THE K SHORTEST PATH PROBLEM – A SURVEY

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Abstract

The K shortest paths problem – a survey

In this talk we survey the K shortest paths problem. Two classes of algorithms are revisited and special emphasis is given to *Deviation Algorithms*. It is also shown how *Deviation Algorithms* can be adapted to the ranking of elementary paths (paths with no repeated nodes) in contrast with *Labeling Algorithms*.

Comparative computational results are also presented.

Contents

Introduction

Given:

- Directed graph $(\mathcal{N}, \mathcal{A})$;
- Initial and terminal nodes $s, t \in \mathcal{N}$ $(s \neq t)$;
- Set of paths from x to $y \to \mathcal{P}_{xy}$;
- Cost of $(i, j) \in \mathcal{A} \to c_{ij} \in \mathbb{R}$;
- Cost of $p \in \bigcup\limits_{x,y \in \mathcal{N}} \mathcal{P}_{xy} \to c(p) = \Sigma_p \, c_{ij}$;

Notes:

- Path of $\mathcal{P}_{st} \to p = \langle s = v_1, v_2, \ldots, v_\ell = t \rangle$, where $v_1, \ldots, v_\ell \in \mathcal{N}$.
- Elementary path of $\mathcal{P}_{st}$ $\to$ path from s to t without repeated intermediate nodes.
- Cycle $\to$ elementary path where $s = t$.
- Given $p \in \mathcal{P}_{ux}$ and $q \in \mathcal{P}_{xv}$, $p \diamond q \in \mathcal{P}_{uv}$ is the $p - q$ concatenation.

Directed network $(\mathcal{N}, \mathcal{A}, \mathrm{c}, s, t)$
$\mathcal{P}^\star \subseteq \mathcal{P}_{st}$
Positive integer K

Problem – Unconstrained problem $(\mathcal{P}^\star = \mathcal{P}_{st})$

To determine $\mathcal{P}_K = \{p_1, \ldots, p_K\}$ $(\mathcal{P}_K \subseteq \mathcal{P}^\star)$ such that:

- $c(p_i) \geq c(p_{i-1}),\ \forall\ i \in \{2, \ldots, K\}$;
- $c(p) \geq c(p_K),\ \forall\ p \in \mathcal{P}^\star - \mathcal{P}_K$;
- p_i is determined after $p_{i-1},\ \forall\ i \in \{2, \ldots, K\}$.

Subproblems – Constrained problem $(\mathcal{P}^\star \subseteq \mathcal{P}_{st})$

Determining paths that satisfy additional constraints $(\mathcal{P}_K \subseteq \mathcal{P}^\star)$:

- The K shortest elementary paths problem;

. . ..

Algorithms for the K shortest paths problem

$$
\begin{array}{l}\text{Optimality Property}\\ \text{generalization}\end{array}
\begin{cases}
\text{Dynamic algorithms} & \begin{cases} \text{Dreyfus} \\ \text{Jiménez and Marzal} \\ \quad \vdots \end{cases} \\[2ex]
\text{Labeling algorithms} & \begin{cases} \text{Shier} \\ \quad \vdots \end{cases} \\[2ex]
\text{Removing algorithms} & \begin{cases} \text{Martins} \\ \text{Azevedo } \textit{et al.} \\ \text{MS} \\ \text{MS – IMP} \end{cases}
\end{cases}
$$

$$
K \text{ shortest paths tree}
\begin{cases}
\text{Every deviation} \quad (c_{ij}, \bar{c}_{ij}) & \begin{cases} \text{Hoffman and Pavley} \\ \text{Eppstein } (\bar{c}_{ij}) \end{cases} \\[2ex]
\text{Shortest deviation} \quad (c_{ij}, \bar{c}_{ij}) & \begin{cases} \text{Elementary} \\ \text{MPS } (\bar{c}_{ij}) \end{cases}
\end{cases}
$$

Algorithms based on the Optimality Property

Generalization for $K > 1$ of the Optimality Property used for $K = 1$:

p_k, is formed by $j^{\underline{\text{th}}}$ shortest sub-paths, with $1 \leq j \leq k$.

The K shortest paths problem verifies the Optimality Property as long as:
$(\mathcal{N}, \mathcal{A}, \mathrm{c}, s, t)$ does not contain negative cycles.

Dynamic algorithms

Algorithms based on the resolution of a functional equation

- Dreyfus (1969), Jiménez and Marzal (1999).

Labeling algorithms

Generalizations of labeling algorithm forms for the $K = 1$ shortest path problem

- Shier (1974), Martins *et al.* (1999).

Removing algorithms

Algorithms based on the sucessive removal of certain paths of $(\mathcal{N}, \mathcal{A}, \mathrm{c}, s, t)$

- Martins (1984), Azevedo *et al.* (1990), Martins and Santos — MS (1995), Martins *et al.* — MS improvement (1999).

The K shortest paths tree

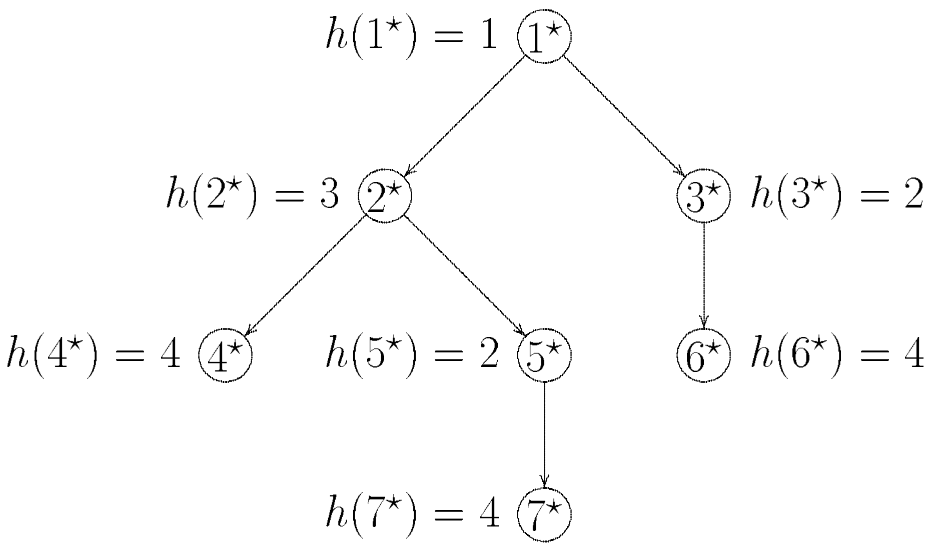
Tree of paths:

> *Pseudo*-tree containing paths between s and t, where each path "deviates" from the others in a node. (Deviation node)

In order to avoid repeated nodes consider:

$$h : \mathcal{N}^\star \longrightarrow \mathbb{N} : h(\mathcal{N}^\star) = h(\mathcal{N}) = \{1, \ldots, n\},$$

- $\mathcal{N}^\star$ is the set of nodes in the paths tree;
- Each node of $\mathcal{N}^\star$ is identified with one node of $\mathcal{N}$.



$$\mathcal{N}^\star = \{1^\star, \ldots, 7^\star\}$$

$$\mathcal{N} = \{1, \ldots, 4\}$$

Tree of the K shortest paths: Tree of paths, $\mathcal{T}_K$, containing only $p_1, \ldots, p_K$.

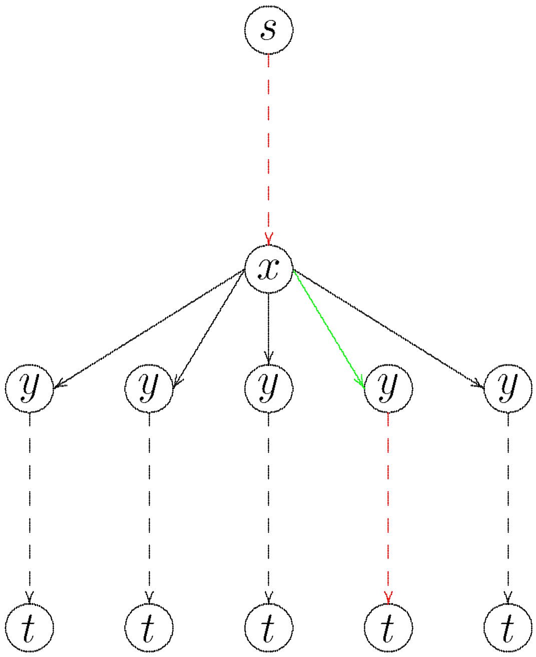
Tree of the K shortest elementary paths: Analogous but containing only the K shortest elementary paths.

For every node x of p_i:

- **Deviation arc** of p_i is $(x, y) \notin p_i$
- **Deviation path** of p_i is

$$sub_{p_i}(s, x) \diamond (x, y) \diamond p_{yt}$$

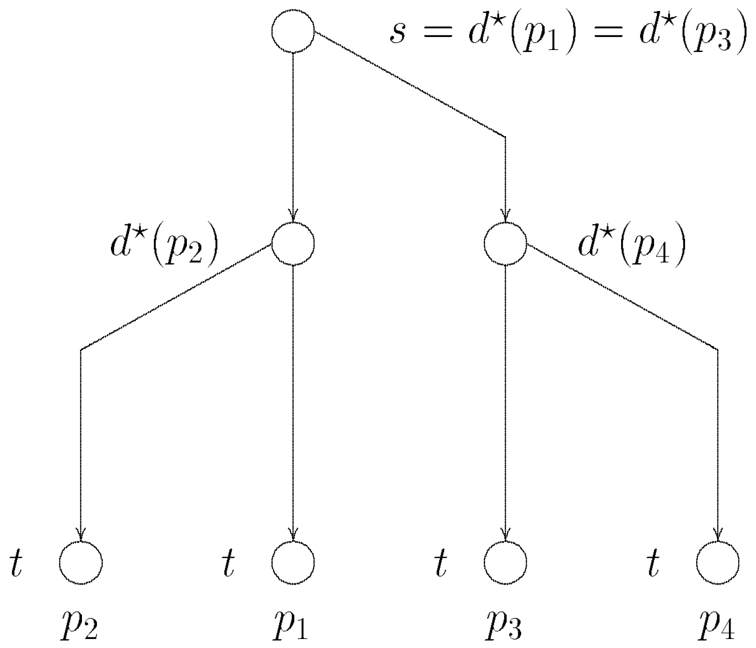
  where p_{yt} is a shortest path from y to t



p_i

Deviation node of p_i:

$$d^\star(p_i) = \begin{cases} s\,, & i = 1 \\ \begin{array}{l} x \in p_i \text{ s.t. } \mathrm{sub}_{p_i}(s,x) \text{ is sub-} \\ \text{-path of } p_1, \ldots, p_{i-1} \text{ with}\,, \\ \text{maximum number of nodes} \end{array} & i > 1 \end{cases}$$



It is intended to determine $\mathcal{P}_K$, that is:

$$\boxed{\text{To determine: } \left\{ \begin{array}{c} \text{Tree } \mathcal{T}_K \\ \text{or} \\ \text{Tree containing } \mathcal{T}_K \end{array} \right.}$$

Notation:

- $\mathcal{T}_t^\star$: Tree of the shortest paths from $i \in \mathcal{N}$ to t.
- $p_{xt}(\mathcal{T}_t^\star)$: Path in $\mathcal{T}_t^\star$ from x to t.

Theorem:
A i-th shortest path, p_i, $i \geq 1$, can be written as

$$p_i = \text{sub}_{p_j}(s, d^\star(p_i)) \diamond (d^\star(p_i), y) \diamond p_{yt}(\mathcal{T}_t^\star)\,,$$

where $y \in \mathcal{N}$ and $j < i$.

Deviation algorithms

$$X = \{\text{candidates to the next shortest path}\}$$

General form:

$p \longleftarrow$ shortest path (elementary path) in $\mathcal{P}$

$X \longleftarrow \{p\}$ /* Initialization */

`For` $i \in \{1, \ldots, K\}$ `Do`

$p_i \longleftarrow$ shortest element of X; $X \longleftarrow X - \{p_i\}$ /* Update X */

Generate new paths (elementary paths) of $\mathcal{P}$ and add them to X

Which new paths (elementary paths) should be generated from p_i?

Ranking paths

Algorithms which determine "every" deviation of p_i at some nodes

- Hoffman e Pavley (1957).

Algorithms that determine the shortest deviation of p_i at some nodes

- Martins *et al.* — Elementary (1997), MPS (1997).

Ranking elementary paths

Algorithms which change $(\mathcal{N}, \mathcal{A}, \mathrm{c}, s, t)$, in order to assure the non determination of non elementary paths.

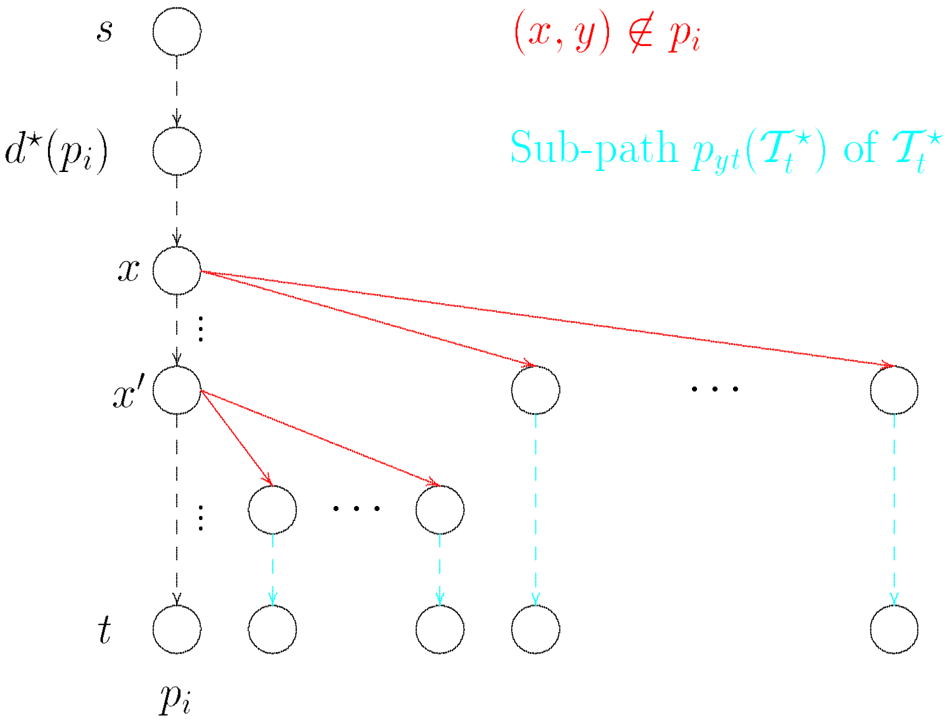
- Yen (1974).

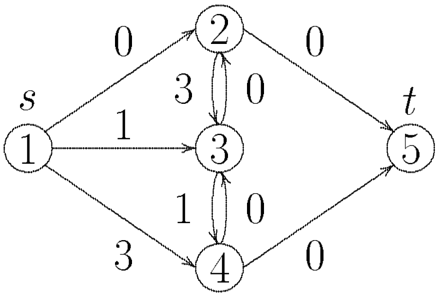
Deviation ranking paths algorithms adaptations

- Martins *et al.* — Elementary (1997), MPS (1997).

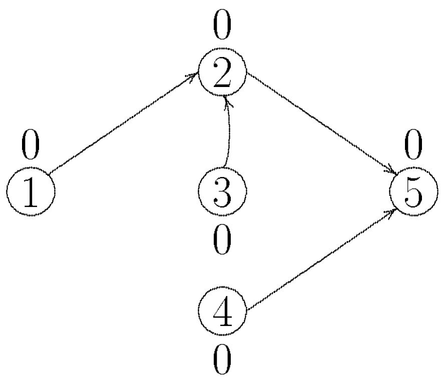
Hoffman and Pavley algorithm (1959)

For any node x of p_i after $d^\star(p_i)$ until t:
Generate **every deviation path** of p_i in x.



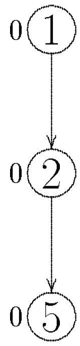


Network $(\mathcal{N}, \mathcal{A}, \mathrm{c}, s, t)$

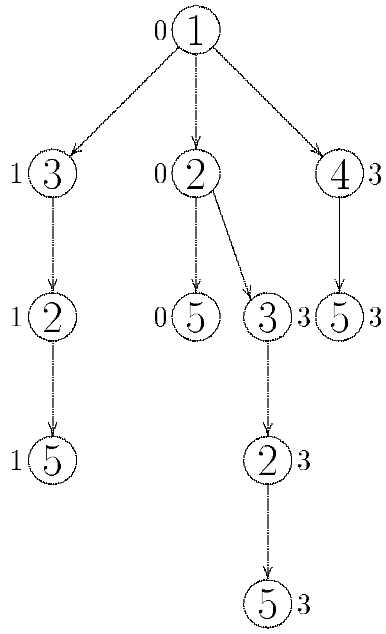


Tree $\mathcal{T}_t^\star$

Hoffman and Pavley



$X = \{\langle 1, 2, 5\rangle\}$



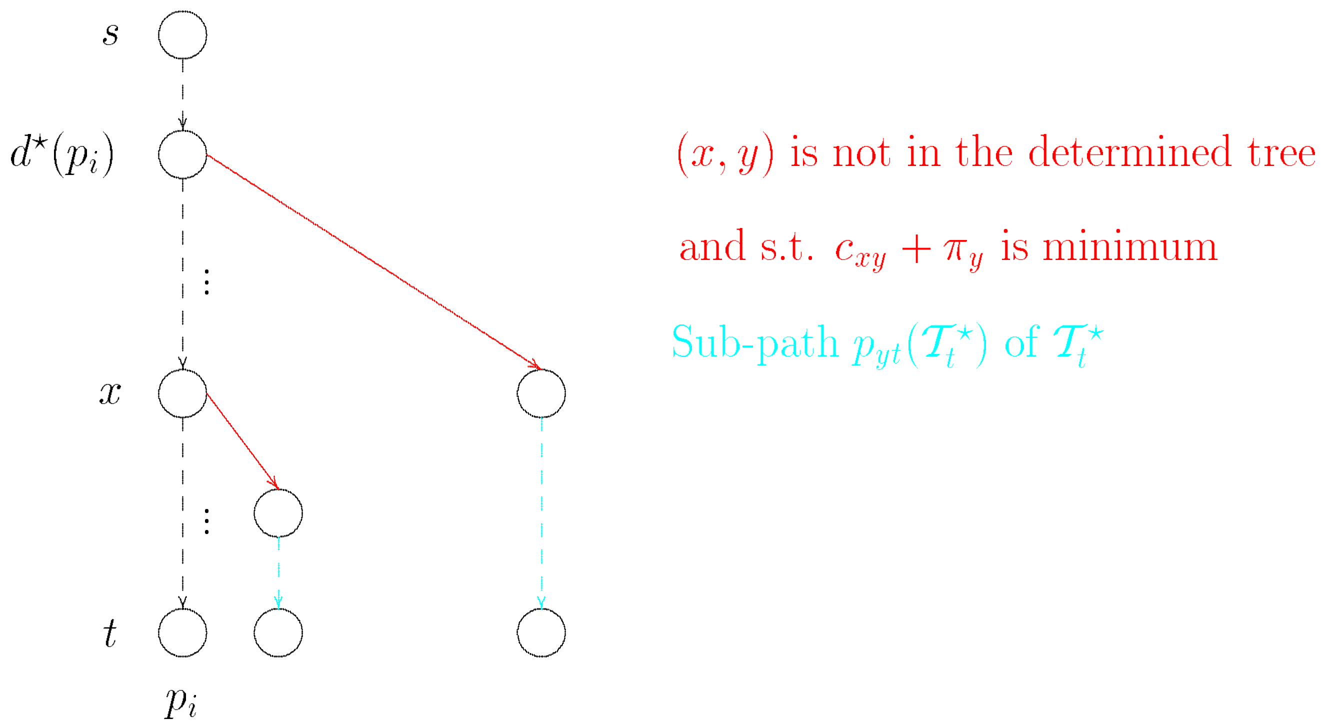
$$p_1 = \langle 1, 2, 5\rangle$$

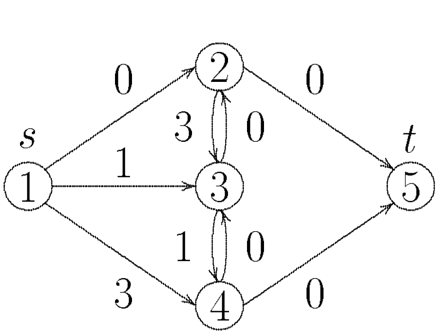
$$X = \{\langle 1, 3, 2, 5\rangle, \langle 1, 4, 5\rangle, \langle 1, 2, 3, 2, 5\rangle\}$$

Elementary algorithm — Martins *et al.* (1997)

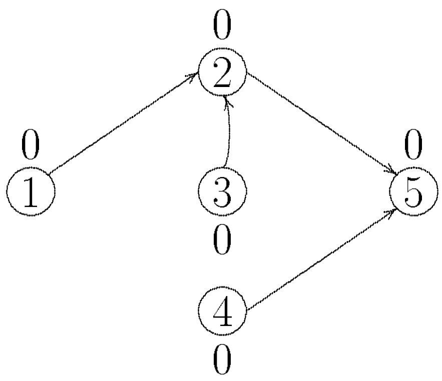
For any node x of p_i from $d^\star(p_i)$ until t:

Generate **only the shortest deviation path** of p_i in x, which has not been determined yet.



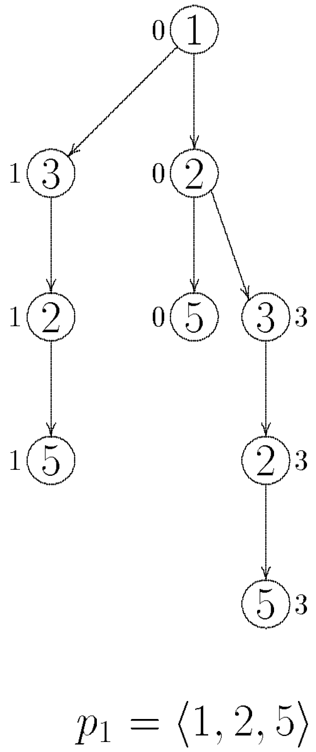


Network $(\mathcal{N}, \mathcal{A}, \mathrm{c}, s, t)$



Tree $\mathcal{T}_t^\star$

Elementary



constrained shortest path problem

$$p_1 = \langle 1, 2, 5 \rangle$$

$$X = \{\langle 1, 3, 2, 5 \rangle, \langle 1, 2, 3, 2, 5 \rangle\}$$

Reduced costs

Notation:

- $\pi_i \rightarrow$ Cost of the path from $i \in \mathcal{N}$ to t in $\mathcal{T}_t^\star$, π_i.

Reduced costs:

- Reduced cost of arc (i,j) related to $\mathcal{T}_t^\star$:

$$\bar{c}_{ij} = \pi_j - \pi_i + c_{ij}$$

Theorem:

1. $c(p) \leq c(q) \Leftrightarrow \bar{c}(p) \leq \bar{c}(q)$
2. $\bar{c}_{ij} \geq 0, \ \forall \ (i,j) \in \mathcal{A}$
3. $\bar{c}_{ij} = 0, \ \forall \ (i,j) \in \mathcal{T}_t^\star$ ($\bar{c}(p_{ij}^\star) = 0, \ \forall \ p_{ij}^\star$ path in $\mathcal{T}_t^\star$)

The reduced costs can be used both in
Hoffman and Pavley algorithm and in the Elementary algorithm.

Using reduced costs
The number of arithmetic operations is decreased.

MPS algorithm — Martins *et al.* (1997)

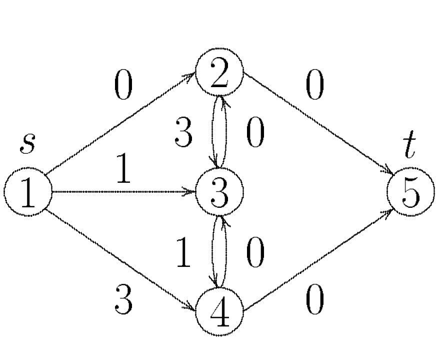
Sorted forward star form:

The set of arcs emerging from a node is sorted by non-decreasing order of their reduced costs.

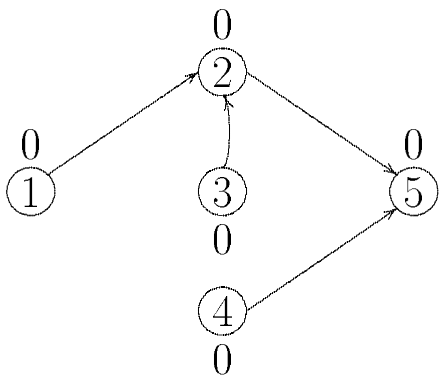
> The determination of the shortest deviation at each node is immediate.

MPS algorithm:

$$\boxed{\left.\begin{array}{c}\text{Elementary algorithm}\\+\\\text{Reduced costs}\\+\\\text{Sorted forward star form}\end{array}\right\}\Rightarrow\ \text{MPS algorithm}}$$

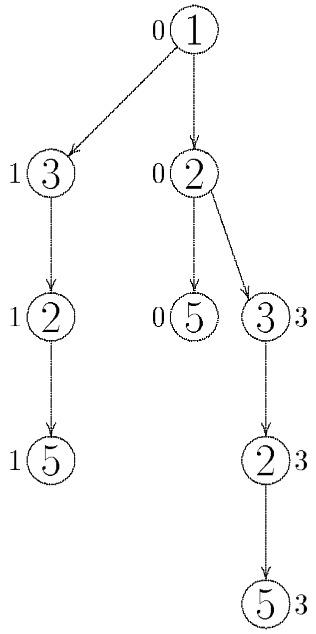


Network $(\mathcal{N}, \mathcal{A}, \mathrm{c}, s, t)$



Tree $\mathcal{T}_t^\star$

MPS



$$\boxed{\begin{array}{c}\text{constrained shortest} \\ \text{path problem} \\ \equiv \\ \text{next arc}\end{array}}$$

$$p_1 = \langle 1, 2, 5 \rangle$$

$$X = \{\langle 1, 3, 2, 5 \rangle, \langle 1, 2, 3, 2, 5 \rangle\}$$

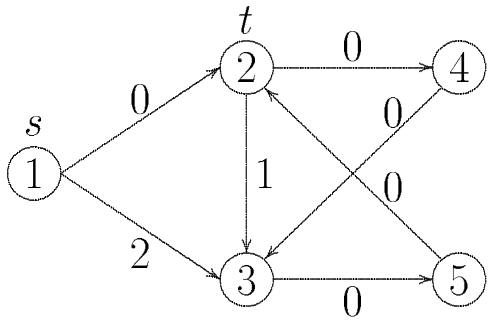
(i,j)	$(1,2)$	$(1,3)$	$(1,4)$	$(2,5)$	$(2,3)$	$(3,2)$	$(3,4)$	$(4,5)$	$(4,3)$
$\bar{c}_{ij}$	0	1	3	0	3	0	2	0	0

$\mathcal{A}$ in the sorted forward star form

Algorithms for the K shortest elementary paths problem

Theorem:
It may exist a kth shortest **elementary path** formed by j-th shortest elementary sub-paths, with $\boxed{j > k}$.

Example:



$$\begin{aligned} p_1 &= \langle 1, 2\rangle, & c(p_1) &= 0 \\ p_2 &= \langle 1, 3, 5, 2\rangle, & c(p_1) &= 2 \end{aligned}$$

$q = \langle 1, 3\rangle$ is an elementary sub-path of p_2

$$\begin{aligned} q_1 &= \langle 1, 2, 4, 3\rangle, & c(q_1) &= 0 \\ q_2 &= \langle 1, 2, 3\rangle, & c(q_2) &= 1 \\ q_3 &= q = \langle 1, 3\rangle, & c(q_3) &= 2 \end{aligned}$$

q is the 3rd elementary shortest path from 1 to 3

Conclusion:
Algorithms based on the Optimality Property cannot be used for ranking elementary paths.

Yen algorithm – Elementary paths ranking (1974)

For any node x of p_i after $d^\star(p_i)$ until t:

Generate **only the shortest deviation path** of p_i in x, which has not been determined yet, assuring that it is an elementary path.

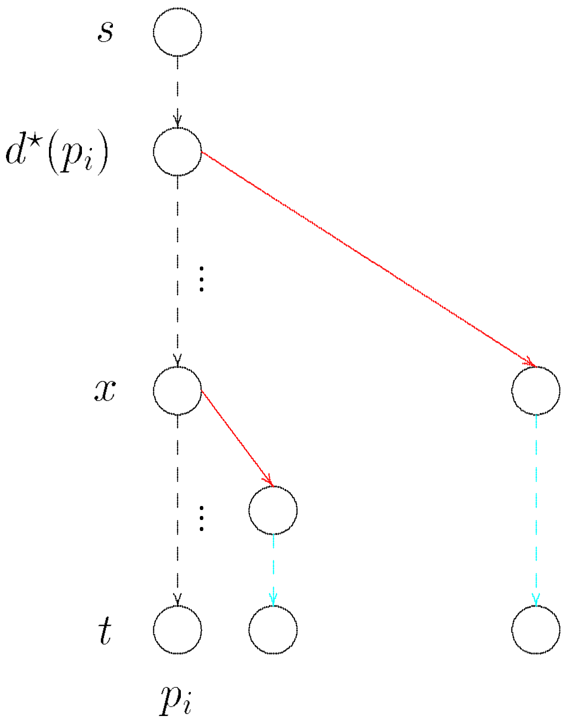
$$\Downarrow$$

Constrained shortest path

To solve this constrained constrained shortest path problem:

1. Arcs and nodes in $\mathrm{sub}_{p_i}(s, x)$ are removed.
2. Arcs starting in x in the tree of paths already determined are also removed.
3. A shortest path problem from x to t in the obtained network is solved.

After this problem's resolution the original network has to be restored.



(x, y) is not in the determined tree

and s.t. $c_{xy} + \pi_y$ is minimum

Sub-path $p_{yt}(\mathcal{T}_t^\star)$ of $\mathcal{T}_t^\star$

Deviation algorithms for ranking elementary paths

Problem:
The elementary paths concatenation may be a path with cycles.

MPS and other deviation algorithms:
Construct the K shortest elementary paths tree minimizing the determination of paths containing cycles.

When analysing a node of p_k:

1. Only nodes until the first one which forms a cycle are considered.

2. The arc chosen is such that its concatenation with the respective sub-path of p_k is still an elementary path.

Comparative analisys

Types of networks

Random networks

Nodes (n)	5000, 10000
Densities (m/n)	1.5, 2, 15, 20

Complete networks

Nodes (n)	100, 250, 500, 1000

Grid networks

Nodes (n)	500, 2500, 4500
($q = w^2/n = n/h^2$)	1, 2, 5, 10

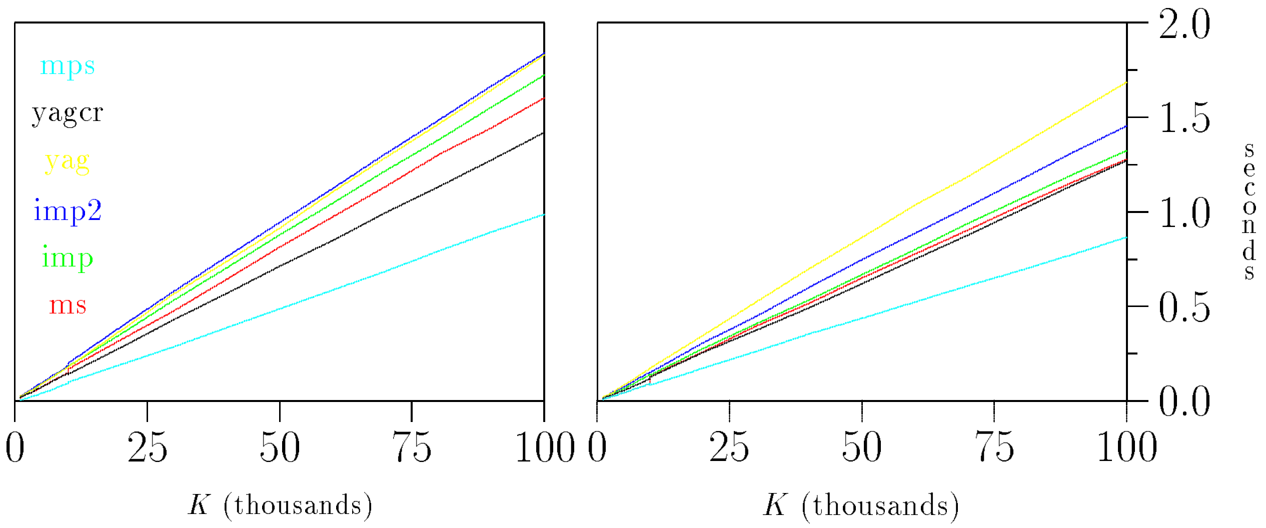
Experimenting conditions

Used machine:

- Pentium II
- 266 MHz
- Linux operating system

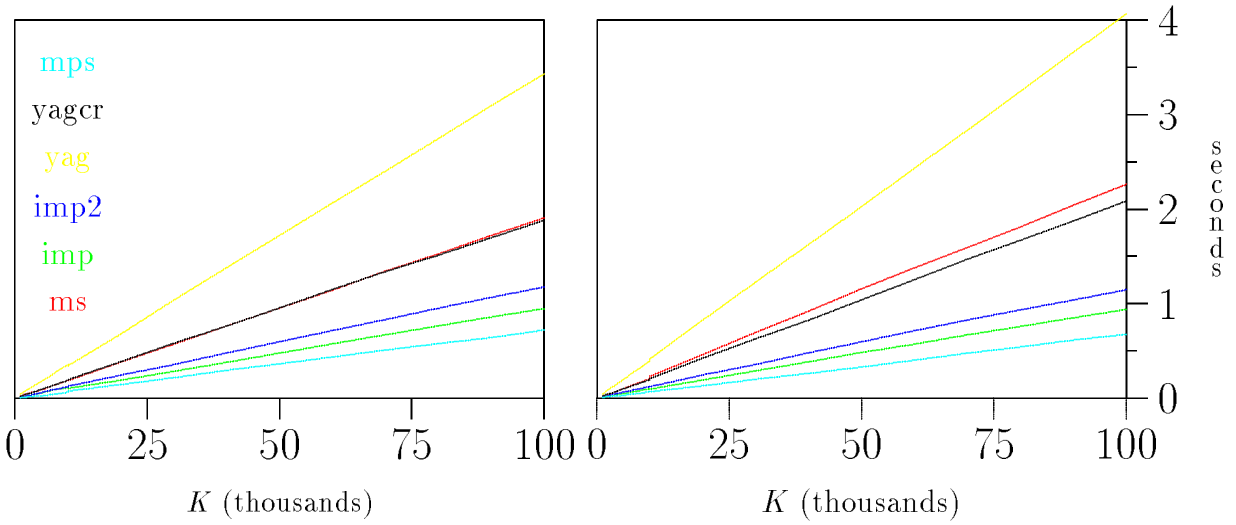
Average execution times for solving:

- 15 problems of each kind;
- 3 types of networks with different dimensions.



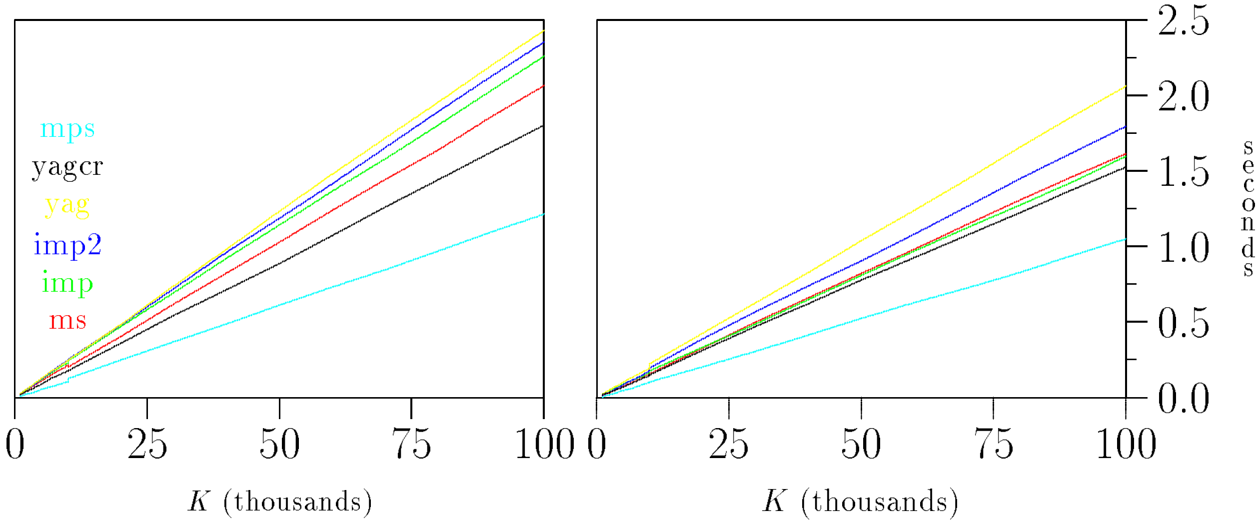
(a) (b)

Random networks with 5000 nodes and (a) 7500 arcs, (b) 10000 arcs



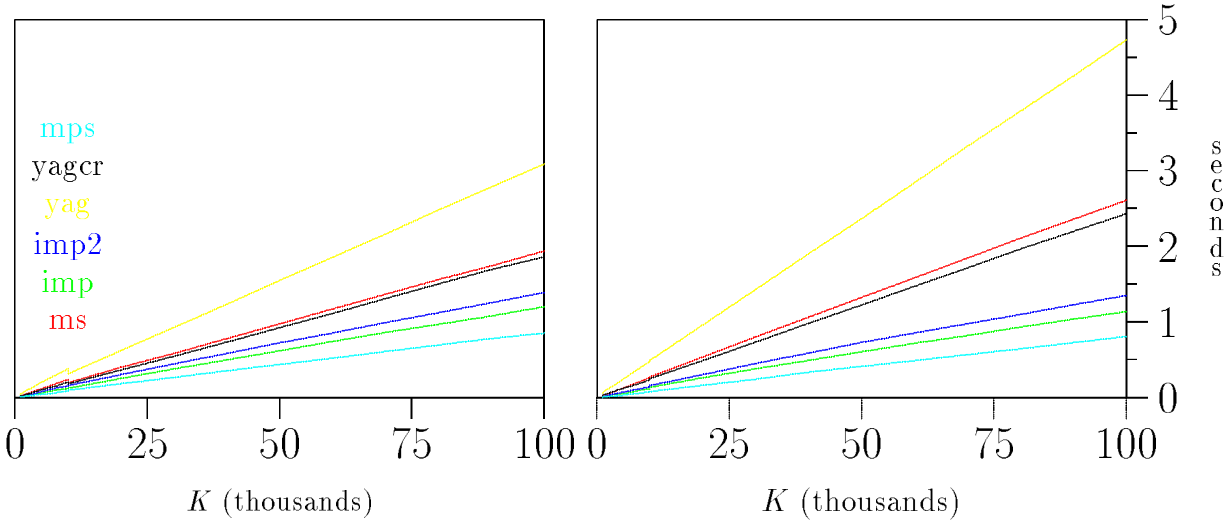
(a) (b)

Random networks with 5000 nodes and (a) 75000 arcs, (b) 100000 arcs



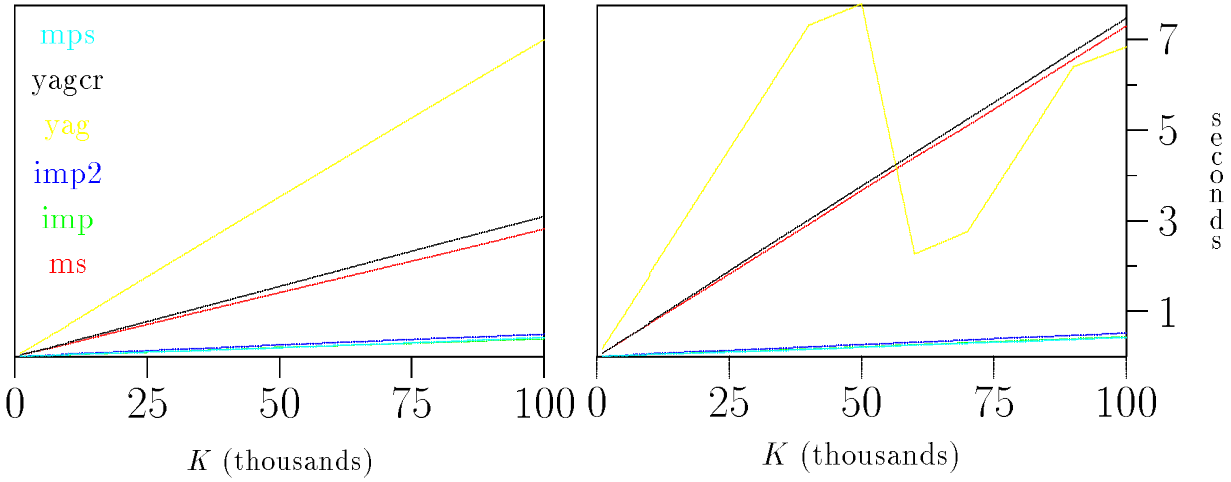
(a) (b)

Random networks with 10000 nodes and (a) 15000 arcs, (b) 20000 arcs

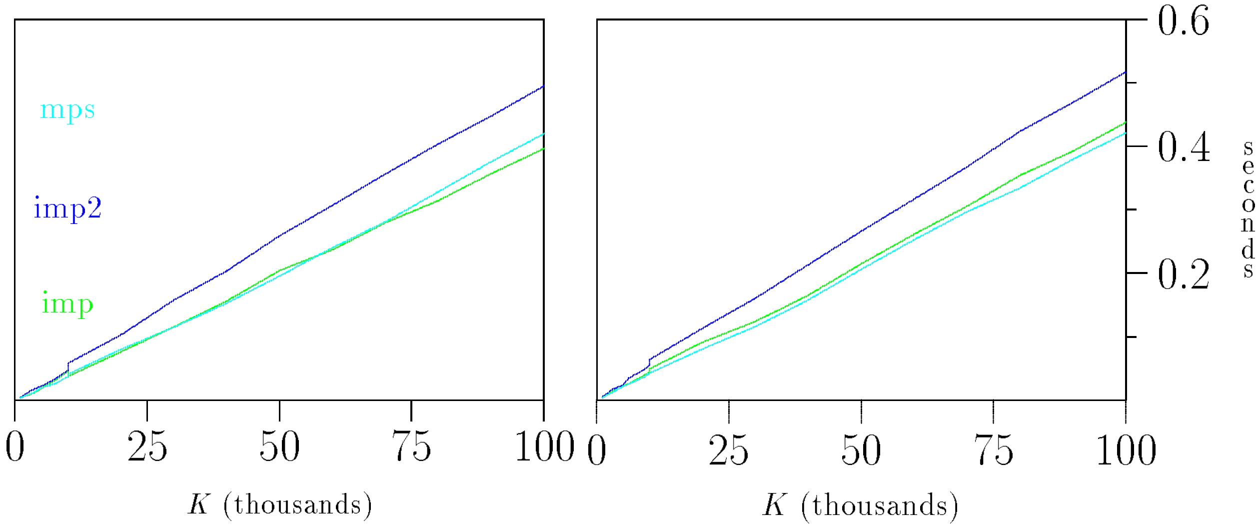


(a) (b)

Random networks with 10000 nodes and (a) 150000 arcs, (b) 200000 arcs

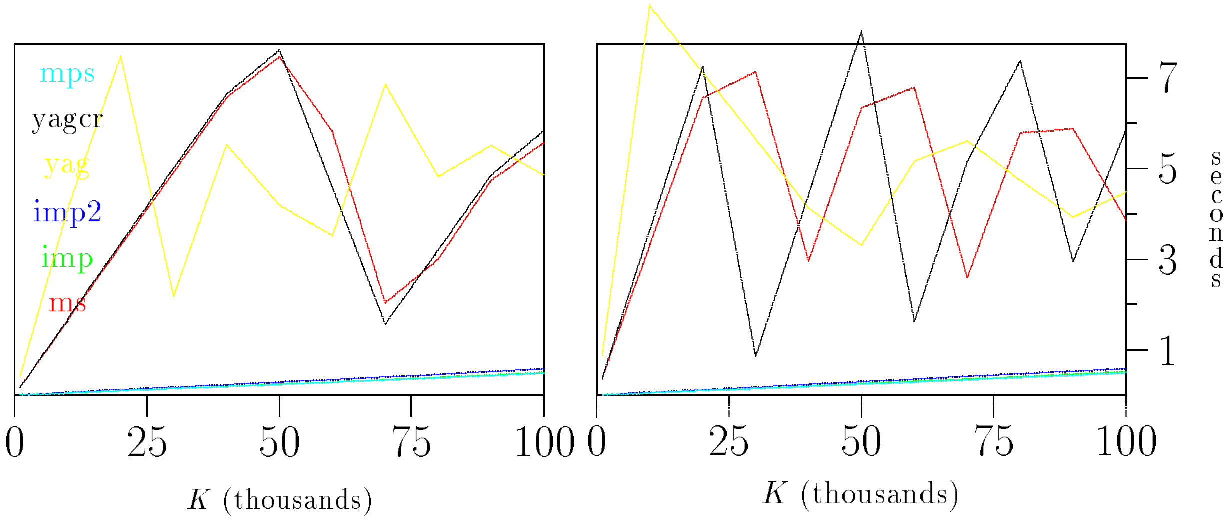


(a) (b)

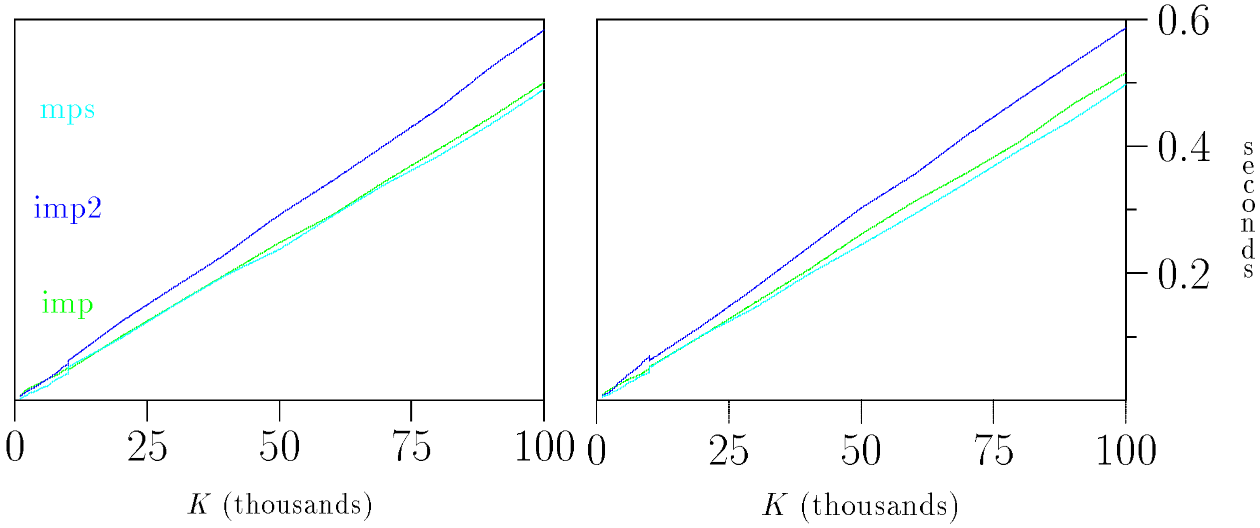


(a) (b)

Complete networks with (a) 100 nodes and (b) 250 nodes

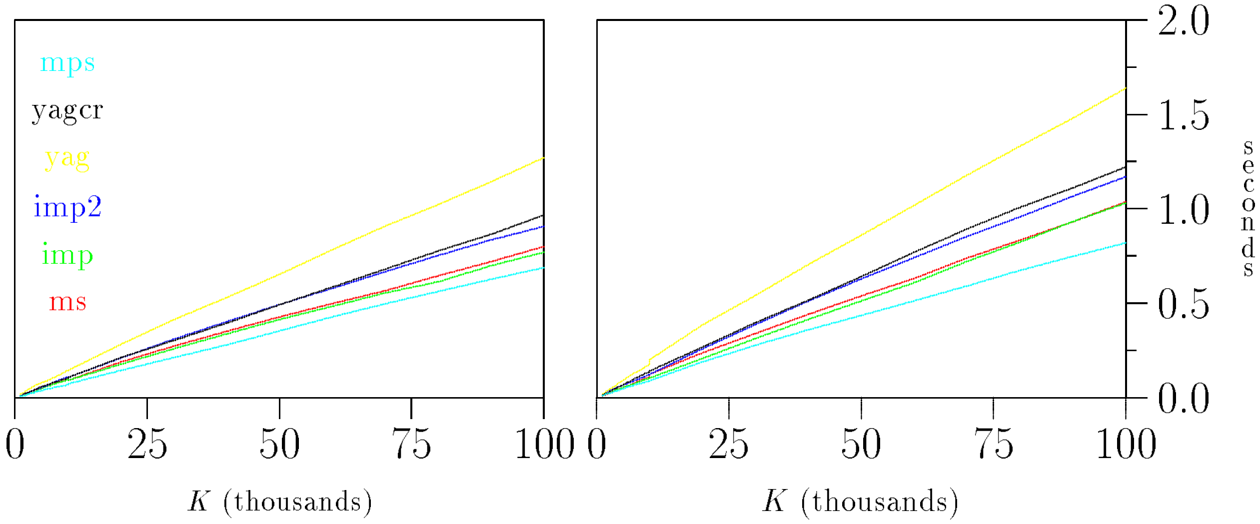


(a) (b)

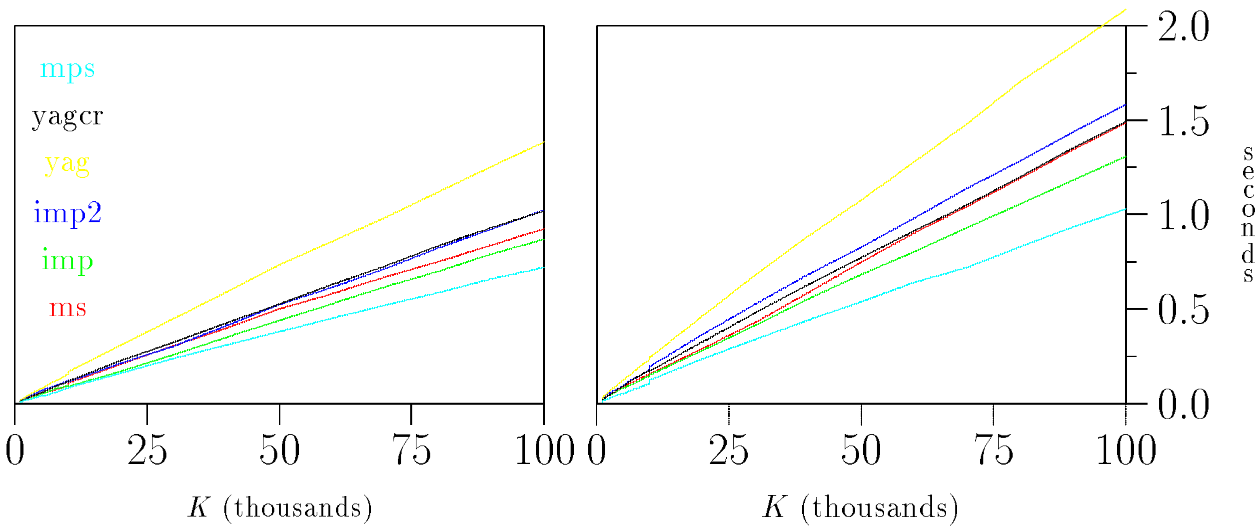


(a) (b)

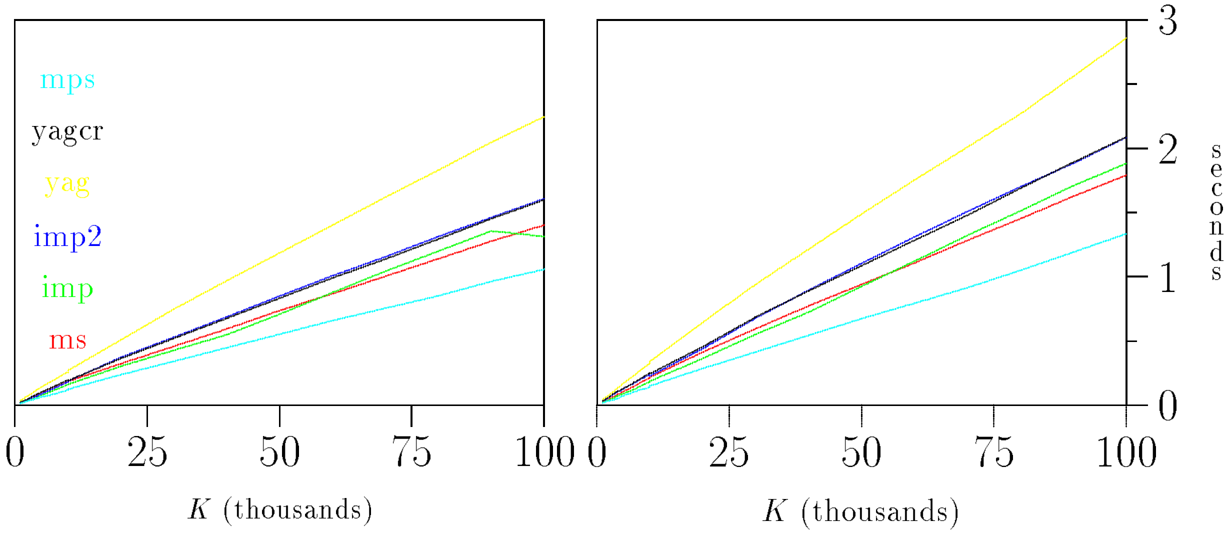
Complete networks with (a) 500 nodes and (b) 1000 nodes



Grid networks with 500 nodes and (a) $q = 1$, (b) $q = 2$

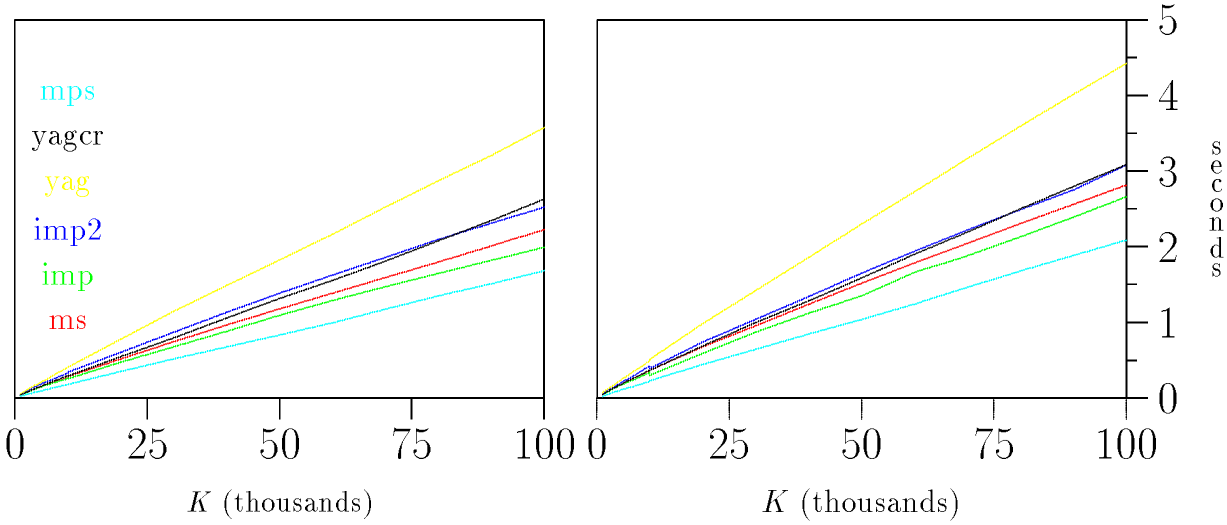


Grid networks with 500 nodes and (a) $q = 5$, (b) $q = 10$



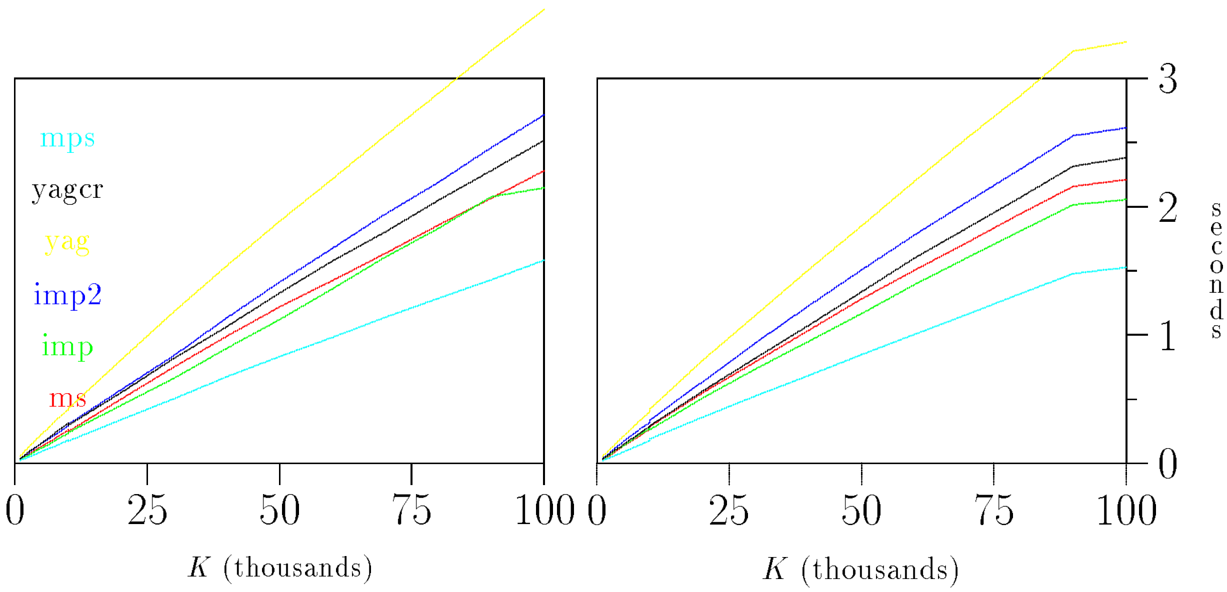
(a) (b)

Grid networks with 2500 nodes and (a) $q = 1$, (b) $q = 2$

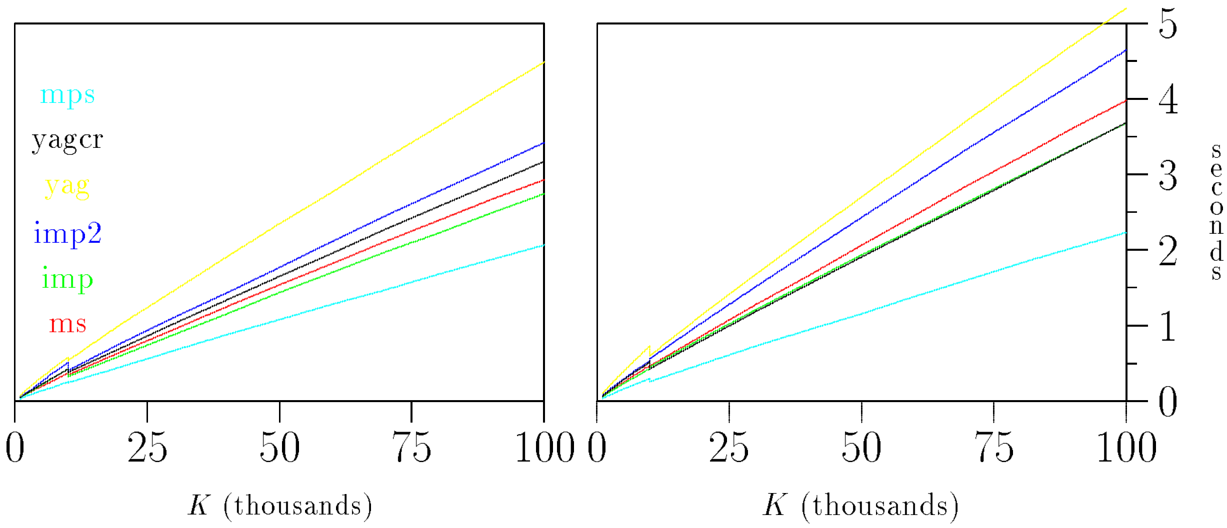


(a) (b)

Grid networks with 2500 nodes and (a) $q = 5$, (b) $q = 10$



Grid networks with 4500 nodes and (a) $q = 1$, (b) $q = 2$



Grid networks with 4500 nodes and (a) $q = 5$, (b) $q = 10$