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An unexpected convergence behaviour in diffusion phenomena in porous media

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Abstract

The aim of this paper is to present a finite difference method on nonuniform grid for quasilinear coupled partial differential equations that can be used to simulate the pressure and the concentration of single phase flows in porous media. The qualitative behaviour of the method is studied and its convergence behaviour is numerically analyzed. An unexpected second convergence order for the pressure and concentration are observed which means that the finite difference method is supraconvergent. As the method introduced can be seen as a fully discrete piecewise linear finite element method, we conclude that such piecewise linear finite element approximations for the pressure and concentration are also second order convergent and consequently such finite element method is superconvergent.

Key words: Pressure, concentration, porous media, finite difference methods, piecewise linear finite element method, supraconvergence, superconvergence.
MSC 2000: AMS 65M06, 65M20, 65M15

1 Introduction

In this paper we study a fully discrete method for the coupled system

$$-(a(c)p_x)_x = q_1 \ \text{ in } (0,1)\times(0,T], \tag{1}$$

$$c_t + (b(c,p_x)c)_x - (d(c,p_x)c_x)_x = q_2 \ \text{ in } (0,1)\times(0,T], \tag{2}$$

with the following boundary conditions

$$p(0,t) = p_\ell(t),\ p(1,t) = p_r(t), t \in (0,T], \tag{3}$$

$$c(0,t) = c_\ell(t),\, c(1,t) = c_r(t), t \in (0,T], \tag{4}$$

and initial conditions

$$c(x,0) = c_0(x), x \in (0,1), p(x,0) = p_0(x), x \in (0,1). \tag{5}$$

The initial boundary value problem (IBVP) (1)-(5) can be used to describe miscible displacement of one incompressible fluid (resident fluid) by another (injected fluid) in one dimensional porous media. In this case, $a(c) = K\mu(c)^{-1}$, $b(c,p_x) = \frac{1}{\phi}v$, $d(c,p_x) = D_m + D_d\frac{1}{\phi}|v|$, $v = -K\mu(c)^{-1}p_x$ denotes the velocity of the fluid mixture, p the pressure of the fluid mixture, c the concentration of the injected fluid, K the permeability of the medium, D_m the molecular diffusion coefficient, D_d the dispersion coefficient and ϕ represents the porosity. The viscosity of the mixture $\mu(c)$ is determined by the commonly used rule $\mu(c) = \mu_0((1-c)+M^{\frac{1}{4}}c)^{-4}$, where M denotes the mobility ratio and μ_0 represents the viscosity of the resident fluid. The two-dimensional or three dimensional versions of this problem with Dirichlet boundary conditions or with Neumann or Robin boundary conditions were largely considered in the literature to study the miscible displacement of one incompressible fluid by another in a porous medium (see for instance [6], [10], [11], [13]).

Piecewise linear finite element method for (1) leads to a first order approximation for the space derivative of p in the L^2-norm. This accuracy deteriorates the numerical approximation for c obtained from (2) if the same method is considered. Several approaches have been considered in the literature to increase the convergence order of the numerical approximation for the velocity. Without be exhaustive we mention the use of cell centered schemes ([14]), mixed finite element methods ([1], [3], [8], [12]), gradient recovery technique ([5] and [9]) and mimetic finite difference approximations which can be seen as a mixed finite element methods with convenient quadrature rules ([2]).

In the present paper we introduce a finite difference method, for the IBVP (1)-(5), defined on a non uniform grid. The qualitative behaviour of the numerical approximations for the pressure and concentration is studied using energy method. Numerically we observe an unexpected second convergence order for the pressure (with respect to a discrete H^1-norm) and concentration (with respect to a discrete L^2-norm). These facts allow us to postulate that we are in presence of a supraconvergent method: it enable us to compute second order convergent approximations for the pressure, for its gradient and for the concentration while its truncation error is only of first order. As such finite difference scheme can be seen as a fully discrete Galerkin method based on piecewise linear approximation and convenient quadrature rules, our numerical results can be also seen as superconveconvergent results.

The paper is organized as follows. In Section 2 we introduce the semi-discretization of problem (1)-(5). The qualitative behaviour of the pressure and concentration are studied in Section 3. An implicit-explicit discrete scheme is studied in Section 4 and fully discrete version of the results proved for the semi-discrete approximations are also established. In

Section 5 we present numerical simulations that allow us to postulate that our method enable us to obtain second order convergent approximations for the pressure and concentration.

2 The semi-discrete approximation

In what follows we introduce the variational formulation of the IBVP (1)-(5). To simplify we assume homogeneous boundary conditions. By $L^2(0,1)$, $H^1(0,1)$ and $H_0^1(0,1)$ we denote the usual Sobolev spaces where we consider the usual inner products $(.,.)_0$ and $(.,.)_1$.

Let V be a Banach space. By $L^2(0,T;V)$ we denote the space of functions $v:(0,T)\to V$ such that

$$\|v\|_{L^2(0,T;V)} = \Big(\int_0^T \|v(t)\|_V^2\,dt\Big)^{1/2}$$

is finite. By $L^\infty(0,T;V)$ we represent the space of functions $v:(0,T)\to V$ such that

$$\|v\|_{L^\infty(0,T;V)} = \operatorname*{ess\,sup}_{[0,T]} \|v(t)\|_V < \infty.$$

We replace the IBVP (1)-(5) by the following variational problem: find $p\in L^\infty(0,T;H^1(0,1))$, $c\in L^2(0,T;H^1(0,1))$ such that $c'\in L^2(0,T;L^2(0,1))$, conditions (3), (4) hold and

$$(a(c(t))p_x(t),w')_0 = (q_1(t),w)_0 \text{ a.e. in } (0,T), \forall w\in H_0^1(0,1), \tag{6}$$

$$\begin{aligned}&(c'(t),w)_0 \quad (d(c(t),p_x(t))c_x(t),w')_0 - (b(c(t),p_x(t))c(t),w')_0\\ &\qquad = (q_2(t),w)_0 \text{ a.e. in } (0,T), \forall w\in H_0^1(0,1).\end{aligned} \tag{7}$$

Let H be a sequence of vectors $h=(h_1,\ldots,h_N)$ such that $\sum_{i=1}^N h_i = 1$ and $h_{max} = \max_i h_i \to 0$. Let $\mathbb{I}_h = \{x_i, i=0,\ldots,N, x_0=0, x_N=1, x_i - x_{i-1} = h_i, i=1,\ldots,N\}$ be a nonuniform partition of $[0,1]$. By $\mathbb{W}_h$ we represent the space of grid functions defined on $\mathbb{I}_h$ and by $\mathbb{W}_{h,0}$ we represent the subspace of $\mathbb{W}_h$ of functions null on the boundary points. Let $\mathbb{P}_h u_h$ be the piecewise linear interpolator of a grid function $u_h\in\mathbb{W}_h$. The space of piecewise linear functions induced by the partition $\mathbb{I}_h$ is denoted by S_h.

The piecewise linear approximations for the pressure and for the concentration are solutions of the finite dimensional coupled variational problem: find $\mathbb{P}_h p_h \in L^\infty(0,T;S_h)$ and $\mathbb{P}_h c_h \in L^2(0,T;S_h)$ such that $\mathbb{P}_h c_h' \in L^2(0,T;S_h)$, boundary conditions (3), (4) hold and

$$(a(\mathbb{P}_h c_h(t))(\mathbb{P}_h p_h)_x(t), \mathbb{P}_h w_h')_0 = (q_1(t),\mathbb{P}_h w_h)_0 \text{ a.e. in } (0,T), \forall w_h\in\mathbb{W}_{h,0}, \tag{8}$$

$$\begin{aligned}&((\mathbb{P}_h c_h)_t(t),\mathbb{P}_h w_h)_0 + (d(\mathbb{P}_h c_h(t),(\mathbb{P}_h p_h)_x(t))(\mathbb{P}_h c_h)_x(t),\mathbb{P}_h w_h')_0\\ &\quad -(b(\mathbb{P}_h c_h(t),(\mathbb{P}_h p_h)_x(t))\mathbb{P}_h c_h(t),\mathbb{P}_h w_h')_0\\ &\quad = (q_2(t),\mathbb{P}_h w_h)_0 \text{ a.e. in } (0,T), \forall w_h\in\mathbb{W}_{h,0}.\end{aligned} \tag{9}$$

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In the space $\mathbb{W}_h$ we consider the norm $\|u_h\|_{1,h}^2 = \|u_h\|_h^2 + \|D_{-x}u_h\|_{h,+}^2$, where D_{-x} denotes the backward finite difference operator with respect to the space variable, $\|.\|_h$ is the norm induced by the inner product

$$(w_h, v_h)_h = \sum_{i=1}^{N} \frac{h_i}{2}\Big(w_h(x_{i-1})v_h(x_{i-1}) + w_h(x_i)v_h(x_i)\Big),\ w_h, v_h \in \mathbb{W}_h,$$

and $\|w_h\|_{h,+} = \Big(\sum_{i=1}^{N} h_i w_h(x_i)^2\Big)^{1/2}$. In what follows we use the notation

$$(w_h, v_h)_{h,+} = \sum_{i=1}^{N} h_i w_h(x_i)v_h(x_i),\ w_h, v_h \in \mathbb{W}_h.$$

The fully discrete (in space) approximations for the pressure and for the concentration are solutions of the following coupled variational problem: find $p_h \in L^\infty(0,T;\mathbb{W}_h)$, $c_h \in L^2(0,T;\mathbb{W}_h)$ such that $c_h' \in L^2(0,T;\mathbb{W}_h)$, and

$$(a_h(t)D_{-x}p_h(t), D_{-x}w_h)_{h,+} = (q_{1,h}(t), w_h)_h \text{ a.e. in } (0,T), \forall w_h \in \mathbb{W}_{h,0}, \tag{10}$$

$$\begin{aligned}(c_h'(t), w_h)_h + (d_h(t)D_{-x}c_h(t), D_{-x}w_h)_{h,+} - (M_h(b_h(t)c_h(t)), D_{-x}w_h)_{h,+} \\ = (q_{2,h}(t), w_h)_h \text{ a.e. in } (0,T), \forall w_h \in \mathbb{W}_{h,0},\end{aligned} \tag{11}$$

$$p_h(x_0,t) = p_\ell(t), p_h(x_N,t) = p_r(t) \text{ a.e. in } (0,T), \tag{12}$$

$$c_h(x_0,t) = c_\ell(t), c_h(x_N,t) = c_r(t) \text{ a.e. in } (0,T), \tag{13}$$

$$c_h(x_i,0) = c_{0,h}(x_i), p_h(x_i,0) = p_{0,h}(x_i),\ i = 1,\dots,N-1. \tag{14}$$

In (10), (11) the following notations were used

$$q_{\ell,h}(x_i,t) = \frac{1}{h_{i+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}} q_\ell(x,t)\,dx, i = 1,\dots,N-1, \ell = 1,2, \tag{15}$$

$h_{i+1/2} = \frac{1}{2}(h_i + h_{i+1})$, $M_h(w_h)(x_i) = \dfrac{1}{2}(w_h(x_{i-1}) + w_h(x_i)), i = 1,\dots,N$. The coefficient functions $a_h(t)$ and $d_h(t)$ are defined by $a_h(x_i,t) = a(M_h(c_h(t))(x_i))$, $d_h(x_i,t) = d(M_h(c_h(t))(x_i), D_{-x}p_h(x_i,t))$ and the grid function $b_h(t)$ is given by

$$b_h(x_i,t) = \begin{cases} b(c_h(x_0,t), D_x p_h(x_0,t)), i = 0, \\ b(c_h(x_i,t), D_h p_h(x_i,t)), i = 1,\dots,N-1, \\ b(c_h(x_N,t), D_{-x}p_h(x_N,t))), i = N, \end{cases}$$

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with

$$D_h p_h(x_i,t) = \frac{1}{h_i + h_{i+1}} (h_i D_{-x} p_h(x_{i+1},t) + h_{i+1} D_{-x} p_h(x_i,t)). \quad (16)$$

In what follows we establish an ordinary differential algebraic coupled system equivalent to the variational problem (10)-(14). In order to do that we introduce the following finite difference operators $(D_c w_h)_i = \frac{w_{i+1} - w_{i-1}}{h_i + h_{i+1}}$,$(D_x w_h)_{i+1/2} = \frac{w_{i+1} - w_i}{h_{i+1}}$, and $(D_x^{1/2} w_h)_i = \frac{w_{i+1/2} - w_{i-1/2}}{h_{i+1/2}}$, where $w_j := w_h(x_j)$ and $w_{j\pm 1/2}$ are used as far as it makes sense. In order to simplify the presentation we also consider that $a_h(x_{i\pm 1/2}, t) = a_h(x_{i\pm 1}, t)$, $d_h(x_{i\pm 1/2}, t) = d_h(x_{i\pm 1}, t)$.

It can be shown that the approximations $p_h(t)$ and $c_h(t)$ are solutions of the following discrete problem:

$$-D_x^{1/2}(a_h(t) D_x p_h(t)) = q_{1,h}(t) \text{ in } \mathbb{I}_h - \{0,1\}, \quad (17)$$

$$c_h'(t) - D_x^{1/2}(d_h(t)\, D_x p_h(t)) + D_c(b_h(t) c_h(t)) = q_{2,h}(t) \text{ in } \mathbb{I}_h - \{0,1\}, \quad (18)$$

with the conditions (12), (13) and (14).

3 Energy estimates for pressure and concentration

We establish now energy estimates for the solution of coupled variational problem (10), (11), or equivalently for coupled finite difference problem (17), (18), under homogeneous Dirichlet boundary conditions, that is, $p_\ell(t) = p_r(t) = c_\ell(t) = c_r(t) = 0$.

We require some smoothness on the solution of the variational problem (10), (11), namely, we assume that $c_h \in C^1(0,T; W_{H,0})$, that is, $c_h, c_h' : [0,T] \to \mathbb{W}_{h,0}$ are continuous when we consider the norm $\|.\|_h$ in $\mathbb{W}_{h,0}$. In what follows we establish energy estimates for the pressure and for the concentration.

Proposition 1 *If $0 < a_0 \le a$ then there exists a positive constant C_p, h independent, such that*

$$\|p_h(t)\|_{1,h} \le C_p \|q_{1,h}(t)\|_h, t \in [0,T]. \quad (19)$$

Proof: Taking in (10) $w_h = p_h(t)$ and considering the Poincaré-Friedrich's inequality $\|w_h\|_h \le \|D_{-x} w_h\|_{h,+}^2$ for $w_h \in \mathbb{W}_{h,0}$, we conclude (19).

■

If

$$\|q_1(t)\|_0 \le C_{q_1}, t \in [0,T], \quad (20)$$

then the sequence $\|p_h(t)\|_{1,h}, h \in H$, satisfies

$$\|p_h(t)\|_{1,h} \le C_p,\ t \in [0,T],\ h \in H, \quad (21)$$

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for some positive constant C_p.
As $|p_h(x_i)| \leq \|p_h(t)\|_{1,h}$, we get $\max_{i=1\dots,N-1} |p_h(x_i,t)| \leq C_q$, that is $\|p_h(t)\|_\infty \leq C_p$. Moreover, as holds the following

$$
\begin{aligned}
a(M_h(c_h(t))(x_{i+1}))D_{-x}p_h(x_{i+1},t) &= \sum_{j=1}^{i} h_{j+1/2} D_x^{(1/2)}(a_h(t)D_{-x}p_h(t))(x_j) \\
&+a(M_h(c_h(t))(x_1))D_{-x}p_h(x_1,t), \\
&= -\sum_{j=1}^{i} h_{j+1/2} q_{1,h}(x_j,t) \\
&+a(M_h(c_h(t))(x_1))D_{-x}p_h(x_1,t),
\end{aligned}
$$

for $i = 1,\dots,N-1$, we conclude

$$
\max_{i=2,\dots,N} |a(M_h(c_h(t))(x_i))D_{-x}p_h(x_i,t)| \leq C_p + |a(M_h(c_h(t))(x_1))| D_{-x}p_h(x_1,t)|,
$$

provided that $q_1 \in L^\infty(0,T;L^2(0,1))$. It is then plausible to admit that, for $0 < a_0 \leq a$ and $h \in H$ with h_{max} small enough, we have

$$
\max_{i=1,\dots,N} |D_{-x}p_h(x_i,t)| \leq C_p, \tag{22}
$$

for some positive constant C_p.

We remark that if we replace the Dirichlet boundary conditions for the pressure by Neumann boundary conditions $p_x(0,t) = p_x(1,t) = 0$ that are discretized by $D_{-x}p_h(x_1,t) = D_{-x}p_h(x_N,t) = 0$, then condition (22) holds.

Proposition 2 *If* $0 < a_0 \leq a$, $0 < d_0 \leq d$, *(22) holds and*

$$
|b(x,y)| \leq C_b|y|, (x,y) \in \mathbb{R}^2, \tag{23}
$$

then

$$
\begin{aligned}
\|c_h(t)\|_h^2 + \int_0^t \|D_{-x}c_h(s)\|_{h,+}^2\, ds &\leq \frac{1}{\min\{1, 2(d_0-\epsilon^2)\}} e^{\left(\frac{1}{2\epsilon^2}C_b^2C_p^2+2\eta^2\right)t} \\
&\Big(\|c_h(0)\|_h^2 + \frac{1}{2\eta^2}\int_0^t \|q_{2,h}(s)\|_h^2\, ds\Big),\ t \in [0,T],
\end{aligned} \tag{24}
$$

$\eta \neq 0$ *is an arbitrary constant and* $\epsilon \neq 0$ *is such that*

$$
d_0 - \epsilon^2 > 0. \tag{25}
$$

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Proof: Taking in (11) w_h replaced by $c_h(t)$, we easily deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|c_h(t)\|_h^2 + d_0\|D_{-x}c_h(t)\|_{h,+}^2 - (M_h(b_h(t)c_h(t)), D_{-x}c_h(t))_{h,+}\\&\leq \frac{1}{4\eta^2}\|q_{2,h}(t)\|_h^2 + \eta^2\|c_h(t)\|_h^2,\end{aligned} \tag{26}$$

for an arbitrary $\eta \neq 0$.
As under the assumptions (22) and (23), we have

$$|(M_h(b_h(t)c_h(t)), D_{-x}c_h(t))_{h,+}| \leq C_bC_p\|c_h(t)\|_h\|D_{-x}c_h(t)\|_{h,+}, \tag{27}$$

from (26) we obtain

$$\frac{d}{dt}\|c_h(t)\|_h^2 + 2(d_0-\epsilon^2)\|D_{-x}c_h(t)\|_{h,+}^2 \leq \Big(\frac{1}{2\epsilon^2}C_b^2C_p^2 + 2\eta^2\Big)\|c_h(t)\|_h^2 + \frac{1}{2\eta^2}\|q_{2,h}(t)\|_h^2,$$

where ϵ, η are nonzero constants.This inequality leads to

$$\begin{aligned}&\|c_h(t)\|_h^2 + 2(d_0-\epsilon^2)\int_0^t \|D_{-x}c_h(s)\|_{h,+}^2\,ds \leq \|c_h(0)\|_h^2\\&+\Big(\frac{1}{2\epsilon^2}C_b^2C_p^2 + 2\eta^2\Big)\int_0^t \|c_h(s)\|_h^2\,ds + \frac{1}{2\eta^2}\int_0^t \|q_{2,h}(s)\|_h^2\,ds.\end{aligned} \tag{28}$$

Finally inequality (24) easily follows from inequality (28).

■

4 An IMEX method

In $[0,T]$ we introduce a uniform grid $\{t_n\}$ with $t_0 = 0, t_M = T$ and $t_j - t_{j-1} = \Delta t$. By D_{-t} we denote the backward finite difference operator with respect to t. Let us suppose that the numerical approximations $p_h^n(x_i)$ and $c_h^n(x_i)$ for $p(x_i,t_n)$ and $c(x_i,t_n)$, respectively, are known. Let $p_h^{n+1}(x_i)$ and $c_h^{n+1}(x_i)$ be the numerical approximations for $p(x_i,t_{n+1})$ and $c(x_i,t_{n+1})$, respectively, defined by the following system

$$(a_h^n D_{-x}p_h^{n+1}, D_{-x}w_h)_{h,+} = (q_{1,h}^{n+1}, w_h)_h,\ w_h \in \mathbb{W}_{h,0}, \tag{29}$$

$$\begin{aligned}&(D_{-t}c_h^{n+1}, w_h)_h + (d_h^{n,n+1}D_{-x}c_h^{n+1}, D_{-x}w_h)_{h,+} - (M_h(b_h^{n,n+1}c_h^{n+1}), D_{-x}w_h)_{h,+}\\&= (q_{2,h}^{n+1}, w_h)_h,\ w_h \in \mathbb{W}_{h,0},\end{aligned} \tag{30}$$

with the boundary conditions

$$p_h^{n+1}(x_0) = p_\ell(t_{n+1}),\ p_h^{n+1}(x_N) = p_r(t_{n+1}), \tag{31}$$

$$c_h^{n+1}(x_0) = c_\ell(t_{n+1}),\, c_h^{n+1}(x_N) = c_r(t_{n+1}), \tag{32}$$

and with the initial conditions

$$c_h^0(x_i) = c_{0,h}(x_i), p_h^0(x_i) = p_{0,h}(x_i),\, i = 1, \dots, N-1. \tag{33}$$

In (29) and (30), $q_{\ell,h}^{n+1}$ is obtained from $q_{\ell,h}(t)$ taking $t = t_{n+1}$, $(\ell = 1, 2)$, the coefficient a_h^n is obtained from $a_h(t)$ replacing $c_h(t)$ by c_h^n, $d_h^{n,n+1}$ and $b_h^{n,n+1}$ are obtained from $d_h(t)$ and $b_h(t)$, respectively, replacing $c_h(t)$ and $p_h(t)$ by c_h^n and p_h^{n+1}, respectively.

We establish in what follows energy estimates for the numerical approximations c_h^{n+1}, p_h^{n+1} defined by (29) and (30). In order to do that we assume that $p_\ell = p_r = c_\ell = c_r = 0$.

Proposition 3 *Under the assumption of Proposition 1, there exists a positive constant C_p which does not depend on h such that*

$$\|p_h^{n+1}\|_{1,h} \le C_p \|q_{1,h}^{n+1}\|_h, n = 0, \dots, M-1. \tag{34}$$

■

If q_1 satisfies (20) then the sequence $\|p_h^n\|_{1,h}, h \in H$, satisfies

$$\|p_h^n\|_{1,h} \le C_p,\, n = 1, \dots, M-1,\, h \in H, \tag{35}$$

for some positive constant C_p.

As in the semi-discrete case, from (35) we conclude $\|p_h^n\|_\infty \le C_p$, $n = 1, \dots, M$, and it is reasonable to assume

$$\max_{i=1,\dots,N} |D_{-x} p_h^n(x_i)| \le C_p,\, n = 1, \dots, M. \tag{36}$$

Proposition 4 *If $0 < a_0 \le a$, $0 < d_0 \le d$, (23) and (36) hold, then c_h^n defined by (29),(30) with homogeneous boundary conditions satisfies*

$$\|c_h^n\|_h^2 + \Delta t \sum_{j=0}^{n} \|D_{-x} c_h^j\|_{h,+}^2 \le \frac{1}{\min\{1-\theta\Delta t, 2(d_0 - \epsilon^2)\}} e^{\frac{\theta n \Delta t}{\min\{1-\theta\Delta t, 2(d_0-\epsilon^2)\}}}$$
$$\Big((1-\theta\Delta t)\|c_h(0)\|_h^2 + 2(d_0 - \epsilon^2)\|D_{-x} c_h^0\|_{h,+}^2 + \frac{1}{2\eta^2}\Delta t \sum_{m=1}^{n} \|q_{2,h}^m\|_h^2\, ds\Big), \tag{37}$$

where

$$\theta = \frac{1}{2\epsilon^2} C_b^2 C_p^2 + 2\eta^2, \tag{38}$$

$\eta \ne 0$ is an arbitrary constant, $\epsilon \ne 0$ is fixed by (25) and Δt satisfies

$$1 - \theta \Delta t > 0. \tag{39}$$

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Proof: Taking in (30) n and w_h replaced by m and c_h^{m+1}, respectively, and following the proof of Proposition 2, it can be shown that

$$\begin{aligned}\|c_h^{m+1}\|_h^2 + 2(d_0-\epsilon^2)\Delta t\|D_{-x}c_h^{m+1}\|_{h,+}^2 &\leq \|c_h^m\|_h^2 + \Big(\tfrac{1}{2\epsilon^2}C_b^2C_p^2 + 2\eta^2\Big)\Delta t\|c_h^{m+1}\|_h^2 \\ &+\tfrac{1}{2\eta^2}\Delta t\|q_{2,h}^{m+1}\|_h^2,\end{aligned} \tag{40}$$

where ϵ, η are nonzero constants. Summing (40) for $m = 0, \ldots, n-1$, we obtain

$$\begin{aligned}\|c_h^n\|_h^2 + 2(d_0-\epsilon^2)\Delta t\sum_{m=1}^{n}\|D_{-x}c_h^m\|_{h,+}^2 &\leq \|c_h^0\|_h^2 + \theta\Delta t\sum_{m=1}^{n}\|c_h^m\|_h^2 \\ &+\frac{1}{2\eta^2}\Delta t\sum_{m=1}^{n}\|q_{2,h}^m\|_h^2\,,\end{aligned} \tag{41}$$

with θ defined by (38).
Inequality (41) can be rewritten in the following equivalent form

$$\begin{aligned}&\|c_h^n\|_h^2 + \Delta t\sum_{m=0}^{n}\|D_{-x}c_h^m\|_{h,+}^2\,ds \leq \frac{\theta\Delta t}{\min\{2(d_0-\epsilon^2), 1-\theta\Delta t\}}\sum_{m=0}^{n-1}\|c_h^m\|_h^2 \\ &+\frac{1}{\min\{2(d_0-\epsilon^2), 1-\theta\Delta t\}}\Big((1-\theta\Delta t)\|c_h^0\|_h^2 + 2(d_0-\epsilon^2)\Delta t\|D_{-x}c_h^0\|_{h,+}^2 \\ &\qquad+\frac{1}{2\eta^2}\Delta t\sum_{m=1}^{n}\|q_{2,h}^m\|_h^2\Big),\end{aligned} \tag{42}$$

provided that ϵ and Δt satisfy (25) and (39), respectively. Using in (42) Gronwall Lemma (Lemma 4.3 of [4]) we deduce (37).

■

5 Convergence order for the pressure and concentration

We illustrate in what follows the behaviour of the fully discrete scheme (29), (30). Let $e_{c,h}^n$ and $e_{p,h}^n$ be the errors for the concentration and for the pressure defined by

$$e_{c,h}^n(x_i) = c_h^n(x_i) - R_h c(x_i, t_n),\ e_{p,h}^n(x_i) = p_h^n(x_i) - R_h p(x_i, t_n), i = 1, \ldots, N-1,$$

where R_h denotes the restriction operator. We would like to show numerically the following estimates

$$\|e_{c,h}^n\|_h^2 + \Delta t\sum_{j=0}^{n}\|D_{-x}e_{c,h}^j)\|_{h,+}^2 = O(h_{max}^4), \tag{43}$$

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and

$$\|e_{p,h}^n\|_{1,h}^2 = O(h_{max}^4), \tag{44}$$

for all n. To do that we consider $\Delta t < h_{max}^2$ in the numerical simulations that we present in what follows.

Example 1 *Let us consider (1)-(5) with* $a(c) = 1 + c(x,t)$, $b(c, p_x) = (c(x,t)p_x(x,t))^2$, $d(c, p_x) = c(x,t) + p_x(x,t) + 2$, q_1, q_2, *the initial and boundary conditions such that this IBVP has a known solution. The numerical approximations* c_h^n *and* p_h^n *were obtained with the IMEX method (29)-(33) with nonuniform grids in* $[0,1]$ *and with* $T = 0.1$ *and* $\Delta t = 10^{-6}$. *The spatial initial grid is arbitrary and the new grid is obtained introducing in* $[x_i, x_{i+1}]$ *the midpoint. In Table 1 we present the errors*

$$Error_c = \max_{n=1,\ldots,M} \left(\|e_{c,h}^n\|_h^2 + \Delta t \sum_{j=0}^{n} \|D_{-x} e_{c,h}^j)\|_{h,+}^2 \right)^{1/2}, \tag{45}$$

$$Error_p = \max_{n=1,\ldots,M} \|D_{-x} e_{p,h}^n\|_{h,+} \tag{46}$$

and the rates $Rate_c$, $Rate_p$ *that were computed by the formula*

$$Rate = \frac{\ln\left(\frac{Error_{h_{max,1}}}{Error_{h_{max,2}}}\right)}{\ln\left(\frac{h_{max,1}}{h_{max,2}}\right)}, \tag{47}$$

where $h_{max,1}$ *and* $h_{max,2}$ *are the maximum step sizes of two consecutive partitions.*

h_{max}	$Error_c$	$Error_p$	$Rate_c$	$Rate_p$
1.3174×10^{-1}	5.5435×10^{-2}	1.1099×10^{-2}	*1.9492*	*1.5048*
6.5869×10^{-2}	1.4355×10^{-2}	3.9113×10^{-3}	*2.0010*	*1.5808*
3.2934×10^{-2}	3.5863×10^{-3}	1.3075×10^{-3}	*2.0024*	*1.8337*
1.6467×10^{-2}	8.9511×10^{-4}	3.6682×10^{-4}	*2.0008*	*1.9296*
8.2336×10^{-3}	2.2366×10^{-4}	9.6288×10^{-5}	*2.0029*	*1.9671*
4.1168×10^{-3}	5.5804×10^{-5}	2.4628×10^{-5}	*2.0109*	*1.9866*
2.0584×10^{-3}	1.3846×10^{-5}	6.2144×10^{-6}	*2.0301*	*2.0015*
1.0292×10^{-3}	3.3899×10^{-6}	1.5520×10^{-6}	-	-

Table 1

The numerical results presented in Table 1 show that $Error_p = O(h_{max}^2)$ *and* $Error_c = O(h_{max}^2)$.

Example 2 *In this example we consider the IBVP (1)-(5) with* $a(c) = \dfrac{1}{1+c}$, $b(c, p_x) = \dfrac{p_x}{1+c}$, $d(c, p_x) = 1 + \dfrac{p_x}{1+c}$, q_1, q_2 *the boundary and initial conditions are such that this IBVP has*

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a known solution. The numerical approximations c_h^n *and* p_h^n *were obtained with the IMEX method (29)-(33) with nonuniform grids and with* $\Delta t = (h_{max}/100)^2$ *and* $T = 10000\Delta t$*. In Table 2 we present the errors defined by (45) and (46) and the rates defined by (47) for the pressure and for the concentration.*

h_{max}	$Error_c$	$Error_p$	$Rate_c$	$Rate_p$
9.4478×10^{-2}	2.7522×10^{-3}	1.0142×10^{-3}	*1.5183*	*1.8416*
4.7239×10^{-2}	9.6078×10^{-4}	2.8297×10^{-4}	*1.7922*	*1.9648*
2.3620×10^{-2}	2.7741×10^{-4}	7.2488×10^{-5}	*1.9739*	*1.9793*
1.1810×10^{-2}	7.0618×10^{-5}	1.8383×10^{-5}	*2.0221*	*1.9901*
5.9049×10^{-3}	1.7386×10^{-5}	4.6274×10^{-6}	*2.0177*	*1.9953*
2.9524×10^{-3}	4.2935×10^{-6}	1.1607×10^{-6}	*2.0089*	*1.9977*
1.4762×10^{-3}	1.0668×10^{-6}	2.9063×10^{-7}		

Table 2

The numerical results presented in Table 2 show that for this example we have $Error_p = O(h_{max}^2)$ *and* $Error_c = O(h_{max}^2)$*.*

In the previous two examples and in a huge number of numerical experiments performed and not included in this paper, we observe an unexpected convergence behaviour of the numerical approximations for the pressure and for the concentration. These facts allow us to postulate that the error estimates (43) and (44) hold. These estimates mean that the finite difference scheme used is supraconvergent or equivalently the correspondent finite element method is supraconvergent.

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