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A Non Fickian single phase flow model

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Abstract

A single phase incompressible flow problem is usually modeled by a system of three equations:a differential equation for the velocity, an algebraic equation linking the velocity and the pressure and a parabolic equation for the concentration depending on the velocity. Some limitations have been pointed out in the literature on the use of a parabolic equations to describe the the concentration evolution, namely related with the use of Fick's law to describe the mass flux. To avoid the pathologic behavior of the classical diffusion equation, non Fickian corrections have been proposed in the literature. In this paper we introduce a new model to describe a single phase incompressible flow problem and its stability will be studied. A numerical method that mimics the continuous model is also studied and some numerical experiments are included.

Key words: Fikian model, Non Fickian model, Diffusion, pressure, concentration, numerical method, stability.

1 Introduction

A single phase incompressible flow problem is usually modeled by a system of three equations: a differential equation for the pressure, an algebraic equation linking the velocity and the pressure and a parabolic equation for the concentration depending on the velocity (see [1], [2], [4], [5], [9] and [10]). This system can be rewritten as a system of an elliptic equation for the pressure and a parabolic equation for the concentration that depends on the gradient of the pressure.

Traditionally, the a diffusion process in a porous medium is described by the parabolic equation

$$\frac{\partial u}{\partial t} + \nabla J = q_2, \tag{1}$$

where u denotes the concentration, J represents the mass flux and q_2 denotes the reaction term. In (1) J can be expressed as

$$J = J_{adv} + J_{dif} + J_{dis}, \tag{2}$$

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where

$$J_{adv} = uv \tag{3}$$

represents the advection mass due to the the fluid velocity v,

$$J_{dif} = -D_m \nabla u \tag{4}$$

denotes the mass flux due to molecular diffusion, being D_m the effective molecular diffusion coefficient, and J_{dis} satisfies the so called Fick's law $J_{dis} = -D_d \nabla u$ and represents the dispersive mass flux associated with random deviations of fluid velocities within the porous space from their macroscopic value v. In the definition of J_{dis}, D_d denotes the dispersive tensor $D_d = \alpha_t \|v\| I + (\alpha_\ell - \alpha_t)\frac{1}{\|v\|} vv^t$ being α_ℓ and α_t the longitudinal and transversal dispersivities.

Combining (1) with (2) we obtain the parabolic equation

$$\frac{\partial u}{\partial t} + \nabla(uv) = \nabla((D_m I + D_d)\nabla u) + q_2, \tag{5}$$

where I is the identity tensor.

Some limitations have been pointed out in the literature on the use of a parabolic equation (5) to describe the concentration evolution (see, for instance, [3], [6], [8]): equation (5) prescribes an infinite speed of propagation for the concentration; it is based on Fick's law for the mass flux which establishes a linear relation between the concentration and dispersive mass flux; the mass flux J is independent of the history of dispersion; in the dispersive tensor the dispersivities coefficients are medium constant and invariant with time and space (often they increase with the distance and/or with time).

To avoid the pathologic behavior of the classical diffusion equation, hyperbolic or non Fickian corrections have been proposed in the literature (see [6], [8]). One possible approach is to consider that the dispersive mass flux satisfies the following differential equation

$$\tau \frac{\partial J_{dis}}{\partial t}(x,t) + J_{dis}(x,t) = -D_d \nabla u(x,t), \tag{6}$$

where τ is a delay parameter ([7]). We remark that the left hand side of (6) is a first order approximation of the left hand side of $J_{dis}(x, t+\tau) = -D_d \nabla u(x,t)$, which means that the dispersion mass flux at the point x and time $t+\tau$ depends on the gradient of the concentration at the same point but at a delayed time. Equation (6) leads to

$$J_{dis}(t) = -\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}} D_d \nabla u(s)\, ds, \tag{7}$$

provided that $J_{dis}(0) = 0$. Combining the partition (2), where J_{adv}, J_{dif} and J_{dis} are given by (3), (4) and (7), respectively, with (1) we obtain the integro-differential equation

$$\frac{\partial u}{\partial t} + \nabla(uv) - \nabla(D_m \nabla u) = \frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}} \nabla(D_d \nabla u)(s)\, ds + f \tag{8}$$

which replaces (5).

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In this paper we consider the following system of equations:

$$-\nabla(A(u)\nabla p) = q_1 \,\text{in}\, (a,b)\times(0,T], \tag{9}$$

$$\begin{aligned}&\frac{\partial u}{\partial t} + \nabla(B(u,\nabla p)u) + \nabla(D_m(u,\nabla p)\nabla u)\\ &= \int_0^t k_{er}(t-s)\nabla(D_d(u,\nabla p)\nabla u)(s)\,ds + q_2 \,\text{in}\, (a,b)\times(0,T],\end{aligned} \tag{10}$$

where q_1 and q_2 are source terms, A, D_m and D_d are smooth enough satisfying the following assumptions:

$$0 < A_0 \leq A(x,y), (x,y) \in \mathbb{R}\times\mathbb{R}, \tag{11}$$

$$|B(x,y)| \leq C_B|y|, (x,y) \in \mathbb{R}\times\mathbb{R}, \tag{12}$$

$$0 < D_{m,0} \leq D_m(x,y), (x,y) \in \mathbb{R}\times\mathbb{R}, \tag{13}$$

$$0 < D_{d,0} \leq D_d(x,y) \leq D_{d,1}|y|, (x,y) \in \mathbb{R}\times\mathbb{R}. \tag{14}$$

In (10) $K_{er}(s)$ denotes a kernel that satisfies some assumptions that will be specified later but it can be in particular defined by $K_{er}(s) = \frac{1}{\tau}e^{-\frac{s}{\tau}}$. System (9), (10) is complemented by Dirichelet boundary conditions

$$p(a,t) = p_a(t), p(b,t) = p_b(t), u(a,t) = u_a(t), u(b,t) = u_b(t),\ t\times]0,T], \tag{15}$$

and initial conditions

$$p(x,0) = p_0(x), u(x,0) = u_0(x), x \in (a,b). \tag{16}$$

The paper is organized as follows. In Section 2 we study the stability of the initial boundary value problem (9), (10), (15), (16). A numerical method that mimics the initial boundary value problem (9), (10), (15), (16) will be presented in Section 3 and its stability will be analized. In Section 4 we include some numerical experiments illustrating the behavior of the pressure and concentration when we replace (5) by (10).

2 Stability analysis

By $H^1(a,b)$ and $H^1_0(a,b)$ we denote the usual Sobolev spaces with the usual norm $\|.\|_1$. By $(.,.)$ we represent the usual inner product defined in $L^2(a,b)$ and $\|.\|$ denotes the norm induced by such inner product. By $L^2(0,T;H^1(a,b))$ we denote the space of functions $v : (0,T) \to H^1(a,b)$ such that $\int_0^T \|v(s)\|_1^2\,ds < \infty$. We also consider the space $\mathcal{W}(0,T) = \{v \in L^2(0,T;H^1(a,b)) : \frac{dv}{dt} \in L^2(0,T;L^2(a,b))\}$, where $L^2(0,T;L^2(a,b))$ is the space of functions $v : (0,T) \to L^2(a,b)$ such that $\int_0^T \|v(s)\|^2\,ds < \infty$.

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We study in what follows the stability of the solution (u,p), $u \in \mathcal{W}(0,T)$, $p \in L^2(0,T;H^1(a,b))$, that satisfies the variational equations

$$(A(u(t))\nabla p(t), \nabla w) = (q_1, w),\ \forall w \in H_0^1(a,b), \tag{17}$$

$$\begin{aligned}(\frac{du}{dt}(t), w) &- (B(u(t), \nabla p(t))u(t), \nabla w) + (D_m(u(t), \nabla p(t))\nabla u(t), \nabla w) \\ &= -(\int_0^t K_{er}(t-s) D_d(u(s), \nabla p(s))\nabla u(s)\, ds, \nabla w) + (q_2(t), w),\ \forall w \in H_0^1(a,b),\end{aligned} \tag{18}$$

almost everywhere in $(0,T]$, the boundary conditions (15) and the initial conditions (16) with $p_0, u_0 \in L^2(a,b)$.

As this section focuses in the stability analysis of initial boundary value problem (9), (10), (15), (16) we assume that $p_a(t) = p_b(t) = u_a(t) = u_b(t) = 0$, $t \in]0,T]$. We also introduce the space $L^2(0,T;H_0^1(a,b))$ which is obtained from $L^2(0,T;H^1(a,b))$ replacing $H^1(a,b)$ by $H_0^1(a,b)$. By $\mathcal{W}_0(0,T)$ we denote the subspace of $\mathcal{W}(0,T)$ that is obtained replacing $H^1(a,b)$ by $H_0^1(a,b)$.

Theorem 1 *Let us suppose that $p_0, u_0 \in L^2(a,b)$, A, B, D_m and D_d satisfy the conditions (11), (12), (13) and (14), respectively. If the solution (u,p) of (17), (18) with initial conditions (16) is in $\mathcal{W}_0(0,T) \times L^2(0,T;H_0^1(a,b))$, then*

$$\|\nabla p(t)\|^2 \le \frac{(b-a)^2}{2A_0^2}\|q_1\|^2, \tag{19}$$

$$\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2\, ds \le e^{-\tilde{C}t}(\|u_0\|^2 + \int_0^t e^{\tilde{C}s} g(s)\, ds) \tag{20}$$

where $g(s) = \dfrac{1}{\min\{1, 2(D_{m,0} - \sigma^2 - \gamma^2)\}}\Big(\|u_0\|^2 + \dfrac{1}{2\eta^2}\displaystyle\int_0^s \|q_2(\mu)\|^2\, d\mu,\Big)$, $\eta \neq 0$ *is an arbitrary constant,* $\sigma \neq 0, \gamma \neq 0$ *satisfy*

$$D_{m,0} - \sigma^2 - \gamma^2 > 0, \tag{21}$$

K_{er} *and* C_{q_1} *are such that*

$$\int_0^t k_{er}(t-s)^2 \|q_1(s)\|^2\, ds \le C_{q_1},\ t \in (0,T], \tag{22}$$

holds, and

$$\tilde{C} = \frac{\max\Big\{D_{d,1} C_{q_1} \frac{(b-a)^2}{2A_0^2}\frac{1}{2\sigma^2}, 2\Big(\eta^2 + (C_B \frac{(b-a)^2}{2A_0^2})^2 \frac{1}{4\gamma^2}\max_{t\in[0,T]}\|q_1(t)\|^2\Big)\Big\}}{\min\Big\{1, 2\Big(D_{m,0} - \sigma^2 - \gamma^2\Big)\Big\}}. \tag{23}$$

Proof: Considering in (17) $w = p(t)$ and using the assumption (11)) we deduce

$$A_0\|\nabla p(t)\|^2 \le \frac{1}{4\epsilon^2}\|q_1(t)\|^2 + \epsilon^2\|p(t)\|^2, \tag{24}$$

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where $\epsilon \neq 0$ is an arbitrary constant.
As the Friedrichs-Poincaré inequality $\|p(t)\|^2 \leq \frac{(b-a)^2}{2}\|\nabla p(t)\|^2$ holds, from (24) we obtain

$$\frac{A_0}{2}\|\nabla p(t)\|^2 + (\frac{A_0}{2} - \epsilon^2\frac{(b-a)^2}{2})\|\nabla p(t)\|^2 \leq \frac{1}{4\epsilon^2}\|q_1\|^2. \tag{25}$$

Then, for ϵ such that $\frac{A_0}{2} - \epsilon^2\frac{(b-a)^2}{2} = 0$ we conclude (19).

Taking in (18) $w = u(t)$ and using (12), (13) and (14) we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 - C_B\|\nabla p(t)\|\|u(t)\|\|\nabla u(t)\| + D_{m,0}\|\nabla u(t)\|^2 \\ \leq \int_0^t |K_{er}(t-s)|D_{d,1}\|\nabla p(s)\|\|\nabla u(s)\|ds\|\nabla u(t)\| + \frac{1}{4\eta^2}\|q_2(t)\|^2 + \eta^2\|u(t)\|^2,\end{aligned} \tag{26}$$

where $\eta \neq 0$ is an arbitrary constant.
It can be shown that

$$\begin{aligned}\int_0^t |K_{er}(t-s)|D_{d,1}\|\nabla p(s)\|\|\nabla u(s)\|ds\|\nabla u(t)\| \\ \leq D_{d,1}^2\frac{(b-a)^2}{2A_0^2}C_{q_1}\frac{1}{4\sigma^2}\int_0^t \|\nabla u(s)\|^2\,ds + \sigma^2\|\nabla u(t)\|^2,\end{aligned} \tag{27}$$

where $\sigma \neq 0$ is an arbitrary constant and C_{q_1} is fixed by (22).

Considering (19) and (27) in (26) we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 - C_B\frac{(b-a)^2}{2A_0^2}\|q_1(t)\|\|u(t)\|\|\nabla u(t)\| + \Big(D_{m,0} - \sigma^2\Big)\|\nabla u(t)\|^2 \\ \leq D_{d,1}C_{q_1}\frac{(b-a)^2}{2A_0^2}\frac{1}{4\sigma^2}\int_0^t \|\nabla u(s)\|^2 ds + \frac{1}{4\eta^2}\|q_2(t)\|^2 + \eta^2\|u(t)\|^2.\end{aligned} \tag{28}$$

Furthermore, as

$$-C_B\frac{(b-a)^2}{2A_0^2}\|q_1(t)\|\|u(t)\|\|\nabla u(t)\| \geq -(C_B\frac{(b-a)^2}{2A_0^2})^2\frac{1}{4\gamma^2}\|q_1(t)\|^2\|u(t)\|^2 - \gamma^2\|\nabla u(t)\|^2,$$

where $\gamma \neq 0$ is an arbitrary constant, from (28) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \Big(D_{m,0} - \sigma^2 - \gamma^2\Big)\|\nabla u(t)\|^2 \leq \frac{1}{4\eta^2}\|q_2(t)\|^2 \\ +D_{d,1}C_{q_1}\frac{(b-a)^2}{2A_0^2}\frac{1}{4\sigma^2}\int_0^t \|\nabla u(s)\|^2 ds + \Big(\eta^2 + (C_B\frac{(b-a)^2}{2A_0^2})^2\frac{1}{4\gamma^2}\|q_1(t)\|^2\Big)\|u(t)\|^2,\end{aligned}$$

that leads to

$$\begin{aligned}\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2\,ds \leq +\tilde{C}\int_0^t \Big(\int_0^s \|\nabla u(\mu)\|^2 d\mu + \|u(s)\|^2\Big)\,ds, \\ +\frac{1}{\min\{1, 2\Big(D_{m,0} - \sigma^2 - \gamma^2\Big)\}}\Big(\|u_0\|^2 + \frac{1}{2\eta^2}\int_0^t \|q_2(s)\|^2\,ds\Big)\end{aligned} \tag{29}$$

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provided that σ and γ are fixed by (21) and $\tilde{C}$ is defined by (23). The proof of (20) is then concluded applying Gronwall Lemma to (29).

■

Theorem 1 can be easily generalized to analyze the stability of the two dimensional or three dimensional versions of the initial boundary value problem (15), (16), (17), (18).

Let us consider the particular non Fickian coupled diffusion model: equation (8),

$$\nabla v = q_1 \,\text{in}\, (a,b) \times (0,T], \tag{30}$$

where v is given by Darcy's law

$$v = -\frac{K}{\mu(u)} \nabla p \,\text{ in}\, (a,b) \times (0,T]. \tag{31}$$

In (31) K denotes the permeability tensor and $\mu(u)$ represents the viscosity. System (9), (10) with $K_{er}(s) = \dfrac{1}{\tau} e^{-\frac{s}{\tau}}$, $A(x,y) = 1, B(x,y) = -\frac{K}{\mu(x)} y$, $D_m(x,y) = D_m(const)$, $D_d(x,y) = D_d|y|$, is a non Fickian version of (5), (30) and (31). The coefficient functions satisfy the conditions (11), (12), (13) and (14) provided that K is bounded and $\mu(x) \geq \mu_0$.

3 The semi-discrete approximation

In this section we introduce the semi-discrete approximation for the variational problem (9), (10) (15), (16). Let $\mathbb{I}_h = \{x_i, i = 0, \ldots, N, x_0 = a, x_N = b, x_i - x_{i-1} = h, i = 1, \ldots, N\}$ be a uniform partition of $[a,b]$. By $\mathbb{P}_h u_h$ we represent the piecewise linear interpolator of a grid function u_h defined in $\mathbb{I}_h$. By $\mathbb{W}_h$ we represent the space of all grid function defined on $\mathbb{I}_h$ and by $\mathbb{W}_{h,0}$ its subspace of all grid function null on the boundary points. The space of piecewise linear functions induced by the partition $\mathbb{I}_h$ is denoted by $S_h = \{\mathbb{P}_h u_h, u_h \in \mathbb{W}_h\}$.

By $L^2(0,T;S_h)$ we denote the subspaces of $L^2(0,T;H^1(a,b))$ that is obtained replacing $H^1(a,b)$ by S_h. We introduce now the piecewise linear approximations for the pressure p and for the concentration u, respectively, $\mathbb{P}_h p_h \in L^2(0,T;S_h)$ and $\mathbb{P}_h u_h \in \{v \in L^2(0,T;S_h) : \dfrac{dv}{dt} \in L^2(0,T;S_h)\}$ such that

$$p_h(x_0,t) = p_a(t), p_h(x_N,t) = p_b(t), u_h(x_0,t) = u_a(t), u_h(x_N,t) = u_b(t), t \in (0,T], \tag{32}$$

$$p_h(x_i,0) = p_0(x_i), u_h(x_i,0) = u_0(x_i), i = 1, \ldots, N-1, \tag{33}$$

and

$$\Big(A(\mathbb{P}_h u_h(t)) \nabla(\mathbb{P}_h p_h)(t), \nabla(\mathbb{P}_h w_h)\Big) = (q_1(t), \mathbb{P}_h w_h),\ w_h \in \mathbb{W}_{h,0}, \tag{34}$$

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$$\begin{aligned}&\Big(\frac{\partial}{\partial t}(\mathbb{P}_h u_h)(t), \mathbb{P}_h w_h\Big) - \Big(B(\mathbb{P}_h u_h(t), \nabla(\mathbb{P}_h p_h)(t))\mathbb{P}_h u_h(t), \nabla(\mathbb{P}_h w_h)\Big)\\ &\quad + \Big(D_m(\mathbb{P}_h u_h(t), \nabla(\mathbb{P}_h p_h)(t))\nabla(\mathbb{P}_h u_h)(t), \nabla(\mathbb{P}_h w_h)\Big)\\ &= -\int_0^t K_{er}(t-s)\Big(D_d(\mathbb{P}_h u_h(s), \nabla(\mathbb{P}_h p_h)(s))\nabla(\mathbb{P}_h u_h)(s), \nabla(\mathbb{P}_h w_h)\Big)ds\\ &\quad + \Big(q_2(t), \mathbb{P}_h w_h\Big), w_h \in \mathbb{W}_{h,0}.\end{aligned} \tag{35}$$

In the space $\mathbb{W}_h$ we consider the norm $\|c_h\|_{1,h}^2 = \|c_h\|_h^2 + \|\nabla(\mathbb{P}_h c_h)\|^2$, where $\|.\|_h$ denotes the norm induced by the inner product

$$(w_h, v_h)_h = \sum_{i=1}^N \frac{h}{2}\Big(w_h(x_{i-1})v_h(x_{i-1}) + w_h(x_i)v_h(x_i)\Big),\ w_h, v_h \in \mathbb{W}_h.$$

Let $L^2(0,T;\mathbb{W}_h)$ be a discrete version of $L^2(0,T;H^1(a,b))$ which is the space of grid functions $v_h : [0,T] \to \mathbb{W}_h$ such that $\displaystyle\int_0^T \|v_h(t)\|_1^2\, dt < \infty$.

We introduce now the fully discrete approximations (in space) $p_h \in L^2(0,T;\mathbb{W}_h)$ and $u_h \in \mathcal{W}_h(0,T) = \{v_h \in L^2(0,T;\mathbb{W}_h) : \dfrac{dv_h}{dt} \in L^2(0,T;\mathbb{W}_h)\}$ as the grid functions that satisfy the conditions (32), (33) and the discrete variational equations

$$(A_h(t)\nabla(\mathbb{P}_h p_h)(t), \nabla(\mathbb{P}_h w_h)) = (q_{1,h}, w_h)_h,\ w_h \in \mathbb{W}_{h,0}, \tag{36}$$

$$\begin{aligned}&\Big(\frac{\partial u_h}{\partial t}(t), w_h\Big)_h - \Big(M_h(B_h(t)u_h(t)), D_{-x}w_h\Big)_{h,+} + \Big(D_{m,h}(t)\nabla(\mathbb{P}_h u_h(t)), \nabla(\mathbb{P}_h w_h)\Big)\\ &= -\int_0^t K_{er}(t-s)\Big(D_{d,h}(s)\nabla(\mathbb{P}_h u_h(s)), \nabla(\mathbb{P}_h w_h)\Big)ds + \Big(q_{2,h}(t), w_h\Big)_h, w_h \in \mathbb{W}_{h,0},\end{aligned} \tag{37}$$

where $M_h(v_h)(x_i) = \dfrac{1}{2}(v_{i-1} + v_i), i = 1, \ldots, N$.

In the previous equations the following notations were used: $D_{-x}w_h(x_i) = \dfrac{w_i - w_{i-1}}{h}$, $i = 1, \ldots, N$, $w_j = w_h(x_j)$, $(v_h, w_h)_{h,+} = \displaystyle\sum_{j=1}^N h v_j w_j$,

$$q_{\ell,h}(x_i, t) = \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} q_\ell(x,t)\, dx, i = 1, \ldots, N-1, \ell = 1, 2, \tag{38}$$

$A_h(x,t)$ and $D_{m,h}(x,t)$ are x piecewise constant functions defined by

$$A_h(x,t) = A(\frac{1}{2}(u_h(x_i,t) + u_h(x_{i+1},t))), \tag{39}$$

$$D_{m,h}(x,t) = D_m(\frac{1}{2}(u_h(x_i,t) + u_h(x_{i+1},t)), D_{-x}p_h(x_{i+1},t)), \tag{40}$$

for $x \in [x_i, x_{i+1})$, and the grid function $B_h(t)$ is given by

$$B_h(x_i,t) = \begin{cases} B(u_h(x_0,t), D_{-x}p_h(x_1,t)), i = 0,\\ B(u_h(x_i,t), D_c p_h(x_i,t)), i = 1, \ldots, N-1,\\ B(u_h(x_N,t), D_{-x}p_h(x_N,t)), i = N,\end{cases} \tag{41}$$

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and D_c will be defined bellow. The definition of the piecewise constant function $D_{d,h}$ is analogous to the definition of $D_{m,h}$.

In what follows we establish an ordinary differential system equivalent to the fully discrete (in space) variational problem (32), (33) (36), (37). In order to do that we introduce the following finite difference operators

$$D_c w_h(x_i) = \frac{w_{i+1} - w_{i-1}}{2h},\ D_x w_h(x_{i+1/2}) = \frac{w_{i+1} - w_i}{h}, D_x^{1/2} w_h(x_i) = \frac{w_{i+1/2} - w_{i-1/2}}{h},$$

where and $w_{j\pm 1/2}$ is used as far it makes sense.

It can be shown that the approximations $p_h(t)$ and $u_h(t)$ are solutions of the discrete problem

$$-D_x^{1/2}(A_h(t) D_x p_h(t)) = q_{1,h}(t) \text{ in } \mathbb{I}_h - \{a, b\}, \tag{42}$$

$$\begin{aligned} &\frac{du_h}{dt}(t) + D_c(B_h(t) u_h(t)) - D_x^{1/2}(D_{m,h}(t) D_x p_h(t)) \\ &\quad = \int_0^t K_{er}(t-s) D_x^{1/2}(D_{d,h}(s) D_x p_h(s)) ds + q_{2,h}(t) \text{ in } \mathbb{I}_h - \{a, b\} \end{aligned} \tag{43}$$

with the conditions (32), (33).

4 Stability of concentration and pressure

We establish now the stability of the coupled variational problem (36), (37) or equivalently the stability of the coupled finite difference problems (42), (43) under Dirichlet boundary conditions, that is $p_a(t) = p_b(t) = u_a(t) = u_b(t) = 0$. Let $C^1([0,T]; W_{h,0})$ be the space of grid functions $u_h : [0,T] \to W_{h,0}$ such that $\frac{du_h}{dt} : [0,T] \to W_{h,0}$ is continuous when in $W_{h,0}$ we consider the norm $\|.\|_h$.

Theorem 2 *If $u_h \in C^1([0,T]; W_{h,0})$ then, under the conditions of Theorem 1,*

$$\|p_h(t)\|_1 \leq \frac{b-a}{A_0} \|q_{1,h}(t)\|_h,\ t \in [0,T]. \tag{44}$$

and

$$\|u_h(t)\|_h^2 + \int_0^t \|\nabla(\mathbb{P}_h u_h)(s)|^2\, ds \leq e^{\tilde{C}t}\Big(\|u_h(0)\|_h^2 + \int_0^t e^{-\tilde{C}s} g_h(s)\ ds\Big), t \in [0,T], \tag{45}$$

provided that

$$\int_0^t k_{er}(t-s)^2 \|q_{1,h}\|_h^2\, ds \leq C_{q_1},\ t \in [0,T]. \tag{46}$$

In (45) $g_h(s)$ is given by

$$g_h(s) = \frac{1}{\min\{1, 2(D_{m,0} - \sigma^2 - \eta^2)\}}\Big(\|u_h(0)\|_h^2 + \frac{1}{2\eta^2}\int_0^s \|q_{2,h}(s)\|_h^2\, ds\Big),$$

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$\eta \neq 0$ is an arbitrary constant, $\tilde{C}$ is defined by

$$\tilde{C} = \frac{\max\left\{C_{q_1} D_{d,1}^2 \frac{(b-a)^2}{A_0^2}\frac{1}{4\sigma^2}, \left(\eta^2 + \frac{C_p^2 C_B^2}{4\gamma^2}\max_{t\in[0,T]}\|q_{1,h}(t)\|_h^2\right\}}{\min\left\{1, 2(D_{m,0}-\sigma^2-\gamma^2)\right\}} \tag{47}$$

and $\sigma \neq 0, \gamma \neq 0$ are such that

$$D_{m,0} - \sigma^2 - \gamma^2 > 0. \tag{48}$$

Proof: As Friedrich's-Poincaré inequality $(b-a)^2\|\nabla(\mathbb{P}_h w_h)\|^2 \geq \|w_h\|_h^2$ holds, the proof of (44) follows the proof of the correspondent continuous inequality (19).
Taking in (37) w_h replaced by $u_h(t)$ we easily deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_h(t)\|_h^2 - (M_h(B_h(t)u_h(t)), D_{-x}u_h(t))_{h,+} + D_{m,0}\|\nabla(\mathbb{P}_h u_h)(t)\|^2 \\ \leq \int_0^t K_{er}(t-s)(D_{d,h}(s)\nabla(\mathbb{P}_h u_h)(s), \nabla(\mathbb{P}_h u_h)(t))ds + \frac{1}{4\eta^2}\|q_{2,h}(t)\|_h^2 + \eta^2\|u_h(t)\|_h^2,\end{aligned} \tag{49}$$

where $\eta \neq 0$ is an arbitrary constant.
As from (12) we have

$$|(M_h(B_h(t)u_h(t)), D_{-x}u_h(t))_{h,+}| \leq 2C_B\|\nabla(\mathbb{P}_h p_h)(t)\|\|u_h(t)\|_h\|\nabla(\mathbb{P}_h u_h)(t)\|,$$

considering (44) we obtain

$$|(M_h(B_h(t)u_h(t)), D_{-x}u_h(t))_{h,+}| \leq C_B^2\frac{(b-a)^2}{A_0^2}\frac{1}{\gamma^2}\|q_{1,h}(t)\|_h^2\|u_h(t)\|_h^2 + \gamma^2\|\nabla(\mathbb{P}_h u_h)(t)\|^2, \tag{50}$$

where $\gamma \neq 0$ is an arbitrary constant.
As in the proof of Theorem 1, it can be shown that

$$\begin{aligned}\int_0^t K_{er}(t-s)|(D_{d,h}(s)\nabla(\mathbb{P}_h u_h(s)), \nabla(\mathbb{P}_h u_h)(t))ds| \\ \leq D_{d,1}^2\frac{(b-a)^2}{A_0^2}C_{q_1}\frac{1}{4\sigma^2}\int_0^t \|\nabla(\mathbb{P}_h u_h)(t)\|^2\, ds + \sigma^2\|\nabla(\mathbb{P}_h u_h)(s)\|^2,\end{aligned} \tag{51}$$

where $\sigma \neq 0$ is an arbitrary constant and C_{q_1} is fixed by (46).
Finally, using (50) and (51) in (49) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_h(t)\|^2 + \left(D_{m,0} - \sigma^2 - \gamma^2\right)\|\nabla(\mathbb{P}_h u_h)(t)\|^2 \leq \frac{1}{4\eta^2}\|q_{2,h}(t)\|_h^2 \\ + C_{q_1}D_{d,1}^2\frac{(b-a)^2}{A_0^2}\frac{1}{4\sigma^2}\int_0^t \|\nabla(\mathbb{P}_h u_h)(s)\|^2 ds + \left(\eta^2 + \frac{C_p^2 C_B^2}{4\gamma^2}\|q_{1,h}(t)\|_h^2\right)\|u_h(t)\|_h^2.\end{aligned} \tag{52}$$

Inequality (52) implies that

$$\begin{aligned}\|u_h(t)\|^2 + \int_0^t \|\nabla(\mathbb{P}_h u_h)(t)\|^2 \leq \tilde{C}\int_0^t \Big(\int_0^s \|\nabla(\mathbb{P}_h u_h)(\mu)\|^2 d\mu + \|u_h(s)\|_h^2\Big)\, ds \\ + \frac{1}{\min\{1, 2(D_{m,0}-\sigma^2-\gamma^2)\}}\Big(\|u_h(0)\|_h^2 + \frac{1}{2\eta^2}\|q_{2,h}(t)\|_h^2\Big),\end{aligned} \tag{53}$$

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where σ and γ are fixed by (48) and $\tilde{C}$ is given by (47). Finally, the inequality (53) leads to (45).

■

Stability results similar to Theorems 1 and 2 hold when q_2 depends on u. In fact we need only to assume that $|q_2(y)| \leq C_{q_2}|y|$.

5 Numerical results

In this section, as an example, we apply the proposed non Fickian model to a single phase incompressible flow problem in a porous media with a point source and sink term. This fluid flow process involves fully miscible displacement of one incompressible fluid by another. In this case p is the pressure of the fluid mixture, u the volumetric concentration of the injected fluid, ϕ the porosity of the medium, K the permeability of the medium, D_m the molecular diffusion coefficient and $D_d(u, \nabla p)$ the mechanical dispersion $D_d(u, \nabla p) = D_d|v|$, where D_d denotes the dispersion coefficient and v represents Darcy's velocity of the fluid mixture which is given by (31). In (31) the viscosity of the mixture $\mu(u)$ is determined by the commonly used rule $\mu(u) = \mu_0((1-u) + M^{\frac{1}{4}}u)^{-4}$, where M is the mobility ratio and μ_0 the viscosity of resident fluid. In (5) the function q_2 is given by $q_2 = u^* q_1$, where u^* is the injected concentration at sources or the concentration u at sinks. The function q_1 is the source and sink terms.

To closed the system (8), (30), (31) we assume natural boundary conditions $v = 0$ on $\{a, b\} \times (0, T]$, $D_m \nabla u + \frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}(D_d \nabla u)\, ds = 0$ on $\{a, b\} \times (0, T]$. In order to compare the Fickian and non Fickian models, we integrate in time the ordinary differential system (36), (37) using the implicit Euler method with a very small step size and discretizing the integral term using the right rectangular rule.

Let $0 = t^0 < t^1 < \ldots < t^N = T$ be a partition of the time interval $[0, T]$ with $\Delta t = t^{n+1} - t^n$ and N the number of time steps. Denote by p_h^n, v_h^n and u_h^n the approximations of p, v and u, respectively, at time level t^n. To compute the numerical approximations at time level t^{n+1} we use the following algorithm:

Step 1: Given u_h^n, solve the finite difference equation

$$-D_x^{1/2}\Big(\frac{K}{\mu(u_h^n)} D_x p_h^{n+1}\Big) = q_1^{n+1}$$

to compute p_h^{n+1};

Step 2: With p_h^{n+1}, compute the velocity of the convective term using the discretization in (41) and v_h^{n+1} in $D_d(v_h^{n+1})$ by

$$v_h^{n+1} = -\tfrac{K}{\mu(u_h^n)} D_{-x} p_h^{n+1};$$

Step 3: Compute u_h^{n+1} using

$$\phi\frac{u_h^{n+1} - u_h^n}{\Delta t} + D_c(u_h^n v_h^{n+1}) - D_x^{1/2}(D_m D_x u_h^n) = \frac{\Delta t}{\tau}\sum_{j=1}^{t_{n+1}} e^{-\frac{t_{n+1}-t_j}{\tau}} D_x^{1/2}(D_d(v_h^j) D_x u_h^j) + q_2^n.$$

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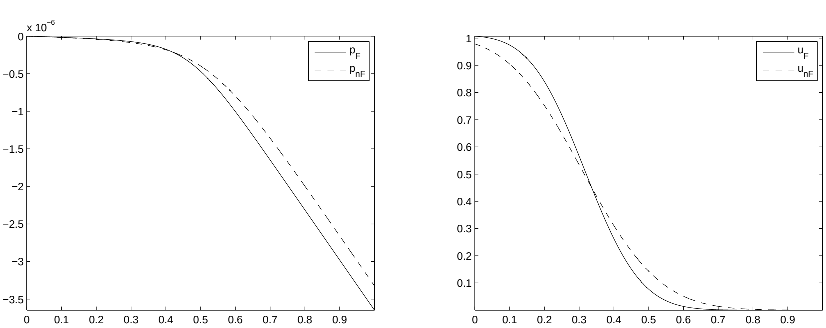
The simulation was performed considering $[a,b]=[0,1]$ and the following parameters: $T=800$, $\phi=1$, $K=60$, $M=41$, $\mu_0=1$, $u^*=5$, $D_m=\phi 10^{-5}$, $D_d=\phi 10^{-3}$, $\tau=0.001$, $h=10^{-4}$, $\Delta t=0.025$ The injection well cover one cell at the left extreme of the interval $[a,b]$ and has a constant injection rate equal to 5. The production well also cover one cell which is located at the right extreme with the production rate equal to -5.

In Figure 1 we plot the numerical approximation for Fickian and non Fickian pressures and concentrations at $t=800$. From the numerical experiments we observe for the Fickian and non Fickian pressure, p_F and p_{nF} respectively, a similar behavior. However for Fickian and non Fickian concentrations u_f, u_{nf}, respectively we observe that $u_F > u_{nF}$ near the injection point and $u_F < u_{nF}$ near the sink point.



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