Departamento de Matemática da Universidade de Coimbra

Métodos Numéricos para Equações com Derivadas Parciais (2009/2010)

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1. Consider the inhomogeneous Dirichlet problem

$$\begin{aligned} -\nabla^T(A(x)\nabla u(x)) + b^T(x)\nabla u(x) + d(x) &= f(x), \text{ for } x \in \Omega, \\ u(x) &= g(x) \text{ for } x \in \partial\Omega, \end{aligned}$$

with bounded and measurable functions $A : \Omega \longrightarrow \mathbb{R}^{n,n}$, $b : \Omega \longrightarrow \mathbb{R}^n$, $d : \Omega \longrightarrow \mathbb{R}$, $f \in L_2(\Omega)$, $g \in L_2(\partial\Omega)$. Let there be a function $w \in H^1(\Omega)$ such that $w(x) = g(x)$ for $x \in \partial\Omega$ and let $\widetilde{u}(x) = u(x) - w(x)$.

   (a) Derive a differential equation for $\widetilde{u}$.

   (b) Derive the weak formulation of the differential equation of (a).

2. Let $\Omega =]0,1[^2$. Consider the finite element space $V_h \subset H^1(\Omega)$ consisting of all continuous piecewise linear functions on a triangulation of Ω obtained from a uniform square mesh of size $h = 1/N$, $N \geq 2$, by dividing each square into two triangles with the diagonal of negative slope. Given that $u \in H^2(\Omega)$ let $I_h u$ denote its continuous piecewise linear interpolant from V_h. You may take it for granted that

$$\|u - I_h u\|_1 \leq K_1 h |u|_2,$$

where K_1 is a positive constant, independent of u, u_h and h. (If you are really ambitious, you may try to prove this, but it is not compulsory; be warned: it is a hard work!).

Now consider the elliptic boundary value problem

$$-\Delta u + u = f(x,y) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $f \in L_2(\Omega)$.

Given that V_h is the finite element space introduced above and u_h denotes the finite element approximation to u in V_h, show that:

   (a) $\|u - u_h\|_1 \leq K_2 h |u|_2$, where K_2 is a positive constant, independent of u, u_h and h;

   (b) $\|u - u_h\|_0 \leq K_2 h^2 |u|_2$, where K_2 is a positive constant, independent of u, u_h and h.