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On linearly related sequences of derivatives of orthogonal polynomials ☆

M.N. de Jesus [a], J. Petronilho [b,*]

[a] Departamento de Matemática, Escola Superior de Tecnologia, Campus Politécnico de Repeses, 3504-510 Viseu, Portugal
[b] CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal

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ABSTRACT

We discuss an inverse problem in the theory of (standard) orthogonal polynomials involving two orthogonal polynomial families $(P_n)_n$ and $(Q_n)_n$ whose derivatives of higher orders m and k (resp.) are connected by a linear algebraic structure relation such as

$$\sum_{i=0}^{N} r_{i,n} P_{n-i+m}^{(m)}(x) = \sum_{i=0}^{M} s_{i,n} Q_{n-i+k}^{(k)}(x)$$

for all $n = 0, 1, 2, \ldots$, where M and N are fixed nonnegative integer numbers, and $r_{i,n}$ and $s_{i,n}$ are given complex parameters satisfying some natural conditions. Let $\mathbf{u}$ and $\mathbf{v}$ be the moment regular functionals associated with $(P_n)_n$ and $(Q_n)_n$ (resp.). Assuming $0 \leqslant m \leqslant k$, we prove the existence of four polynomials Φ_{M+m+i} and Ψ_{N+k+i}, of degrees $M+m+i$ and $N+k+i$ (resp.), such that

$$D^{k-m}(\Phi_{M+m+i}\mathbf{u}) = \Psi_{N+k+i}\mathbf{v} \quad (i = 0, 1),$$

the $(k-m)$th-derivative, as well as the left-product of a functional by a polynomial, being defined in the usual sense of the theory of distributions. If $k = m$, then $\mathbf{u}$ and $\mathbf{v}$ are connected by a rational modification. If $k = m+1$, then both $\mathbf{u}$ and $\mathbf{v}$ are semiclassical linear functionals, which are also connected by a rational modification. When $k > m$, the Stieltjes transform associated with $\mathbf{u}$ satisfies a non-homogeneous linear ordinary differential equation of order $k-m$ with polynomial coefficients.

1. Introduction

This paper deals with a special type of inverse problem in the theory of orthogonal polynomials (OP's). We characterize the regular moment linear functionals $\mathbf{u}$ and $\mathbf{v}$, corresponding to given monic orthogonal polynomial sequences (OPS's), say $(P_n)_n$ and $(Q_n)_n$ (resp.), whose derivatives of orders m and k (resp.) fulfill the following linear algebraic structure relation

$$\sum_{i=0}^{N} r_{i,n} P_{n-i+m}^{(m)}(x) = \sum_{i=0}^{M} s_{i,n} Q_{n-i+k}^{(k)}(x) \tag{1}$$

for all $n \geqslant \max\{M, N\}$, where M and N are fixed nonnegative integer numbers, and $r_{i,n}$ and $s_{i,n}$ are given complex numbers satisfying some natural conditions, with the conventions $r_{i,n} = s_{i,n} = 0$ if $n < i$.

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* Corresponding author.
E-mail addresses: mnasce@mat.estv.ipv.pt (M.N. de Jesus), josep@mat.uc.pt (J. Petronilho).

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The structure relation (1) appears naturally in the framework of the theory of Sobolev OP's, when the above moment linear functionals $\mathbf{u}$ and $\mathbf{v}$ admit integral representations in terms of two positive Borel measures $d\mu_1$ and $d\mu_0$ (resp.) which are taken to define a Sobolev inner product of the form

$$\langle f, g\rangle_S = \int_{-\infty}^{+\infty} fg\, d\mu_0 + \lambda \int_{-\infty}^{+\infty} f'g'\, d\mu_1. \tag{2}$$

Here, it is assumed that $\lambda > 0$ and $d\mu_0$ and $d\mu_1$ are supported on an interval $I \subset \mathbb{R}$ (the support of each of these measures being an infinite set) and with non-vanishing absolutely continuous components. This kind of orthogonality in Sobolev spaces has attracted considerable attention, specially after an important work by A. Iserles, P.E. Koch, S.P. Nørsett, and J.M. Sanz-Serna [8] who demonstrate how this theory can be used for efficient evaluation of Sobolev–Fourier coefficients. For instance, in [9] K.H. Kwon, J.H. Lee, and F. Marcellán studied Sobolev OP's arising from a relation as (1) with $(N, M, m, k) = (0, 2, 0, 1)$, and the case $(N, M, m, k) = (1, 1, 0, 1)$ has been completely studied by A.M. Delgado and F. Marcellán in [7]. We also remark that the situation $m = 0$, $k = 1$ and $N = 0$ (with arbitrary M) was studied in [12] by F. Marcellán, A. Martínez-Finkelshtein and J. Moreno-Balcázar as an extension of the concept of coherence.

Assuming, without loss of generality, that $0 \leqslant m \leqslant k$, we prove the existence of four polynomials Φ_{M+m+i} and Ψ_{N+k+i}, of degrees $M + m + i$ and $N + k + i$ (resp.), such that

$$D^{k-m}(\Phi_{M+m+i}\mathbf{u}) = \Psi_{N+k+i}\mathbf{v}, \quad i = 0, 1,$$

the $(k - m)$th-derivative, as well as the left-product of a functional by a polynomial, being defined in the usual sense of the theory of distributions. If $k = m$, this implies that $\mathbf{u}$ and $\mathbf{v}$ are connected by a rational modification. If $k = m + 1$, then $\mathbf{u}$ and $\mathbf{v}$ are semiclassical linear functionals which are also connected by a rational modification.

The structure of the paper is as follows. In Section 2 we review some basic tools concerning the general theory of OP's, focusing our attention on some topics of the algebraic theory of OP's, as well as on the theory of semiclassical OPS's. In Section 3 we give the main result of the paper, characterizing the moment linear functionals such that the corresponding OPS's fulfill the above (up to normalization) linear structure relation (1). In Section 4 we state a relation between the Stieltjes transforms associated with these moment linear functionals. In particular, when $k > m$, we will show that the Stieltjes transform associated with the moment linear functional with respect to which $(P_n)_n$ is orthogonal satisfies a nonhomogeneous linear ordinary differential equation of degree $k - m$ with polynomial coefficients. Finally, the particular situation $k = m + 1$ is studied in detail in Section 5, since it goes into the theory of semiclassical OPS's. We also give an illustrative example showing that the proofs of our results are constructive.

2. Remarks on semiclassical orthogonal polynomials

In this section we make a review on the so-called semiclassical OPS's, but first let us recall some basic tools concerning algebraic (topological) aspects in the theory of OP's. (See [6,11,14–16,19].)

2.1. Topological aspects

The linear space of polynomials with complex coefficients will be denoted by $\mathcal{P}$. In $\mathcal{P}$ we consider the topology of the strict inductive limit of the sequence $(\mathcal{P}_n)_n$, where $\mathcal{P}_n$ represents the Banach space of all polynomials of degree at most n with the topology induced by the norm

$$\|q\|_n := \sum_{k=0}^{n} |a_k|, \qquad q \equiv \sum_{k=0}^{n} a_k x^k \in \mathcal{P}_n.$$

The topological dual space of $\mathcal{P}$ will be represented by $\mathcal{P}'$ and we consider on it the strong dual topology (which coincides with the weak one). This topology can be also characterized by the system of semi-norms $\{|.|_n\colon n \in \mathbb{N}_0\}$, where

$$|\mathbf{u}|_n = \sup_{0\leqslant m\leqslant n} |u_m|, \quad n = 0, 1, 2, \ldots,$$

being $u_n := \langle \mathbf{u}, x^n\rangle$ the moment of order n of $\mathbf{u}$. Since this system of semi-norms is countable, $\mathcal{P}'$ becomes a metrizable space and this fact can be used to prove the following property:

$$\mathcal{P}' = \mathcal{P}^*,$$

where $\mathcal{P}^*$ denotes the algebraic dual of $\mathcal{P}$. The usefulness of this property comes from the fact that it allows us to give a sense to any expansion of any functional $\mathbf{u} \in \mathcal{P}^*$ as

$$\mathbf{u} = \sum_{n\geqslant 0} \lambda_n \mathbf{a}_n, \quad \lambda_n := \langle \mathbf{u}, q_n\rangle,$$

in the sense of the weak dual topology in $\mathcal{P}'$, where $(q_n)_n$ denotes any simple set of polynomials (i.e., $\deg q_n = n$ for all n) and $(\mathbf{a}_n)_n$ is its corresponding dual basis in $\mathcal{P}'$, so that

$$\langle \mathbf{a}_n, q_m\rangle := \begin{cases} 1, & \text{if } n=m,\\ 0, & \text{if } n\neq m \end{cases} \quad (n,m=0,1,2,\ldots).$$

In the next, we introduce some operations in $\mathcal{P}$ and in $\mathcal{P}'$. Let $\mathbf{u}\in\mathcal{P}'$ and $\phi\in\mathcal{P}$. The left-multiplication of the functional $\mathbf{u}$ by the polynomial ϕ, denoted $\phi\mathbf{u}$, is the functional defined by

$$\langle \phi\mathbf{u}, q\rangle := \langle \mathbf{u}, \phi q\rangle, \quad q\in\mathcal{P}.$$

The right-multiplication of the functional $\mathbf{u}$ by the polynomial ϕ, denoted $\mathbf{u}\phi$, is the polynomial defined by

$$(\mathbf{u}\phi)(x) := \left\langle \mathbf{u}_\xi, \frac{x\phi(x)-\xi\phi(\xi)}{x-\xi}\right\rangle,$$

where the subscript ξ in $\mathbf{u}_\xi$ means that $\mathbf{u}$ acts in functions of the variable ξ. We notice that, setting $\phi(x)=\sum_{i=0}^n c_i x^i$, then

$$(\mathbf{u}\phi)(x) = \sum_{i=0}^n \left(\sum_{j=i}^n c_j u_{j-i}\right) x^i,$$

where $u_\nu := \langle \mathbf{u}, x^\nu\rangle$ is the moment of order ν of $\mathbf{u}$, for ever nonnegative integer number ν. Finally, the (distributional) derivative of the functional $\mathbf{u}$ is the functional $D\mathbf{u}\in\mathcal{P}'$ defined by

$$\langle D\mathbf{u}, q\rangle := -\langle \mathbf{u}, q'\rangle, \quad q\in\mathcal{P}.$$

If $\mathbf{u}\in\mathcal{P}'$ and $(P_n)_n$ is a simple set of polynomials, then $(P_n)_n$ is called an orthogonal polynomial sequence (OPS) with respect to $\mathbf{u}$ if

$$\langle \mathbf{u}, P_nP_m\rangle = \begin{cases} \nu_n, & \text{if } n=m,\\ 0, & \text{if } n\neq m \end{cases} \quad (n,m=0,1,2,\ldots)$$

where $(\nu_n)_n$ is a sequence of nonzero complex numbers. If there exists an OPS with respect to $\mathbf{u}$ then $\mathbf{u}$ is called regular or quasi-definite. Since any monic OPS $(P_n)_n$ is a simple set of polynomials then we can consider the corresponding dual basis, $(\mathbf{a}_n)_n$. It is a very important fact that each element of this dual basis admits the following explicit representation

$$\mathbf{a}_n = \frac{P_n}{\langle \mathbf{u}, P_n^2\rangle}\mathbf{u}, \quad n=0,1,2,\ldots. \tag{3}$$

2.2. Semiclassical orthogonal polynomials

A functional $\mathbf{u}\in\mathcal{P}'$ is called semiclassical if the following two conditions hold:

(i) $\mathbf{u}$ is regular;
(ii) there exist two polynomials ϕ and ψ, with $\deg\psi\geqslant 1$, such that

$$D(\phi\mathbf{u}) = \psi\mathbf{u}. \tag{4}$$

Under such conditions, the class of $\mathbf{u}$ is the (unique) nonnegative integer number s defined by

$$s := \min_{(\phi,\psi)\in\mathcal{A}} \max\{\deg\phi-2, \deg\psi-1\},$$

where $\mathcal{A}$ is the set of all pairs of polynomials (ϕ,ψ), with $\deg\psi\geqslant 1$ satisfying the distributional differential equation (4).

We also say that an OPS associated with a semiclassical linear functional is a semiclassical OPS (of class s, if the class of $\mathbf{u}$ is s).

When $s=0$, $\mathbf{u}$ is called a classical functional. If $(P_n)_n$ is an OPS associated with a classical functional, then it is called a classical OPS. It is well known that up to a linear change of the variables, we obtain Hermite polynomials, $(H_n)_n$, in the case $\phi\equiv$ const; Laguerre polynomials, $(L_n^{(\alpha)})_n$, in the case $\deg\phi=1$; Jacobi polynomials, $(P_n^{(\alpha,\beta)})_n$, in the case $\deg\phi=2$ and ϕ with distinct roots; and Bessel polynomials, $(B_n^{(\alpha)})_n$, in the case $\deg\phi=2$ with a double root. Furthermore, we can take for ϕ and ψ the canonical forms in Table 1.

Remark 2.1. To check the regularity of a given moment linear functional in $\mathcal{P}'$ may be an hard task. However, for classical moment linear functionals we notice the following useful criteria [13]: given a nonzero functional $\mathbf{u}\in\mathcal{P}'$, then it is classical if and only if there exist two nonzero polynomials $\phi\in\mathcal{P}_2$ and $\psi\in\mathcal{P}_1$ such that Eq. (4) holds and

$$\psi'_{n/2}\neq 0, \qquad \phi\left(-\frac{\psi_n(0)}{\psi'_n}\right)\neq 0 \quad \text{for all } n=0,1,2,\ldots,$$

where $\psi_\nu(x) := \psi(x)+\nu\phi'(x)$ for any real number ν.

Table 1
Classification of the classical OPS

P_n	ϕ	ψ	Restrictions
H_n	1	$-2x$	–
$L_n^{(\alpha)}$	x	$-x+\alpha+1$	$\alpha \neq -n,\ n \geqslant 1$
$P_n^{(\alpha,\beta)}$	$1-x^2$	$-(\alpha+\beta+2)x+\beta-\alpha$	$\alpha \neq -n,\ \beta \neq -n,\ \alpha+\beta+1 \neq -n,\ n \geqslant 1$
$B_n^{(\alpha)}$	x^2	$(\alpha+2)x+2$	$\alpha \neq -n,\ n \geqslant 2$

Remark 2.2. Usually, in the literature, it is imposed in the definition of semiclassical functional that any pair (ϕ, ψ) satisfying (4) must be an admissible pair, in the sense that the condition

$$na + p \neq 0 \quad \text{for all } n = 0, 1, 2, \ldots$$

must hold, where a and p are the leading coefficients of ϕ and ψ (resp.). J.C. Medem [18] found a regular functional $\mathbf{u}$ and a pair of non-admissible polynomials (ϕ, ψ) satisfying (4). For this reason we dropped the admissibility condition in the above definition of semiclassical functional. We notice, however, that it always holds for classical functionals [17], since in this case it is equivalent to the condition $\psi'_{n/2} \neq 0$ for all $n = 0, 1, 2, \ldots$, stated in Remark 2.1. (In fact, this is one of the reasons why the condition was imposed in the semiclassical case.)

In practice, if $\mathbf{u} \in \mathcal{P}'$ is a semiclassical functional, then we know that it satisfies Eq. (4) for some pair of polynomials (ϕ, ψ), with $\deg \psi \geqslant 1$. For such a pair (ϕ, ψ), the integer number $\widetilde{s} := \max\{\deg \phi - 2, \deg \psi - 1\}$ needs not to be the class of $\mathbf{u}$ (in this situation we only can say that $\mathbf{u}$ is semiclassical of class at most $\widetilde{s}$). However, when a pair (ϕ, ψ) fulfils (4), this upper bound $\widetilde{s}$ for the class of $\mathbf{u}$ can be improved (reduced) in order to get the class. This can be done applying an algorithm which can be constructed on the basis of the following property [16]: if $\mathbf{u}$ is a semiclassical functional which satisfies (4) for a certain pair (ϕ, ψ), then the class of $\mathbf{u}$ is $\max\{\deg \phi - 2, \deg \psi - 1\}$ if and only if

$$\prod_{c \in \mathcal{Z}_\phi} \left(\left|\psi(c) - \phi'(c)\right| + \left|\left\langle \mathbf{u}, \theta_c \psi - \theta_c^2 \phi \right\rangle\right| \right) > 0, \tag{5}$$

where $\mathcal{Z}_\phi := \{c\colon\ \phi(c) = 0\}$ and

$$\theta_c q(x) := \frac{q(x) - q(c)}{x - c}, \quad q \in \mathcal{P}. \tag{6}$$

The algorithm to improve (reduce) the class of a given semiclassical functional, once a pair (ϕ, ψ) has been founded fulfilling (4), can then be described as follows. If this pair satisfies (5) then the class of $\mathbf{u}$ is $\max\{\deg \phi - 2, \deg \psi - 1\}$. If not, pick a zero d of ϕ such that

$$\psi(d) - \phi'(d) = 0 \quad \text{and} \quad \left\langle \mathbf{u}, \theta_d \psi - \theta_d^2 \phi \right\rangle = 0 \tag{7}$$

and define $\widetilde{\phi} := \theta_d \phi$ and $\widetilde{\psi} := \theta_d \psi - \theta_d^2 \phi$. Then $\mathbf{u}$ satisfies $D(\widetilde{\phi}\mathbf{u}) = \widetilde{\psi}\mathbf{u}$ and $\widetilde{s} := \max\{\deg \widetilde{\phi} - 2, \deg \widetilde{\psi} - 1\} < \max\{\deg \phi - 2, \deg \psi - 1\}$. Hence, if (5) is fulfilled for the polynomials $\widetilde{\phi}$ and $\widetilde{\psi}$, then $\widetilde{s}$ is the class of $\mathbf{u}$. If not, pick a zero e of $\widetilde{\phi}$ such that (7) holds for these polynomials $\widetilde{\phi}$ and $\widetilde{\psi}$ (with d replaced by e) and proceed successively as before.

To finish this section, we point out that semiclassical functionals may be characterized by several different ways (see [16]). One of the most useful characterizations involves the (formal) Stieltjes series $S_{\mathbf{u}}(z)$ associated with the given moment linear functional $\mathbf{u}$, which is defined as

$$S_{\mathbf{u}}(z) := -\sum_{n \geqslant 0} \frac{u_n}{z^{n+1}}.$$

In fact, a regular functional $\mathbf{u} \in \mathcal{P}'$ is semiclassical if and only if there exist polynomials ϕ, C and D such that $S_{\mathbf{u}}$ is a (formal) solution of the first order nonhomogeneous linear ordinary differential equation

$$\phi(z) S'_{\mathbf{u}}(z) = C(z) S_{\mathbf{u}}(z) + D(z). \tag{8}$$

Further, if ϕ, C and D are co-prime then the class of $\mathbf{u}$ is

$$s = \max\{\deg C - 1, \deg D\}.$$

Remark 2.3. It is well known that if $\mathbf{u} \in \mathcal{P}'$ is semiclassical and satisfies (4) then $S_{\mathbf{u}}$ satisfies (8) where ϕ is the same polynomial appearing in (4) and

$$C(z) = \psi(z) - \phi'(z), \qquad D(z) = (\mathbf{u}\theta_0 \psi)(z) - (\mathbf{u}\theta_0 \phi)'(z).$$

3. Main result

In the sequel the following notation will be useful

$$P_n^{[m]}(x) := \frac{P_{n+m}^{(m)}(x)}{(n+1)_m} \quad (n, m = 0, 1, 2, \ldots),$$

where $P_{n+m}^{(m)} \equiv \frac{d^m}{dx^m} P_{n+m}$ and $(\alpha)_n$ denotes the Pochhammer symbol: for any complex number α and n a nonnegative integer number,

$$(\alpha)_0 := 1, \qquad (\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1).$$

Notice that $P_n^{[m]}(x)$ is a monic polynomial in x of degree n. If $n < 0$ we set $P_n^{[m]}(x) := 0$.

Theorem 3.1. *Let* **u** *and* **v** *be two regular moment linear functionals in $\mathcal{P}'$, and let $(P_n)_n$ and $(Q_n)_n$ be the corresponding MOPS's. Assume that there exist two nonnegative integer numbers N and M, and complex numbers $r_{i,n}$ and $s_{j,n}$ ($i = 1, \ldots, N$; $j = 1, \ldots, M$; $n = 0, 1, \ldots$), with the conventions $r_{i,n} = 0$ if $n < i$ and $s_{j,n} = 0$ if $n < j$, such that*

$$P_n^{[m]}(x) + \sum_{i=1}^{N} r_{i,n} P_{n-i}^{[m]}(x) = Q_n^{[k]}(x) + \sum_{j=1}^{M} s_{j,n} Q_{n-j}^{[k]}(x) \tag{9}$$

holds for all $n = 0, 1, 2, \ldots$. Without loss of generality, we assume $0 \leqslant m \leqslant k$. Define a matrix $A_{N+M} := [a_{i,j}]_{i,j=1}^{N+M}$ of order $N+M$ as

$$a_{i,j} = \begin{cases} r_{j-i,j-1}, & \text{if } 1 \leqslant i \leqslant M \wedge i \leqslant j \leqslant N+i, \\ s_{j-i+M,j-1}, & \text{if } M+1 \leqslant i \leqslant M+N \wedge i-M \leqslant j \leqslant i, \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

with the convention $r_{0,\kappa} = s_{0,\nu} = 1$ ($\kappa = 0, \ldots, M-1$; $\nu = 0, \ldots, N-1$). Assume that the following conditions are fulfilled

$$r_{N,M+N+i} s_{M,M+N+i} \neq 0 \quad (i = 0, 1), \qquad \det A_{N+M} \neq 0.$$

Then there exist polynomials Φ_{M+m+i} and Ψ_{N+k+i}, of degrees $M+m+i$ and $N+k+i$ (resp.), such that

$$D^{k-m}(\Phi_{M+m+i}\mathbf{u}) = \Psi_{N+k+i}\mathbf{v}, \quad i = 0, 1. \tag{11}$$

Proof. Let $(\mathbf{a}_n)_n$ and $(\mathbf{b}_n)_n$ be the dual bases associated with the simple sets $(P_n)_n$ and $(Q_n)_n$, respectively. Then one can write

$$\mathbf{a}_n = \frac{P_n}{\langle \mathbf{u}, P_n^2 \rangle}\mathbf{u}, \qquad \mathbf{b}_n = \frac{Q_n}{\langle \mathbf{v}, Q_n^2 \rangle}\mathbf{v} \tag{12}$$

for all $n = 0, 1, 2, \ldots$. According to (9), set

$$R_n(x) := \sum_{i=0}^{N} r_{i,n} P_{n-i}^{[m]}(x) = \sum_{i=0}^{M} s_{i,n} Q_{n-i}^{[k]}(x), \tag{13}$$

with the convention $r_{0,n} = s_{0,n} = 1$ for all $n = 0, 1, 2, \ldots$. Then $(R_n)_n$ is a simple set of polynomials. Let $(\mathbf{c}_n)_n$, $(\mathbf{e}_n)_n$, and $(\mathbf{d}_n)_n$ be the dual basis associated with the simple sets $(Q_n^{[k]})_n$, $(P_n^{[m]})_n$, and $(R_n)_n$, respectively. Expanding $\mathbf{e}_n$ in terms of the basis $(\mathbf{d}_n)_n$, we can write

$$\mathbf{e}_n = \sum_{j \geqslant 0} \lambda_{n,j} \mathbf{d}_j, \quad n = 0, 1, 2, \ldots, \tag{14}$$

where, according to (13),

$$\lambda_{n,j} = \langle \mathbf{e}_n, R_j \rangle = \sum_{i=0}^{N} r_{i,j} \langle \mathbf{e}_n, P_{j-i}^{[m]}(x) \rangle = \begin{cases} r_{j-n,j}, & \text{if } n \leqslant j \leqslant n+N, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, (14) reduces to

$$\mathbf{e}_n = \sum_{j=n}^{n+N} r_{j-n,j} \mathbf{d}_j, \quad n = 0, 1, 2, \ldots. \tag{15}$$

Similarly, expanding $\mathbf{c}_n$ in terms of the basis $(\mathbf{d}_n)_n$, we find

$$\mathbf{c}_n = \sum_{j=n}^{n+M} s_{j-n,j} \mathbf{d}_j, \quad n = 0, 1, 2, \ldots. \tag{16}$$

Now, consider Eq. (15) for $n = 0, 1, \ldots, M-1$ and Eq. (16) for $n = 0, 1, \ldots, N-1$ to get the following system of linear equations

$$A_{M+N}\begin{bmatrix} \mathbf{d}_0 \\ \vdots \\ \mathbf{d}_{M-1} \\ \mathbf{d}_M \\ \vdots \\ \mathbf{d}_{M+N-1} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_0 \\ \vdots \\ \mathbf{e}_{M-1} \\ \mathbf{c}_0 \\ \vdots \\ \mathbf{c}_{N-1} \end{bmatrix}, \tag{17}$$

where $A_{M+N} = [a_{ij}]_{i,j=1}^{N+M}$, the a_{ij}'s being defined in (10). Since we assume $\det(A_{M+N}) \neq 0$, solving (17) with respect to $\mathbf{d}_i$ we get

$$\mathbf{d}_i = \ell_{i,0}\mathbf{e}_0 + \cdots + \ell_{i,M-1}\mathbf{e}_{M-1} + \ell_{i,M}\mathbf{c}_0 + \cdots + \ell_{i,M+N-1}\mathbf{c}_{N-1} \tag{18}$$

for all $i = 0, 1, \ldots, M+N-1$, where $\ell_{i,j}$ $(j = 0, 1, \ldots, M+N-1)$ are some constants.

Consider now the system of two equations, one of which is (15) with $n = M$ and the other one is (16) with $n = N$. Multiplying the first one of these two equations by $s_{M,M+N}$ and the second one by $r_{N,M+N}$, and then subtracting the resulting equations, we get

$$s_{M,M+N}\mathbf{e}_M - r_{N,N+M}\mathbf{c}_N = \ell_1\mathbf{d}_K + \cdots + \ell_{M+N-K}\mathbf{d}_{N+M-1}, \tag{19}$$

where $K := \min\{N, M\}$ and $\ell_1, \ldots, \ell_{M+N-K}$ are constants. Then, substitute $\mathbf{d}_K, \ldots, \mathbf{d}_{N+M-1}$ given by (18) in the right-hand side of (19) to arrive at

$$\alpha_0\mathbf{e}_0 + \cdots + \alpha_{M-1}\mathbf{e}_{M-1} + s_{M,M+N}\mathbf{e}_M = \beta_0\mathbf{c}_0 + \cdots + \beta_{N-1}\mathbf{c}_{N-1} + r_{N,N+M}\mathbf{c}_N, \tag{20}$$

where $\alpha_0, \ldots, \alpha_{M-1}, \beta_0, \ldots, \beta_{N-1}$ are constants. Taking the kth derivative on both sides of (20) and taking into account the relations

$$D^k\mathbf{c}_n = (-1)^k(n+1)_k\mathbf{b}_{n+k}, \tag{21}$$

$$D^m\mathbf{e}_n = (-1)^m(n+1)_m\mathbf{a}_{n+m}, \tag{22}$$

since we are assuming $m \leqslant k$, according to (12) we get

$$D^{k-m}(\Phi_{M+m}\mathbf{u}) = \Psi_{N+k}\mathbf{v}, \tag{23}$$

where

$$\Phi_{M+m}(x) := (-1)^m\frac{(M+1)_m s_{M,N+M}}{\langle \mathbf{u}, P_{M+m}^2\rangle}x^{M+m} + \pi_{M+m-1}(x),$$

$$\Psi_{N+k}(x) := (-1)^k\frac{(N+1)_k r_{N,N+M}}{\langle \mathbf{v}, Q_{N+k}^2\rangle}x^{N+k} + \widetilde{\pi}_{N+k-1}(x),$$

with $\pi_{M+m-1} \in \mathcal{P}_{M+m-1}$ and $\widetilde{\pi}_{N+k-1} \in \mathcal{P}_{N+k-1}$. It is clear that Φ_{M+m} and Ψ_{N+k} are polynomials of degrees $M+m$ and $N+k$, respectively.

Next, consider a new system with two equations, one of which is (15) with $n = M+1$ and the other one is (16) with $n = N+1$. Multiplying the first one of these equations by $s_{M,M+N+1}$ and the second one by $r_{N,M+N+1}$ and then subtracting the resulting equations, we get

$$s_{M,M+N+1}\mathbf{e}_{M+1} - r_{N,N+M+1}\mathbf{c}_{N+1} = \widetilde{\ell}_1\mathbf{d}_{K+1} + \cdots + \widetilde{\ell}_{M+N-K}\mathbf{d}_{N+M}, \tag{24}$$

where $\widetilde{\ell}_1, \ldots, \widetilde{\ell}_{M+N-K}$ are again some constants. But, from (15) with $n = M$ we can write

$$\mathbf{d}_{N+M} = \frac{1}{r_{N,N+M}}\left(\mathbf{e}_M - \sum_{j=M}^{M+N-1} r_{j-M,j}\mathbf{d}_j\right),$$

and so, the right-hand side of (24) becomes a linear combination of the functionals $\mathbf{d}_K, \ldots, \mathbf{d}_{N+M-1}$, hence, using (18) we get

$$\widetilde{\alpha}_0\mathbf{e}_0 + \cdots + \widetilde{\alpha}_{M-1}\mathbf{e}_{M-1} + s_{M,M+N+1}\mathbf{e}_{M+1} = \widetilde{\beta}_0\mathbf{c}_0 + \cdots + \widetilde{\beta}_{N-1}\mathbf{c}_{N-1} + r_{N,N+M+1}\mathbf{c}_{N+1}, \tag{25}$$

where $\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{M-1}, \widetilde{\beta}_0, \ldots, \widetilde{\beta}_{N-1}$ are constants. Taking the kth derivative of both sides of (25) and taking into account (21), (22) and (12), we find

$$D^{k-m}(\Phi_{M+m+1}\mathbf{u}) = \Psi_{N+k+1}\mathbf{v}, \tag{26}$$

with

$$\Phi_{M+m+1}(x) := (-1)^m \frac{(M+2)_m s_{M,M+N+1}}{\langle \mathbf{u}, P^2_{M+m+1}\rangle} x^{M+m+1} + \pi_{M+m}(x),$$
$$\Psi_{N+k+1}(x) := (-1)^k \frac{(N+2)_k r_{N,N+M+1}}{\langle \mathbf{v}, Q^2_{N+k+1}\rangle} x^{N+k+1} + \widetilde{\pi}_{N+k}(x),$$

where $\pi_{M+m} \in \mathcal{P}_{M+m}$ and $\widetilde{\pi}_{N+k} \in \mathcal{P}_{N+k}$. This completes the proof. $\square$

Remark 3.1. Setting $k=m=0$ in Theorem 3.1 gives the main result in [20]: $\mathbf{u}$ and $\mathbf{v}$ are connected by a rational modification:

$$\Phi_M \mathbf{u} = \Psi_N \mathbf{v}.$$

See also [1] and [2] were some related problems were studied.

Remark 3.2. Theorem 3.1 gives the solution of the inverse problem associated with the linear algebraic structure relation (9). Similar inverse problems, involving orthogonal polynomials and their derivatives, have been studied previously in many contexts, both from the algebraic as well as from the analytical point of view. For instance, in [4,5] S. Bonan, D.S. Lubinsky, and P. Nevai characterized all positive Borel measures associated with OPS's $(P_n)_n$ and $(R_n)_n$ such that there exists nonnegative integer numbers s and t, and a polynomial ϕ of degree t, such that the structure relation

$$\phi(x) R'_{n+1}(x) = \sum_{i=n-s}^{n+t} \lambda_{n,i} P_i(x), \quad n \geqslant s,$$

holds, where the $\lambda_{n,i}$'s are real numbers such that $\lambda_{n,n-i} \neq 0$ for all $n \geqslant s$, with the convention $\lambda_{n,i}=0$ if $i<0$. The main result concerning this structure relation, stated in [5], is that the involved measures are connected by a rational modification plus an atomic measure with finite support. In the setting of the theory of formal OP's it can be shown that both $(P_n)_n$ as well as $(R_n)_n$ are semiclassical OPS's (see [10]).

4. Relation between the formal Stieltjes series

In order to explore the meaning of the differential relations (11) between the linear functionals $\mathbf{u}$ and $\mathbf{v}$ in Theorem 3.1, we will deduce some extra information concerning their moments. This will be done given the relation between the associated formal Stieltjes series,

$$S_{\mathbf{u}}(z) := -\sum_{n\geqslant 0} \frac{u_n}{z^{n+1}}, \qquad S_{\mathbf{v}}(z) := -\sum_{n\geqslant 0} \frac{v_n}{z^{n+1}},$$

where $u_n := \langle \mathbf{u}, x^n\rangle$ and $v_n := \langle \mathbf{v}, x^n\rangle$ are the moments of order n for $\mathbf{u}$ and $\mathbf{v}$ (resp.). Notice that the (formal) derivative of order j of $S_{\mathbf{u}}(z)$ is

$$S_{\mathbf{u}}^{(j)}(z) := (-1)^{j+1} \sum_{n\geqslant 0} (n+1)_j \frac{u_n}{z^{n+1+j}}.$$

We also note that given $\phi \in \mathcal{P}$, and being $\theta_0\phi$ defined according with (6) with $c=0$, then $\mathbf{u}\theta_0\phi$ is the polynomial defined by

$$(\mathbf{u}\theta_0\phi)(z) = \left\langle \mathbf{u}_\xi, \frac{z(\theta_0\phi)(z) - \xi(\theta_0\phi)(\xi)}{z-\xi} \right\rangle.$$

Theorem 4.1. *Under the conditions of Theorem* 3.1 *the two relations*

$$\Psi_{N+k+i}(z) S_{\mathbf{v}}(z) - \big(\Phi_{M+m+i}(z) S_{\mathbf{u}}(z)\big)^{(k-m)} = B(z;i), \quad i=0,1, \tag{27}$$

hold, where $B(\cdot;0)$ and $B(\cdot;1)$ are polynomials in z, given explicitly by

$$B(z;i) = (\mathbf{u}\theta_0\Phi_{M+m+i})^{(k-m)}(z) - (\mathbf{v}\theta_0\Psi_{N+k+i})(z), \quad i=0,1,$$

being $\deg B(\cdot;i) \leqslant i-1-k+\max\{M+2m, N+2k\}$ *for* $i=0,1$.

Proof. We will prove (27) for $i=0$. The case $i=1$ can be handled *mutatis mutandis*. From (11) with $i=0$ we get

$$\big\langle D^{k-m}(\Phi_{M+m}\mathbf{u}), x^n\big\rangle = \big\langle \Psi_{N+k}\mathbf{v}, x^n\big\rangle, \quad n=0,1,2,\ldots.$$

Setting

$$\Phi_{M+m}(z) = \sum_{\nu=0}^{M+m} a_\nu z^\nu, \qquad \Psi_{N+k}(z) = \sum_{\nu=0}^{N+k} b_\nu z^\nu,$$

the above relation gives the following linear relation between the moments of the linear functionals $\mathbf{u}$ and $\mathbf{v}$:

$$(-1)^{k-m} \sum_{\nu=0}^{M+m} a_\nu (n-k+m+1)_{k-m} u_{n-k+m+\nu} = \sum_{\nu=0}^{N+k} b_\nu v_{n+\nu}$$

for all $n = 0, 1, 2, \ldots$. Multiplying both sides of this equality by z^{-n-1} and summing for $n = 0, 1, 2, \ldots$ we obtain

$$(-1)^{k-m} \sum_{\nu=0}^{M+m} a_\nu \sum_{n\geqslant 0} (n-k+m+1)_{k-m} \frac{u_{n-k+m+\nu}}{z^{n+1}} = \sum_{\nu=0}^{N+k} b_\nu \sum_{n\geqslant 0} \frac{v_{n+\nu}}{z^{n+1}}. \tag{28}$$

Now, we see that

$$\sum_{n\geqslant 0} \frac{v_{n+\nu}}{z^{n+1}} = -z^\nu \left(S_{\mathbf{v}}(z) + \sum_{n=0}^{\nu-1} \frac{v_n}{z^{n+1}} \right), \quad \nu = 0, \ldots, N+k, \tag{29}$$

and, for $\nu = 0, \ldots, M+m$, we also have

$$\begin{aligned} &z^{k-m-\nu} \sum_{n\geqslant 0} (n-k+m+1)_{k-m} \frac{u_{n-k+m+\nu}}{z^{n+1}} \\ &\quad = \sum_{n\geqslant 0} (n+1)_{k-m} \frac{u_{n+\nu}}{z^{n+\nu+1}} \\ &\quad = \sum_{j=0}^{k-m} \binom{k-m}{j} (-\nu)_{k-m-j} \sum_{n\geqslant 0} (n+\nu+1)_j \frac{u_{n+\nu}}{z^{n+\nu+1}} \\ &\quad = \sum_{j=0}^{k-m} \binom{k-m}{j} (-\nu)_{k-m-j} \left((-1)^{j+1} z^j S_{\mathbf{u}}^{(j)}(z) - \sum_{n=0}^{\nu-1} (n+1)_j \frac{u_n}{z^{n+1}} \right), \end{aligned} \tag{30}$$

the second equality being justified by taking $p = k-m$, $\alpha = -\nu$ and $\beta = n+\nu+1$ in the following analogue of the binomial theorem (see [3, p. 70])

$$(\alpha+\beta)_p = \sum_{j=0}^{p} \binom{p}{j} (\alpha)_{p-j} (\beta)_j,$$

which holds for all complex numbers α and β and any nonnegative integer number p. Substituting (29) and (30) in (28), and taking into account that for a given polynomial ϕ of degree p, say $\phi(z) = \sum_{\nu=0}^{p} c_\nu z^\nu$, $\mathbf{u}\theta_0\phi$ admits the explicit representations

$$(\mathbf{u}\theta_0\phi)(z) = \sum_{\nu=0}^{p-1} c_{\nu+1} \sum_{n=0}^{\nu} u_n z^{\nu-n} = \sum_{\nu=0}^{p-1} \left(\sum_{n=\nu}^{p-1} c_{n+1} u_{n-\nu} \right) z^\nu,$$

then the desired result follows after straightforward computations. □

Theorem 4.2. *Under the conditions of Theorem 4.1, if $k > m$ then*

$$\sum_{\nu=0}^{k-m} A_\nu(z) S_{\mathbf{u}}^{(\nu)}(z) = B(z), \tag{31}$$

so that $S_{\mathbf{u}}(z)$ is a (formal) solution of a nonhomogeneous linear ordinary differential equation of order $k-m$ with polynomial coefficients, given by

$$\begin{aligned} A_\nu(z) &:= \binom{k-m}{\nu} \left[\Psi_{N+k}(z) \Phi_{M+m+1}^{(k-m-\nu)}(z) - \Psi_{N+k+1}(z) \Phi_{M+m}^{(k-m-\nu)}(z) \right], \\ B(z) &:= \Psi_{N+k+1}(z) B(z;0) - \Psi_{N+k}(z) B(z;1), \end{aligned}$$

being

$$\deg A_\nu \leqslant N + M + 2m + 1 + \nu, \quad \nu = 0, \ldots, k - m,$$

$$\deg B \leqslant N + \max\{M + 2m, N + 2k\}.$$

Proof. Consider the two equations resulting from (27) for $i = 0$ and $i = 1$. Multiplying both sides of the first one of these equations by $\Psi_{n+k+1}(z)$ and those of the second one by $\Psi_{n+k}(z)$, and then subtracting the resulting equations, we get

$$\Psi_{N+k}(z)(\Phi_{M+m+1} S_{\mathbf{u}})^{(k-m)}(z) - \Psi_{N+k+1}(z)(\Phi_{M+m} S_{\mathbf{u}})^{(k-m)}(z) = B(z),$$

from which (31) follows, taking into account Leibniz rule for the higher derivatives of a product. □

Remark 4.1. We notice that the restriction $k > m$ needs not to be made explicitly in the statement of the theorem. However, if $k = m$ then Eq. (31) becomes trivial, since then one can show that $A_0(z) = B(z) \equiv 0$.

Remark 4.2. If we are able to solve the ODE (31) to find $S_{\mathbf{u}}(z)$, then we also get $S_{\mathbf{v}}(z)$ immediately from Eq. (27).

5. The case $k = m + 1$

5.1. Semiclassical character

The case $k = m + 1$ in Theorem 3.1 is of particular interest, since in this situation both $\mathbf{u}$ and $\mathbf{v}$ are semiclassical moment linear functionals. We notice that if $k = m + 1$ then Theorem 4.2 shows that $S_{\mathbf{u}}$ satisfies an ODE of the type (8), hence it follows immediately that $\mathbf{u}$ is a semiclassical functional. We will show that $\mathbf{v}$ is also a semiclassical functional and, furthermore, each one of these functionals is a rational perturbation of the other one. We begin with the following

Theorem 5.1. *Under the conditions of Theorem 3.1, if $k = m + 1$ and $(P_n)_n \equiv (Q_n)_n$, so that $\mathbf{u}$ and $\mathbf{v}$ coincide up to a constant factor, then*

$$D(\Phi_{M+m}\mathbf{u}) = \Psi_{N+m+1}\mathbf{u},$$

hence $\mathbf{u}$ is semiclassical of class at most $\max\{M + m - 2, N + m\}$.

Remark 5.1. The special case when $m = 0$, $N = 0$ and $M = 2$ in Theorem 5.1, gives the well-known characterization for classical OP's which states that the classical OPS's are the only OPS's such that each polynomial in the sequence is a linear combination of the derivatives of three consecutive polynomials of the same family (see e.g. Marcellán et al. [11]).

When $(P_n)_n \not\equiv (Q_n)_n$ the following proposition holds.

Theorem 5.2. *Under the conditions of Theorem 3.1, if $k = m + 1$, then $\mathbf{u}$ and $\mathbf{v}$ are semiclassical moment linear functionals of classes at most $N + M + 2m$ and $N + 3M + 4m$ (resp.), which are also connected by a rational modification. More precisely, we have*

$$\Lambda\mathbf{u} = \Phi\mathbf{v}, \tag{32}$$

$$D(\Phi\mathbf{u}) = \Psi\mathbf{u}, \tag{33}$$

$$D(\widetilde{\Phi}\mathbf{v}) = \widetilde{\Psi}\mathbf{v}, \tag{34}$$

where

$$\Lambda := \Phi_{M+m}\Phi'_{M+1+m} - \Phi_{M+1+m}\Phi'_{M+m} \in \mathcal{P}_{2(M+m)},$$

$$\Phi := \Phi_{M+m}\Psi_{N+2+m} - \Phi_{M+1+m}\Psi_{N+1+m} \in \mathcal{P}_{N+M+2m+2}$$

$$\Psi := \Psi'_{N+2+m}\Phi_{M+m} - \Psi'_{N+1+m}\Phi_{M+1+m} \in \mathcal{P}_{N+M+2m+1},$$

$$\widetilde{\Phi} := \Lambda\Phi \in \mathcal{P}_{N+3M+4m+2}, \qquad \widetilde{\Psi} := 2\Lambda'\Phi + \Lambda(\Psi - \Phi') \in \mathcal{P}_{N+3M+4m+1}.$$

Proof. From Theorem 3.1 we have

$$D(\Phi_{M+m}\mathbf{u}) = \Psi_{N+1+m}\mathbf{v}, \qquad D(\Phi_{M+1+m}\mathbf{u}) = \Psi_{N+2+m}\mathbf{v}, \tag{35}$$

from which (32) and (33) are easily deduced. To prove (34), notice first that, using (32),

$$D(\widetilde{\Phi}\mathbf{v}) = D\big(\Lambda(\Phi\mathbf{v})\big) = \Lambda'\Phi\mathbf{v} + \Lambda D(\Lambda\mathbf{u}),$$

and since

$$\begin{aligned}D(\Lambda\mathbf{u}) &= D\big(\Phi'_{M+1+m}(\Phi_{M+m}\mathbf{u})\big) - D\big(\Phi'_{M+m}(\Phi_{M+1+m}\mathbf{u})\big)\\ &= \Phi''_{M+1+m}\Phi_{M+m}\mathbf{u} + \Phi'_{M+1+m}\Psi_{N+1+m}\mathbf{v} - \Phi''_{M+m}\Phi_{M+1+m}\mathbf{u} - \Phi'_{M+m}\Psi_{N+2+m}\mathbf{v}\\ &= \Lambda'\mathbf{u} + (\Psi - \Phi')\mathbf{v},\end{aligned}$$

the desired result follows. □

Remark 5.2. It remains an open problem to know whether **u** and **v** still remain semiclassical moment linear functionals in the case $k > m+1$.

Remark 5.3. The numbers $N+M+2m$ and $N+3M+4m$ are upper bounds for the classes of the semiclassical functionals **u** and **v** in Theorem 5.2 (resp.). For concrete families of OPS's these numbers may not coincide with the class of the functionals **u** and **v**, hence they may be improved (reduced) by applying the reduction process described in Section 2. An example illustrating this situation is given in below.

Remark 5.4. Often, the regular linear functionals **u** and **v** in Theorem 5.2 are positive-definite, so there exists two positive Borel measures $\mathrm{d}\mu_1$ and $\mathrm{d}\mu_2$, whose supports are infinite sets, such that **u** and **v** admit the integral representations (see, e.g., [6, Chapter II])

$$\langle\mathbf{u}, f\rangle = \int_{-\infty}^{+\infty} f\,\mathrm{d}\mu_1, \qquad \langle\mathbf{v}, f\rangle = \int_{-\infty}^{+\infty} f\,\mathrm{d}\mu_2$$

for all $f \in \mathcal{P}$. Under the conditions of Theorem 5.2 assume, in addition to the positive-definitiveness of **u** and **v**, that all zeros of the polynomial Λ are real and distinct, say, $x_1 < x_2 < \cdots < x_\nu$ ($\nu := \deg\Lambda \geqslant 1$), and that all these zeros of Λ lie out the convex-hull of the support of $\mathrm{d}\mu_1$. Then, it can be shown that Eq. (32) leads to the following relation between the measures $\mathrm{d}\mu_1$ and $\mathrm{d}\mu_2$:

$$\mathrm{d}\mu_1(x) = \left|\frac{\Phi(x)}{\Lambda(x)}\right| d\mu_2(x) + \sum_{i=1}^{\deg\Lambda} M_i\delta(x - x_i), \tag{36}$$

provided

$$M_i := u_0 - \frac{1}{\Lambda'(x_i)}\left(\sum_{j=0}^{\deg\Phi-1}\frac{v_j}{j!}(\theta_{x_i}\Phi)^{(j)}(0) + \Phi(x_i)F(x_i;\mathrm{d}\mu_2)\right) \geqslant 0$$

for all $i = 1, 2, \ldots, \deg\Lambda$. Here, u_0 and v_j denote the moments of orders 0 and j for the measures $\mathrm{d}\mu_1$ and $\mathrm{d}\mu_2$ (resp.), θ_{x_i} is the operator defined as in (6), and $F(\cdot;\mathrm{d}\mu_2)$ is the Stieltjes transform associated with the measure $\mathrm{d}\mu_2$,

$$F(z;\mathrm{d}\mu_2) := \int_{-\infty}^{+\infty}\frac{\mathrm{d}\mu_2(x)}{x - z}, \quad z \in \mathbb{C}\setminus \mathrm{co}\big(\mathrm{supp}(\mathrm{d}\mu_2)\big),$$

where $\mathrm{co}(K)$ means the convex-hull of a subset $K \subset \mathbb{R}$. Of course, if all the zeros of the polynomial Φ are real and distinct, then an equation analogous to (36) can be written expressing $\mathrm{d}\mu_2$ in terms of $\mathrm{d}\mu_1$. Formula (36) is certainly known, but we did not find an available reference.

5.2. Some final remarks

The aim of these final remarks is to point out that the proofs of Theorems 3.1 and 5.2 are constructive and, moreover, to illustrate the fact mentioned above that the upper bounds for the classes of the moment linear functionals in Theorem 5.2 may be improved for concrete families of OP's. These considerations will be illustrated by an example. We consider two sequences of monic OP's $(P_n)_n$ and $(Q_n)_n$ and assume that it is known *a priori* that they fulfill the following structure relation

$$P_n + r_n P_{n-1} = \frac{Q'_{n+1}}{n+1} + s_n\frac{Q'_n}{n}, \quad n = 1, 2, \ldots,$$

where $(r_n)_n$ and $(s_n)_n$ are sequences of real numbers. We also assume that the r_i's and s_i's are known for $i = 1, 2, 3$, say

$$r_1 = \frac{1}{6}, \qquad r_2 = \frac{4}{15}, \qquad r_3 = \frac{9}{28}, \qquad s_1 = -\frac{1}{3}, \qquad s_2 = -\frac{2}{5}, \qquad s_3 = -\frac{3}{7},$$

as well as the first five polynomials of the $(Q_n)_n$ family:

$$Q_0(x)=1, \qquad Q_1(x)=x, \qquad Q_2(x)=x^2-\frac{1}{3},$$
$$Q_3(x)=x^3-\frac{3}{5}x, \qquad Q_4(x)=x^4-\frac{6}{7}x^2+\frac{3}{35}.$$

We will show that Theorems 3.1 and 5.2 can be applied in order to completely characterize the two OPS $(P_n)_n$ and $(Q_n)_n$. Notice first that this example corresponds to a situation where $k=1$, $m=0$ and $N=M=1$. Therefore, going to the proof of Theorem 3.1, we compute

$$\Phi_1(x)=-\frac{8}{3}x+\frac{8}{3}, \qquad \Psi_2(x)=-6x^2-4x+2,$$
$$\Phi_2(x)=-\frac{225}{16}x^2+\frac{225}{8}x-\frac{225}{16}, \qquad \Psi_3(x)=-\frac{675}{16}x^3+\frac{675}{16}x,$$

and so from (33) and (34) in Theorem 5.2 we get

$$D\big(\big(x^2-1\big)^2\mathbf{u}\big)=6\big(x^2-1\big)\left(x-\frac{1}{3}\right)\mathbf{u},$$
$$D\big((x+1)^2(x-1)^4\mathbf{v}\big)=6(x+1)(x-1)^3\left(x+\frac{1}{3}\right)\mathbf{v}.$$

Therefore, $\mathbf{u}$ and $\mathbf{v}$ are semiclassical moment linear functionals of classes at most 2 and 4, respectively. However, applying the algorithm of reduction of the class described in Section 2, after some straightforward computations we can show that

$$D\big(\big(1-x^2\big)\mathbf{u}\big)=2(1-2x)\mathbf{u}, \qquad D\big(\big(1-x^2\big)\mathbf{v}\big)=-2x\mathbf{v}.$$

As a consequence, $\mathbf{u}$ and $\mathbf{v}$ are, in fact, classical moment linear functionals. Furthermore, we see from Table 1 that P_n is a monic Jacobi polynomial of degree n and Q_n is the monic Legendre polynomial of degree n:

$$P_n\equiv\widehat{P}_n^{(0,2)}, \qquad Q_n\equiv\widehat{P}_n^{(0,0)}\equiv\widehat{L}_n.$$

We notice that this illustrative example is a particular situation of the family $J_{1,2}$ considered in [7, p. 251].

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