Departamento de Matemática da Universidade de Coimbra

Métodos Numéricos para Equações com Derivadas Parciais (2009/2010)

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Let us consider a porous media domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$. If we denote by k the conductivity tensor (permeability divided by fluid viscosity), the equation that models the pressure p (in stationary regime) may be written in the form

$$\begin{cases} -\mathrm{div}(k\nabla p) = f, & \text{in } \Omega, \\ \\ p = g, & \text{in } \partial\Omega, \end{cases}$$

where f is a given source term. In some situations it is the flux (or Darcy velocity) $-k\nabla p$ that is of primary interest. However, in the standard finite element method the derivatives, and hence the flux, are approximated lower order than the solution p. A way to overcome this disadvantage is to consider is to consider a so called mixed formulation of this problem. Here the flux u is introduced as a separate dependent variable whose approximation is sought in a different finite element space that the solution itself. This may be done in a such way that the flux is approximated to the same order of accuracy as p.

In the mixed form, the problem is to find (u, p) such that

$$\begin{cases} \mathrm{div}(u) = f, & \text{in } \Omega, \\ \\ u = -k\nabla p, & \text{in } \Omega, \\ \\ p = g, & \text{in } \partial\Omega, \end{cases}$$

where k is a symmetric, positive definite second-order tensor with components in $L^\infty(\Omega)$.

1. Prove that the week formulation of this problem is: find u in V and p in W such that

$$\begin{cases} (k^{-1}u, v) = (p, \mathrm{div}(v)) - (g, v^T n)_{\partial\Omega}, & v \in V, \\ \\ (\mathrm{div}(u), w) = (f, w), & w \in W, \end{cases}$$

   where n is the outward unit normal in $\partial\Omega$, $(\cdot,\cdot)$ is the L^2 inner product

$$V = H(\mathrm{div};\Omega) = \{v \in (L^2(\Omega))^2, \mathrm{div}(v) \in L^2(\Omega)\}$$

   with norm $\|v\|_V^2 = \|v\|^2 + \|\mathrm{div}(v)\|^2$, and

$$W = L^2(\Omega).$$

2. Solve the problem in the unit square considering

$$k = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

   and the true solution

$$p = x^3 y + y^4 + \sin x \cos y.$$

   The values of f and g are obtained according to the equation.