1 Introduction to optimization models in Data Science and Learning

2 Stochastic gradient descent for Stochastic Optimization

3 Multi-Objective Optimization

4 Stochastic Multi-Objective Optimization
   - The stochastic multi-gradient algorithm and assumptions
   - Convergence rates in the strongly convex and convex cases

5 Implementation and numerical results

6 Conclusions and future directions
A data set for analysis involving optimization is typically of the form

\[ D = \{(a_j, y_j), j = 1, \ldots, N\} \]

where the \( a_j \)'s vectors are features or attributes

the \( y_j \)'s vectors are labels or observation or responses.
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The analysis consists of finding a prediction function \( \phi(a_j) \) such that

\[ \phi(a_j) \simeq y_j, \quad j = 1, \ldots, N \]

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Often, \( \phi \) is parameterized \( \phi(x) = \phi(a; x) \). The parameters are \( x \).
Notes:

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2. When the $y_j$'s are reals, one has a regression problem.

3. When the $y_j$'s lie in a finite set $\{1, ..., M\}$, one has a classification problem. $M=2$ leads to binary classification.

4. The labels may be null. In that case, one may want: to group the $a_j$'s in clusters (clusterization) or to identify a low-dimensional subspace (or a collection of) where the $a_j$'s lie (subspace identification). The labels may have to be learned while learning $\phi$. 
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4. The labels may be **null**. In that case, one may want:
   - to group the $a_j$’s in clusters (**clusterization**) or to identify a low-dimensional subspace (or a collection of)
   - where the $a_j$’s lie (**subspace identification**).

   **The labels may have to be learned while learning $\phi$.**
Data is assumed to be clean for optimization, but still:

i) \((a_j, y_j)\)'s could be noisy or corrupted.

ii) Some \(a_j\) or \(y_j\)'s could be missing.

iii) Data could arrive in streaming fashion (\(\phi\) must be learned online).
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Thus, \(\phi\) has to be robust to changes in the data set.

Such data analysis is often referred to as machine learning or data mining.

predictive or supervised learning (when labels exist)

unsupervised learning (when labels are null): extract interesting information from data
The correct/accurate match of the data is typically quantified by a loss function $\ell(a, y; \phi(x))$ and thus learning can be formulated as

$$\min_x \sum_{j=1}^{N} \ell(a_j, y_j; \phi(x))$$
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To avoid overfitting such a model to a sample $D$, one often adds a regularizer

$$\min_x \sum_{j=1}^N \ell(a_j, y_j; \phi(x)) + g(x)$$

so that $\phi$ is not so sensitive to changes in $D$. 
The general form of optimization models in DS is

$$\min_{x \in \mathbb{R}^n} f(x) + g(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is smooth.
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where $f : \mathbb{R}^n \to \mathbb{R}$ is smooth.

$\nabla f$ is at least Lipschitz continuous.

$g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex, proper and closed.

$g$ is typically non-smooth.
Classical example: logistic regression for binary classification
Classical example: **logistic regression for binary classification**

Consider a data set with $y \in \{-1, 1\}$

\[
\phi(x) = r^\top a - b
\]

A linear classifier. One aims to seek for an **optimal parameter** $x = (r, b)$ such that

- $r^\top a_j - b \geq 0$ when $y_j = 1$
- $r^\top a_j - b < 0$ when $y_j = -1$

$\forall j = 1, ..., N$ (\ding{55})

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Classical example: logistic regression for binary classification

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\phi(x) = \phi(a; r, b) = r^\top a - b \quad \text{is a linear classifier.}
$$

One aims to seek for an optimal parameter $x = (r, b)$ such that

$$
\begin{cases}
  r^\top a_j - b \geq 0 & \text{when } y_j = 1 \\
  r^\top a_j - b < 0 & \text{when } y_j = -1
\end{cases}
\quad \forall j = 1, \ldots, N \quad (*)
$$
Logistic regression can be motivated by two arguments.

1. Smoothing/convexifying the true loss.

What we want to do is

$$\min 1(\text{sign}(z) \neq y)$$

where $$z = \phi(x)$$.

\[ y = +1 \]
Smoothing/convexifying to **Hinge loss (SVM)**

\[ y = +1 \]

\[ \max\{0, -(+1)z\} \]

Smoothing to **Logistic loss**

\[ y = +1 \]

\[ \log(1 + e^{-(+1)z}) \]
Probabilistic argument: (\#) is equivalent to

\[ \sigma(\phi(x)) = \frac{1}{1 + e^{-\phi(x)}} \begin{cases} \geq 0.5 & \text{when } y = 1 \\ < 0.5 & \text{when } y = -1 \end{cases} \]
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Think of it as the probability

\[ P(y = 1|a) = \sigma(\phi(x)) = \frac{1}{1 + e^{(-1)\phi(x)}} \]
\[ P(y = -1|a) = 1 - \sigma(\phi(x)) = \frac{1}{1 + e^{-(+1)\phi(x)}} \]
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\[ P(y = 1|a) = \sigma(\phi(x)) = \frac{1}{1 + e^{-(+1)\phi(x)}} \]
\[ P(y = -1|a) = 1 - \sigma(\phi(x)) = \frac{1}{1 + e^{-(-1)\phi(x)}} \]

Assuming data points are i.i.d., the maximum likelihood problem is

\[ \max \prod_{j=1}^{N} P(y_j|a_j) = \prod_{j=1}^{N} \frac{1}{1 + e^{-y_j\phi(x)}} \]
prob. of correct prediction
Then, the negative log likelihood leads to the loss minimization problem

\[
\min_{r,b} \frac{1}{N} \sum_{j=1}^{N} \ell_L(a_j, y_j; \phi(x))
\]

with the smooth convex logistic loss function

\[
\ell_L(a_j, y_j; \phi(x)) = \log(1 + e^{-y_j(r^T a_j - b)})
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Then, the negative log likelihood leads to the loss minimization problem

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$$\ell_L(a_j, y_j; \phi(x)) = \log(1 + e^{-y_j(r^T a_j - b)})$$

Binary classification is formulated as the optimization problem

$$\min_{r,b} \frac{1}{N} \sum_{j=1}^{N} \ell_L(a_j, y_j; \phi(x)) + \frac{\lambda}{2} \|r\|_2^2$$
Presentation outline

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The general goal now is to minimize the expected risk of misclassification

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R(x) = \int \ell(a, y; \phi(x)) dP_\Omega(a, y)
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When \(P_{\Omega}\) is unknown, only having a sample \(D\) leads to minimizing the empirical risk of misclassification

\[
R_N(x) = \frac{1}{N} \sum_{j=1}^{N} \ell(a_j, y_j; \phi(x))
\]

as an estimation of \(R(x)\).
For simplicity, let $f(x) = \ell(a, y; \phi(x))$. 
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\( w \) is the \textbf{random seed variable} representing a sample. (A set of realizations \( \{w[j]\}_{j=1}^{N} \) corresponds to a sample set \( \{(a_j, y_j), j = 1, \ldots, N\} \).)
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The general objective could be written as

\[
R(x) = \begin{cases} 
R(x) = \mathbb{E}[f(x; w)] & \text{Online setting (expected risk)} \\
R_N(x) = \frac{1}{N} \sum_{j=1}^{N} f_j(x) & \text{Finite sum setting (empirical risk)}
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where \( f_j(x) = f(x; w[j]) \) is the loss associated with the \( j \)-th sample.
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Both can be minimized by the stochastic gradient method.
The stochastic gradient method

Algorithm 1 Stochastic Gradient (SG) method (Robbins and Monro 1951)

1: Choose an initial point $x_0 \in \mathbb{R}^n$.
2: for $k = 0, 1, \ldots$ do
3: Compute a stochastic gradient $g(x_k; w_k)$.
4: Choose a step-size $\alpha_k > 0$.
5: Set the new iterate $x_{k+1} = x_k - \alpha_k g(x_k; w_k)$.
6: end for
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The stochastic gradient \( g(x_k; w_k) \) could be

\[
\nabla f(x_k; w_k) = \nabla f_{j_k}(x_k) \quad \text{simple or basic SG}
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$$\nabla f(x_k; w_k) = \nabla f_{j_k}(x_k) \quad \text{simple or basic SG}$$

$$\frac{1}{|B_k|} \sum_{j \in B_k} \nabla f(x_k; w_{k,j}) \quad \text{mini-batch SG (}|B_k| \text{ is the mini-batch size)}$$
1. The sequence of realizations \( \{w_k\} \) are i.i.d. sample of \( w \).

2. Function \( R \) has Lipschitz continuous gradient.

3. The stochastic gradient is an unbiased estimate
   \[
   \mathbb{E}_{w_k}[g(x_k; w_k)] = \nabla R(x_k)
   \]

4. The stochastic gradient has bounded variance
   \[
   \nabla_{w_k}[g(x_k; w_k)] \leq M + \mathcal{O}(\|\nabla R(x_k)\|^2)
   \]

5. Function \( R \) is bounded below: \( R(x) \geq R_*, \forall x \). \( R_* = R(x_*) \) where \( x_* \) is a minimizer of \( R \).

\( \mathbb{E}[\cdot] \) denotes the expected value taken w.r.t. the joint distribution of \( \{w_k\} \).
The strongly convex case (Sacks 1958)

Consider a diminishing step-size sequence (e.g., $\alpha_k \approx O(1/k)$), one has

$$\mathbb{E}[R(x_k)] - R_* \leq O(1/k)$$
Convergence rates for SG method

The strongly convex case (Sacks 1958)

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The convex case (Nemirovski and Yudin 1978)

Consider a diminishing step-size sequence (e.g., \( \alpha_k \simeq O(1/\sqrt{k}) \)), one has

\[
E[R_{\text{best}}^k] - R_* \leq O(1/\sqrt{k})
\]

where \( R_{\text{best}}^k = \min_{1 \leq i \leq k} R(x_i) \).
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Multi-Objective Optimization (MOO) deals with multiple potentially conflicting objectives simultaneously.
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We consider smooth MOO problems of the general form

\[
\min \ H(x) = (h_1(x), \ldots, h_m(x))
\]
\[
\text{s.t. } x \in X
\]

where \( h_i : \mathbb{R}^n \rightarrow \mathbb{R} \) are smooth functions and \( X \) is a feasible region.
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\end{align*}$$

where $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are smooth functions and $\mathcal{X}$ is a feasible region.

Definition of optimality for MOO

- $x$ dominates $y$ if $H(x) < H(y)$ componentwise, $\forall x, y \in \mathcal{X}$.
- $x$ is a Pareto minimizer if it is not dominated by any other point in $\mathcal{X}$.

Also called a non-dominated or efficient point.
Denote $\mathcal{P}$ as the set of Pareto minimizers.

**Pareto front** is a mapping from $\mathcal{P}$ to function value space $\mathbb{R}^m$, i.e.,

$$H(\mathcal{P}) = \{ H(x) : x \in \mathcal{P} \}$$

**Figure:** Pareto front of problem SP1
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**Figure**: Pareto front of problem SP1

**The goal of MOO**: find the set of Pareto minimizers and hence Pareto front to define the best trade-off among several competing criteria.
Pareto first-order stationary condition: $x$ is a Pareto stationary point if

$$\nexists \ d \in \mathbb{R}^n \ such \ that \ \begin{bmatrix} \nabla h_1(x)^\top \\ \vdots \\ \nabla h_m(x)^\top \end{bmatrix} d < 0$$

Note: when all the functions are convex, $x \in P$ iff $x$ is Pareto first-order stationary.
Necessary condition for Pareto minimizers

Pareto first-order stationary condition: \( x \) is a Pareto stationary point if

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Or equivalently, if the convex hull of \( \nabla h_i(x) \)'s contains the origin

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\exists \lambda \in \Delta^m \text{ such that } \sum_{i=1}^{m} \lambda_i \nabla h_i(x) = 0
\]

where \( \Delta^m = \{ \lambda : \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0, \forall i = 1, \ldots, m \} \) is a simplex set.
**Necessary condition for Pareto minimizers**

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**Note:** when all the functions are convex, $x \in \mathcal{P}$ iff $x$ is Pareto first-order stationary.
Overview of methods for MOO

Methods with a priori preferences (Ehrgott 2005; Miettinen 2012)

- **Weighted-sum method**

  \[
  \min \ S(x; a) = \sum_{i=1}^{m} a_i f_i(x), \quad \text{where } a \in \Delta^m
  \]

  **Limitations:** hard to preselect weights for different magnitudes; cannot find Pareto minimizers in non-convex regions.

- **\(\epsilon\)-constraint method**

  \[
  \min \ f_i(x) \quad \text{s.t. } f_j(x) \leq \epsilon_j, \quad \forall j \neq i
  \]

  where \(\epsilon_j \geq \min_{x \in \mathcal{X}} f_j(x)\) are upper bounds.

  **Limitations:** inappropriate upper bounds lead to infeasibility.

**Common issues:** output single non-dominated point at one run, produce poorly distributed Pareto front with multiple runs.
Overview of methods for MOO

Methods with *a posteriori* preferences

**Population-based heuristic methods**: no convergence proofs, e.g.,
- NSGA-II (genetic algorithms) (Deb et al. 2002).
- AMOSA (simulated annealing) (Bandyopadhyay et al. 2008).

**Convergent methods**: proved convergence to Pareto stationary points
- multi-gradient method (Fliege and Svaiter 2000).
- Newton’s method for MOO (Fliege et al. 2009).
- direct multi-search algorithm (Custódio et al. 2011), etc.

**Superiority**: able to construct the whole well-spread Pareto front.
The multi-gradient algorithm iterates: \( x_{k+1} = x_k + \alpha_k d_k \).
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Subproblem 1 (Fliege and Svaiter 2000):

\[
d_k \in \arg\min_{d \in \mathbb{R}^n} \max_{1 \leq i \leq m} \left\{ -\nabla h_i(x_k)^\top d \right\} + \frac{1}{2} \|d\|^2
\]

Note: \( d_k = 0 \in \mathbb{R}^n \) iff \( x_k \) is Pareto stationary.
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**Subproblem 2:** dual problem of Subproblem 1

\[
\lambda_k \in \arg\min_{\lambda \in \mathbb{R}^m} \left\| \sum_{i=1}^m \lambda_i \nabla h_i(x_k) \right\|^2 \quad \text{s.t.} \quad \lambda \in \Delta^m
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Then, \( d_k = -\sum_{i=1}^m (\lambda_k)_i \nabla h_i(x_k) \) is a common descent direction.
The multi-gradient algorithm iterates: $x_{k+1} = x_k + \alpha_k d_k$.

Subproblem 1 (Fliege and Svaiter 2000):

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Subproblem 2: dual problem of Subproblem 1

$$\lambda_k \in \underset{\lambda \in \mathbb{R}^m}{\text{argmin}} \left\| \sum_{i=1}^m \lambda_i \nabla h_i(x_k) \right\|^2 \text{ s.t. } \lambda \in \Delta^m$$

Then, $d_k = -\sum_{i=1}^m (\lambda_k)_i \nabla h_i(x_k)$ is a common descent direction.

Note: when $m = 1$, one recovers $d_k = -\nabla h_1(x_k)$. 
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Consider a **stochastic multi-objective** problem

$$\min \quad F(x) = (f_1(x), \ldots, f_m(x)) = (\mathbb{E}[f_1(x; w)], \ldots, \mathbb{E}[f_m(x; w)])$$

s.t. \quad x \in \mathcal{X}$$

where \( w \in \mathbb{R}^{m \times p} \) are random parameters obeying a certain distribution.
Consider a stochastic multi-objective problem

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s.t. $$x \in \mathcal{X}$$

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Subproblem 3: replace $$\nabla f_i(x_k)$$ by an estimate $$g_i(x_k; w_k)$$ in Subproblem 2

$$\lambda^g(x_k; w_k) \in \arg\min \limits_{\lambda \in \mathbb{R}^m} \left\| \sum_{i=1}^m \lambda_i g_i(x_k; w_k) \right\|^2$$

s.t. $$\lambda \in \Delta^m$$

$$g(x_k; w_k) = \sum_{i=1}^m (\lambda^g_k)_i g_i(x_k; w_k)$$ denotes the stochastic multi-gradient.
Consider a stochastic multi-objective problem

$$\min \ F(x) = (f_1(x), \ldots, f_m(x)) = (\mathbb{E}[f_1(x; w)], \ldots, \mathbb{E}[f_m(x; w)])$$

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where \( w \in \mathbb{R}^{m \times p} \) are random parameters obeying a certain distribution.

Subproblem 3: replace \( \nabla f_i(x_k) \) by an estimate \( g_i(x_k; w_k) \) in Subproblem 2

$$\lambda^g(x_k; w_k) \in \arg\min_{\lambda \in \mathbb{R}^m} \left\| \sum_{i=1}^{m} \lambda_i g_i(x_k; w_k) \right\|^2$$

s.t. \( \lambda \in \Delta^m \)

\( g(x_k; w_k) = \sum_{i=1}^{m} (\lambda^g_k)_i g_i(x_k; w_k) \) denotes the stochastic multi-gradient.

The stochastic multi-gradient (SMG) algorithm: \( x_{k+1} = x_k - \alpha_k g(x_k; w_k) \)
The stochastic multi-gradient method

Consider using an orthogonal projection $P$ when minimize over a closed and convex set $\mathcal{X} \subseteq \mathbb{R}^n$. 

Algorithm 1

Stochastic Multi-Gradient (SMG) Algorithm

1: Choose an initial point $x_0 \in \mathbb{R}^n$ and a step-size sequence $\{\alpha_k\}_{k \in \mathbb{N}} > 0$.

2: for $k = 0, 1, \ldots$

3: Compute the stochastic gradients $g_i(x_k; w_k)$ for $i = 1, \ldots, m$.

4: Solve Subproblem 3 to obtain the stochastic multi-gradient $g(x_k; w_k) = \sum_{i=1}^{m} (\lambda g_k)_i g_i(x_k; w_k)$, with $\lambda g_k \in \Delta_m$.

5: Update the next iterate $x_{k+1} = P_X(x_k - \alpha_k g(x_k; w_k))$.

6: end for
The stochastic multi-gradient method

Consider using an orthogonal projection $P$ when minimize over a closed and convex set $\mathcal{X} \subseteq \mathbb{R}^n$.

**Algorithm 1** Stochastic Multi-Gradient (SMG) Algorithm

1: Choose an initial point $x_0 \in \mathbb{R}^n$ and a step-size sequence $\{\alpha_k\}_{k \in \mathbb{N}} > 0$.
2: for $k = 0, 1, \ldots$ do
3: Compute the stochastic gradients $g_i(x_k; w_k)$ for $i = 1, \ldots, m$.
4: Solve Subproblem 3 to obtain the stochastic multi-gradient
   \[ g(x_k; w_k) = \sum_{i=1}^{m} (\lambda^g_k)_i g_i(x_k; w_k), \text{ with } \lambda^g_k \in \Delta^m. \]
5: Update the next iterate $x_{k+1} = P\mathcal{X}(x_k - \alpha_k g(x_k; w_k))$.
6: end for
Consider the case $m = 2, n = 2$.

$g_1$ and $g_2$ are two stochastic multi-gradients by solving Subproblem 3. They are estimates of the true multi-gradient $g$ (Subproblem 1 or 2).
Denote

\[ S(x; \lambda) = \sum_{i=1}^{m} \lambda_i f_i(x) \]
\[ \nabla_x S(x; \lambda) = \sum_{i=1}^{m} \lambda_i \nabla f_i(x) \]
Denote

\[ S(x; \lambda) = \sum_{i=1}^{m} \lambda_i f_i(x) \]
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Even under the classical assumption in SG

\[ \mathbb{E}_w[g_i(x; w)] = \nabla f_i(x), \ \forall i = 1, \ldots, m \]

it turns out that \( g(x; w) \) is a biased estimate, i.e.,

\[ \mathbb{E}_w[g(x; w)] \neq \nabla_x S(x; \lambda) \]

where \( \lambda \) are the true coefficients from Subproblem 2.
Biasedness of the stochastic multi-gradient

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\[ \mathbb{E}_w[g(x; w)] \neq \nabla_x S(x; \lambda) \]

where \( \lambda \) are the true coefficients from Subproblem 2. We also have

\[ \mathbb{E}_w[g(x; w)] \neq \mathbb{E}_w[\nabla_x S(x; \lambda^g)] \]
The biasedness using either the true coefficients $\lambda$ or $\lambda^g$ decreases as batch size increases and eventually vanishes in the full batch setting.
Assumptions for convergence

1. All objective functions \( f_i : \mathbb{R}^n \rightarrow \mathbb{R} \) are continuously differentiable with Lipschitz continuous gradients \( \nabla f_i \).

2. The feasible region \( \mathcal{X} \subseteq \mathbb{R}^n \) is bounded.

3. (Unbiasedness) \( \mathbb{E}_w[g_i(x; w)] = \nabla f_i(x) \).

4. (Bound on the biasedness) There exist \( M_1, M_F > 0 \) such that

\[
\|\mathbb{E}_w[g(x; w) - \nabla_x S(x; \lambda^g)]\| \leq \alpha (M_1 + M_F \|\mathbb{E}_w[\nabla_x S(x; \lambda^g)]\|)
\]

(can be guaranteed by dynamic sampling)

5. (Bound on the second moment) There exist \( G, G_V > 0 \) such that

\[
\mathbb{E}_w[\|g(x; w)\|^2] \leq G^2 + G_V^2 \|\mathbb{E}_w[\nabla_x S(x; \lambda^g)]\|^2
\]
Sublinear rate in the strongly convex case

Theorem (Individual $f_i$ are strongly convex ($c$ is max of constants))

Assume $\lambda_t \to \lambda_*$. Let $x_* \in \mathcal{P}$ be associated with $\lambda_*$. Considering a diminishing step-size sequence $\alpha_t = \frac{2}{c(t+1)}$, we have

$$
\min_{t=1, \ldots, k} \mathbb{E}[S(x_t; \lambda_t)] - \mathbb{E}[S(x_*; \bar{\lambda}_k)] \leq \mathcal{O}(1/k)
$$

where $\bar{\lambda}_k = \sum_{t=1}^{k} \frac{t}{\sum_{t=1}^{k} t} \lambda_t \in \Delta^m$.

Then, we have $\min_{1 \leq t \leq k} \mathbb{E}[S(x_t; \lambda_t)] \to \mathbb{E}[S(x_*; \lambda_*)]$. 

Assume $\lambda_k$ being a better approximation to $\lambda_*$. We know $\nabla_x S(x_*; \lambda_*)^\top (x_k - x_*) \geq 0$. 

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[->] (-2,0) -- (2,0) node[right] {$f_1$};
\draw[->] (0,-2) -- (0,2) node[above] {$f_2$};
\draw[blue, thick] (-1,-1) .. controls (1,-1) .. (1,1);
\filldraw[red] (1,1) circle (2pt) node[above] {$\lambda_k$};
\filldraw[red] (0,0) circle (2pt) node[above] {$\lambda_*$};
\end{tikzpicture}
\end{figure}
Assume $\lambda_k$ being a better approximation to $\lambda_*$. We know $\nabla_x S(x_*; \lambda_*)^\top (x_k - x_*) \geq 0$.

**Assumption**

*For any $x_k \in X$, one has*

$$\nabla_x S(x_*; \lambda_k)^\top (x_k - x_*) \geq 0$$

*where $\lambda_k$ are the true coefficients by solving Subproblem 2.*
Theorem (SL&LNV 2019)

Let $x_*$ be the Pareto minimizer associated with $\lambda_*$. Considering a step-size sequence $\alpha_k = \gamma/k$ with $\gamma > 1/2c$, we have

$$\mathbb{E}[\|x_k - x_*\|^2] \leq O(1/k)$$

and

$$\mathbb{E}[S(x_k; \lambda_*)] - \mathbb{E}[S(x_*; \lambda_*)] \leq O(1/k)$$
The Pareto-Front version of SMG is designed to obtain a complete Pareto front in a single run.

**Algorithm 2** Pareto-Front Stochastic Multi-Gradient (PF-SMG) Algorithm

1: Generate a list of starting points $L_0$. Select $r, p, q \in \mathbb{N}$.

2: for $k = 0, 1, \ldots$ do

3: \hspace{1em} Set $L_{k+1} = L_k$.

4: \hspace{1em} for each point $x$ in the list $L_{k+1}$ do

5: \hspace{2em} for $t = 1, \ldots, r$ do

6: \hspace{3em} Add $x + w^t$ to the list $L_{k+1}$ where $w^t$ is a realization of $w_k$.

7: \hspace{1em} for each point $x$ in the list $L_{k+1}$ do

8: \hspace{2em} for $t = 1, \ldots, p$ do

9: \hspace{3em} Apply $q$ iterations of the SMG algorithm starting from $x$.

10: \hspace{2em} Add the final output point $x_q$ to the list $L_{k+1}$.

11: Remove all the dominated points from $L_{k+1}$.
$r = 3, p = 2$

Generate starting points
$r = 3, p = 2$

Add perturbed points
$r = 3, p = 2$

Apply multiple times SMG
$r = 3, p = 2$

Remove dominated points
After certain number of iterations

\[ r = 3, p = 2 \]
Motivation: consider a set of data points in 2D

Data points labeled by ♦ and ● may be collected from different sources/groups.
**Motivation:** consider a set of data points in 2D

\[ y = 1 \]
\[ y = -1 \]

Data points labeled by ♦ and ● may be collected from different sources/groups.

**Key idea:** design a MOO problem to identify the existence of bias in data and define the best trade-off if data bias exists.
Recall the logistic (prediction) loss function

\[
f(r, b) = \frac{1}{N} \sum_{j=1}^{N} \log(1 + e^{-y_j(r^\top a_j - b)}) + \frac{\lambda}{2} \|r\|^2
\]

where \(\{(a_j, y_j)\}_{j=1}^{N}\) are i.i.d. feature/label pairs sampled from a certain joint probability distribution of \((A, Y)\).

Testing data sets are selected from LIBSVM (Chang and Lin 2011).

Split a data set into two groups according to a binary feature. Let \(J_1\) and \(J_2\) be two index sets. A two-objective problem is constructed as

\[
\min_{r,b} (f_1(r, b), f_2(r, b))
\]

where

\[
f_i(r, b) = \frac{1}{|J_i|} \sum_{j \in J_i} \log(1 + e^{-y_j(r^\top a_j - b)}) + \frac{\lambda_i}{2} \|r\|^2
\]
Approximated Pareto fronts for the multi-objective logistic regression problems:

(a) *heart*

(b) *australian*

(c) *svmguide3*

(d) *german.numer*
Consistently, wider Pareto fronts of *heart* and *svmguide3* indicate higher distinction between two groups.
Consistently, wider Pareto fronts of *heart* and *svmguide3* indicate higher distinction between two groups.

**Two implications:**

- Given groups of data instances for the same problem, one can evaluate the bias by observing the range of Pareto fronts.
- New data instance (of unknown group) can be classified more accurately by selecting a set of nondominated points.
Presentation outline

1. Introduction to optimization models in Data Science and Learning
2. Stochastic gradient descent for Stochastic Optimization
3. Multi-Objective Optimization
4. Stochastic Multi-Objective Optimization
   - The stochastic multi-gradient algorithm and assumptions
   - Convergence rates in the strongly convex and convex cases
5. Implementation and numerical results
6. Conclusions and future directions
Established sublinear convergence rates, $O(1/k)$ for strongly convex and $O(1/\sqrt{k})$ for convex case in terms of a weighted sum function.

Designed PF-SMG algorithm that is robust and efficient to generate well-spread and sufficiently accurate Pareto fronts.

In logistic binary classification, developed a novel tool for identifying bias among potentially different sources of data.

Future directions

- Have a (probabilistic?) result for determining the whole Pareto front.
- Investigate variance reduction techniques.
- Deal with nonconvexity, nonsmoothness, general constraints, . . .

- Expand use of MOO in machine learning:
  - Conflicting robotic learning.