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Sobolev orthogonal polynomials and (M, N)-coherent pairs of measures

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ABSTRACT

We introduce the notion of $(M, N)-$coherent pair of measures as a generalization of the concept of coherent pair of measures introduced by Iserles et al. [A. Iserles, P.E. Koch, S.P. Nørsett, J.M. Sanz-Serna, On polynomials orthogonal with respect to certain Sobolev inner products, J. Approx. Theory 65(2) (1991) 151–175], and subsequently generalized by several authors. A pair of measures $(\mathrm{d}\mu_0, \mathrm{d}\mu_1)$ is called (M, N)-coherent if the corresponding orthogonal polynomial sequences $(P_n)_n$ and $(Q_n)_n$ (resp.) satisfy a (non-zero) structure relation such as

$$\sum_{i=0}^{N} r_{i,n} P_{n-i}(x) = \sum_{i=0}^{M} s_{i,n} Q'_{n-i+1}(x)$$

for all $n = 0, 1, 2, \ldots$, where M and N are fixed non-negative integer numbers, and $r_{i,n}$ and $s_{i,n}$ are given real parameters satisfying some natural conditions. We prove that the regular moment linear functionals associated to an (M, N)-coherent pair are semiclassical and they are related by a rational modification (in the usual sense of distribution theory). We also discuss the converse statement. Under the assumption that $(\mathrm{d}\mu_0, \mathrm{d}\mu_1)$ form an (M, N)-coherent pair, we study the sequence $(S_n^{\lambda})_n$ of the monic orthogonal polynomials with respect to the Sobolev inner product

$$\langle f, g\rangle_{\lambda} := \int_{-\infty}^{+\infty} fg \, \mathrm{d}\mu_0 + \lambda \int_{-\infty}^{+\infty} f'g' \, \mathrm{d}\mu_1,$$

defined in the space of all polynomials with real coefficients, where $\lambda \geq 0$. An efficient algorithm is stated to compute the coefficients in the Fourier–Sobolev type series $f(x) \sim \sum_{n=0}^{\infty} c_n^{\lambda} S_n^{\lambda}(x)$ with respect to $\langle \cdot, \cdot \rangle_{\lambda}$ for suitable smooth functions f such that $f \in L^2_{\mu_0}(\mathbb{R})$ and $f' \in L^2_{\mu_1}(\mathbb{R})$. Finally, some illustrative computational examples are presented.

1. Introduction

Throughout this paper, we use the notations OP, OPS, and MOPS for orthogonal polynomial, orthogonal polynomial sequence, and monic orthogonal polynomial sequence (respectively). The theory of the so-called Sobolev OPs has been a subject of great research interest. These are sequences of polynomials, $(S_n)_n$, such that S_n has degree n for each n, and $(S_n)_n$ is an orthogonal set with respect to a Sobolev inner product, i.e., an inner product defined in the space $\mathcal{P}$ of all polynomials as

$$\langle f, g\rangle_S := \sum_{k=0}^{m} \langle f^{(k)}, g^{(k)}\rangle_k, \quad f, g \in \mathcal{P},$$

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and extended to an appropriate space of functions (e.g. the completion of $\mathcal{P}$ with respect to the norm induced by this inner product). For a brief history on the theory of Sobolev OPs we refer to the paper [1] by Meijer and the doctoral dissertation in [2], as well as the introduction in the paper [3] by Gautschi and Zhang. A large list of references on this subject was compiled in [4].

In this paper, we consider the sequence $(S_n^\lambda)_n$ of the monic Sobolev OPs with respect to a special Sobolev inner product of the form

$$\langle f, g\rangle_\lambda = \int_{-\infty}^{+\infty} f\, g\, \mathrm{d}\mu_0 + \lambda \int_{-\infty}^{+\infty} f'\, g'\, \mathrm{d}\mu_1, \tag{1}$$

under the assumption that $(\mathrm{d}\mu_0, \mathrm{d}\mu_1)$ is an (M, N)-coherent pair of measures in the following sense: if $(P_n)_n$ and $(Q_n)_n$ denote the standard MOPSs with respect to the measures $\mathrm{d}\mu_1$ and $\mathrm{d}\mu_0$ (resp.), then $(\mathrm{d}\mu_0, \mathrm{d}\mu_1)$ is called an (M, N)-coherent pair if a linear algebraic structure relation such as

$$\sum_{i=0}^{N} r_{i,n}\, P_{n-i}(x) = \sum_{i=0}^{M} s_{i,n} \frac{Q'_{n-i+1}(x)}{n-i+1} \tag{2}$$

holds for all $n = 0, 1, 2, \ldots$, where M and N are fixed non-negative integers, and $r_{i,n}$ and $s_{i,n}$ are real parameters satisfying some appropriate conditions.

The case $(M, N) = (1, 0)$ corresponds to the concept of coherence introduced in [5] by Iserles et al., which has been extensively studied by several authors, both from the algebraic and the analytical point of view. The case $(M, N) = (2, 0)$ corresponds to the concept of generalized coherence introduced in [6,7], also studied in [8]. When $(M, N) = (k + 1, 0)$, with k a fixed nonnegative integer number, one obtains the so-called k-coherence, introduced in [9]. Marcellán and Delgado [10] made a very detailed study of the case $(M, N) = (1, 1)$, by determining all the $(1, 1)$-coherent pairs of measures (see also [11]).

Among other results, we show that the standard MOPSs $(P_n)_n$ and $(Q_n)_n$ associated to an (M, N)-coherent pair of measures belong to the semiclassical class, thus generalizing to (M, N)-coherent pairs of measures, a well-known property for coherent pairs. In fact, this is a consequence of the results in [12] where we considered a more general relation than (2), namely a structure relation involving derivatives of arbitrary orders m and k such as

$$\sum_{i=0}^{N} r_{i,n}\, P_{n-i+m}^{(m)}(x) = \sum_{i=0}^{M} s_{i,n} Q_{n-i+k}^{(k)}(x) \tag{3}$$

for all $n = 0, 1, 2, \ldots$. If we take $k = m + 1$ in (3) then, by [12, Theorem 5.1], both $(P_n)_n$ and $(Q_n)_n$ are semiclassical OPSs. (M, N)-coherence fits into this situation, since it corresponds to take $m = 0$ and $k = 1$ in (3).

One of our main purposes is the effective computation, in the framework of (M, N)-coherence, of the Fourier–Sobolev series for functions f living in appropriate function spaces in terms of the Sobolev OPs $(S_n^\lambda)_n$,

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f_n}{s_n} S_n^\lambda(x), \tag{4}$$

where $s_n := \|S_n^\lambda\|_\lambda^2 \equiv \langle S_n^\lambda, S_n^\lambda\rangle_\lambda$ and $f_n := \langle f, S_n^\lambda\rangle_\lambda$. Among other results, we state that both $(s_n)_n$ and $(f_n)_n$ satisfy the non-homogeneous linear difference equation with variable coefficients of order $K := \max\{M, N\}$, which allows us to give an algorithm for the computation of the Fourier–Sobolev coefficients $c_n^\lambda := f_n/s_n$ in (4). This extends to (M, N)-coherent pairs, the algorithms presented in [5,8] for coherent and generalized coherent pairs.

The structure of the paper is as follows. In Section 2, we recall some basic tools concerning the general theory of standard OPs. In Section 3, we review some results stated in our previous work [12] concerning the analysis of the structure relation (3). In Section 4, we focus on the notion of (M, N)-coherent pair, in the more general setting of moment linear functions (not only for positive Borel measures), characterizing explicitly the moment linear functionals such that the corresponding OPSs fulfill the structure relation (2). In Section 5, we consider Sobolev OPs arising from (M, N)-coherent pairs, and in Section 6 we concentrate on the effective computation of the Fourier–Sobolev series (4). Finally, in Section 7, we present some examples of the developed theory.

As a last remark we mention that the asymptotic properties of coherent pairs have been a subject of considerable research along the years. In a future work we will consider the asymptotic aspects in the framework of (M, N)-coherence.

2. Basic tools

All the notions and results presented in this section can be found, e.g., in Refs. [13–15]. The space of all polynomials with complex coefficients will be denoted by $\mathcal{P}$ and the corresponding dual space by $\mathcal{P}^*$. If $\mathbf{u} \in \mathcal{P}^*$, then a sequence of polynomials $(P_n)_n$ is called an orthogonal polynomial sequence (OPS) with respect to $\mathbf{u}$ if $\deg P_n = n$ for all $n = 0, 1, 2, \ldots$ and

$$\langle \mathbf{u}, P_n P_m\rangle = \begin{cases} k_n, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \quad (n, m = 0, 1, 2, \ldots)$$

where $(k_n)_n$ is a sequence of nonzero complex numbers, and $\langle\cdot,\cdot\rangle$ is the duality bracket. $\mathbf{u}$ is regular or quasi-definite if there exists an OPS with respect to it.

Let $\mathbf{u} \in \mathcal{P}^*$ (not necessarily regular) and $\phi \in \mathcal{P}$. The left-multiplication of the functional $\mathbf{u}$ by the polynomial ϕ is the linear functional, denoted by $\phi\mathbf{u}$, defined in the usual sense of distribution theory by

$$\langle \phi\mathbf{u}, q\rangle := \langle \mathbf{u}, \phi q\rangle, \quad q \in \mathcal{P}.$$

The (distributional) derivative of $\mathbf{u}$ is the functional $D\mathbf{u} \in \mathcal{P}^*$ defined by

$$\langle D\mathbf{u}, q\rangle := -\langle \mathbf{u}, q'\rangle, \quad q \in \mathcal{P}.$$

A functional $\mathbf{u} \in \mathcal{P}^*$ is called semiclassical if $\mathbf{u}$ is regular and there exist two polynomials ϕ and ψ, with $\deg\psi \geq 1$, such that

$$D(\phi\mathbf{u}) = \psi\mathbf{u}. \tag{5}$$

If $\mathbf{u}$ is semiclassical, the class of $\mathbf{u}$ is the nonnegative integer number

$$s := \min_{(\phi,\psi)\in\mathcal{A}} \max\{\deg\phi - 2, \deg\psi - 1\},$$

where $\mathcal{A}$ is the set of all pairs of polynomials (ϕ,ψ), with $\deg\psi \geq 1$, which fulfill the distributional differential equation (5). We also say that an OPS associated to a semiclassical linear functional is a semiclassical OPS. When $s = 0$ we obtain the classical OPSs of Hermite, Laguerre, Jacobi and Bessel.

Any MOPS $(P_n)_n$ with respect to a functional $\mathbf{u}$ can be characterized by a three-term recurrence relation (TTRR)

$$\begin{aligned} &P_0 = 1, \qquad P_1(x) = x - \beta_0 \\ &P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n = 1, 2, \ldots \end{aligned} \tag{6}$$

where $(\beta_n)_n$ and $(\gamma_n)_n$ are sequences of complex numbers with $\gamma_n \neq 0$ for all $n = 1, 2, \ldots$. When β_n is real and $\gamma_n > 0$ for all n then there exists a positive Borel measure $\mathrm{d}\mu$, whose support is an infinite subset of the real line and with finite moments of all orders, such that the corresponding regular functional admits an integral representation such as

$$\langle \mathbf{u}, q\rangle = \int_{\mathbb{R}} q\,\mathrm{d}\mu, \quad q \in \mathcal{P}.$$

In such a case we say that $\mathbf{u}$ is positive-definite and $(P_n)_n$ is orthogonal in the positive-definite sense, or $(P_n)_n$ is orthogonal w.r.t. the measure $\mathrm{d}\mu$.

3. Linearly related sequences of derivatives of OPs

Direct and inverse problems (in the sense of the theory of OPs) involving linear relations between two OPSs, or their derivatives of fixed orders, have been studied by several authors. For instance, we mention the works [16–20,12,21]. In this section, we recall some results on this subject stated in [12], which will be applied later to (M,N)-coherent pairs. For simplicity, we write

$$P_n^{[m]}(x) := \frac{P_{n+m}^{(m)}(x)}{(n+1)_m} \quad (n, m = 0, 1, 2, \ldots),$$

where $P_{n+m}^{(m)} \equiv \frac{\mathrm{d}^m}{\mathrm{d}x^m}P_{n+m}$ and $(a)_n$ denotes the Pochhammer symbol: for $a > 0$ and n a nonnegative integer number,

$$(a)_0 := 1, \quad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1),$$

where Γ denotes the Gamma function. Notice that $P_n^{[m]}(x)$ is a monic polynomial in x of degree n. If $n < 0$ we set $P_n^{[m]}(x) := 0$.

We begin by recalling the following general result.

Theorem 3.1 (*[12, Theorem 3.1]*). *Let $\mathbf{u}$ and $\mathbf{v}$ be two regular linear functionals in $\mathcal{P}^*$, and let $(P_n)_n$ and $(Q_n)_n$ be the corresponding MOPSs. Assume that there exist two nonnegative integer numbers N and M, and complex numbers $r_{i,n}$ and $s_{j,n}$ $(i = 1,\ldots,N; j = 1,\ldots,M; n = 0,1,\ldots)$, with the conventions $r_{i,n} = 0$ if $n < i$ and $s_{j,n} = 0$ if $n < j$, such that*

$$P_n^{[m]}(x) + \sum_{i=1}^{N} r_{i,n}P_{n-i}^{[m]}(x) = Q_n^{[k]}(x) + \sum_{j=1}^{M} s_{j,n}Q_{n-j}^{[k]}(x) \tag{7}$$

holds for all $n = 0, 1, 2, \ldots$. Define the matrix $A_{N+M} \equiv \left[a_{i,j}\right]_{i,j=1}^{N+M}$ as

$$a_{i,j} = \begin{cases} r_{j-i,j-1}, & \text{if } 1 \leq i \leq M \text{ and } i \leq j \leq N+i \\ s_{j-i+M,j-1}, & \text{if } M+1 \leq i \leq M+N \text{ and } i-M \leq j \leq i \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

with $r_{0,\kappa} = s_{0,\nu} = 1$ $(\kappa = 0, \ldots, M-1;\ \nu = 0, \ldots, N-1)$. *Assume that*

$$r_{N,M+N+i}\, s_{M,M+N+i} \neq 0 \quad (i = 0, 1) \quad \text{and} \quad \det A_{N+M} \neq 0.$$

Without loss of generality, suppose $0 \leq m \leq k$. *Then there exist polynomials* Φ_{M+m+i} *and* Ψ_{N+k+i}, *of degrees* $M + m + i$ *and* $N + k + i$, *such that*

$$D^{k-m}\left(\Phi_{M+m+i}\mathbf{u}\right) = \Psi_{N+k+i}\mathbf{v}, \quad i = 0, 1. \tag{9}$$

The case $k = m + 1$ in Theorem 3.1 is of particular interest, because in this situation both $\mathbf{u}$ and $\mathbf{v}$ are semiclassical moment linear functionals. Indeed, we have the following.

Theorem 3.2 (*[12, Theorem 5.1]*)**.** *Under the conditions of Theorem 3.1, if* $k = m + 1$ *and* $(P_n)_n \equiv (Q_n)_n$ *(so that* $\mathbf{u}$ *and* $\mathbf{v}$ *coincide up to a constant factor), then*

$$D(\Phi_{M+m}\mathbf{u}) = \Psi_{N+m+1}\mathbf{u};$$

hence $\mathbf{u}$ *is semiclassical of class at most* $\max\{M + m - 2, N + m\}$.

Theorem 3.3 (*[12, Theorem 5.2]*)**.** *Under the conditions of Theorem 3.1, if* $k = m + 1$ *and* $(P_n)_n \not\equiv (Q_n)_n$, *then* $\mathbf{u}$ *and* $\mathbf{v}$ *are semiclassical linear functionals of classes at most* $N + M + 2m$ *and* $N + 3M + 4m$ *(respectively), which are also connected by a rational modification. More precisely, we have*

$$\Lambda\mathbf{u} = \Phi\mathbf{v}, \tag{10}$$

$$D\left(\Phi\mathbf{u}\right) = \Psi\mathbf{u}, \tag{11}$$

$$D\left(\widetilde{\Phi}\mathbf{v}\right) = \widetilde{\Psi}\mathbf{v}, \tag{12}$$

where, being $\mathcal{P}_n$ *the set of all polynomials of degree less than or equal to n,*

$$\Lambda := \Phi_{M+m}\Phi'_{M+1+m} - \Phi_{M+1+m}\Phi'_{M+m} \in \mathcal{P}_{2(M+m)},$$

$$\Phi := \Phi_{M+m}\Psi_{N+2+m} - \Phi_{M+1+m}\Psi_{N+1+m} \in \mathcal{P}_{N+M+2m+2},$$

$$\Psi := \Psi'_{N+2+m}\Phi_{M+m} - \Psi'_{N+1+m}\Phi_{M+1+m} \in \mathcal{P}_{N+M+2m+1},$$

$$\widetilde{\Phi} := \Lambda\Phi \in \mathcal{P}_{N+3M+4m+2}, \qquad \widetilde{\Psi} := 2\Lambda'\Phi + \Lambda(\Psi - \Phi') \in \mathcal{P}_{N+3M+4m+1}.$$

Remark 3.1. The numbers $N+M+2m$ and $N+3M+4m$ are upper bounds for the classes of the semiclassical functionals $\mathbf{u}$ and $\mathbf{v}$ in Theorem 3.3. For concrete families of OPSs, the classes of $\mathbf{u}$ and $\mathbf{v}$ can be computed applying a well known reduction process described in [14].

4. (M, N)-coherent pairs

4.1. Definition

Definition 4.1. Let $\mathbf{u}$ and $\mathbf{v}$ be two regular linear functionals in $\mathcal{P}^*$ and let $(P_n)_n$ and $(Q_n)_n$ be the corresponding MOPSs (respectively). Let M and N be nonnegative integer numbers. We call $(\mathbf{v}, \mathbf{u})$ an (M, N)-coherent pair if there exist complex numbers $r_{1,n}, \ldots, r_{N,n}$ and $s_{1,n}, \ldots, s_{M,n}$, such that the structure relation

$$P_n(x) + \sum_{i=1}^{N} r_{i,n}P_{n-i}(x) = \frac{Q'_{n+1}(x)}{n+1} + \sum_{j=1}^{M} s_{j,n}\frac{Q'_{n-j+1}(x)}{n-j+1} \tag{13}$$

holds for all integer numbers $n = 0, 1, 2, \ldots$, with the conditions

$$r_{N,n} \neq 0 \quad \text{if } n \geq N; \qquad s_{M,n} \neq 0 \quad \text{if } n \geq M, \tag{14}$$

and the conventions

$$r_{i,n} = 0 \quad \text{if } n < i \leq N; \qquad s_{j,n} = 0 \quad \text{if } n < j \leq M. \tag{15}$$

Under these conditions we also call $((Q_n)_n, (P_n)_n)$ an (M, N)-coherent pair.

Remark 4.1. In the positive-definite case, both functionals $\mathbf{u}$ and $\mathbf{v}$ admit integral representations in terms of positive Borel measures $d\mu_1$ and $d\mu_0$ (respectively). Under such conditions, being $(\mathbf{v}, \mathbf{u})$ an (M, N)-coherent pair, we say that $(d\mu_0, d\mu_1)$ is an (M, N)-coherent pair of measures.

As pointed out in the introduction, when $(M, N) = (1, 0)$ we get the concept of coherence introduced in [5]. One of the most important results on coherent pairs is due to Meijer [22] who found all the possible coherent pairs of measures, by showing that at least one of the measures in a coherent pair must be a classical measure and then giving a complete classification of the coherent pairs (see also [23]). As a nice extension of this result, Marcellán and Delgado [10] determined all the $(1, 1)$-coherent pairs of measures. Indeed, these authors first showed that at least one of the moment linear functionals associated to a $(1, 1)$-coherent pair of measures is semiclassical of class at most one, and then to describe all the $(1, 1)$-coherent pairs they only needed to consider the cases when one of the functionals is classical or semiclassical of class one.

4.2. What is beyond (M, N)-coherent pairs?

In [24], the authors raised the question "*What is beyond coherent pairs of orthogonal polynomials?*" The answer is [23,24,22]: coherent pairs belong to the semiclassical families of OPSs and the associated linear functionals are related by a rational factor. Furthermore [22]: at least one of these families is classical. Concerning (M, N)-coherent pairs, as an immediate consequence of Theorem 3.3 for $m = 0$, we may state the following proposition.

Theorem 4.1. *Let $(\mathbf{v}, \mathbf{u})$ be an (M, N)-coherent pair. Then the linear functionals $\mathbf{u}$ and $\mathbf{v}$ are semiclassical of classes at most $N + M$ and $N + 3M$ (respectively), provided that the condition $\det A_{N+M} \neq 0$ holds, where A_{N+M} is the matrix of order $N + M$ defined as in (8). Further, the functionals $\mathbf{u}$ and $\mathbf{v}$ are related by a rational factor.*

Remark 4.2. When $(M, N) = (0, 0)$, Theorem 4.1 implies Hahn's characterization for the classical functionals [25], and when $(M, N) = (2, 0)$ it gives Theorem 3.3 in [8]. When $(M, N) = (1, 1)$ some results in [10] are recovered.

Theorem 4.1 shows that if $(\mathbf{v}, \mathbf{u})$ is an (M, N)-coherent pair then both $\mathbf{u}$ and $\mathbf{v}$ are semiclassical functionals and they are related by a rational factor (up to some natural conditions). The next result shows that the converse statement holds under some appropriate conditions.

Theorem 4.2. *Let $\mathbf{u}$ and $\mathbf{v}$ be two regular functionals. Suppose that $\mathbf{u}$ is semiclassical and $\mathbf{u}$ and $\mathbf{v}$ are related by a rational factor, so there exist monic polynomials Φ and Ω, and nonzero polynomials Ψ and Λ, with $\deg \Psi \geq 1$, such that*

$$D(\Phi \mathbf{u}) = \Psi \mathbf{u}, \tag{16}$$

$$\Lambda \mathbf{u} = \Omega \mathbf{v}. \tag{17}$$

Let $(P_n)_n$ and $(Q_n)_n$ be the MOPS with respect to $\mathbf{u}$ and $\mathbf{v}$ (respectively), and set

$$p := \deg \Phi, \qquad q := \deg \Psi, \qquad \ell := \deg \Omega, \qquad t := \deg \Lambda.$$

Then the structure relation (13) holds with the conventions (15), being M and N given by

$$M := t + 2p + \ell, \qquad N := s + p + 2\ell. \tag{18}$$

Moreover, if $q \geq p - 1$ then conditions (14) hold, provided whenever $q = p - 1$ the condition $p - b \notin \mathbb{N} \cup \{0\}$ is fulfilled, where b denotes the leading coefficient of Ψ, so that $(\mathbf{v}, \mathbf{u})$ is an (M, N)-coherent pair and if $q < p - 1$ then the following conditions hold:

$$r_{N,n} \neq 0 \quad \text{if } n \geq N + p + 1; \qquad s_{M,n} \neq 0 \quad \text{if } n \geq M. \tag{19}$$

Proof. From the general theory of semiclassical OPs (see e.g. [14]) we know that the MOPS $(P_n)_n$ w.r.t. $\mathbf{u}$ fulfills a structure relation of the form

$$\Phi P'_{n+1} = \sum_{i=n-s}^{n+p} a_{n,i} P_i, \quad n \geq 0, \tag{20}$$

where $s := \max\{p - 2, q - 1\}$ and the $a_{n,i}$'s are complex numbers fulfilling the convention $a_{n,i} = 0$ if $i < 0$. Notice that, using (16) and the rules of the distributional calculus, for all $n \geq s$ we may write

$$\langle \mathbf{u}, P_{n-s}^2 \rangle a_{n,n-s} = \langle \mathbf{u}, \Phi P'_{n+1} P_{n-s} \rangle = -\langle \mathbf{u}, \left(\Phi P'_{n-s} + \Psi P_{n-s} \right) P_{n+1} \rangle.$$

Hence

$$a_{n,n-s} = \begin{cases} -b\, \gamma_{n+1,s+1}, & \text{if } q > p - 1 \\ -(n - s + b)\, \gamma_{n+1,s+1}, & \text{if } q = p - 1 \\ -(n - s)\, \gamma_{n+1,s+1}, & \text{if } q < p - 1 \end{cases}$$

for all $n \geq s$, where

$$\gamma_{m,k} := \frac{\langle \mathbf{u}, P_m^2 \rangle}{\langle \mathbf{u}, P_{m-k}^2 \rangle} = \prod_{i=m-k+1}^{m} \gamma_i, \quad m \geq k \tag{21}$$

(if $k = 0$, empty product equals one), $(\gamma_n)_{n\geq 1}$ being the sequence of parameters appearing in the three-term recurrence relation (6) satisfied by $(P_n)_n$. Moreover, $a_{n,n+p} = n + 1$ for all $n \geq 0$. Now, from (17) we deduce

$$\Omega P_n = \sum_{i=n-t}^{n+\ell} b_{n,i} Q_i, \quad n \geq 0, \tag{22}$$

where the $b_{n,i}$'s are complex numbers fulfilling $b_{n,i} = 0$ whenever $i < 0$ (by convention). Moreover, $b_{n,n+\ell} = 1$ for all $n \geq 0$, and it is easy to see that $b_{n,n-t} = \lambda \langle \mathbf{u}, P_n^2 \rangle / \langle \mathbf{v}, Q_{n-t}^2 \rangle \neq 0$ for all $n \geq t$, where λ denotes the leading coefficient of Λ. Notice also that

$$\Phi Q_n = \sum_{j=n-p}^{n+p} c_{n,j} Q_j, \quad n \geq 0, \tag{23}$$

where the $c_{n,j}$'s are complex numbers satisfying $c_{n,j} = 0$ if $j < 0$, being $c_{n,n+p} = 1$ for all $n \geq 0$, and $c_{n,n-p} = \langle \mathbf{v}, Q_n^2 \rangle / \langle \mathbf{v}, Q_{n-p}^2 \rangle \neq 0$ for all $n \geq p$. Next, consider the obvious equality

$$(\Omega \Phi P_{n+1})' = (\Omega \Phi)' P_{n+1} + \Omega \Phi P'_{n+1}, \quad n \geq 0. \tag{24}$$

Then, using (22) and (23) on the left-hand side of (24), we obtain

$$(\Omega \Phi P_{n+1})' = \sum_{i=n+1-t}^{n+1+\ell} b_{n+1,i} \, (\Phi Q_i)' = \sum_{i=n+1-t}^{n+1+\ell} b_{n+1,i} \sum_{j=i-p}^{i+p} c_{i,j} Q'_j.$$

Hence

$$(\Omega \Phi P_{n+1})' = \sum_{i=n-t-p}^{n+\ell+p} d_{n,i} \, \frac{Q'_{i+1}}{i+1}, \quad n \geq 0, \tag{25}$$

where $d_{n,i} = 0$ if $i < 0$, $d_{n,n+\ell+p} = n + 1 + \ell + p \neq 0$ for all $n \geq 0$, and

$$d_{n,n-t-p} = (n + 1 - t - p) b_{n+1,n+1-t} c_{n+1-t,n+1-t-p} \neq 0, \quad n \geq t + p. \tag{26}$$

On the other hand, we may write

$$\Omega P_n = \sum_{j=n-\ell}^{n+\ell} e_{n,j} P_j, \quad n \geq 0,$$

where $e_{n,j} = 0$ if $j < 0$, $e_{n,n+\ell} = 1$ for $n \geq 0$, $e_{n,n-\ell} = \gamma_{n,\ell} \neq 0$ for all $n \geq \ell$, and $\gamma_{n,\ell}$ is defined by (21). Therefore, taking into account (20), we deduce

$$\Omega \Phi P'_{n+1} = \sum_{i=n-s}^{n+p} a_{n,i} \Omega P_i = \sum_{i=n-s}^{n+p} a_{n,i} \sum_{j=i-\ell}^{i+\ell} e_{i,j} P_j = \sum_{i=n-s-\ell}^{n+\ell+p} f_{n,i} P_i, \tag{27}$$

where $f_{n,i} = 0$ if $i < 0$, $f_{n,n+\ell+p} = n+1$ for all $n \geq 0$, and $f_{n,n-s-\ell} = a_{n,n-s} e_{n-s,n-s-\ell}$ for all $n \geq s+\ell$. Indeed, for all $n \geq s+\ell$,

$$f_{n,n-s-\ell} = \begin{cases} -b \, \gamma_{n+1,s+1} \gamma_{n-s,\ell}, & \text{if } q > p-1 \\ -(n-s+b) \, \gamma_{n+1,s+1} \gamma_{n-s,\ell}, & \text{if } q = p-1 \\ -(n-s) \, \gamma_{n+1,s+1} \gamma_{n-s,\ell}, & \text{if } q < p-1. \end{cases}$$

On the other hand, similarly as before, we can write

$$(\Omega \Phi)' P_{n+1} = \sum_{i=n-\ell-p+2}^{n+\ell+p} g_{n,i} P_i, \quad n \geq 0, \tag{28}$$

where empty sum equals zero if $\ell + p = 0$, and whenever $\ell + p \geq 1$ the following holds: $g_{n,i} = 0$ if $i < 0$, $g_{n,n+\ell+p} = \ell + p$ for all $n \geq 0$, and $g_{n,n-\ell-p+2} = (\ell + p) \, \gamma_{n+1,\ell+p-1} \neq 0$ for all $n \geq \ell + p - 2$.

As a consequence, substituting the right-hand sides of (25), (27) and (28) in (24), and taking into account that $s \ge p-2$, we obtain

$$\sum_{i=n-s-\ell}^{n+\ell+p} h_{n,i} P_i = \sum_{i=n-t-p}^{n+\ell+p} d_{n,i} \frac{Q'_{i+1}}{i+1}, \quad n \ge 0, \tag{29}$$

where $h_{n,i} = 0$ if $i < 0$, and $h_{n,n+\ell+p} := f_{n,n+\ell+p} + g_{n,n+\ell+p} = n+1+\ell+p$ for all $n \ge 0$. Furthermore, for $n \ge s+\ell$,

$$\begin{aligned} h_{n,n-\ell-s} &= \begin{cases} f_{n,n-\ell-s}, & \text{if } s > p-2 \\ f_{n,n-\ell-s} + g_{n,n-\ell-s}, & \text{if } s = p-2 \end{cases} \\ &= \begin{cases} -b\,\gamma_{n+1,s+1}\gamma_{n-s,\ell}, & \text{if } q > p-1 \\ -(n-s-\ell-p+b)\,\gamma_{n+1,s+1}\gamma_{n-s,\ell}, & \text{if } q = p-1 \\ -(n-s-\ell-p)\,\gamma_{n+1,s+1}\gamma_{n-s,\ell}, & \text{if } q < p-1, \end{cases} \end{aligned}$$

and so, taking into account that we are assuming the hypothesis $p-b \notin \mathbb{N}_0$ whenever $q = p-1$, as well as the definition (21) of the $\gamma_{m,k}$'s, we deduce

$$\begin{aligned} &h_{n,n-s-\ell} \ne 0, \quad n \ge s+\ell, \ \text{if } q \ge p-1; \\ &h_{n,n-s-\ell} \ne 0, \quad n \ge s+\ell+p+1, \ \text{if } q < p-1. \end{aligned} \tag{30}$$

Finally, we may rewrite (29) as (13), with M and N as in (18), and

$$\begin{aligned} r_{i,n} &:= h_{n-p-\ell,n-i}/(n+1), \quad 1 \le i \le N, \ n \ge 0; \\ s_{j,n} &:= d_{n-p-\ell,n-j}/(n+1), \quad 1 \le j \le M, \ n \ge 0. \end{aligned}$$

Notice that (26) and (30) ensure that conditions (14) hold if $q \ge p-1$, and conditions (19) hold if $q < p-1$. It is also clear that relations (15) hold. This completes the proof. □

Remark 4.3. The proof of Theorem 4.2 is constructive. Moreover, by direct inspection of the proof, one sees that when $\Omega \equiv \Phi$ it simplifies and in place of (18) we can take $M = t+p$ and $N = s+p$.

Remark 4.4. Notice that Eqs. (16) and (17) imply

$$D(\widetilde{\Phi}\mathbf{v}) = \widetilde{\Psi}\mathbf{v}, \qquad \widetilde{\Phi} := \Lambda\Omega\Phi, \qquad \widetilde{\Psi} := \left(2\Lambda'\Phi + \Lambda\Psi\right)\Omega,$$

so, under the conditions of Theorem 4.2, **v** is also a semiclassical functional. Thus, Theorems 4.1 and 4.2 show that, up to certain natural conditions, a structure relation like (13) with the conventions (15) holds if and only if the moment linear functionals with respect to which $(P_n)_n$ and $(Q_n)_n$ are orthogonal are semiclassical functionals related by a rational factor.

5. Sobolev OPs arising from (M, N)-coherent pairs

In this section, we assume that $\mathrm{d}\mu_0$ and $\mathrm{d}\mu_1$ are given positive Borel measures, with finite moments of all orders and such that their supports coincide with the same interval $I \subset \mathbb{R}$. Then we may define the following Sobolev inner product in the space $\mathcal{P}$ of all polynomials with real coefficients:

$$\langle f, g\rangle_\lambda = \int_{-\infty}^{+\infty} f\,g\,\mathrm{d}\mu_0 + \lambda \int_{-\infty}^{+\infty} f'\,g'\,\mathrm{d}\mu_1. \tag{31}$$

It is assumed that $\lambda \ge 0$. Completion of $\mathcal{P}$ with respect to the norm

$$\|\cdot\|_\lambda := \langle\cdot,\cdot\rangle_\lambda^{1/2}$$

leads to the appropriate Sobolev space of functions. Notice that (31) can be rewritten as

$$\langle f, g\rangle_\lambda = \langle f, g\rangle_0 + \lambda\langle f', g'\rangle_1,$$

where $\langle\cdot,\cdot\rangle_0$ and $\langle\cdot,\cdot\rangle_1$ are the inner products induced by $\mathrm{d}\mu_0$ and $\mathrm{d}\mu_1$, i.e.,

$$\langle f, g\rangle_i = \int_{-\infty}^{+\infty} f\,g\,\mathrm{d}\mu_i, \quad i = 0, 1. \tag{32}$$

Let $(S_n^\lambda)_n$ be the monic Sobolev OPS for the Sobolev inner product $\langle\cdot,\cdot\rangle_\lambda$, obtained by applying the Gram–Schmidt process to the canonical basis of $\mathcal{P}$. Our next result extends to (M, N)-coherent pairs, a fundamental algebraic property known for coherent, generalized coherent and k-coherent pairs of measures, as stated in [5,6,8,9].

Theorem 5.1. *Let* $(\mathrm{d}\mu_0, \mathrm{d}\mu_1)$ *be an* (M, N)*-coherent pair of measures, so that* (13) *holds for all* $n = 0, 1, \ldots$*. Set* $K := \max\{M, N\}$*. Then there exist numbers* $t_{1,n}^{\lambda}, \ldots, t_{K,n}^{\lambda}$ *such that*

$$Q_{n+1}(x) + \sum_{j=1}^{M} s_{j,n} \frac{n+1}{n-j+1} Q_{n-j+1}(x) = S_{n+1}^{\lambda}(x) + \sum_{j=1}^{K} t_{j,n}^{\lambda} S_{n-j+1}^{\lambda}(x) \tag{33}$$

for all $n = 0, 1, 2, \ldots$*, with*

$$t_{j,n}^{\lambda} = 0 \quad \text{if } n < j \le K. \tag{34}$$

Furthermore, the following holds:

(i) *if* $N \neq M$ *then* $t_{K,n}^{\lambda} \neq 0$ *for all* $n \ge K$*;*
(ii) *if* $N = M (= K)$ *then* $t_{K,n}^{\lambda} \neq 0$ *for all* $n \ge K$ *if and only if* λ *satisfies*

$$\lambda \neq -\frac{s_{K,n}}{r_{K,n}} \frac{\|Q_{n-K+1}\|_0^2}{(n-K+1)^2 \, \|P_{n-K}\|_1^2} \quad \text{for all } n \ge K. \tag{35}$$

Proof. The proof is similar to the proof of [9, Proposition 1]. It is well known that the coefficients of any Sobolev OP S_n^{λ} are rational functions of λ, where the numerator and the denominator in each of these rational functions are polynomials in λ of the same degree; hence there exists

$$R_n(x) := \lim_{\lambda \to +\infty} S_n^{\lambda}(x), \quad n = 0, 1, 2, \ldots.$$

For each n, R_n is a polynomial of degree n. Further [26]

$$\langle R_n, 1 \rangle_0 = 0, \quad n \ge 1, \tag{36}$$

$$\langle R_{n+1}', x^m \rangle_1 = 0, \quad 0 \le m \le n-1, \quad n \ge 1. \tag{37}$$

The last equality may be justified as follows. Taking into account the relation

$$\left\langle S_{n+1}^{\lambda}, \frac{x^{m+1}}{m+1} \right\rangle_0 + \lambda \left\langle \left(S_{n+1}^{\lambda}\right)', x^m \right\rangle_1 = \left\langle S_{n+1}^{\lambda}, \frac{x^{m+1}}{m+1} \right\rangle_{\lambda} = 0, \quad 0 \le m \le n-1,$$

we obtain

$$\left\langle \left(S_{n+1}^{\lambda}\right)', x^m \right\rangle_1 = -\frac{1}{\lambda} \left\langle S_{n+1}^{\lambda}, \frac{x^{m+1}}{m+1} \right\rangle_0, \quad 0 \le m \le n-1,$$

and so we deduce

$$\langle R_{n+1}', x^m \rangle_1 = -\lim_{\lambda \to +\infty} \frac{1}{\lambda} \left\langle S_{n+1}^{\lambda}, \frac{x^{m+1}}{m+1} \right\rangle_0 = 0, \quad 0 \le m \le n-1.$$

From (37) we get

$$R_{n+1}' = (n+1) P_n, \quad n \ge 1. \tag{38}$$

Therefore (13) can be rewritten as

$$\frac{R_{n+1}'}{n+1} + \sum_{i=1}^{N} r_{i,n} \frac{R_{n-i+1}'}{n-i+1} = \frac{Q_{n+1}'}{n+1} + \sum_{j=1}^{M} s_{j,n} \frac{Q_{n-j+1}'}{n-j+1}$$

for all $n = 0, 1, 2, \ldots$. Integrating both sides of this equality and taking into account (36) we find

$$\frac{R_{n+1}}{n+1} + \sum_{i=1}^{N} r_{i,n} \frac{R_{n-i+1}}{n-i+1} = \frac{Q_{n+1}}{n+1} + \sum_{j=1}^{M} s_{j,n} \frac{Q_{n-j+1}}{n-j+1} \tag{39}$$

for all $n = 0, 1, 2, \ldots$. Now, the Fourier expansion of the left-hand side of (39) in the basis $\{S_{\nu}^{\lambda}\}_{\nu=0}^{n+1}$ gives

$$\frac{R_{n+1}}{n+1} + \sum_{i=1}^{N} r_{i,n} \frac{R_{n-i+1}}{n-i+1} = \frac{S_{n+1}^{\lambda}}{n+1} + \sum_{k=1}^{n+1} \frac{t_{k,n}^{\lambda}}{n+1} S_{n-k+1}^{\lambda},$$

for all $n = 0, 1, 2, \ldots$, where the numbers $t^{\lambda}_{k,n}$ $(k = 1, \ldots, n+1)$ are given by

$$\|S^{\lambda}_{n-k+1}\|^2_{\lambda} \frac{t^{\lambda}_{k,n}}{n+1} = \left\langle \frac{R_{n+1}}{n+1} + \sum_{i=1}^{N} r_{i,n} \frac{R_{n-i+1}}{n-i+1}, S^{\lambda}_{n-k+1} \right\rangle_{\lambda}. \tag{40}$$

From (39), (37) and (38) we deduce

$$\|S^{\lambda}_{n-k+1}\|^2_{\lambda} \frac{t^{\lambda}_{k,n}}{n+1} = \sum_{j=1}^{M} \frac{s_{j,n}}{n-j+1} \left\langle Q_{n-j+1}, S^{\lambda}_{n-k+1} \right\rangle_0 + \lambda \sum_{i=1}^{N} r_{i,n} \left\langle P_{n-i}, \left(S^{\lambda}_{n-k+1}\right)' \right\rangle_1 \tag{41}$$

for all $n = 0, 1, 2, \ldots$ and $k = 1, \ldots, n+1$. Therefore, for fixed n, we derive $t^{\lambda}_{k,n} = 0$ for all $k > K$. This proves (33) for all $n = 0, 1, 2 \ldots$. Now, starting with (40), which holds for all $n \geq 0$, and taking into account (36) and (38), one easily sees that (34) must hold. Finally, for $k = K := \max\{M, N\}$ we get

$$\frac{t^{\lambda}_{K,n}}{n+1} = \begin{cases} \dfrac{s_{M,n}}{n-M+1} \dfrac{\|Q_{n-M+1}\|_0^2}{\|S^{\lambda}_{n-M+1}\|^2_{\lambda}} & \text{if } N < M \\ \lambda\, r_{N,n}(n-N+1) \dfrac{\|P_{n-N}\|_1^2}{\|S^{\lambda}_{n-N+1}\|^2_{\lambda}} & \text{if } N > M \\ \dfrac{s_{N,n}\,\|Q_{n-N+1}\|_0^2 + \lambda\, r_{N,n}(n-N+1)^2\, \|P_{n-N}\|_1^2}{(n-N+1)\,\|S^{\lambda}_{n-N+1}\|^2_{\lambda}} & \text{if } N = M, \end{cases} \tag{42}$$

from which (i) and (ii) are deduced. □

Remark 5.1. Condition (35) holds, for instance, if $\lambda > 0$ and $s_{K,n} r_{K,n} > 0$ for all $n \geq K$.

6. Computation of the Fourier–Sobolev coefficients

Let us consider the following Sobolev space of smooth functions

$$W^{1,2}[I, \mathrm{d}\mu_0, \mathrm{d}\mu_1] := \left\{ f : I \to \mathbb{R} \mid f \in L^2_{\mu_0}(I),\ f' \in L^2_{\mu_1}(I) \right\},$$

where I is a given open interval. Every function $f \in W^{1,2}[I, \mathrm{d}\mu_0, \mathrm{d}\mu_1]$ generates a Fourier–Sobolev series with respect to the monic Sobolev OPs $(S^{\lambda}_n)_n$,

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f_n}{s_n} S^{\lambda}_n(x), \tag{43}$$

where

$$f_n \equiv f_n(\lambda) := \langle f, S^{\lambda}_n \rangle_{\lambda}, \qquad s_n \equiv s_n(\lambda) := \langle S^{\lambda}_n, S^{\lambda}_n \rangle_{\lambda} = \|S^{\lambda}_n\|^2_{\lambda} \tag{44}$$

for all $n = 0, 1, 2, \ldots$. An efficient algorithm for computing the f_n's and s_n's, in the case when $(\mathrm{d}\mu_0, \mathrm{d}\mu_1)$ is a coherent or a generalized coherent pair of measures, was provided in [5,8]. Here we extend these algorithms to the general situation when $(\mathrm{d}\mu_0, \mathrm{d}\mu_1)$ is an (M, N)-coherent pair of measures. This algorithm is a consequence on the next two results, which show how to compute the sequences $(f_n)_n$ and $(s_n)_n$ in (44); hence the Fourier–Sobolev coefficients $c^{\lambda}_n := f_n/s_n$, for every n. The proofs of these propositions are based on the algebraic property stated in Theorem 5.1.

Theorem 6.1. *The sequence $(f_n)_n$, given by (44), satisfies*

$$f_{n+1} + \sum_{j=1}^{K} t^{\lambda}_{j,n} f_{n-j+1} = u_n \quad (n = 0, 1, 2, \ldots), \tag{45}$$

where $u_n \equiv u^{\lambda}_n(f)$ is defined by

$$u_n := \left\langle f, Q_{n+1} + \sum_{j=1}^{M} \widetilde{s}_{j,n} Q_{n-j+1} \right\rangle_0 + \lambda(n+1) \left\langle f', P_n + \sum_{i=1}^{N} r_{i,n} P_{n-i} \right\rangle_1, \tag{46}$$

with $\widetilde{s}_{j,n} := (n+1)s_{j,n}/(n-j+1)$.

Proof. By Theorem 5.1, we have

$$\langle f, S_{n+1}^{\lambda}\rangle_0 = \left\langle f, Q_{n+1} + \sum_{j=1}^{M} \frac{(n+1)s_{j,n}}{n-j+1} Q_{n-j+1} \right\rangle_0 - \sum_{j=1}^{K} t_{j,n} \langle f, S_{n-j+1}^{\lambda}\rangle_0 \tag{47}$$

for all $n = 0, 1, 2, \ldots$, with the conventions (15) and (34), where, for simplicity, we wrote $t_{j,n}$ instead of $t_{j,n}^{\lambda}$. On the other hand, by (13) and using again Theorem 5.1, we see that

$$(n+1)\left(P_n + \sum_{i=1}^{N} r_{i,n} P_{n-i}\right) = (S_{n+1}^{\lambda})' + \sum_{j=1}^{K} t_{j,n} (S_{n-j+1}^{\lambda})'$$

for all $n = 0, 1, 2, \ldots$. Hence

$$\langle f', (S_{n+1}^{\lambda})'\rangle_1 = (n+1) \left\langle f', P_n + \sum_{i=1}^{N} r_{i,n} P_{n-i} \right\rangle_1 - \sum_{j=1}^{K} t_{j,n} \langle f', (S_{n-j+1}^{\lambda})'\rangle_1 \tag{48}$$

for all $n = 0, 1, 2, \ldots$. From (47) and (48) we obtain (45). □

Notice that (45) is a non-homogeneous linear difference equation of order K with variable coefficients. Therefore, the f_n's may be computed recursively, provided we know how to compute the $t_{j,n}^{\lambda}$'s. We remark that the u_n's are known, since they are directly computed in terms of the data (the function f, the parameter λ, and the (M, N)-coherence relation (13)). Next we show that the $t_{j,n}$'s, together with the s_n's, satisfy a system of $K + 1$ difference equations from which they can be computed (hence also the f_n's, and so *a fortiori* the Fourier–Sobolev coefficients). We make the convention

$$t_{0,n}^{\lambda} = 1 \quad \text{if } n \geq 0. \tag{49}$$

Theorem 6.2. *The relations*

$$s_{n-K+\ell+1}\, t_{K-\ell,n}^{\lambda} + \sum_{i=1}^{\ell} t_{i,n-K+\ell}^{\lambda}\, t_{K-\ell+i,n}^{\lambda} s_{n-K+\ell-i+1} = c_{\ell,n} \tag{50}$$

hold for all $\ell = 0, 1, \ldots, K$ *and* $n = 0, 1, 2, \ldots$, *where*

$$c_{\ell,n} := \sum_{i=K-\ell}^{M} \widetilde{s}_{i,n} \widetilde{s}_{i-K+\ell,n-K+\ell} \|Q_{n-i+1}\|_0^2 + \lambda(n+1)(n-K+\ell+1) \sum_{i=K-\ell}^{N} r_{i,n} r_{i-K+\ell,n-K+\ell} \|P_{n-i}\|_1^2 \tag{51}$$

and $\widetilde{s}_{i,n} := (n+1)s_{i,n}/(n-i+1)$.

Proof. Setting $k = K - \ell$ in (41) we obtain

$$s_{n-K+\ell+1}\, t_{K-\ell,n}^{\lambda} = \sum_{j=K-\ell}^{M} \widetilde{s}_{j,n} \langle Q_{n-j+1}, S_{n-K+\ell+1}^{\lambda}\rangle_0 + \lambda \sum_{j=K-\ell}^{N} (n+1) r_{j,n} \langle P_{n-j}, (S_{n-K+\ell+1}^{\lambda})'\rangle_1 \tag{52}$$

for all $\ell = 0, 1, \ldots, K$. Using the definition (13) of (M, N)-coherence and Theorem 5.1, from (52) we deduce, after straightforward computations, that

$$\begin{aligned} s_{n-K+\ell+1}\, t_{K-\ell,n}^{\lambda} = & \sum_{j=K-\ell}^{M} \sum_{i=0}^{M} \widetilde{s}_{j,n} \widetilde{s}_{i,n-K+\ell} \langle Q_{n-j+1}, Q_{n-K+\ell-i+1}\rangle_0 \\ & - \sum_{j=K-\ell}^{M} \sum_{i=1}^{K} \widetilde{s}_{j,n} t_{i,n-K+\ell}^{\lambda} \langle Q_{n-j+1}, S_{n-K+\ell-i+1}^{\lambda}\rangle_0 \\ & + \lambda(n+1)(n-K+\ell+1) \sum_{j=K-\ell}^{N} \sum_{i=0}^{N} r_{j,n} r_{i,n-K+\ell} \langle P_{n-j}, P_{n-K+\ell-i}\rangle_1 \\ & - \lambda(n+1) \sum_{j=K-\ell}^{N} \sum_{i=1}^{K} r_{j,n} t_{i,n-K+\ell}^{\lambda} \langle P_{n-j}, (S_{n-K+\ell-i+1}^{\lambda})'\rangle_1 \end{aligned} \tag{53}$$

for all $\ell = 0, 1, \ldots, K$. Next, the orthogonality implies that the first and the third terms on the right-hand side of (53) are equal to

$$\sum_{j=K-\ell}^{M} \widetilde{s}_{j,n}\widetilde{s}_{j-K+\ell,n-K+\ell}\|Q_{n-j+1}\|_0^2$$

and

$$\lambda(n+1)(n-K+\ell+1)\sum_{j=K-\ell}^{N} r_{j,n}r_{j-K+\ell,n-K+\ell}\|P_{n-j}\|_1^2.$$

Furthermore, the second term on the right-hand side of (53) is equal to

$$\sum_{i=1}^{\ell} t^{\lambda}_{i,n-K+\ell}\sum_{j=K-\ell+i}^{M} \widetilde{s}_{j,n}\langle Q_{n-j+1}, S^{\lambda}_{n-K+\ell+1-i}\rangle_0.$$

To prove this, notice that $\langle Q_{n-j+1}, S^{\lambda}_{n-K+\ell-i+1}\rangle_0 = 0$ if $j < K-\ell+i$ or if $i > M-K+\ell$ (for all $\ell \in \{0, \ldots, K\}$, $i \in \{1, \ldots, K\}$ and $j \in \{K-\ell, \ldots, M\}$); hence the second term in (53) is, in fact, equal to

$$\sum_{i=1}^{M-K+\ell} t^{\lambda}_{i,n-K+\ell}\sum_{j=K-\ell+i}^{M} \widetilde{s}_{j,n}\langle Q_{n-j+1}, S^{\lambda}_{n-K+\ell-i+1}\rangle_0 = \sum_{i=1}^{\ell} t^{\lambda}_{i,n-K+\ell}\sum_{j=K-\ell+i}^{M} \widetilde{s}_{j,n}\langle Q_{n-j+1}, S^{\lambda}_{n-K+\ell-i+1}\rangle_0,$$

where the last equality follows from

$$\sum_{i=M-K+\ell+1}^{\ell} t^{\lambda}_{i,n-K+\ell}\sum_{j=K-\ell+i}^{M} \widetilde{s}_{j,n}\langle Q_{n-j+1}, S^{\lambda}_{n-K+\ell-i+1}\rangle_0 = 0.$$

This relation can be stated by distinguishing the two possible cases $K = M$ and $K = N$. In the same way we can show that the fourth term on the right-hand side of (53) is equal to

$$\sum_{i=1}^{\ell} t^{\lambda}_{i,n-K+\ell}\sum_{j=K-\ell+i}^{N} r_{j,n}\langle P_{n-j}, (S^{\lambda}_{n-K+\ell+1-i})'\rangle_1.$$

It follows that (53) can be rewritten as

$$\begin{aligned} s_{n-K+\ell+1}\, t^{\lambda}_{K-\ell,n} = {} & \sum_{i=K-\ell}^{M} \widetilde{s}_{i,n}\widetilde{s}_{i-K+\ell,n-K+\ell}\|Q_{n-i+1}\|_0^2 + \lambda(n+1)(n-K+\ell+1)\sum_{i=K-\ell}^{N} r_{i,n}r_{i-K+\ell,n-K+\ell}\|P_{n-i}\|_1^2 \\ & - \sum_{i=1}^{\ell} t^{\lambda}_{i,n-K+\ell}\left(\sum_{j=K-\ell+i}^{M} \widetilde{s}_{j,n}\langle Q_{n-j+1}, S^{\lambda}_{n-K+\ell+1-i}\rangle_0 + \lambda(n+1)\sum_{j=K-\ell+i}^{N} r_{j,n}\langle P_{n-j}, (S^{\lambda}_{n-K+\ell+1-i})'\rangle_1\right). \end{aligned}$$

According to (52), the expression between the large brackets in the last sum is equal to $t^{\lambda}_{K-\ell+i,n}s_{n-K+\ell-i+1}$. Thus, we arrive at the desired result. □

Remark 6.1. Taking $\ell = K$ in (50) and setting $t_{j,n} \equiv t^{\lambda}_{j,n}$ we find that $(s_n)_n$ satisfies the non-homogeneous linear difference equation of variable coefficients of order K

$$s_{n+1} + \sum_{j=1}^{K} t^2_{j,n}s_{n+1-j} = c_n \quad (n = 0, 1, 2, \ldots), \tag{54}$$

where

$$c_n \equiv c_{K,n} := \sum_{i=0}^{M} \widetilde{s}^2_{i,n}\|Q_{n+1-i}\|_0^2 + \lambda(n+1)^2\sum_{i=0}^{N} r^2_{i,n}\|P_{n-i}\|_1^2 > 0.$$

We may now state bounds for the norm of the Sobolev polynomial S^{λ}_n arising from an (M, N)-coherent pair of measures $(d\mu_0, d\mu_1)$. In fact, setting

$$k_n := \|P_n\|_1^2, \qquad k'_n := \|Q_n\|_0^2; , \qquad s_n := \|S^{\lambda}_n\|_{\lambda}^2,$$

we may state the following.

Corollary 6.3. *The inequalities*

$$k'_n + \lambda n^2 k_{n-1} \le s_n \le \sum_{j=0}^{M} \widetilde{s}^{\,2}_{j,n-1} k'_{n-j} + \lambda n^2 \sum_{j=0}^{N} r^2_{j,n-1} k_{n-1-j} \tag{55}$$

hold for all $n = 0, 1, 2, \ldots$, *with* $\widetilde{s}_{j,n-1} = n s_{j,n-1}/(n-j)$ *for all* j *and* n.

Proof. The first inequality in (55) follows from the extremal property for OPs (we argue as in the proof of [26, Theorem 2]; cf. also [27, Theorem 3.1.2]):

$$s_n = \|S^\lambda_n\|^2_0 + \lambda \|(S^\lambda_n)'\|^2_1 \ge \|Q_n\|^2_0 + \lambda n^2 \|P_{n-1}\|^2_1.$$

On the other hand, since $c_n > 0$ for all n, from (54) we deduce $s_{n+1} \le c_n$ for all n. Thus, after changing n into $n-1$, the second inequality in (55) follows, taking into account that the right-hand side of (55) equals c_{n-1}. □

Making the change of variables $\ell = K - j$ in Eq. (50) and then, in the resulting formula, changing n into $n+j-1$, we find

$$s_n\, t^\lambda_{j,n+j-1} = c_{K-j,n+j-1} - \sum_{i=1}^{K-j} t^\lambda_{i,n-1} t^\lambda_{i+j,n-1+j} s_{n-i} \tag{56}$$

for all $j = 0, 1, \ldots, K$ and $n = 0, 1, 2, \ldots$. Relations (56) are the basis of the construction of the following algorithm to compute all the Sobolev norms s_n's as well as all the linking coefficients $t^\lambda_{j,n}$'s in the algebraic relation (33), for all $j = 0, 1, \ldots, K$ and $n = 0, 1, 2, \ldots$. As a consequence, and taking into account Theorem 6.1, the algorithm allows us to compute the Fourier–Sobolev coefficients $c^\lambda_n := f_n/s_n$ appearing in (43).

Algorithm 6.1. The Fourier–Sobolev coefficients appearing in (43) may be computed using the following algorithm:

- *starting data*. We first write down the initial conditions

  $$t_{0,n} := 1, \qquad t_{j,n} := 0 \quad \text{if } j > K \text{ or } n < j \le K \ (n = 0, 1, 2, \ldots).$$

- *step* 1. Using the starting data and taking $n = 1$ in (56), we compute s_1 and the diagonal elements $t_{j,j}$ for all $j = 1, \ldots, K$. In fact, we find $s_1 t_{j,j} = c_{K-j,j}$ for all $j = 0, 1, \ldots, K$, so that

  $$\begin{aligned} s_1 &= c_{K,0} = \|Q_1\|^2_0 + \lambda \|P_0\|^2_1, \\ t_{j,j} &= c_{K-j,j}/s_1, \quad j = 1, 2, \ldots, K. \end{aligned} \tag{57}$$

- *step* 2. Using the starting data, the information given in step 1 and taking $n = 2$ in (56), we compute

  $$\begin{aligned} s_2 &= c_{K,1} - t^2_{1,1} s_1, \\ t_{j,j+1} &= \left(c_{K-j,j+1} - t_{1,1} t_{j+1,j+1} s_1\right)/s_2, \quad j = 1, 2, \ldots, K. \end{aligned} \tag{58}$$

- *step* 3. Using the starting data, the information in steps 1 and 2, and taking $n = 3$ in (56), we compute

  $$\begin{aligned} s_3 &= c_{K,2} - t^2_{1,2} s_2 - t^2_{2,2} s_1, \\ t_{j,j+2} &= \left(c_{K-j,j+2} - t_{1,2} t_{j+1,j+2} s_2 - t_{2,2} t_{j+2,j+2} s_1\right)/s_3, \quad j = 1, \ldots, K. \end{aligned} \tag{59}$$

- Go on until *step* r: For any fixed positive integer number r, using the information obtained until step $r-1$ and taking $n = r$ in (56), we compute s_r and $t_{j,j+r-1}$ for all $j = 1, \ldots, K$.
- *last steps*. Steps 1 to r give s_n and $t_{j,j+n-1}$ for all $j = 1, \ldots, K$ and $n = 1, 2, \ldots, r$. Since r may be chosen arbitrarily, then the parameters s_n and $t_{j,n}$ may be computed for all $j = 1, \ldots, K$ and $n = 1, 2, \ldots$. Hence the f_n's may be computed recurrently from (45), for all $n = 1, 2, \ldots$. This gives all the Fourier–Sobolev coefficients f_n/s_n appearing in (43).

Remark 6.2. For coherent pairs, Iserles et al. [5] remarked that "to evaluate the Fourier–Sobolev coefficients there is absolutely no need whatsoever to form Sobolev-orthogonal polynomials S^λ_n explicitly!". The previous algorithm shows that a similar observation holds for (M, N)-coherent pairs.

7. Some special cases

In this section, we apply Algorithm 6.1 to some special cases, namely, when (M, N) is one of the pairs $(1, 0)$, $(1, 1)$, $(2, 0)$ or $(2, 1)$.

7.1. (1, 1)*-coherence*

In this case $N = M = K = 1$, and relations (13) and (33) reduce to

$$P_n + r_{1,n}P_{n-1} = \frac{Q'_{n+1}}{n+1} + s_{1,n}\frac{Q'_n}{n}, \tag{60}$$

$$Q_{n+1} + \widetilde{s}_{1,n}Q_n = S^{\lambda}_{n+1} + t_{1,n}S^{\lambda}_n \tag{61}$$

for all $n = 0, 1, 2, \ldots$, being

$$\widetilde{s}_{1,0} = 0, \qquad \widetilde{s}_{1,n} := s_{1,n}\,\frac{n+1}{n} \quad (n = 1, 2, 3, \ldots). \tag{62}$$

Thus, after straightforward computations, from (56) we obtain the following system of difference equations

$$\begin{cases} t_{1,n}s_n = a_n \\ s_{n+1} = b_n - t_{1,n}a_n \end{cases} \tag{63}$$

for all $n = 0, 1, 2, \ldots$, where $a_n \equiv a_n(\lambda)$ and $b_n \equiv b_n(\lambda)$ are defined by

$$\begin{aligned} a_n &:= \widetilde{s}_{1,n}\|Q_n\|_0^2 + \lambda\, n(n+1)r_{1,n}\|P_{n-1}\|_1^2 \\ b_n &:= \|Q_{n+1}\|_0^2 + \widetilde{s}^{\,2}_{1,n}\|Q_n\|_0^2 + \lambda(n+1)^2\left(\|P_n\|_1^2 + r^2_{1,n}\|P_{n-1}\|_1^2\right). \end{aligned} \tag{64}$$

In order to solve (63), notice that the first equation in (63) gives $t_{1,n} = a_n/s_n$, and substituting this into the second equation we get

$$s_{n+1} - b_n + \frac{a_n^2}{s_n} = 0, \quad n = 0, 1, 2, \ldots. \tag{65}$$

This gives all the s_n's recurrently and hence also all the $t_{1,n}$'s. Then the Fourier–Sobolev coefficients can be computed. However, we can say something more. Motivated by the theory of continued fractions, we introduce the MOPS $(\pi_n)_n \equiv (\pi_n(\cdot;\lambda))_n$ characterized by the three-term recurrence relation

$$\pi_{n+1}(x;\lambda) = (x + b_n)\pi_n(x;\lambda) - a_n^2\pi_{n-1}(x;\lambda), \quad n = 0, 1, 2, \ldots, \tag{66}$$

with initial conditions $\pi_{-1}(x;\lambda) = 0$ and $\pi_0(x;\lambda) = 1$, where $a_n \equiv a_n(\lambda)$ and $b_n \equiv b_n(\lambda)$ are given by (64). Notice that, when a_n and b_n are real numbers, then $a_n^2 > 0$ for all $n = 1, 2, \ldots$; hence $(\pi_n)_n$ is orthogonal with respect to some positive Borel measure. Thus we may state the following proposition:

Theorem 7.1. *Let* $((Q_n), (P_n))$ *be a* (1, 1)*-coherent pair, so that* (60) *holds, and assume that* $(P_n)_n$ *and* $(Q_n)_n$ *are orthogonal w.r.t. positive Borel measures* $d\mu_1$ *and* $d\mu_0$*. Consider the associated Sobolev inner product* (31) *and let* (S^{λ}_n) *be the corresponding Sobolev OPS. Then the following holds:*

(i) *The (square of the) Sobolev-norm of* S^{λ}_n *is given by*

$$s_n = \|S^{\lambda}_n\|^2_{\lambda} = \frac{\pi_n(0;\lambda)}{\pi_{n-1}(0;\lambda)} \quad (n = 1, 2, \ldots), \tag{67}$$

where $(\pi_n)_n$ *is a MOPS, orthogonal w.r.t. some positive Borel measure, characterized by the three-term recurrence relation* (66)*, the recurrence coefficients* $a_n \equiv a_n(\lambda)$ *and* $b_n \equiv b_n(\lambda)$ *being given by* (64)*. Further, setting*

$$k_n := \|P_n\|_1^2, \qquad k'_n := \|Q_n\|_0^2,$$

we have that s_n *satisfies the inequalities*

$$k'_n + \lambda n^2 k_{n-1} \le s_n \le k'_n + \widetilde{s}^{\,2}_{1,n-1}k'_{n-1} + \lambda n^2\left(k_{n-1} + r^2_{1,n-1}k_{n-2}\right) \tag{68}$$

for all $n = 1, 2, 3, \ldots.$

(ii) *Relation* (61) *holds with*

$$t_{1,n} = a_n\,\frac{\pi_{n-1}(0;\lambda)}{\pi_n(0;\lambda)} \quad (n = 0, 1, 2, \ldots). \tag{69}$$

(iii) *Let* $f \in W^{1,2}[I, d\mu_0, d\mu_1]$*. Having determined* s_n *and* $t_{1,n}$ *from* (67) *and* (69)*, then the Fourier–Sobolev coefficients in the expansion*

$$f(x) \sim \sum_{n=0}^{\infty}\frac{f_n}{s_n}\,S^{\lambda}_n(x) \tag{70}$$

can be computed recurrently by solving the first order linear difference equation

$$f_{n+1} + t_{1,n} f_n = u_n \quad (n = 0, 1, 2, \ldots), \tag{71}$$

where $u_n \equiv u_n(\lambda)$ *is defined by*

$$u_n := \langle f, Q_{n+1} + \widetilde{s}_{1,n} Q_n \rangle_0 + \lambda\,(n+1)\,\langle f', P_n + r_{1,n} P_{n-1} \rangle_1 \tag{72}$$

for all $n = 0, 1, 2, \ldots$, *and* $(\widetilde{s}_{1,n})_n$ *is given by* (62).

7.2. Coherence

As remarked before, coherence is $(1, 0)$-coherence. In this case $N = 0$ and $M = K = 1$, and relations (13) and (33) reduce to

$$P_n = \frac{Q'_{n+1}}{n+1} + s_{1,n} \frac{Q'_n}{n}, \qquad Q_{n+1} + \widetilde{s}_{1,n}\, Q_n = S^{\lambda}_{n+1} + t_{1,n} S^{\lambda}_n$$

for all $n = 0, 1, 2, \ldots$, being $(\widetilde{s}_{1,n})_n$ as in (62). In this case one can easily state a result similar to Theorem 7.1. In fact, such a result follows immediately by taking (formally) $r_{1,n} = 0$ in all formulas in Theorem 7.1. This gives the results stated in [5, Section 6]. We notice that, taking $r_{1,n} = 0$ in (68), we obtain the inequalities for the Sobolev norm stated in [26, Theorem 2] for coherent pairs. Further, when we take $r_{1,n} = 0$ in (64) we see that a_n is independent of λ and $b_n \equiv b_n(\lambda)$ becomes a linear function of λ. Therefore, for any fixed x, $\pi_n(x; \lambda)$ can be regarded as a polynomial in λ of degree n. In particular,

$$\pi_{n+1}(0; \lambda) = b_n(\lambda)\pi_n(0; \lambda) - a_n^2 \pi_{n-1}(0; \lambda),$$

$$b_n(\lambda) = (n+1)^2 \|P_n\|_1^2 (\lambda + c_n), \quad c_n := \frac{\|Q_{n+1}\|_0^2 + \widetilde{s}^2_{1,n} \|Q_n\|_0^2}{(n+1)^2 \|P_n\|_1^2} > 0.$$

Therefore, setting

$$\widetilde{\gamma}_n(\lambda) := \pi_n(0; \lambda)/\epsilon_n, \quad \epsilon_n := (n!)^2 \prod_{j=0}^{n-1} \|P_j\|_1^2 \; (n = 1, 2, \ldots),$$

$(\widetilde{\gamma}_n(\lambda))_n$ becomes a MOPS in the variable λ, satisfying

$$\widetilde{\gamma}_{n+1}(\lambda) = (\lambda + c_n)\widetilde{\gamma}_n(\lambda) - d_n \widetilde{\gamma}_{n-1}(\lambda)$$

for all $n = 0, 1, 2, \ldots$, with initial conditions $\widetilde{\gamma}_{-1}(\lambda) = 0$ and $\widetilde{\gamma}_0(\lambda) = 1$, and

$$d_n := \frac{s^2_{1,n}\, \|Q_n\|_0^4}{n^4\, \|P_n\|_1^2\, \|P_{n-1}\|_1^2} > 0.$$

Hence $(\widetilde{\gamma}_n(\lambda))_n$ is a MOPS with respect to some positive Borel measure. $(\widetilde{\gamma}_n(\lambda))_n$ is the OPS introduced in the proof of Theorem 1 in [28] to show that

$$t_{1,n} = \frac{\gamma_{n-1}(\lambda)}{\gamma_n(\lambda)} \quad (n = 0, 1, 2, \ldots), \tag{73}$$

$(\gamma_n(\lambda))_n$ being an OPS defined as an appropriate normalization of $(\widetilde{\gamma}_n(\lambda))_n$, say $\widetilde{\gamma}_n(\lambda) = \theta_n \gamma_n(\lambda)$, with $(\theta_n)_n$ a sequence of nonzero real numbers. Indeed, if $(\theta_n)_n$ is chosen so that $\theta_0 = 1$ and $\theta_n/\theta_{n-1} = a_n \epsilon_{n-1}/\epsilon_n$ for all $n = 0, 1, 2, \ldots$, then we see that (73) follows from (69), implying Theorem 1 in [28].

7.3. $(2, 1)$-coherence

In this case, $N = 1$ and $M = K = 2$, so relations (13) and (33) reduce to

$$P_n + r_{1,n} P_{n-1} = \frac{Q'_{n+1}}{n+1} + s_{1,n} \frac{Q'_n}{n} + s_{2,n} \frac{Q'_{n-1}}{n-1}, \tag{74}$$

$$Q_{n+1} + \widetilde{s}_{1,n} Q_n + \widetilde{s}_{2,n} Q_{n-1} = S^{\lambda}_{n+1} + t_{1,n} S^{\lambda}_n + t_{2,n} S^{\lambda}_{n-1} \tag{75}$$

for all $n = 0, 1, 2, \ldots$, where

$$\begin{aligned} &\widetilde{s}_{1,0} = \widetilde{s}_{2,0} = \widetilde{s}_{2,1} = 0, \\ &\widetilde{s}_{1,n} := s_{1,n} \frac{n+1}{n} \quad \text{if } n \geq 1, \qquad \widetilde{s}_{2,n} := s_{2,n} \frac{n+1}{n-1} \quad \text{if } n \geq 2. \end{aligned} \tag{76}$$

Hence, applying (56), after straightforward computations we obtain

$$\begin{cases} t_{2,n}s_{n-1} = a_n \\ t_{1,n}s_n = b_n - a_n t_{1,n-1} \\ s_{n+1} = c_n - b_n t_{1,n} + a_n(t_{1,n}t_{1,n-1} - t_{2,n}) \end{cases} \tag{77}$$

for all $n = 0, 1, 2, \ldots$, where a_n, $b_n \equiv b_n(\lambda)$ and $c_n \equiv c_n(\lambda)$ are defined by

$$\begin{aligned} a_n &:= \widetilde{s}_{2,n}\, \|Q_{n-1}\|_0^2 \\ b_n &:= \widetilde{s}_{1,n}\|Q_n\|_0^2 + \widetilde{s}_{2,n}\widetilde{s}_{1,n-1}\|Q_{n-1}\|_0^2 + \lambda n(n+1)r_{1,n}\|P_{n-1}\|_1^2 \\ c_n &:= \|Q_{n+1}\|_0^2 + \widetilde{s}^2_{1,n}\|Q_n\|_0^2 + \widetilde{s}^2_{2,n}\|Q_{n-1}\|_0^2 + \lambda(n+1)^2\left(\|P_n\|_1^2 + r^2_{1,n}\|P_{n-1}\|_1^2\right). \end{aligned} \tag{78}$$

We now assume $t_{1,n} \neq 0$ for all $n \geq 1$. Then, from the second equations in (77) we see that also $b_n - a_n t_{1,n-1} \neq 0$ for all $n \geq 1$. Hence from the first and the second equations in (77) we may write $t_{2,n} = a_n/s_{n-1} = a_n t_{1,n-1}/(b_{n-1} - a_{n-1}t_{1,n-2})$ and $s_{n+1} = (b_{n+1} - a_{n+1}t_{1,n})/t_{1,n+1}$. Replacing these expressions in the third equation in (77) we obtain

$$t_{1,n+1} = \frac{\left(b_{n+1} - a_{n+1}t_{1,n}\right)\left(b_{n-1} - a_{n-1}t_{1,n-2}\right)}{\left[c_n - \left(b_n - a_n t_{1,n-1}\right)t_{1,n}\right]\left(b_{n-1} - a_{n-1}t_{1,n-2}\right) - a_n^2 t_{1,n-1}} \tag{79}$$

for all $n = 2, 3, \ldots$. We notice that the denominator on the right-hand side of (79) never vanishes since, using (77), one easily sees that it is equal to $s_{n+1}s_{n-1}t_{1,n-1}$ for all $n \geq 2$. From (79) we determine recurrently all the $t_{1,n}$'s, taking into account the initial conditions

$$t_{1,0} = 0, \qquad t_{1,1} = b_1/s_1, \qquad t_{1,2} = (s_1 b_2 - a_2 b_1)/(s_1 c_1 - b_1^2).$$

Notice that s_1 is computed by using (57). We then determine all the s_n's from the second equation in (77), and then the $t_{2,n}$'s from the first one. Hence, for a given $f \in W^{1,2}[I, d\mu_0, d\mu_1]$, the Sobolev–Fourier coefficients f_n/s_n in the expansion (43) can be computed recurrently using (45) applied to this particular situation, i.e., from the second order linear difference equation

$$f_{n+1} + t_{1,n}f_n + t_{2,n}f_{n-1} = u_n \quad (n = 0, 1, 2, \ldots), \tag{80}$$

where

$$u_n = \langle f, Q_{n+1} + \widetilde{s}_{1,n}Q_n + \widetilde{s}_{2,n}Q_{n-1}\rangle_0 + \lambda\,(n+1)\,\langle f', P_n + r_{1,n}P_{n-1}\rangle_1. \tag{81}$$

Further, since $((Q_n), (P_n))$ is a $(2, 1)$-coherent pair then the Sobolev-norm $s_n := \|S_n^\lambda\|_\lambda^2$ satisfies the inequalities

$$k'_n + \lambda n^2 k_{n-1} \leq s_n \leq k'_n + \widetilde{s}^2_{1,n-1}k'_{n-1} + \widetilde{s}^2_{2,n-1}k'_{n-2} + \lambda n^2\left(k_{n-1} + r^2_{1,n-1}k_{n-2}\right)$$

for all $n = 2, 3, \ldots$, where $k_n := \|P_n\|_1^2$ and $k'_n := \|Q_n\|_0^2$.

7.4. $(2, 0)$-*coherence*

In this case relations (13) and (33) reduce to

$$\begin{aligned} &P_n = \frac{Q'_{n+1}}{n+1} + s_{1,n}\frac{Q'_n}{n} + s_{2,n}\frac{Q'_{n-1}}{n-1}, \\ &Q_{n+1} + \widetilde{s}_{1,n}Q_n + \widetilde{s}_{2,n}Q_{n-1} = S^\lambda_{n+1} + t_{1,n}S^\lambda_n + t_{2,n}S^\lambda_{n-1} \end{aligned} \tag{82}$$

for all $n = 0, 1, 2, \ldots$, with $(\widetilde{s}_{1,n})_n$ and $(\widetilde{s}_{2,n})_n$ as in (76). The Fourier–Sobolev coefficients can be computed as in the $(2, 1)$-coherent case. Indeed, the formulas for the $(2, 0)$-coherent case follow by taking (formally) $r_{1,n} = 0$ in the formulas for the $(2, 1)$-coherent case. This leads to the algorithm presented in [8, Section 5].

8. Fourier–Sobolev series: numerical experiments

In order to illustrate the algorithm presented above, in this section we give some examples involving the construction of the Fourier–Sobolev series with respect to Sobolev inner products of the type (31) where $(d\mu_0, d\mu_1)$ is some concrete (M, N)-coherent pair of measures.

Example 1. Consider the Jacobi weight

$$d\mu^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta\chi_{(-1,1)}(x)\,dx,$$

which is well defined provided $\alpha, \beta > -1$. The corresponding MOPS will be denoted by $(\widehat{P}_n^{(\alpha,\beta)})_n$. Choosing α and β such that $\alpha, \beta > 1$, we may consider measures $d\mu_0$ and $d\mu_1$ as

$$d\mu_0 := d\mu^{\alpha-1,\beta-2}, \qquad d\mu_1 := d\mu^{\alpha-2,\beta} \quad (\alpha, \beta > 1).$$

Table 1
The first 20 Fourier–Sobolev coefficients for function f in Example 1.

n	$t_{1,n}$	$t_{2,n}$	s_n	f_n	f_n/s_n
0	0	0	1	−0.302	−0.302
1	−0.095	0	0.204	0.252	1.235
2	0.015	0.025	0.075	0.001	0.013
3	0.077	0.015	0.023	0.024	1.043
4	0.123	0.011	0.007	0.002	0.286
5	0.159	0.009	0.002	0.004	2
6	0.188	0.007	0.001	4.9×10^{-4}	0.49
7	0.212	0.006	1.6×10^{-4}	7.5×10^{-4}	4.688
8	0.233	0.005	4.6×10^{-5}	1.2×10^{-4}	2.609
9	0.251	4.4×10^{-3}	1.3×10^{-5}	1.5×10^{-4}	11.539
10	0.266	3.8×10^{-3}	3.6×10^{-6}	3.0×10^{-5}	8.333
11	0.280	3.4×10^{-3}	1.0×10^{-6}	3.1×10^{-5}	31
12	0.292	3.0×10^{-3}	2.7×10^{-7}	7.2×10^{-6}	26.667
13	0.302	2.7×10^{-3}	7.6×10^{-8}	6.9×10^{-6}	90.79
14	0.312	2.4×10^{-3}	2.1×10^{-8}	1.7×10^{-6}	80.952
15	0.321	2.2×10^{-3}	5.7×10^{-9}	1.5×10^{-6}	263.158
16	0.329	2.0×10^{-3}	1.5×10^{-9}	4.1×10^{-7}	273.333
17	0.336	1.8×10^{-3}	4.1×10^{-10}	3.4×10^{-7}	829.268
18	0.343	1.6×10^{-3}	1.1×10^{-10}	9.9×10^{-8}	900
19	0.349	1.5×10^{-3}	3.0×10^{-11}	7.8×10^{-8}	2600

Then $(d\mu_0, d\mu_1)$ is a $(2, 1)$-coherent pair of measures, being

$$P_n := \widehat{P}_n^{(\alpha-2,\beta)}, \qquad Q_n := \widehat{P}_n^{(\alpha-1,\beta-2)}.$$

In fact, this follows from the relation (see p. 390 in [21] and compare with (19) therein, taking into account that the relation $(\widehat{P}_{n+1}^{(a,b)})' = (n+1)\widehat{P}_n^{(a+1,b+1)}$ holds for the monic Jacobi polynomials)

$$\widehat{P}_n^{(\alpha-2,\beta)} + r_{1,n}\widehat{P}_{n-1}^{(\alpha-2,\beta)} = \frac{\left(\widehat{P}_{n+1}^{(\alpha-1,\beta-2)}\right)'}{n+1} + s_{1,n}\frac{\left(\widehat{P}_{n}^{(\alpha-1,\beta-2)}\right)'}{n} + s_{2,n}\frac{\left(\widehat{P}_{n-1}^{(\alpha-1,\beta-2)}\right)'}{n-1}$$

$(n = 0, 1, 2, \ldots)$, where

$$r_{1,n} := \frac{2n(n+\alpha-2)}{(2n+\alpha+\beta-3)(2n+\alpha+\beta-2)},$$
$$s_{1,n} := -\frac{4n(n+\beta-1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta-3)},$$
$$s_{2,n} := \frac{4n(n-1)(n+\beta-2)(n+\beta-1)}{(2n+\alpha+\beta-4)(2n+\alpha+\beta-3)^2(2n+\alpha+\beta-2)}.$$

Now, consider the function $f : (-1, 1) \to \mathbb{R}$ defined by

$$f(x) := \frac{\sin(x)}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

Since $f \in L^2_{\mu_0}(-1, 1)$ if $\alpha > 1$ and $\beta > 2$, and $f' \in L^2_{\mu_1}(-1, 1)$ if $\alpha > 4$ and $\beta > 2$, then choosing, for instance, $(\alpha, \beta) = (5, 4)$, the theory developed in the previous sections may be used to determine the Fourier–Sobolev series of f with respect to the Sobolev MOPS associated to the Sobolev inner product (31) defined by the $(2, 1)$-coherent pair

$$(d\mu_0, d\mu_1) \equiv \left(d\mu^{4,2}, d\mu^{3,4}\right).$$

In fact, for this choice, applying Algorithm 6.1, or the results obtained in Section 7.3, we determine the Fourier–Sobolev coefficients of f (with the help of MAPLE), as well as the sequences $(t_{1,n})_n$ and $(t_{2,n})_n$ appearing in (75). In particular, if $\lambda = 0.1$, we obtain the values contained in Table 1 from which we may determine the first 20 Fourier–Sobolev coefficients f_n/s_n.

Example 2. We analyze in an alternative way (using the theory developed in the previous sections) an example presented by Iserles et al. in [29, Section 4]. Let

$$d\mu(x) := \chi_{(-1,1)}(x)\, dx$$

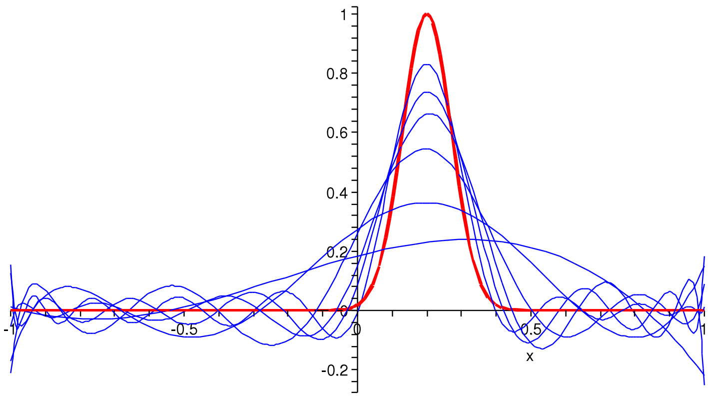

Fig. 1. Graphs of f (bold) and of the partial sums of degrees $n = 3, 6, 9, 12, 15, 18$ of the Fourier–Legendre series of f.

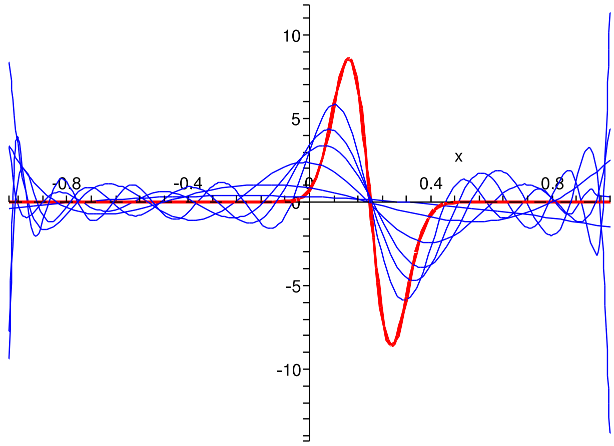

Fig. 2. Graphs of f' (bold) and of the derivatives of the partial sums of degrees $n = 3, 6, 9, 12, 15, 18$ of the Fourier–Legendre series of f.

(the Lebesgue measure supported on $(-1, 1)$, so that the corresponding monic OPs are Legendre polynomials, $P_n := \widehat{P}_n^{(0,0)}$). Then $(d\mu_0, d\mu_1) := (d\mu, d\mu)$ is a $(2, 0)$-coherent pair. This fact is a consequence of the following well known relation for the (monic) Legendre polynomials:

$$P_n = \frac{P'_{n+1}}{n+1} - \frac{nP'_{n-1}}{(2n-1)(2n+1)} \quad (n = 0, 1, 2, \ldots).$$

Let $f : (-1, 1) \to \mathbb{R}$ be the function considered in [29, Section 4], defined by

$$f(x) := e^{-100\left(x-\frac{1}{5}\right)^2}, \quad -1 < x < 1.$$

Obviously, $f \in L^2_{\mu_0}(-1, 1)$ and $f' \in L^2_{\mu_1}(-1, 1)$. Therefore the theory developed in the previous sections applies to find the Fourier–Sobolev series of f w.r.t. the monic Sobolev OPS associated to the Sobolev inner product (31) defined by the $(2, 0)$-coherent pair $(d\mu, d\mu)$. In fact, if we proceed as in Example 1, applying Algorithm 6.1, or the results contained in Section 7.4, we find the Fourier–Sobolev coefficients of f, as well as the sequences $(t_{1,n})_n$ and $(t_{2,n})_n$ appearing in (82). Therefore, we may obtain the four plots contained in [29, Figure 1]: a first plot (Fig. 1) including the graphs of f and of the partial sums of degrees $n = 3, 6, 9, 12, 15, 18$ of the Fourier–Legendre series of f (which coincide with the Fourier–Sobolev series for $\lambda = 0$), in the interval $[-1, 1]$; a second plot (Fig. 2) including the graphs of f' and of the derivatives of the partial sums of degrees $n = 3, 6, 9, 12, 15, 18$ of the Fourier–Legendre series of f and a third (Fig. 3) and a fourth (Fig. 4) plots including the graphical representations corresponding to the previous two ones but w.r.t. the partial sums of the Fourier–Sobolev series for $\lambda = 0.01$.

Remark 8.1. We considered many examples for different choices of λ ("large" and "small") and we found numerical evidences that the results do not change significatively. However, we have not a mathematical explanation for this behavior, so the role that the parameter λ plays in the Fourier–Sobolev series remains open. For instance, Fig. 5 presents a plot of the function difference between the two partial sums of degree 18 for the Fourier–Sobolev series for f corresponding to $\lambda = 0.01$

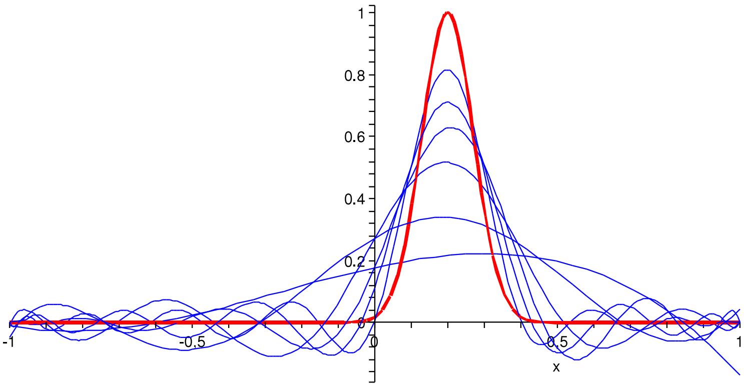

Fig. 3. Graphs of f (bold) and of the partial sums of degrees $n = 3, 6, 9, 12, 15, 18$ of the Fourier–Sobolev series of f for $\lambda = 0.01$.

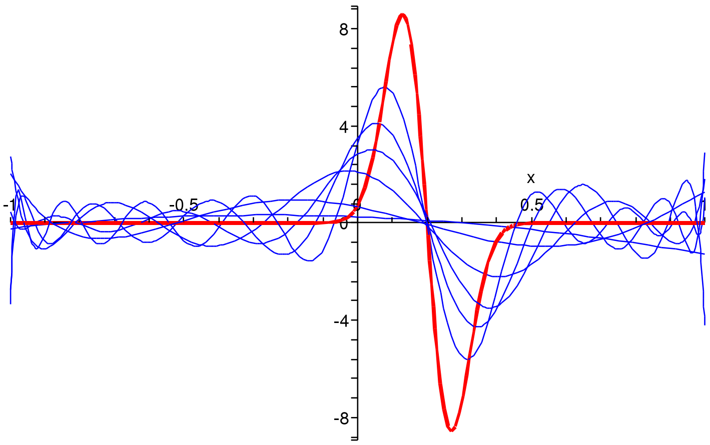

Fig. 4. Graphs of f' (bold) and of the derivatives of the partial sums of degrees $n = 3, 6, 9, 12, 15, 18$ of the Fourier–Sobolev series of f for $\lambda = 0.01$.

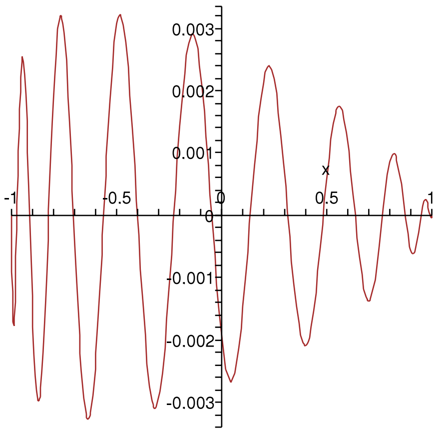

Fig. 5. Graph of the function difference between the partial sums of degree 18 of the two Fourier–Sobolev series for f corresponding to $\lambda = 0.01$ and $\lambda = 10^6$ (respectively).

and $\lambda = 10^6$ (respectively). Similarly, Fig. 6 presents a plot of the function difference between the derivatives of the partial sums of degree 18 of the two Fourier–Sobolev series of f corresponding to the same choices of λ.

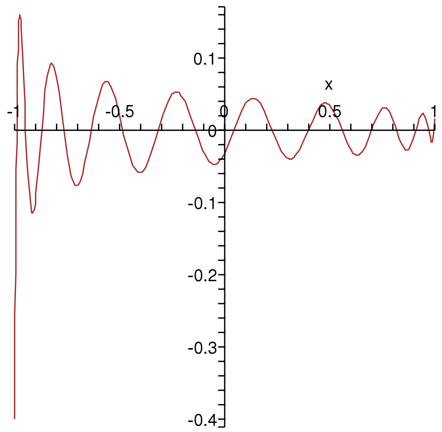

Fig. 6. Graph of the function difference between the derivatives of the partial sums of degree 18 of the two Fourier–Sobolev series of f corresponding to $\lambda = 0.01$ and $\lambda = 10^6$ (respectively).

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