Risk and Decision Making (Part II - Game Theory) Exam
MIT/Portugal 3 pages
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1. Consider the following non-cooperative bimatrix game, where Player 1 chooses one of the three rows and Player 2 chooses one of the three columns:

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \left[\begin{array}{ccc} (-1,1) & (0,2) & (0,2) \\ (2,1) & (1,-1) & (0,0) \\ (0,0) & (1,1) & (1,2) \end{array}\right] \end{array}.$$

   (a) Find the safety levels and the maxmin strategies for both players.

   (b) Find as many strategic equilibria as you can, including the mixed one.

Solution:

(a) The game is defined by the following matrices,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

In the A matrix, the top row is dominated by the second row. Hence, we may assume Player I plays the following game

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Now, we see that middle column is dominated by the third column (*i.e.*, Player I's opponent prefers to play the third column rather than the second). The resulting matrix is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, since this is a two-by-two nonzero sum game without a saddle point, the minimax strategies are the equalizing strategies. Player I's safety level is 2/3, and his maxmin strategy is $p = (0, 1/3, 2/3)$.
Now, to compute Player II's safety level, we must consider the matrix

$$B^T = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 2 & 0 & 2 \end{bmatrix}.$$

In the B^T matrix, the first column (by the second) and, then, the second row is dominated (by the third). The resulting matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Again, since this is a two-by-two nonzero sum game without a saddle point, the minimax strategies are the equalizing strategies. Player II's safety level is 2/3, and her maxmin strategy is $q = (2/3, 0, 1/3)$.

(b) The top row of the matrix A is dominated by the second row. Hence, we may assume that Player I will never choose the top row. Then, the game becomes

$$\begin{array}{c} \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \left[\begin{array}{ccc} (2,1) & (1,-1) & (0,0) \\ (0,0) & (1,1) & (1,2) \end{array} \right] \end{array}.$$

Now, we see that Player II will never choose C_2 because her payoffs are always strictly higher if she chooses C_2. Hence, the game becomes

$$\begin{array}{c} \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_3 \end{array} \\ \left[\begin{array}{cc} (2,1) & (0,0) \\ (0,0) & (1,2) \end{array} \right] \end{array},$$

the Battle of the Sexes. There are two PSE's, one at (second row,first column), and the other at (third row, third column).

There is therefore a third SE given by the equalizing strategies in the Battle of the Sexes. Let's find it. The game is defined by the pair of matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let (p_1, q_1) be a pair of equalizing strategies for B, which are defined by

$$p_1 = \left(\frac{2}{3}, \frac{1}{3} \right), \qquad q_1 = (\cdot, \cdot).$$

Denoting by $g_2(p,q)$ the average payoff to Player II if Player I uses the mixed strategy p and Player II uses the mixed strategy q then, from the definition of equalizing strategy,

$$g_2(p_1, q_1) = g_2(p_1, q), \qquad \text{for every mixed strategy } q.$$

Now, let (p_2, q_2) be a pair of equalizing strategies for

$$A^T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

which are defined by

$$p_2 = \left(\frac{1}{3}, \frac{2}{3} \right), \qquad q_2 = (\cdot, \cdot).$$

Denoting by $g_1(p,q)$ the average payoff to Player I if Player I uses the mixed strategy p and Player II uses the mixed strategy q then, from the definition of equalizing strategy,

$$g_1(q_2, p_2) = g_1(q, p_2), \qquad \text{for every mixed strategy } q.$$

Hence, (p_1, p_2) is a Nash equilibrium. In other words, the mixed Nash equilibrium is to play

$$\left(\frac{2}{3} R_2 + \frac{1}{3} R_3; \frac{1}{3} C_1 + \frac{2}{3} C_3 \right).$$

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2. Suppose in the Cournot duopoly model that the two firms have different production costs and different set-up costs. Suppose Player I's cost of producing x is $x+2$, and II's cost of producing y is $3y+1$. Suppose also that the price function is $P(x,y) = 17 - x - y$, where x and y are the amounts produced by I and II respectively. What is the equilibrium production, and what are the players' equilibrium profits?

Solution: The player's profits are

$$u_1(x,y) = x(17 - x - y) - (x+2), \qquad u_2(x,y) = y(17 - x - y) - (3y+1).$$

To find the equilibrium production, we set the partial derivatives to zero:

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= 16 - 2x - y = 0 \\ \frac{\partial u_1}{\partial y} &= 14 - x - 2y = 0 \end{aligned}$$

which gives $(x,y) = (6,4)$ as the equilibrium production. The equilibrium profits are

$$(u_1(6,4), u_2(6,4)) = (34, 15).$$

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3. Consider the cooperative TU bimatrix game:

$$\begin{bmatrix} (1,5) & (2,2) & (0,1) \\ (4,2) & (1,0) & (2,1) \\ (5,0) & (2,3) & (0,0) \end{bmatrix}.$$

   (a) Find the TU-values.
   (b) Find the associated side payment.
   (c) Find the optimal threat strategies.

Solution:

(a) The maximum total payoff is $\sigma = 6$, with payoff (1, 5)[1] The difference matrix is

$$\begin{bmatrix} -4 & 0 & -1 \\ 2 & 1 & 1 \\ 5 & -1 & 0 \end{bmatrix}.$$

which has a saddle point in the second row and third column. Hence, $\delta = 1$ so that the TU solution is

$$\varphi = \left(\frac{\sigma+\delta}{2}, \frac{\sigma-\delta}{2}\right) = \left(\frac{7}{2}, \frac{5}{2}\right).$$

(b) Once the game is played and the outcome is $(1,5)$, Player II should pay $5/2$ to Player I.

(c) The threat strategies are $p = (0,1,0)$ and $q = (0,0,1)$.

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[1] or $(4,2)$, alternatively.