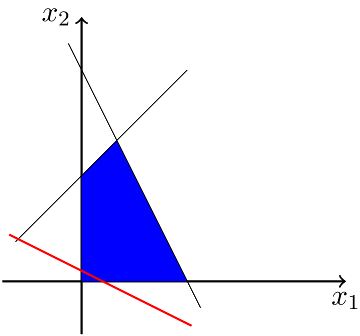

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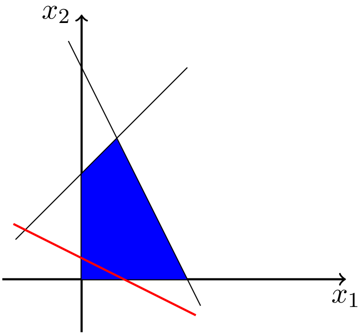

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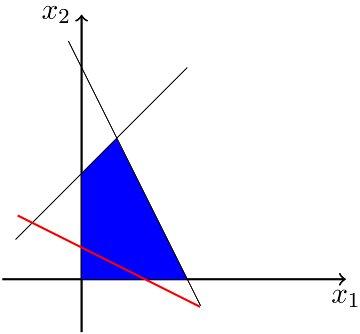

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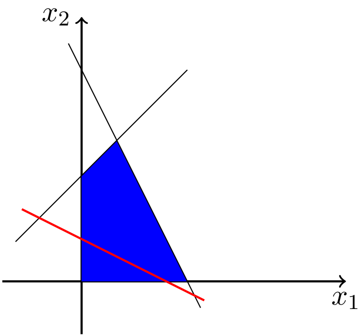

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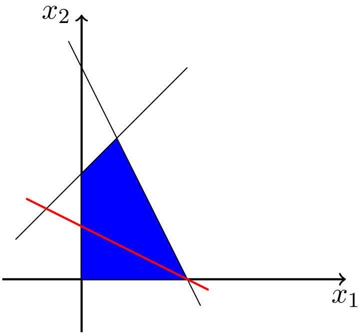

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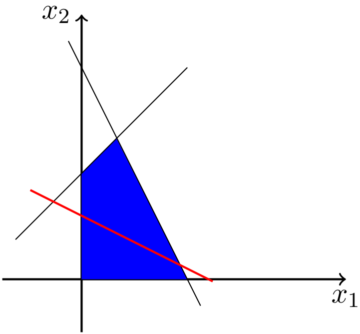

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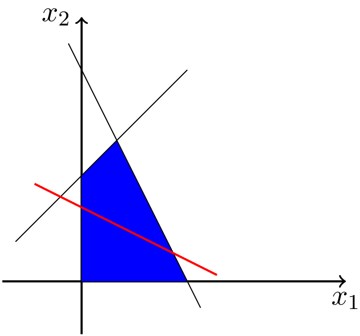

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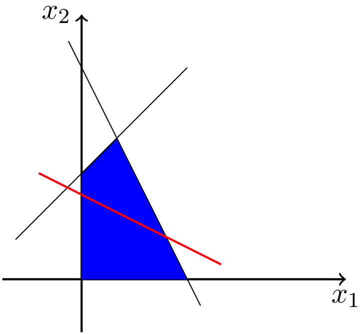

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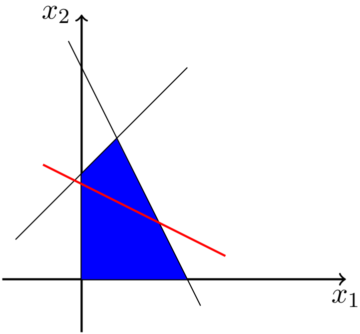

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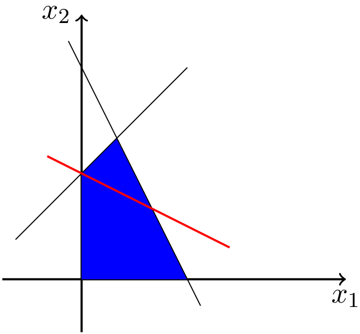

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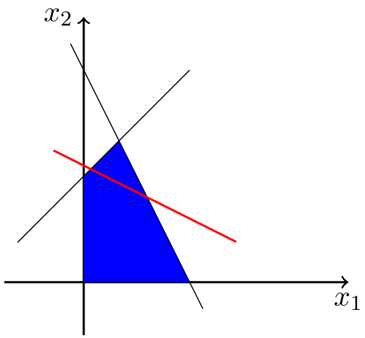

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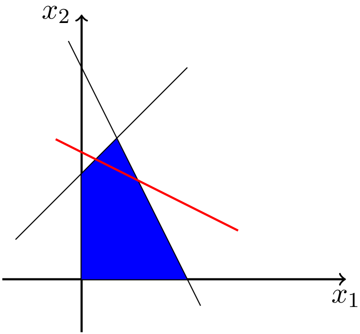

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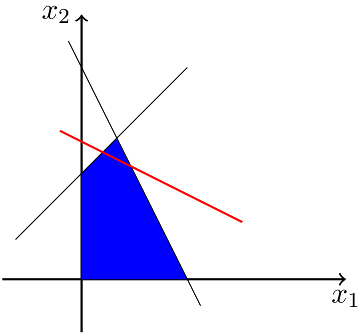

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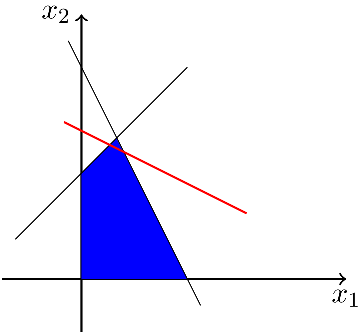

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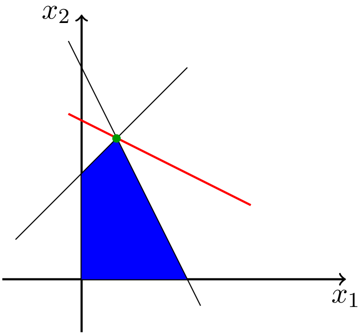

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The simplex algorithm was developed by George Dantzig in 1947.

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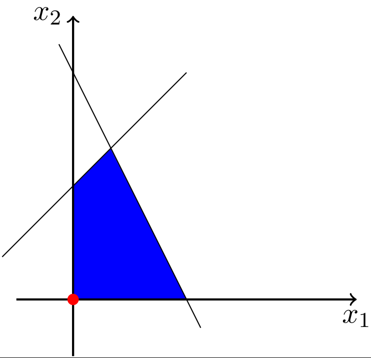
| BV | $x_1$ | $x_2$ | $s_1$ | $s_2$ | RHS |
|---|---|---|---|---|---|
| $s_1$ | $-1$ | $\textcircled{1}$ | 1 | 0 | $2 \leftarrow$ |
| $s_2$ | 2 | 1 | 0 | 1 | 4 |
| $z$ | $-1$ | $-2\uparrow$ | 0 | 0 | 0 |



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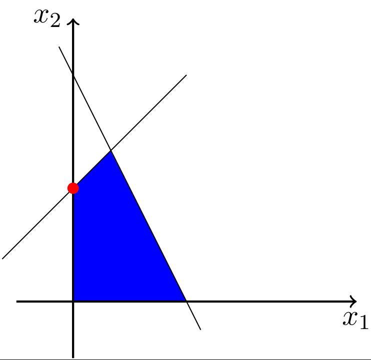
|---|---|---|---|---|---|
| $x_2$ | $-1$ | 1 | 1 | 0 | 2 |
| $s_2$ | (3) | 0 | $-1$ | 1 | 2 ← |
| $z$ | $-3$ ↑ | 0 | 2 | 0 | 4 |



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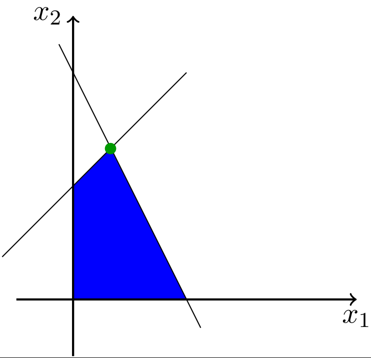
|---|---|---|---|---|---|
| $x_2$ | $0$ | $1$ | $2/3$ | $1/3$ | $8/3$ |
| $x_1$ | $1$ | $0$ | $-1/3$ | $1/3$ | $2/3$ |
| $z$ | $0$ | $0$ | $1$ | $3$ | $6$ |



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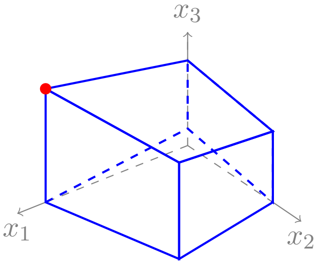
Assume we want to minimize $x_3$ in the following region.

The simplex algorithm meets $2^3 - 1$ corner points before reaching the optimal one:



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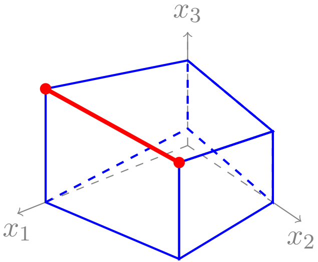

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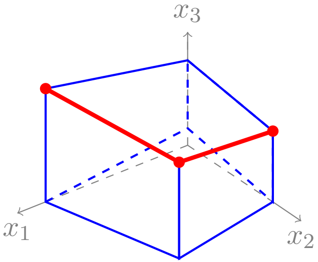

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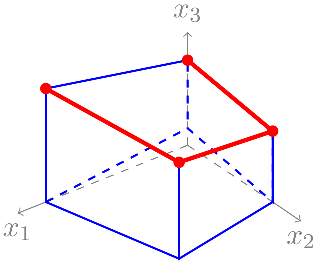

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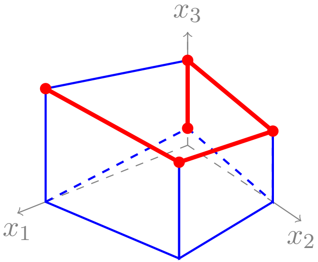

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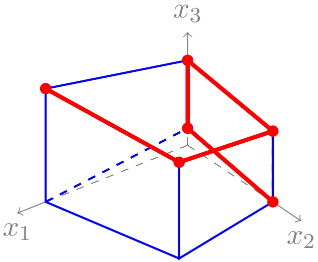

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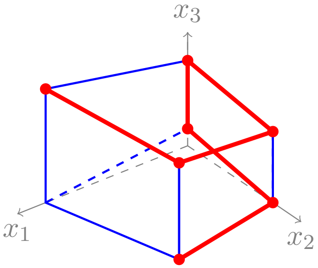

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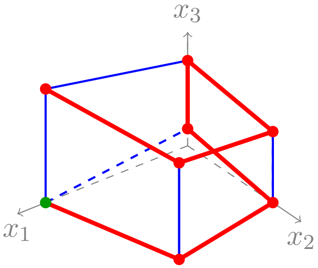

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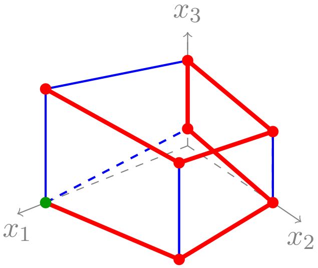


In $n$ dimensions, the cost is $2^n - 1$, which shows an exponential-time complexity.

# The breakthrough

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In 1984, Narendra Karmarkar introduced a polynomial-time algorithm for solving LP problems.

His discovery received a huge media coverage. The news appeared in the front page of the New York Times:

### *Breakthrough in Problem Solving*

By JAMES GLEICK

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

The discovery, which is to be formally published next month, is already circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

**Faster Solutions Seen**

These problems are fiendishly complicated systems, often with thousands of variables. They arise in a variety of commercial and government applications, ranging from allocating time on a communications satellite to routing millions of telephone calls over long distances, or whenever a limited, expensive resource must be spread most efficiently among competing users. And investment companies use them in creating portfolios with the best mix of stocks and bonds.

The Bell Labs mathematician, Dr. Narendra Karmarkar, has devised a radically new procedure that may speed the routine handling of such problems by businesses and Government agencies and also make it possible to tackle problems that are now far out of reach.

"This is a path-breaking result," said Dr, Ronald L. Graham, director of mathematical sciences for Bell Labs in Murray Hill, N.J.

"Science has its moments of great progress, and this may well be one of them."

Because problems in linear programming can have billions or more possible answers, even high-speed computers cannot check every one. So computers must use a special procedure, an algorithm, to examine as few answers as possible before finding the best one — typically the one that minimizes cost or maximizes efficiency.

A procedure devised in 1947, the simplex method, is now used for such problems,

Continued on Page A19, Column 1

THE NEW YORK TIMES, November 19, 1984

Karmarkar at Bell Labs: an equation to find a new way through the maze

### Folding the Perfect Corner

*A young Bell scientist makes a major math breakthrough*

Every day 1,200 American Airlines jets crisscross the U.S., Mexico, Canada and the Caribbean, stopping in 110 cities and bearing over 80,000 passengers. More than 4,000 pilots, copilots, flight personnel, maintenance workers and baggage carriers are shuffled among the flights; a total of 3.6 million gal. of high-octane fuel is burned. Nuts, bolts, altimeters, landing gears and the like must be checked at each destination. And while performing these scheduling gymnastics, the company must keep a close eye on costs, projected revenue and profits.

Like American Airlines, thousands of companies must routinely untangle the myriad variables that complicate the efficient distribution of their resources. Solving such monstrous problems requires the use of an abstruse branch of mathematics known as linear programming. It is the kind of math that has frustrated theoreticians for years, and even the fastest and most powerful computers have had great difficulty juggling the bits and pieces of data. Now Narendra Karmarkar, a 28-year-old Indian-born mathematician at Bell Laboratories in Murray Hill, N.J., after only a years' work has cracked the puzzle of linear programming by devising a new algorithm, a step-by-step mathematical formula. He has translated the procedure into a program that should allow computers to track a greater combination of tasks than ever before and in a fraction of the time.

Unlike most advances in theoretical mathematics, Karmarkar's work will have an immediate and major impact on the real world. "Breakthrough is one of the most abused words in science," says Ronald Graham, director of mathematical sciences at Bell Labs. "But this is one situation where it is truly appropriate."

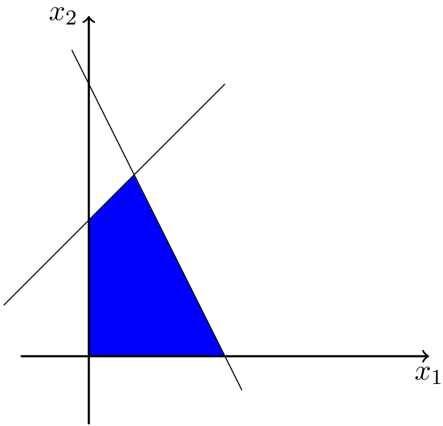
Before the Karmarkar method, linear equations could be solved only in a cumbersome fashion, ironically known as the simplex method, devised by Mathematician George Dantzig in 1947. Problems are conceived of as giant geodesic domes with thousands of sides. Each corner of a facet on the dome

TIME MAGAZINE, December 3, 1984

# Interior point method (central path)

The iterates, instead, follow a path inside the feasible region.

$$\begin{aligned} \text{maximize} \quad & {\color{red} x_1 + 2x_2} \\ \text{subject to} \quad & -x_1 + x_2 \le 2 \\ & 2x_1 + x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$



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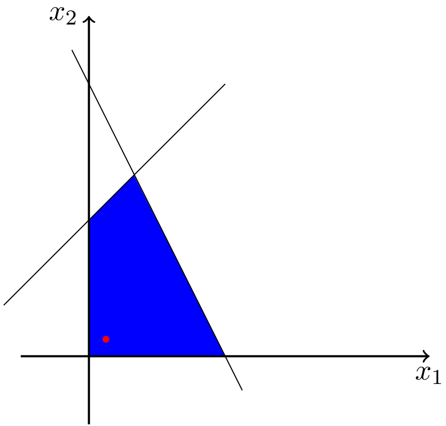

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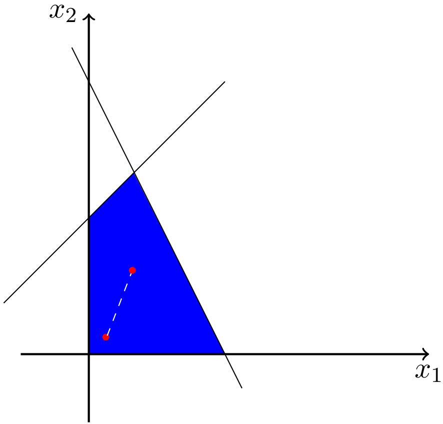

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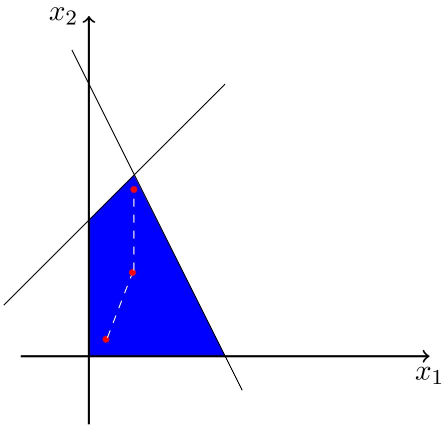

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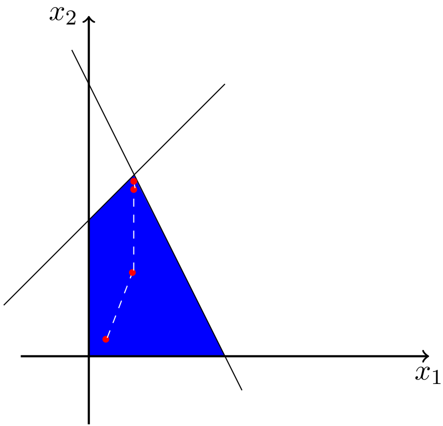

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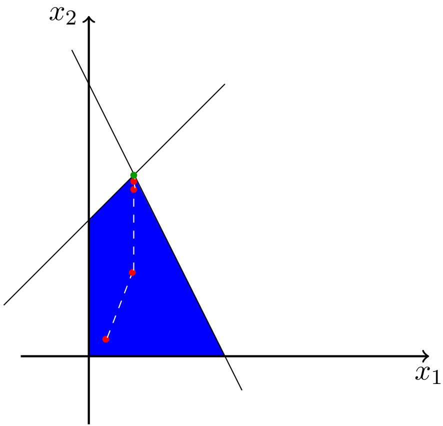

# LP polynomial-time complexity

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- Later, the above bound on the # iterations was improved to $\mathcal{O}(n^{0.5})$.
- The complexity of LP is thus $\mathcal{O}(n^3)$.

# The Kepler conjecture

**Conjecture (Kepler)**

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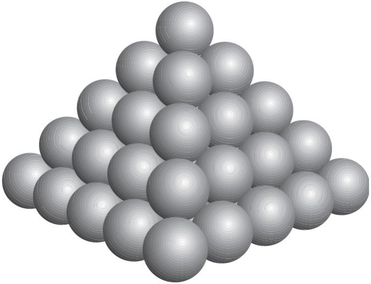

FCC packing

The optimal density is $\frac{\pi}{\sqrt{18}} \approx 0.74$.

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- In 1998, Thomas Hales, following L. F. Tóth in 1953, and assisted by his graduate student Samuel Ferguson, announced the proof!

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The reviewers were not able to completely verify the computer programming part...

# Presentation outline

1. Importance of Optimization (illustration by Linear Programming)
2. Application of Optimization (illustration by Classification)
3. Classes of Optimization Problems
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## Example

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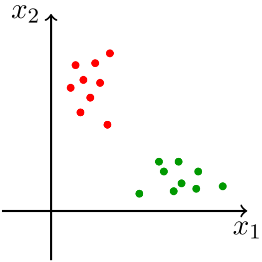
## Example

- *(Medicine) $n$ patients with malignant tumors and $m$ patients with benign tumors. Based on the observed data, classify a new patient.*
- *(Pattern Recognition) $n$ photos of cats and $m$ photos of dogs. Have a device telling us whether a new photo is a cat or a dog.*

# SVM (linear classifier)

Consider the following two classes of data. If we can separate the data, then we can decide about a new case.



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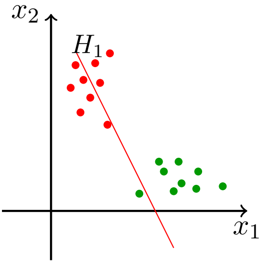


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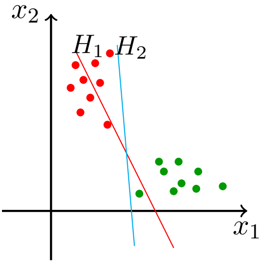


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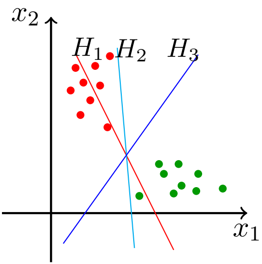


- $H_1$ does not separate the data.
- $H_2$ separates data with a small margin.
- $H_3$ separates data with a maximum margin.

# SVM (linear classifier)

Assume that the following linearly separable data points have been given:

$$\mathcal{D} = \{(x^i, y^i) \mid x^i \in \mathbb{R}^p, y^i \in \{-1, 1\}, i = 1, \cdots, n\},$$

where $y^i = 1$ and $y^i = -1$ represents the two classes that each $x^i$ belongs to.

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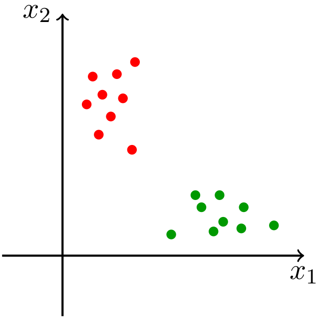
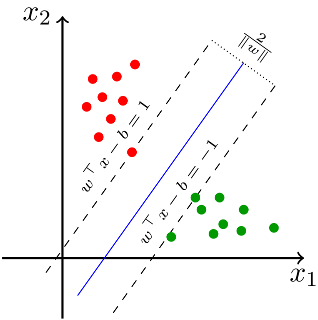
Below we have been given a set of $n = 18$ linearly separable data points in $\mathbb{R}^2$ for classification.



# SVM (linear classifier)

We need to maximize the distance between the hyperplanes $w^\top x - b = 1$ and $w^\top x - b = -1$ in a way that no data point falls between them:



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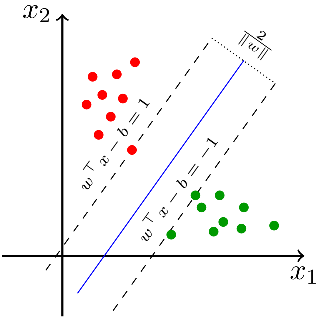


$$w^\top x - b \ \geq \ 1 \text{ when } x^i \text{ belongs to the class } y^i = 1$$

or

$$w^\top x' - b \ \leq \ -1 \text{ when } x^i \text{ belongs to the class } y^i = -1.$$

# SVM (linear classifier)

So, to maximize the distance we need to minimize $\|w\|$. Therefore, we have the following problem:

$$\begin{aligned} \underset{w,b}{\text{minimize}} \quad & \frac{1}{2}\|w\|^2 \\ \text{subject to} \quad & y^i(w^\top x - b) \geq 1, \text{ for any } i = 1, \ldots, n, \end{aligned}$$

which is a quadratic optimization/programming problem.

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# Classes of Optimization problems

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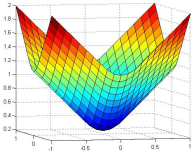
Linear Programming (LP)

Quadratic Programming (QP)

Conic Optimization ($f$ linear, $\Omega$ = Polyhedron LP $\cap$ Cone)

Convex Optimization ($f$ and $\Omega$ convex)

$$\begin{array}{ll} \text{min} & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$




non-linear $f$

$\longrightarrow$ Nonlinear Programming.

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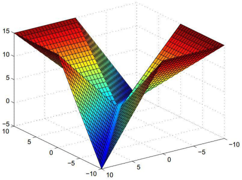
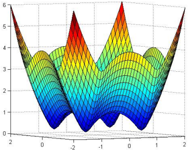



non-differentiable $f$

$\longrightarrow$ Non-differentiable Optimization.

$$\begin{array}{ll} \text{min} & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$




non-convex $f$

$\longrightarrow$ Global Optimization.

$$\begin{array}{ll} \text{min} & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$

$f$ can be a vectorial function

$$F(x) \ = \ (f_1(x), \ldots, f_m(x))$$

$\longrightarrow$ MultiObjective Optimization.

# Classes of Optimization problems

$$\begin{array}{ll} \text{min} & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$

The variables can take only discrete values

$$x \in \mathbb{Z}^n$$

$\longrightarrow$ Combinatorial/Discrete Optimization.

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When not rigorous, it is a HEURISTIC.

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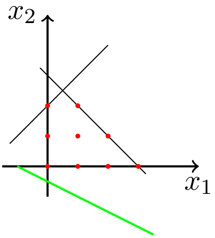
maximize $x_1 + 2x_2$

subject to $-x_1 + x_2 \leq 2$

$x_1 + x_2 \leq 3$

$x_1, x_2 \in \mathbb{Z}^+$



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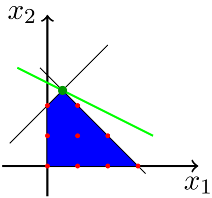


The solution of the LP relaxation is not integer.

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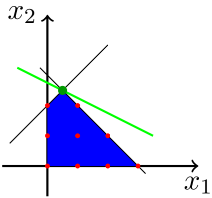


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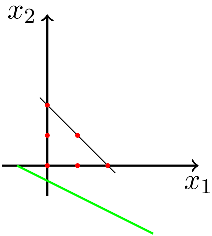

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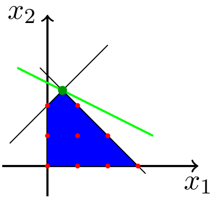


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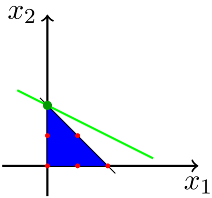


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When $A$ is TU and $b$ is an integer vector, then every vertex of

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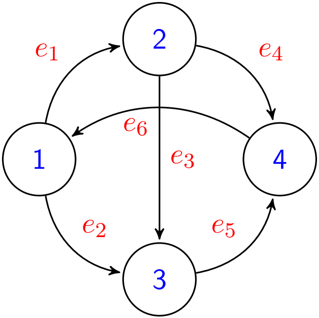
is integer (i.e., the polyhedron is integral).

# A TU example

The incidence matrix of



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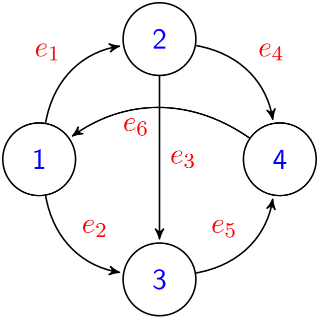


is

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & +1 \\ +1 & 0 & -1 & -1 & 0 & 0 \\ 0 & +1 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & +1 & -1 \end{bmatrix}.$$

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$$\text{subject to} \quad \sum_{u} x_{u\to v} - \sum_{w} x_{v\to w} = \begin{cases} 0 & \text{if } v \neq s,t \\ -1 & \text{if } v = s \\ 1 & \text{if } v = t \end{cases}$$

$$x_{u\to v} \geq 0$$

$$x_{u\to v} \in \{0,1\}$$

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\end{aligned}$$

can be solved polynomially (since the solution of the LP relaxation is integer).

# Convexity (local/global and uniqueness)

Local/Global:

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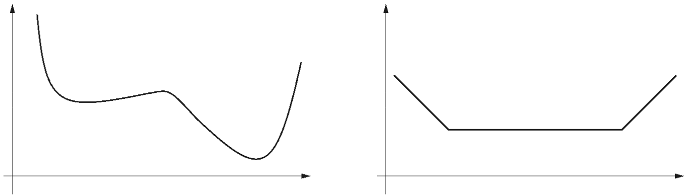

(a) $f$ not convex

(b) $f$ not strictly convex

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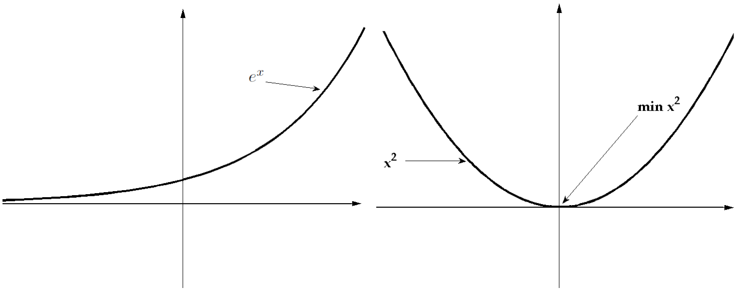


(a) no minimizer (b) $\exists^1$ minimizer

# Non-smooth calculus (directional derivative)

## Definition

*For $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ Lipschitz continuous near $x$, the Clarke generalized directional derivative is:*

$$f^\circ(x; d) = \limsup_{y \to x \; t \downarrow 0} \frac{f(y + td) - f(y)}{t}.$$

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What does this $\limsup$ exactly mean?

$$\limsup_{y \to x \; t \downarrow 0} \frac{f(y+td) - f(y)}{t} \;=\; \lim_{\epsilon \downarrow 0} \; \sup_{\|y-x\| \le \epsilon, 0 < t \le \epsilon} \left\{ \frac{f(y+td) - f(y)}{t} \right\}.$$

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*Let $f$ be Lipschitz cont. near $x$. The Clarke subdifferential is given by:*

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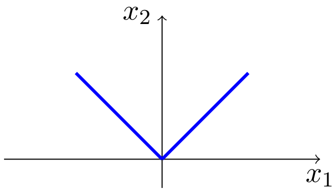

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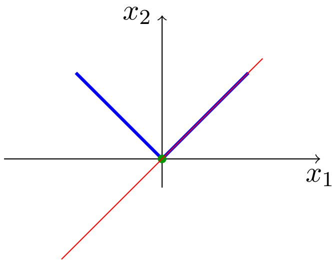

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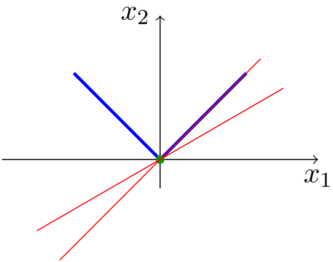

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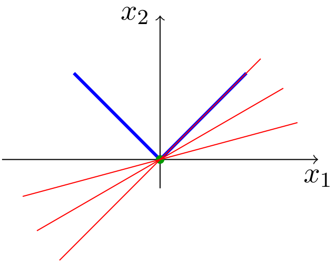

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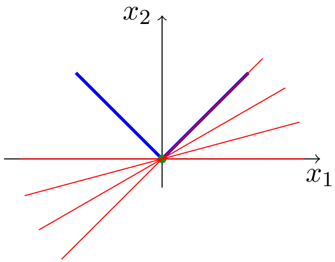

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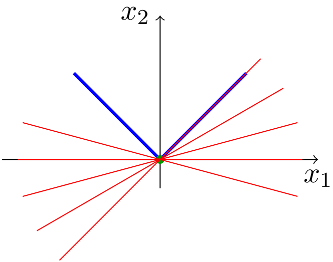

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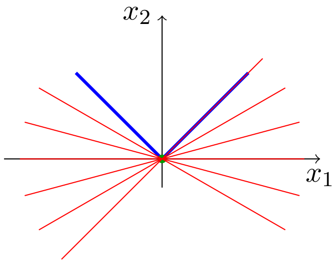

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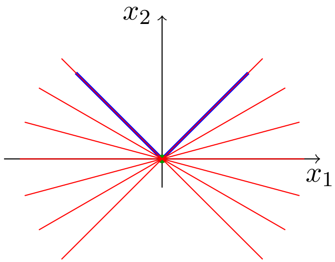


At the origin, $\partial f(0) = [-1, 1]$.

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If $f$ is increasing from $x$, along $d$, then

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**First order stationarity (Clarke)**

*If $x_*$ is a local minimizer, $f^\circ(x_*; v) \geq 0,\ \forall v \in \mathbb{R}^n$ or, equivalently, $0 \in \partial f(x_*)$.*

# Example of use of non-smooth calculus in Optimization

Suppose $x_k \to x_*$, $\alpha_k \to 0 \in \mathbb{R}$, and $d_k \to d$ for some infinite sequence $K$. Then:

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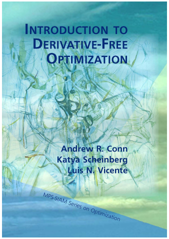
... if an Optimization algorithm can generate a subsequence of $K$ such that

$$f(x_k + \alpha_k d_k) \geq f(x_k).$$

# Luis Nunes Vicente

http://www.mat.uc.pt/~lnv




Introduction to Derivative-Free Optimization

My research interests include the development and analysis of numerical methods for large-scale nonlinear programming, sparse optimization, PDE constrained optimization problems, and derivative-free optimization problems, and applications in computational sciences, engineering, and finance.