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On orthogonal polynomials obtained via polynomial mappings

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Abstract

Let $(p_n)_n$ be a given monic orthogonal polynomial sequence (OPS) and k a fixed positive integer number such that $k \geq 2$. We discuss conditions under which this OPS originates from a polynomial mapping in the following sense: to find another monic OPS $(q_n)_n$ and two polynomials π_k and θ_m, with degrees k and m (resp.), with $0 \leq m \leq k-1$, such that

$$p_{nk+m}(x) = \theta_m(x) q_n(\pi_k(x)) \quad (n = 0, 1, 2, \ldots).$$

In this work we establish algebraic conditions for the existence of a polynomial mapping in the above sense. Under such conditions, when $(p_n)_n$ is orthogonal in the positive-definite sense, we consider the corresponding inverse problem, giving explicitly the orthogonality measure for the given OPS $(p_n)_n$ in terms of the orthogonality measure for the OPS $(q_n)_n$. Some applications and examples are presented, recovering several known results in a unified way.
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1. Introduction and preliminaries

Let $(p_n)_n$ be a given monic orthogonal polynomial sequence (OPS) and k a fixed positive integer number such that $k \geq 2$. The purpose of this work is to analyze conditions under which

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this OPS originates from a polynomial mapping in the following sense: to find another monic OPS $(q_n)_n$ and two polynomials π_k and θ_m, with degrees k and m (resp.), with $0 \le m \le k-1$, such that

$$p_{nk+m}(x) = \theta_m(x) q_n(\pi_k(x)), \quad n = 0, 1, 2, \ldots. \tag{1.1}$$

The study of the theoretical aspects as well as applications of the OPS's $(p_n)_n$ and $(q_n)_n$ which can be related by a polynomial transformation as in (1.1) has been a subject of considerable activity in the past few decades, especially due to the interesting applications that we can find in several domains (e.g., in physics, chemistry, operator theory, potential theory, and matrix theory). Far from giving an exhaustive list, we mention here specially the works by Bessis and Moussa [8], Geronimo and Van Assche [15,14], Charris and Ismail [9], Charris et al. [10], Peherstorfer [27,26], and Totik [34]. From the algebraic point of view, the problem was first considered for $(k, m) = (3, 0)$ by Barrucand and Dickinson [7], and also for the cases $k = 2$ and $k = 3$ by Marcellán and Petronilho [21–23]. We notice that most of the works in the literature deal with situations where $m = 0$ or $m = k - 1$, i.e., the polynomial mapping is such that θ_m is either a constant or a polynomial of degree $k - 1$. From the analytical point of view, when we have orthogonality in the positive-definite sense, one of the most important properties satisfied by OPS's fulfilling (1.1) is that $(p_n)_n$ is orthogonal with respect to a positive measure whose support is contained in a union of at most k intervals, and mass points may appear in between these intervals (as expected, according to the results in the above references).

Let us recall the definition of some determinants which will play a fundamental role in the sequel. According to the so-called Favard theorem (also known as the spectral theorem for orthogonal polynomials), any given monic OPS, $(p_n)_n$, can be characterized by a three-term recurrence relation. Fixing an integer number $k \ge 2$, this recurrence relation can be given by general blocks (with k equations) of recurrence relations of the type [9,10]

$$\begin{aligned} &(x - b_n^{(j)}) p_{nk+j}(x) = p_{nk+j+1}(x) + a_n^{(j)} p_{nk+j-1}(x), \\ &\quad j = 0, 1, \ldots, k-1;\ n = 0, 1, 2, \ldots, \end{aligned} \tag{1.2}$$

with $a_n^{(j)} \in \mathbb{C} \setminus \{0\}$ and $b_n^{(j)} \in \mathbb{C}$ for all n and j, and satisfying initial conditions

$$p_{-1}(x) = 0, \qquad p_0(x) = 1. \tag{1.3}$$

Without loss of generality, we will take $a_0^{(0)} = 1$, and polynomials p_j of degree $j \le -1$ will always be defined as the zero polynomial. Next introduce determinants $\Delta_n(i, j; \cdot)$ as in [9,10], such that

$$\Delta_n(i, j; x) := \begin{cases} 0 & \text{if } j < i - 2 \\ 1 & \text{if } j = i - 2 \\ x - b_n^{(i-1)} & \text{if } j = i - 1 \end{cases} \tag{1.4}$$

and, if $j \ge i \ge 1$,

$$\Delta_n(i, j; x) := \begin{vmatrix} x - b_n^{(i-1)} & 1 & 0 & \ldots & 0 & 0 \\ a_n^{(i)} & x - b_n^{(i)} & 1 & \ldots & 0 & 0 \\ 0 & a_n^{(i+1)} & x - b_n^{(i+1)} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & x - b_n^{(j-1)} & 1 \\ 0 & 0 & 0 & \ldots & a_n^{(j)} & x - b_n^{(j)} \end{vmatrix}, \tag{1.5}$$

for every $n = 0, 1, 2, \ldots$. Taking into account that $\Delta_n(i, j; \cdot)$ is a polynomial whose degree may exceed k, and since in (1.2) the $a_n^{(j)}$'s and $b_n^{(j)}$'s were defined only for $0 \le j \le k-1$, we adopt the convention

$$b_n^{(k+j)} := b_{n+1}^{(j)}, \qquad a_n^{(k+j)} := a_{n+1}^{(j)} \tag{1.6}$$

for all $i, j = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$, and so the useful equality

$$\Delta_n(k+i, k+j; \cdot) = \Delta_{n+1}(i, j; \cdot) \tag{1.7}$$

holds for every $i, j = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$.

The structure of the paper is as follows. In Section 2 algebraic conditions for the existence of a polynomial mapping in the sense of (1.1) are established. Under such conditions, in Section 3 the positive-definite case is analyzed, so that we show that the orthogonality measure for the given OPS $(p_n)_n$ can be explicitly obtained from the one for the OPS $(q_n)_n$. In Sections 4 and 5 some examples of application of the previous results are presented, recovering several known results in an unified way. This includes a description of the so called sieved ultraspherical polynomials of the second kind introduced by Al-Salam et al. [1], as well as a classical result of Geronimus [16,17] concerning orthogonal polynomials such that the coefficients in the corresponding three-term recurrence relation are periodic sequences of period k. In both cases $(q_n)_n$ is a sequence of classical Chebyshev polynomials of the second kind (suitably shifted and rescaled). In Section 6 an example in a situation corresponding to $k = 5$ and $m = 1$ is analyzed in detail. Finally, in Section 7, we discuss a result of A. Máté, P. Nevai, and W. Van Assche concerning the support of the measure associated with an OPS with limit-periodic recurrence coefficients and the spectrum of the related self-adjoint Jacobi operator.

At this point we would like to point out that the subject "OPS's obtained via polynomial mappings" is nowadays a classical topic in the theory of OP's, and so it is not surprising that many of the results presented here (specially in the last sections) have been discovered previously by many authors, but we believe that the unified approach presented here may be of interest. Besides, in most cases our proofs of the known results are quite different from the original ones.

2. OPS's obtained via polynomial mappings

The following proposition gives necessary and sufficient conditions for the existence of a polynomial mapping in the sense of (1.1).

Theorem 2.1. *Let $(p_n)_n$ be a monic OPS characterized by the general block of recurrence relations (1.2). Fix $r \in \mathbb{C}$, $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$, with $0 \le m \le k-1$. Then, there exist polynomials π_k and θ_m of degrees k and m (resp.) and a monic OPS $(q_n)_n$ such that $q_1(0) = -r$ and*

$$p_{kn+m}(x) = \theta_m(x) q_n(\pi_k(x)), \quad n = 0, 1, 2, \ldots \tag{2.1}$$

if and only if the following four conditions hold:

(i) $b_n^{(m)}$ *is independent of n for $n = 0, 1, 2, \ldots$;*
(ii) $\Delta_n(m+2, m+k-1; x)$ *is independent of n for $n = 0, 1, 2, \ldots$;*
(iii) $\Delta_0(m+2, m+k-1; \cdot)$ *is divisible by $\Delta_0(1, m-1; \cdot)$, i.e., there exists a polynomial η_{k-1-m} of degree $k-1-m$ such that*

$$\Delta_0(m+2, m+k-1; x) = \Delta_0(1, m-1; x)\eta_{k-1-m}(x);$$

(iv) *$r_n(x)$ is independent of x for all $n = 1, 2, \ldots$, where*

$$r_n(x) := a_n^{(m+1)} \Delta_n(m+3, m+k-1; x) - a_0^{(m+1)} \Delta_0(m+3, m+k-1; x) + a_n^{(m)} \Delta_{n-1}(m+2, m+k-2; x) - a_0^{(m)} \Delta_0(1, m-2; x)\eta_{k-1-m}(x).$$

Under such conditions, the polynomials θ_m and π_k are explicitly given by

$$\begin{aligned} \theta_m(x) &= \Delta_0(1, m-1; x) \equiv p_m(x), \\ \pi_k(x) &= \Delta_0(1, m; x)\eta_{k-1-m}(x) - a_0^{(m+1)} \Delta_0(m+3, m+k-1; x) + r, \end{aligned} \tag{2.2}$$

and the monic OPS $(q_n)_n$ is generated by the three-term recurrence relation

$$q_{n+1}(x) = (x - r_n)q_n(x) - s_n q_{n-1}(x), \quad n = 0, 1, 2, \ldots \tag{2.3}$$

with initial conditions $q_{-1}(x) = 0$ and $q_0(x) = 1$, where

$$r_0 := r, \qquad r_n := r + r_n(0), \qquad s_n := a_n^{(m)} a_{n-1}^{(m+1)} \ldots a_{n-1}^{(m+k-1)} \tag{2.4}$$

for all $n = 1, 2, \ldots$. Moreover, for each $j = 0, 1, \ldots, k-1$ and all $n = 0, 1, 2, \ldots$,

$$p_{kn+m+j+1}(x) = \frac{1}{\eta_{k-1-m}(x)} \Bigg\{ \Delta_n(m+2, m+j; x) q_{n+1}(\pi_k(x)) + \left(\prod_{i=1}^{j+1} a_n^{(m+i)} \right) \Delta_n(m+j+3, m+k-1; x) q_n(\pi_k(x)) \Bigg\}. \tag{2.5}$$

Proof. Split the first m equations in the first block of recurrence relations (1.2); we rewrite (1.2) with the initial conditions (1.3) as

$$\begin{aligned} &p_{-1}(x) = 0, \qquad p_0(x) = 1, \\ &p_{j+1}(x) = (x - b_0^{(j)}) p_j(x) - a_0^{(j)} p_{j-1}(x), \quad j = 0, 1, \ldots, m-1, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &(x - b_n^{(m+j)}) p_{nk+m+j}(x) = p_{nk+m+j+1}(x) + a_n^{(m+j)} p_{nk+m+j-1}(x), \\ &\quad j = 0, 1, \ldots, k-1; n = 0, 1, 2, \ldots, \end{aligned} \tag{2.7}$$

with the conventions (1.6). Now we consider (2.7) as new blocks, and we proceed by applying a technique developed in [9,10] to these blocks. Hence, rewrite (2.7) as a system in matrix form:

$$V_k \begin{pmatrix} p_{nk+m+1} \\ p_{nk+m+2} \\ \vdots \\ p_{nk+m+k-2} \\ p_{nk+m+k-1} \\ p_{nk+m-1} \end{pmatrix} = \begin{pmatrix} a_n^{(m+1)} p_{nk+m} \\ 0 \\ \vdots \\ 0 \\ p_{(n+1)k+m} \\ (x - b_n^{(m)}) p_{nk+m} \end{pmatrix}, \tag{2.8}$$

where $V_k \equiv V_k(m,n;x)$ is a matrix of order k defined by

$$V_k := \begin{pmatrix} x-b_n^{(m+1)} & -1 & 0 & & & & \\ -a_n^{(m+2)} & x-b_n^{(m+2)} & -1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -a_n^{(m+k-2)} & x-b_n^{(m+k-2)} & -1 & 0 \\ & & 0 & -a_n^{(m+k-1)} & x-b_n^{(m+k-1)} & 0 \\ 1 & & 0 & 0 & 0 & a_n^{(m)} \end{pmatrix}.$$

Notice that V_k differs from a tridiagonal matrix only for the "1" in entry $(k,1)$. Solving system (2.8) for $p_{nk+m+j+1}$ in terms of p_{nk+m} and $p_{(n+1)k+m}$, by Cramer's rule we obtain

$$\begin{aligned} &\Delta_n(m+2,m+k-1;x)p_{nk+m+j+1}(x) = \Delta_n(m+2,m+j;x)p_{(n+1)k+m}(x) \\ &\quad + a_n^{(m+1)}\dots a_n^{(m+j+1)}\Delta_n(m+j+3,m+k-1;x)p_{nk+m}(x), \\ &\quad j=0,1,\dots,k-2,\ n=0,1,2,\dots \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &a_n^{(m)}\Delta_n(m+2,m+k-1;x)p_{nk+m-1}(x) = \Big\{(x-b_n^{(m)})\Delta_n(m+2,m+k-1;x) \\ &\quad - a_n^{(m+1)}\Delta_n(m+3,m+k-1;x)\Big\}p_{nk+m}(x) - p_{(n+1)k+m}(x), \\ &\quad n=0,1,2,\dots. \end{aligned} \tag{2.10}$$

Set $j=k-2$ in (2.9) and then replace n by $n-1$ to find

$$\begin{aligned} &\Delta_{n-1}(m+2,m+k-1;x)p_{nk+m-1}(x) = a_{n-1}^{(m+1)}\dots a_{n-1}^{(m+k-1)}p_{(n-1)k+m}(x) \\ &\quad + \Delta_{n-1}(m+2,m+k-2;x)p_{nk+m}(x), \quad n=1,2,3,\dots. \end{aligned} \tag{2.11}$$

Assume first that conditions (i)–(iv) are fulfilled. Then, from (2.10) and (2.11), and taking into account hypotheses (i) and (ii), we obtain

$$\begin{aligned} p_{(n+1)k+m}(x) &= \Big\{(x-b_0^{(m)})\Delta_0(m+2,m+k-1;x) \\ &\quad - a_n^{(m)}\Delta_{n-1}(m+2,m+k-2;x) - a_n^{(m+1)}\Delta_n(m+3,m+k-1;x)\Big\}p_{nk+m}(x) \\ &\quad - a_n^{(m)}a_{n-1}^{(m+1)}\dots a_{n-1}^{(m+k-1)}p_{(n-1)k+m}(x), \quad n=1,2,3,\dots, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} p_{(n+1)k+m}(x) = &\Big\{(x-b_0^{(m)})\Delta_0(m+2,m+k-1;x) - a_0^{(m)}\Delta_0(1,m-2;x)\eta_{k-1-m}(x) \\ &- a_0^{(m+1)}\Delta_0(m+3,m+k-1;x) - r_n(x)\Big\}p_{nk+m}(x) \\ &- a_n^{(m)}a_{n-1}^{(m+1)}\dots a_{n-1}^{(m+k-1)}p_{(n-1)k+m}(x), \quad n=1,2,3,\dots. \end{aligned}$$

From this, since, by hypothesis (iii),

$$\begin{aligned} &(x-b_0^{(m)})\Delta_0(m+2,m+k-1;x) - a_0^{(m)}\Delta_0(1,m-2;x)\eta_{k-1-m}(x) \\ &\quad = ((x-b_0^{(m)})\Delta_0(1,m-1;x) - a_0^{(m)}\Delta_0(1,m-2;x))\eta_{k-1-m}(x) \\ &\quad = \Delta_0(1,m;x)\eta_{k-1-m}(x), \end{aligned}$$

we deduce, using hypothesis (iv), that

$$p_{(n+1)k+m}(x) = (\pi_k(x) - r_n)p_{nk+m}(x) - s_n p_{(n-1)k+m}(x) \tag{2.12}$$

for all $n = 1, 2, 3, \ldots$, where π_k is defined by (2.2) and r_n and s_n are defined by (2.4). Equality (2.12) is still true for $n = 0$. In fact, set $j = n = 0$ in (2.9) and use $p_{m+1}(x) = (x - b_0^{(m)})p_m(x) - a_0^{(m)} p_{m-1}(x)$ to derive

$$\begin{aligned} p_{k+m}(x) &= \Big\{(x - b_0^{(m)})\Delta_0(m+2, m+k-1; x) \\ &\quad - a_0^{(m+1)}\Delta_0(m+3, m+k-1; x)\Big\}\, p_m(x) \\ &\quad - a_0^{(m)}\Delta_0(m+2, m+k-1; x)p_{m-1}(x) \\ &= \Delta_0(1, m-1; x)\Big\{(x - b_0^{(m)})\Delta_0(m+2, m+k-1; x) \\ &\quad - \ a_0^{(m+1)}\Delta_0(m+3, m+k-1; x) - a_0^{(m)}\eta_{k-1-m}(x)p_{m-1}(x)\Big\} \\ &= \theta_m(x)(\pi_k(x) - r) = \theta_m(x)q_1(\pi_k(x)), \end{aligned}$$

where θ_m is defined by (2.2). We notice that in the above computations for the second equality we have used hypothesis (iii) and the third one can be justified also by (iii) as follows:

$$\begin{aligned} &(x - b_0^{(m)})\Delta_0(m+2, m+k-1; x) - a_0^{(m+1)}\ \Delta_0(m+3, m+k-1; x) \\ &\quad - a_0^{(m)}\eta_{k-1-m}(x)p_{m-1}(x) \\ &= [(x - b_0^{(m)})p_m(x) - a_0^{(m)}\, p_{m-1}(x)]\, \eta_{k-1-m}(x) - a_0^{(m+1)}\ \Delta_0(m+3, m+k-1; x) \\ &= p_{m+1}(x)\eta_{k-1-m}(x) - a_0^{(m+1)}\ \Delta_0(m+3, m+k-1; x) \\ &= \pi_k(x) - r. \end{aligned}$$

Thus (2.12) is true for all $n = 0, 1, 2, \ldots$, from which it follows by induction that (2.1) holds. We also notice that (2.5) is an immediate consequence of (2.9) and (2.1), taking into account hypothesis (iii).

Conversely, suppose that there exist polynomials π_k and θ_m, of degrees k and m (resp.), with $0 \le m \le k-1$, and a monic OPS $(q_n)_n$ such that $(p_n)_n$ satisfies (2.1), and let us prove that the four conditions (i)–(iv) must hold for all $n = 0, 1, 2, \ldots$. Setting $n = 0$ and $n = 1$ in (2.1) we find

$$\theta_m(x) = p_m(x) = \Delta_0(1, m-1; x), \qquad p_{m+k}(x) = \theta_m(x)(\pi_k(x) - r).$$

But, since

$$\begin{aligned} p_{m+k}(x) &= \Delta_0(1, k+m-1; x) \\ &= \Delta_0(1, m-1; x)\Delta_0(m+1, m+k-1; x) \\ &\quad - a_0^{(m)}\Delta_0(1, m-2; x)\Delta_0(m+2, m+k-1; x), \end{aligned}$$

we obtain

$$\begin{aligned} \Delta_0(1, m-1; x)(\pi_k(x) - r) &= \Delta_0(1, m-1; x)\Delta_0(m+1, m+k-1; x) \\ &\quad - a_0^{(m)}\Delta_0(1, m-2; x)\Delta_0(m+2, m+k-1; x), \end{aligned}$$

and so, since $\Delta_0(1, m-1; \cdot) \equiv p_m$ and $\Delta_0(1, m-2; x) \equiv p_{m-1}$ cannot have common zeros, we see that $\Delta_0(m+2, m+k-1; x)$ must be divisible by $\Delta_0(1, m-1; \cdot)$, so condition (iii) holds.

As a consequence, we also conclude that

$$\pi_k(x) = \Delta_0(m+1, m+k-1; x) - a_0^{(m)} \Delta_0(1, m-2; x)\eta_{k-1-m}(x) + r,$$

which can be rewritten as in (2.2) (see Remark 2.3). Since $(q_n)_n$ is a monic OPS then it fulfills a three-term recurrence as (2.3), where $(r_n)_n$ and $(s_n)_n$ are some sequences of complex numbers with $s_n \neq 0$ for all $n = 1, 2, \ldots$. Changing x into $\pi_k(x)$ in this recurrence relation and then multiplying both sides of the resulting equation by $\theta_m(x)$, we find

$$p_{(n+1)k+m}(x) = (\pi_k(x) - r_n) p_{nk+m}(x) - s_n p_{(n-1)k+m}(x) \tag{2.13}$$

for all $n = 0, 1, 2, \ldots$. On the other hand, from (2.10) and (2.11) we obtain

$$\begin{aligned} &\Delta_{n-1}(m+2, m+k-1; x) p_{(n+1)k+m}(x) \\ &\quad = \Big\{ (x - b_n^{(m)}) \Delta_{n-1}(m+2, m+k-1; x) \Delta_n(m+2, m+k-1; x) \\ &\qquad - a_n^{(m+1)} \Delta_{n-1}(m+2, m+k-1; x) \Delta_n(m+3, m+k-1; x) \\ &\qquad - a_n^{(m)} \Delta_{n-1}(m+2, m+k-2; x) \Delta_n(m+2, m+k-1; x) \Big\} p_{nk+m}(x) \\ &\qquad - a_n^{(m)} a_{n-1}^{(m+1)} \ldots a_{n-1}^{(m+k-1)} \Delta_n(m+2, m+k-1; x) p_{(n-1)k+m}(x) \end{aligned} \tag{2.14}$$

for all $n = 1, 2, \ldots$. Substituting (2.13) in the left-hand side of (2.14) we derive

$$\begin{aligned} &\Big\{ s_n \Delta_{n-1}(m+2, m+k-1; x) \\ &\qquad - a_n^{(m)} a_{n-1}^{(m+1)} \ldots a_{n-1}^{(m+k-1)} \Delta_n(m+2, m+k-1; x) \Big\} p_{(n-1)k+m}(x) \\ &\quad = \Big\{ (\pi_k(x) - r_n) \Delta_{n-1}(m+2, m+k-1; x) \\ &\qquad - (x - b_n^{(m)}) \Delta_{n-1}(m+2, m+k-1; x) \Delta_n(m+2, m+k-1; x) \\ &\qquad + a_n^{(m+1)} \Delta_{n-1}(m+2, m+k-1; x) \Delta_n(m+3, m+k-1; x) \\ &\qquad + a_n^{(m)} \Delta_{n-1}(m+2, m+k-2; x) \Delta_n(m+2, m+k-1; x) \Big\} p_{nk+m}(x) \end{aligned}$$

for all $n = 1, 2, \ldots$. Looking at this equality we see that the left-hand side is a polynomial of degree at most $nk + m - 1$, while the right-hand side is either zero or a polynomial of degree at least $nk + m$, and so we may conclude that both polynomials inside {} in the two sides of the last equality must be zero; hence

$$s_n \Delta_{n-1}(m+2, m+k-1; x) = a_n^{(m)} a_{n-1}^{(m+1)} \ldots a_{n-1}^{(m+k-1)} \Delta_n(m+2, m+k-1; x) \tag{2.15}$$

for all $n = 1, 2, \ldots$, and

$$\begin{aligned} &(\pi_k(x) - r_n) \Delta_{n-1}(m+2, m+k-1; x) \\ &\quad = (x - b_n^{(m)}) \Delta_{n-1}(m+2, m+k-1; x) \Delta_n(m+2, m+k-1; x) \\ &\qquad - a_n^{(m+1)} \Delta_{n-1}(m+2, m+k-1; x) \Delta_n(m+3, m+k-1; x) \\ &\qquad - a_n^{(m)} \Delta_{n-1}(m+2, m+k-2; x) \Delta_n(m+2, m+k-1; x) \end{aligned} \tag{2.16}$$

for all $n = 1, 2, \ldots$. Since $\Delta_{n-1}(m+2, m+k-1; \cdot)$ and $\Delta_n(m+2, m+k-1; \cdot)$ are monic polynomials it follows immediately from (2.15) that the expression for s_n given by (2.4) holds and, as a consequence, also

$$\Delta_{n-1}(m+2, m+k-1; x) = \Delta_n(m+2, m+k-1; x)$$

for every $n = 1, 2, \ldots$, which proves that condition (ii) holds. Then from (2.16) we obtain

$$\begin{aligned} r_n &= \pi_k(x) - (x - b_n^{(m)})\Delta_0(m+2, m+k-1; x) \\ &\quad + a_n^{(m+1)}\Delta_n(m+3, m+k-1; x) + a_n^{(m)}\Delta_{n-1}(m+2, m+k-2; x) \\ &= (b_n^{(m)} - b_0^{(m)})\Delta_0(m+2, m+k-1; x) + r + r_n(x) \end{aligned} \tag{2.17}$$

for all $n = 1, 2, \ldots$, where the last equality may be justified by using the expression (2.2) for $\pi_k(x)$, already proved (see also Remark 2.3), and taking into account the definition of $r_n(x)$. Therefore, since the left-hand side of (2.17) is independent of x, it follows that the right-hand side also must be independent of x, and so, taking into account that $\Delta_0(m+2, m+k-1; x)$ is a monic polynomial of degree $k-1$ and $r_n(x)$ is a polynomial of degree at most $k-2$, we conclude that $b_n^{(m)} - b_0^{(m)} = 0$ for all $n = 1, 2, \ldots$, i.e., condition (i) holds. Finally, (iv) and the expressions for r_n as in (2.4) are immediate consequences of (i) and (2.17). This completes the proof. $\square$

Remark 2.2. If $m = 0$ then hypothesis (iii) in Theorem 2.1 is always fulfilled, since in such a case we see that $\theta_m(x) = \theta_0(x) \equiv 1$, where $\eta_{k-1-m}(x) = \eta_{k-1}(x) = \Delta_0(2, k-1; x)$. On the other hand, if $m = k-1$, then hypotheses (ii) and (iii) together are equivalent to

$$\Delta_n(1, k-2; x) = \Delta_0(1, k-2; x) \equiv \theta_{k-1}(x), \quad n = 0, 1, 2, \ldots.$$

In fact, if $m = k-1$ then hypothesis (ii) and equality (1.7) give $\Delta_n(m+2, m+k-1; x) = \Delta_0(m+2, m+k-1; x) = \Delta_0(k+1, k+k-2; x) = \Delta_1(1, k-2; x) = \theta_{k-1}(x)$, the last equality being justified by hypothesis (iii); in such a case, of course, we have $\eta_{k-1-m}(x) = \eta_0(x) \equiv 1$.

Remark 2.3. It follows from the proof of Theorem 2.1 that the polynomial π_k also admits the following alternative representations:

$$\begin{aligned} \pi_k(x) &= r + (x - b_0^{(m)})\Delta_0(m+2, m+k-1; x) \\ &\quad - a_0^{(m+1)}\,\Delta_0(m+3, m+k-1; x) - a_0^{(m)}\eta_{k-1-m}(x)p_{m-1}(x) \\ &= \Delta_0(m+1, m+k-1; x) - a_0^{(m)}\Delta_0(1, m-2; x)\eta_{k-1-m}(x) + r. \end{aligned} \tag{2.18}$$

Remark 2.4. In order to have independence of x, the leading coefficient in the polynomial $r_n(x)$ must vanish; hence under the conditions of the theorem, if $k > 2$ it follows that

$$a_n^{(m+1)} + a_n^{(m)} = \begin{cases} a_0^{(m+1)} + a_0^{(m)} & \text{if } m \in \{1, \ldots, k-1\} \\ a_0^{(1)} & \text{if } m = 0 \end{cases}$$

must hold for all $n = 1, 2, \ldots$ (i.e., the left-hand side of this equality must be independent of n).

3. Computation of the orthogonality measure

In this section we will analyze the so-called positive-definite case, which corresponds to the situation when the OPS $(p_n)_n$ in Theorem 2.1 is orthogonal with respect to a positive Borel measure (and so also $(q_n)_n$ is an OPS in the positive-definite sense). Our aim is to determine the orthogonality measure for the sequence $(p_n)_n$ in terms of the orthogonality measure for the sequence $(q_n)_n$. For this we first establish a relation between an appropriate subsequence of the monic OPS of the numerator polynomials, $(p_n^{(1)})_n$, and the polynomials of the monic OPS's $(q_n)_n$

and $(q_n^{(1)})_n$ (see Lemma 3.3 below). This will give us the relation between the Stieltjes transforms corresponding to the sequences $(p_n)_n$ and $(q_n)_n$ and so the relation between their orthogonality measures will follow by Markov's theorem. In fact, we need the following improved version of Markov's theorem, due to W. Van Assche (stated here for monic OPS's).

Lemma 3.1 *([36]). Let $(p_n)_n$ be a monic OPS, orthogonal with respect to some positive measure $\mathrm{d}\sigma$. If the moment problem for $\mathrm{d}\sigma$ is determined, then*

$$-u_0 \lim_{n\to+\infty} \frac{p_{n-1}^{(1)}(z)}{p_n(z)} = \int_{\mathbb{R}} \frac{\mathrm{d}\sigma(x)}{x-z} =: F(z;\mathrm{d}\sigma) \tag{3.1}$$

uniformly on compact subsets of $\mathbb{C}\setminus(X_1\cup X_2)$.

Here $u_0 := \int_{\mathbb{R}} \mathrm{d}\sigma(x)$ (the first moment of the measure $\mathrm{d}\sigma$) and, denoting by $x_{n,1} < \cdots < x_{n,n}$ the zeros of the polynomial $p_n(x)$ and setting

$$Z_1 := \{x_{n,j} \mid 1 \le j \le n,\ n = 1,2,3,\ldots\},$$

the sets X_1 and X_2 are defined by (cf. [11, p. 61])

$$X_1 := Z_1' \equiv \{\text{accumulation points of } Z_1\},$$
$$X_2 := \{x \in Z_1 \mid p_n(x) = 0 \text{ for infinitely many } n\}.$$

We notice the following chain of inclusions:

$$\mathrm{supp}(\mathrm{d}\psi) \subset X_1 \cup X_2 \subset [\xi_1,\eta_1] \subset \mathrm{co}(\mathrm{supp}(\mathrm{d}\psi)), \tag{3.2}$$

where $\mathrm{co}(\mathrm{supp}(\mathrm{d}\psi))$ denotes the convex hull of the set $\mathrm{supp}(\mathrm{d}\psi)$, i.e., the smallest closed interval containing $\mathrm{supp}(\mathrm{d}\psi)$, and $[\xi_1,\eta_1]$ is the true interval of orthogonality of the sequence $(p_n)_n$ (cf. [11, p. 29]). The function $F(\cdot;\mathrm{d}\psi)$ is called the Stieltjes transform of the measure $\mathrm{d}\psi$.

We need also the following result which has been stated in [24]. The proof is based on the ideas presented in [15].

Lemma 3.2 *([24]). Let τ be a given distribution function with $\mathrm{supp}(\mathrm{d}\tau) \subset [\xi,\eta]$, $-\infty < \xi < \eta < +\infty$. Let T be a real and monic polynomial of degree $k \ge 2$ such that the derivative T' has $k-1$ real and distinct zeros, denoted in increasing order by $y_1 < y_2 < \cdots < y_{k-1}$. Assume that either $T(y_{2i-1}) \ge \eta$ and $T(y_{2i}) \le \xi$ (for all possible i) if k is odd, or $T(y_{2i-1}) \le \xi$ and $T(y_{2i}) \ge \eta$ if k is even. Let A and B be two real and monic polynomials such that $\deg(A) = k-1-m$ and $\deg(B) = m$, with $0 \le m \le k-1$. Assume also that the zeros of AB are real and distinct, AB and T' have the same sign in each point of the set $T^{-1}([\xi,\eta])$ and, if $m \ge 1$,*

$$\int_\xi^\eta \frac{\mathrm{d}\tau(y)}{|y - T(b_j)|} < +\infty$$

for $j = 1,\ldots,m$, where $b_1, b_2, \ldots, b_m$ denote the zeros of B. Let

$$F(z) := \frac{1}{B(z)}[A(z)F(T(z);\mathrm{d}\tau) - L_{m-1}(z)], \quad z \in \mathbb{C}\setminus T^{-1}([\xi,\eta]),$$

where $L_{m-1}(z) := \sum_{j=1}^m \lambda_j B(z)/(z-b_j)$, $\lambda_j := A(b_j)F(T(b_j);\mathrm{d}\tau)/B'(b_j)$ for all $j = 1,\ldots,m (L_0(z) \equiv 0)$, i.e., L_{m-1} is the Lagrange interpolating polynomial of degree $m-1$

that coincides with $A(z)F(T(z); \mathrm{d}\tau)$ at the zeros of B. Then, F is the Stieltjes transform of the distribution σ defined by

$$\mathrm{d}\sigma(x) := \left|\frac{A(x)}{B(x)}\right| \frac{\mathrm{d}\tau(T(x))}{T'(x)},$$

whose support is contained in the set $T^{-1}([\xi, \eta])$ (a union of k intervals).

The next proposition gives a relation between the associated polynomials of the first kind of the sequences $(p_n)_n$ and $(q_n)_n$. We notice that F. Peherstorfer stated essentially the same relation (cf. [27, Theorem 3.4]) on the basis of his previous results contained in [26], considering as the starting point that one knows a priori the relation between the moment linear functionals associated with $(p_n)_n$ and $(q_n)_n$ (i.e., from the viewpoint of a "direct problem"). Our proof does not assume previous knowledge of the relation between the orthogonality measures (so we follow the "inverse problem" point of view).

Lemma 3.3. *Under the conditions of Theorem 2.1,*

$$p^{(1)}_{nk+m-1}(x) = \Delta_0(2, m-1; x)q_n(\pi_k(x)) + \left(\prod_{j=1}^{m} a_0^{(j)}\right) \eta_{k-1-m}(x) q^{(1)}_{n-1}(\pi_k(x))$$

for all $n = 0, 1, 2, \ldots$.

Proof. We know the three-term recurrence relation (2.3) for the q_n's, also satisfied by the numerator polynomials $q^{(1)}_{n-1}$'s, with $q^{(1)}_{-1} \equiv 0$ and $q^{(1)}_0 \equiv 1$. According to (2.1), the contraction of the three-term recurrence relation for the p_n's must yield (2.3) with argument $\pi_k(x)$. The numerator polynomials $p^{(1)}_{n-1}$ satisfy the same three-term recurrence relation as p_n, so $p^{(1)}_{nk+m-1}(x)$ must be a linear combination of the two solutions of (2.3) with argument $\pi_k(x)$; hence

$$p^{(1)}_{nk+m-1}(x) = \alpha(x)q_n(\pi_k(x)) + \beta(x)q^{(1)}_{n-1}(\pi_k(x)), \quad n = 0, 1, 2, \ldots \tag{3.3}$$

with $\alpha(x)$ and $\beta(x)$ two polynomials in x, independent of n. In fact, to prove (3.3), consider the following difference equation of second order:

$$y_{n+1}(x) = (\pi_k(x) - r_n)y_n(x) - s_n y_{n-1}(x), \quad n = 1, 2, \ldots, \tag{3.4}$$

r_n and s_n being defined as in (2.4) and x is regarded as a parameter. According to (2.3), the solution of (3.4) which satisfies the initial conditions $y_0(x) = 1$ and $y_1(x) = \pi_k(x) - r_0$ is given by $y_n(x) = q_n(\pi_k(x))$ ($n = 0, 1, 2, \ldots$). On the other hand, since the sequence of the associated polynomials of the first kind $\{q^{(1)}_{n-1}\}_{n=1}^{\infty}$ also satisfies the same recurrence (2.3) as is satisfied by $\{q_n\}_{n=0}^{\infty}$, but with initial conditions $y_0(x) = 1$ and $y_1(x) = x - r_1$, then the solution of (3.4) which satisfies the initial conditions $y_0(x) = 1$ and $y_1(x) = \pi_k(x) - r_1$ is given by $y_n(x) = q^{(1)}_{n-1}(\pi_k(x))$ ($n = 0, 1, 2, \ldots$). These two solutions of (3.4) are linearly independent since their Wronskian never vanishes. In fact, by a well-known relation linking the polynomials of a given monic OPS and the corresponding numerator polynomials (see [11, p. 86]), we can write

$$\begin{vmatrix} q_n(\pi_k(x)) & q^{(1)}_{n-1}(\pi_k(x)) \\ q_{n+1}(\pi_k(x)) & q^{(1)}_n(\pi_k(x)) \end{vmatrix} = s_1 s_2 \cdots s_n \neq 0, \quad n = 1, 2, \ldots.$$

Consequently, the general solution of the difference equation (3.4) is

$$y_n(x) = C_1(x)q_n(\pi_k(x)) + C_2(x)q_{n-1}^{(1)}(\pi_k(x)), \quad n = 0, 1, 2, \ldots,$$

where $C_1(x)$ and $C_2(x)$ are independent of n. Therefore, to prove (3.3), we only need to show that $y_n(x) = p_{nk+m-1}^{(1)}(x)$ also satisfies (3.4). In order to prove this, denote by $\mathbf{u}$ the moment linear functional with respect to which $(p_n)_n$ is orthogonal. Then, using (2.1) and setting $u_0 := \langle \mathbf{u}, 1\rangle$, we can write (the subscript y in $\mathbf{u}_y$ means that $\mathbf{u}$ acts on functions of the variable y)

$$\begin{aligned}
p_{nk+m-1}^{(1)}(x) &= \frac{1}{u_0}\left\langle \mathbf{u}_y, \frac{p_{nk+m}(x) - p_{nk+m}(y)}{x-y}\right\rangle \\
&= \frac{1}{u_0}\left\langle \mathbf{u}_y, \frac{\theta_m(x)q_n(\pi_k(x)) - \theta_m(y)q_n(\pi_k(y))}{x-y}\right\rangle \\
&= \frac{1}{u_0}\left\langle \mathbf{u}_y, \frac{\theta_m(x) - \theta_m(y)}{x-y}\right\rangle q_n(\pi_k(x)) \\
&\quad + \frac{1}{u_0}\left\langle \mathbf{u}_y, \theta_m(y)\frac{q_n(\pi_k(x)) - q_n(\pi_k(y))}{x-y}\right\rangle \\
&= p_{m-1}^{(1)}(x)q_n(\pi_k(x)) + R_n(x),
\end{aligned}$$

where

$$R_n(x) := \frac{1}{u_0}\left\langle \mathbf{u}_y, \theta_m(y)\frac{q_n(\pi_k(x)) - q_n(\pi_k(y))}{x-y}\right\rangle, \quad n = 0, 1, 2, \ldots.$$

Hence, to prove (3.3), it is sufficient to show that $y_n(x) = R_n(x)$ is also a solution of the difference equation (3.4). In fact, using the three-term recurrence relation (2.3) for $(q_n)_n$, we derive

$$\begin{aligned}
&\pi_k(x)q_n(\pi_k(x)) - \pi_k(x)q_n(\pi_k(y)) \\
&\quad = [q_{n+1}(\pi_k(x)) - q_{n+1}(\pi_k(y))] + r_n\,[q_n(\pi_k(x)) - q_n(\pi_k(y))] \\
&\qquad + s_n[q_{n-1}(\pi_k(x)) - q_{n-1}(\pi_k(y))] - [\pi_k(x) - \pi_k(y)]q_n(\pi_k(y)).
\end{aligned}$$

Hence,

$$\begin{aligned}
\pi_k(x)R_n(x) &= \frac{1}{u_0}\left\langle \mathbf{u}_y, \theta_m(y)\frac{\pi_k(x)q_n(\pi_k(x)) - \pi_k(x)q_n(\pi_k(y))}{x-y}\right\rangle \\
&= R_{n+1}(x) + r_nR_n(x) + s_nR_{n-1}(x) \\
&\quad - \frac{1}{u_0}\left\langle \mathbf{u}_y, \frac{\pi_k(x) - \pi_k(y)}{x-y}\,\theta_m(y)q_n(\pi_k(y))\right\rangle
\end{aligned}$$

for every $n = 0, 1, 2, \ldots$. On the other hand, $\theta_m(y)q_n(\pi_k(y)) = p_{nk+m}(y)$ and $\varrho_{k-1}(y;x) := (\pi_k(x) - \pi_k(y))/(x-y)$ is a polynomial on the variable y of degree $k-1$, so we can write

$$\left\langle \mathbf{u}_y, \frac{\pi_k(x) - \pi_k(y)}{x-y}\,\theta_m(y)q_n(\pi_k(y))\right\rangle = \langle \mathbf{u}_y, \varrho_{k-1}(y;x)p_{nk+m}(y)\rangle = 0$$

for every $n = 1, 2, \ldots$, the last equality being justified by the orthogonality of $(p_n)_n$ with respect to $\mathbf{u}$. Therefore we conclude that $R_n(x)$ satisfies (3.4), implying that $p_{nk+m-1}^{(1)}(x)$ is of the form indicated in (3.3). But, for $n = 0$ we see from (3.3) that

$$\alpha(x) = p_{m-1}^{(1)}(x) = \Delta_0(2, m-1; x),$$

and using (2.2) with $n = 1$, we find

$$\begin{aligned}\beta(x) &= p^{(1)}_{k+m-1}(x) - \alpha(x)(\pi_k(x) - r) \\ &= \Delta_0(2, k+m-1; x) - \Delta_0(2, m-1; x)\{p_{m+1}(x)\eta_{k-1-m}(x) \\ &\quad - a_0^{(m+1)}\,\Delta_0(m+3, m+k-1; x)\} \\ &= \Delta_0(2, m; x)\Delta_0(m+2, m+k-1; x) - \Delta_0(2, m-1; x)p_{m+1}(x)\eta_{k-1-m}(x) \\ &= (\Delta_0(2, m; x)p_m(x) - \Delta_0(2, m-1; x)p_{m+1}(x))\,\eta_{k-1-m}(x) \\ &= (p^{(1)}_m(x)p_m(x) - p^{(1)}_{m-1}(x)p_{m+1}(x))\eta_{k-1-m}(x) \\ &= \left(\prod_{j=1}^{m} a_0^{(j)}\right)\eta_{k-1-m}(x),\end{aligned}$$

which completes the proof. $\square$

We are ready to analyze the case where $(p_n)_n$ is a monic OPS in the positive-definite sense. This is equivalent to saying that the conditions

$$b_n^{(j)} \in \mathbb{R}, \qquad a_n^{(j)} > 0 \quad (j = 0, 1, \ldots, k-1;\ n = 0, 1, 2 \ldots)$$

hold. For simplicity, here and in the rest of the paper, we choose the parameter r in Theorem 2.1 to be

$$r = 0.$$

Under these conditions, since

$$p_m(x) = \theta_m(x), \qquad p_{k+m}(x) = \theta_m(x)q_1(\pi_k(x)) = \theta_m(x)\pi_k(x),$$

it follows that both θ_m and π_k have all zeros real and distinct (recall that in the positive-definite case all the zeros of any orthogonal polynomial p_n are real and distinct). So, by Rolle's theorem, also π_k' has $k - 1$ real and distinct zeros.

Theorem 3.4. *Under the conditions of Theorem 2.1, choose $r = 0$ and assume that $(p_n)_n$ is a monic OPS in the positive-definite sense, orthogonal with respect to some measure $\mathrm{d}\sigma$. Then $(q_n)_n$ is also a monic OPS in the positive-definite sense, orthogonal with respect to a measure $\mathrm{d}\tau$. Further, assume that the following four conditions hold:*

(i) $\mathrm{supp}(\mathrm{d}\tau)$ *is a compact set, so*

$$-\infty < \xi := \min \mathrm{supp}(\mathrm{d}\tau) < \eta := \max \mathrm{supp}(\mathrm{d}\tau) < +\infty;$$

(ii) *if $m \geq 1$,*

$$\int_\xi^\eta \frac{\mathrm{d}\tau(x)}{|x - \pi_k(z_i)|} < \infty, \quad i = 1, 2, \ldots, m,$$

where $z_1 < z_2 < \cdots < z_m$ are the zeros of θ_m;

(iii) *either $\pi_k(y_{2i-1}) \geq \eta$ and $\pi_k(y_{2i}) \leq \xi$ (for all possible i) if k is odd, or $\pi_k(y_{2i-1}) \leq \xi$ and $\pi_k(y_{2i}) \geq \eta$ if k is even, where $y_1 < \cdots < y_{k-1}$ denote the zeros of π_k';*

(iv) *$\theta_m\eta_{k-1-m}$ and π_k' have the same sign at each point of the set $\pi_k^{-1}([\xi, \eta])$.*

Then the Stieltjes transforms $F(\cdot; \mathrm{d}\sigma)$ *and* $F(\cdot; \mathrm{d}\tau)$ *are related by*

$$F(z; \mathrm{d}\sigma) = \frac{-v_0\, \Delta_0(2, m-1; z) + \left(\prod_{j=1}^{m} a_0^{(j)}\right) \eta_{k-1-m}(z) F(\pi_k(z); \mathrm{d}\tau)}{\theta_m(z)},$$
$$z \in \mathbb{C} \setminus (\pi_k^{-1}([\xi, \eta]) \cup \{z_1, \dots, z_m\}),$$

where the normalization condition $v_0 := \int_\xi^\eta \mathrm{d}\tau = \int_{\mathrm{supp}(\mathrm{d}\sigma)} \mathrm{d}\sigma =: u_0$ *is assumed. Further, the measure* $\mathrm{d}\sigma$ *can be obtained from* $\mathrm{d}\tau$ *by*

$$\mathrm{d}\sigma(x) = \sum_{i=1}^{m} M_i\, \delta(x - z_i)\mathrm{d}x + \left|\frac{\eta_{k-1-m}(x)}{\theta_m(x)}\right| \frac{\mathrm{d}\tau(\pi_k(x))}{\pi_k'(x)} \tag{3.5}$$

(up to constant factors), where if $m \geq 1$

$$M_i := \frac{v_0\, \Delta_0(2, m-1; z_i) / \left(\prod_{j=1}^{m} a_0^{(j)}\right) - \eta_{k-1-m}(z_i) F(\pi_k(z_i); \mathrm{d}\tau)}{\theta_m'(z_i)} \geq 0 \tag{3.6}$$

for all $i = 1, \dots, m$. *Notice that the support of* $\mathrm{d}\sigma$ *is contained in the set*

$$\pi_k^{-1}([\xi, \eta]) \cup \{z_1, \dots, z_m\},$$

a union of k *intervals and* m *possible mass points.*

Proof. By Markov's theorem (Lemma 3.1) and taking into account Lemma 3.3 and Theorem 2.1, for every $z \in \mathbb{C} \setminus \pi_k^{-1}([\xi, \eta]) \cup \{z_1, \dots, z_m\}$, we deduce

$$\begin{aligned} F(z; \mathrm{d}\sigma) &= -u_0 \lim_{n\to\infty} \frac{p^{(1)}_{nk+m-1}(z)}{p_{nk+m}(z)} \\ &= \frac{-v_0\, \Delta_0(2, m-1; z) + \left(\prod_{j=1}^{m} a_0^{(j)}\right) \eta_{k-1-m}(z) F(\pi_k(z); \mathrm{d}\tau)}{\theta_m(z)} \\ &= \frac{-v_0\, \Delta_0(2, m-1; z) + \left(\prod_{j=1}^{m} a_0^{(j)}\right) L_{m-1}(z)}{\theta_m(z)} \\ &\quad + \left(\prod_{j=1}^{m} a_0^{(j)}\right) \frac{\eta_{k-1-m}(z) F(\pi_k(z); \mathrm{d}\tau) - L_{m-1}(z)}{\theta_m(z)} \\ &= \left(\prod_{j=1}^{m} a_0^{(j)}\right) \left(\sum_{i=1}^{m} \frac{M_i}{z_i - z} + \frac{\eta_{k-1-m}(z) F(\pi_k(z); \mathrm{d}\tau) - L_{m-1}(z)}{\theta_m(z)}\right), \end{aligned}$$

where L_{m-1} is the Lagrange interpolator polynomial of degree $m-1$ that coincides with $\eta_{k-1-m} F(\pi_k(\cdot); \mathrm{d}\tau)$ at the zeros of θ_m. Hence we obtain the representation (3.5) from Lemma 3.2, with $A = \eta_{k-1-m}$ and $B = \theta_m$.

It remains to prove (3.6), i.e., $M_i \geq 0$ for all $i = 1, \ldots, m$. Let $(\widetilde{p}_n)_n$ be the orthonormal sequence corresponding to the monic OPS $(p_n)_n$, so

$$\widetilde{p}_n(x) = \left(u_0 \prod_{i=1}^{n} \gamma_i\right)^{-\frac{1}{2}} p_n(x), \quad n = 0, 1, 2, \ldots, \tag{3.7}$$

where $\gamma_{nk+j} := a_n^{(j)}$ for all $j = 0, 1, \ldots, k-1$ and $n = 0, 1, 2, \ldots$. Recall that

$$p_{n+1}(x) p_{n+1}^{(1)}(x) - p_{n+2}(x) p_n^{(1)}(x) = \prod_{j=1}^{n+1} \gamma_j, \quad n = 0, 1, 2, \ldots. \tag{3.8}$$

Changing n to $nk + m - 1$ in (3.8) and setting $x = z_i$ in the resulting equation, we find

$$p_{nk+m+1}(z_i) p_{nk+m-1}^{(1)}(z_i) = -\prod_{j=1}^{nk+m} \gamma_j = -\prod_{j=1}^{nk} \gamma_j \prod_{j=1}^{m} a_n^{(j)} \tag{3.9}$$

for all $n = 0, 1, 2, \ldots$ and each fixed $i = 1, \ldots, m$. Further, taking $j = m$ in (1.2) and then setting again $x = z_i$ in the resulting equation, we obtain

$$a_n^{(m)} = -\frac{p_{nk+m+1}(z_i)}{p_{nk+m-1}(z_i)} \quad (i = 1, \ldots, m;\ n = 0, 1, 2, \ldots). \tag{3.10}$$

(Notice that $p_{nk+m-1}(z_i) \neq 0$ because z_i is a zero of p_{nk+m} for every n and the orthogonal polynomials p_{nk+m} and p_{nk+m-1} cannot have common zeros.) Combining relations (3.7), (3.9) and (3.10), we deduce

$$\widetilde{p}_{kn}^{2}(z_i) = \frac{p_{kn}^2(z_i)}{u_0 \prod_{j=1}^{nk} \gamma_j} = \frac{p_{kn}^2(z_i) \prod_{j=1}^{m-1} a_n^{(j)}}{u_0 p_{nk+m-1}(z_i) p_{nk+m-1}^{(1)}(z_i)} \tag{3.11}$$

for all $n = 0, 1, 2, \ldots$ and $i = 1, \ldots, m$. Now, by Lemma 3.3, for all x such that $q_n(\pi_k(x)) \neq 0$ we can write

$$u_0 p_{nk+m-1}^{(1)}(x) = \left(\prod_{j=1}^{m} a_0^{(j)}\right) q_n(\pi_k(x)) F_n(x), \tag{3.12}$$

where

$$F_n(x) := u_0 \Delta_0(2, m-1; x) \Big/ \left(\prod_{j=1}^{m} a_0^{(j)}\right) - \eta_{k-1-m}(x) \left(-u_0 \frac{q_{n-1}^{(1)}(\pi_k(x))}{q_n(\pi_k(x))}\right).$$

Since $p_{kn+m}(x) = \theta_m(x) q_n(\pi_k(x))$ we derive $p'_{kn+m}(z_i) = \theta'_m(z_i) q_n(\pi_k(z_i))$ for all $i = 1, \ldots, m$. From this, since $p'_{kn+m}(z_i) \neq 0$ (because p_{nk+m} and p'_{nk+m} do not have common zeros) and $\theta'_m(z_i) \neq 0$, it follows that also $q_n(\pi_k(z_i)) \neq 0$ for all $n = 0, 1, 2, \ldots$ and $i = 1, \ldots, m$. Henceforth, from (3.12) we obtain

$$u_0 p_{nk+m-1}^{(1)}(z_i) = \left(\prod_{j=1}^{m} a_0^{(j)}\right) p'_{kn+m}(z_i) F_n(z_i) / \theta'_m(z_i)$$

for all $n = 0, 1, 2, \ldots$ and $i = 1, \ldots, m$. Substituting into (3.11) we obtain

$$\widetilde{p}_{kn}^{2}(z_i) = \frac{p_{kn}^{2}(z_i)}{p_{nk+m-1}(z_i)\,p'_{kn+m}(z_i)} \cdot \frac{\prod_{j=1}^{m-1} a_n^{(j)}}{\prod_{j=1}^{m} a_0^{(j)}} \cdot \frac{\theta'_m(z_i)}{F_n(z_i)} \tag{3.13}$$

for all $n = 0, 1, 2, \ldots$ and $i = 1, \ldots, m$. Now, using the inequality (cf. [11, p. 24]) $p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x) > 0$ ($n = 0, 1, 2, \ldots$), changing n into $nk + m - 1$ and setting $x = z_i$ we deduce $p_{nk+m-1}(z_i)p'_{nk+m}(z_i) > 0$ for all $n = 0, 1, 2, \ldots$ and $i = 1, \ldots, m$. Hence, from (3.13) it follows that $\theta'_m(z_i)/F_n(z_i) > 0$ for all $n = 0, 1, 2, \ldots$ and $i = 1, \ldots, m$. Therefore,

$$M_i = \lim_{n\to+\infty} \frac{F_n(z_i)}{\theta'_m(z_i)} \geq 0$$

for all $i = 1, \ldots, m$. $\square$

Remark 3.5. Under the conditions of Theorem 3.4, if $d\tau$ is an absolutely continuous measure with density w_τ (a weight function), then the absolutely continuous part of $d\sigma$ has density

$$w_\sigma(x) := \left|\frac{\eta_{k-1-m}(x)}{\theta_m(x)}\right| w_\tau(\pi_k(x))$$

with support contained in a union of at most k closed intervals, and mass points may appear at the zeros of θ_m.

4. The case $m = 0$

4.1. The case $m = 0$

As a first application involving the results of the previous sections, we will analyze the case $k \geq 2$ and $m = 0$, and we will see that in this case Theorem 2.1 gives conditions for the existence of a polynomial mapping in the sense described by Geronimo and Van Assche in [15]. In fact, such conditions were given by Charris et al. in [10], although there they were not stated explicitly as a theorem.

Theorem 4.1. *Let $(p_n)_n$ be a monic OPS characterized by the general block of recurrence relations (1.2), and define $\Delta_0(x) := 0$ and*

$$\Delta_n(x) := a_n^{(0)}\Delta_{n-1}(2, k-2; x) + a_n^{(1)}\Delta_n(3, k-1; x) - a_0^{(1)}\Delta_0(3, k-1; x) \tag{4.1}$$

for $n = 1, 2, 3, \ldots$. Assume that, for all $n = 0, 1, 2, \ldots$, the following two conditions hold:

(i) *$b_n^{(0)}$ and $\Delta_n(2, k-1; x)$ are independent of n for every x;*
(ii) *$\Delta_n(x)$ is independent of x for every n.*

Fix $b, c \in \mathbb{C}$, with $c \neq 0$, and define a polynomial T (of degree k) as

$$T(x) := c\,(\Delta_0(1, k-1; x) - b).$$

Let $(\widetilde{q}_n)_n$ be the monic OPS generated by the three-term recurrence relation

$$\widetilde{q}_{n+1}(x) = (x - \widetilde{r}_n)\widetilde{q}_n(x) - \widetilde{s}_n\widetilde{q}_{n-1}(x), \quad n = 0, 1, 2, \ldots$$

with initial conditions $\widetilde{q}_{-1}(x) = 0$ *and* $\widetilde{q}_0(x) = 1$, *where*

$$\widetilde{r}_n := c(\Delta_n(0) - b), \qquad \widetilde{s}_n := c^2 a_n^{(0)} a_{n-1}^{(1)} \dots a_{n-1}^{(k-1)}.$$

Then

$$p_{kn}(x) = c^{-n} \widetilde{q}_n(T(x)), \quad n = 0, 1, 2, \dots$$

so $(p_n)_n$ *is obtained from* $(\widetilde{q}_n)_n$ *via the polynomial mapping* T. *Further, for each* $j = 1, 2, \dots, k-1$ *and all* $n = 0, 1, 2 \dots$,

$$p_{kn+j}(x) = \frac{c^{-n}}{\Delta_0(2, k-1; x)} \Big\{ c^{-1} \Delta_n(2, j-1; x) \widetilde{q}_{n+1}(T(x)) + a_n^{(1)} a_n^{(2)} \dots a_n^{(j)} \Delta_n(j+2, k-1; x) \widetilde{q}_n(T(x)) \Big\}.$$

Proof. As we have just remarked, this proposition follows from the results in [10], although it was not stated explicitly as a theorem. The result can be deduced from Theorem 2.1. In fact, taking $m = 0 = r$ in the theorem we see that

$$p_{kn+j}(x) = \frac{1}{\Delta_0(2, k-1; x)} \Big\{ \Delta_n(2, j-1; x) q_{n+1}(\pi_k(x)) + a_n^{(1)} a_n^{(2)} \dots a_n^{(j)} \Delta_n(j+2, k-1; x) q_n(\pi_k(x)) \Big\}$$

for all $n = 0, 1, 2, \dots$ and $j = 0, 1, \dots, k-1$, where

$$\pi_k(x) = \Delta_0(1, k-1; x)$$

and $(q_n)_n$ is the monic OPS characterized by the three-term recurrence relation with recurrence coefficients

$$r_0 := 0, \qquad r_n := \Delta_n(0), \qquad s_n := a_n^{(0)} a_{n-1}^{(1)} a_{n-1}^{(2)} \dots a_{n-1}^{(k-1)}$$

for all $n = 1, 2, \dots$. Thus Theorem 4.1 follows on noticing that

$$\widetilde{q}_n(x) = c^n q_n(c^{-1}x + b), \qquad \widetilde{r}_n = c(r_n - b), \qquad \widetilde{s}_n = c^2 s_n$$

for all n and $T(x) = c(\pi_k(x) - b)$. □

As an application of Theorem 3.4 we obtain the following proposition, which is essentially a result stated by Geronimo and Van Assche [15].

Theorem 4.2. *Under the conditions of Theorem 4.1, assume that* $(p_n)_n$ *is a monic OPS in the positive-definite sense, orthogonal with respect to some measure* $d\sigma$. *Then* $(\widetilde{q}_n)_n$ *is also a monic OPS in the positive-definite sense, orthogonal with respect to a measure* $d\tau$. *Furthermore, assume that the normalization conditions* $\int_{\mathbb{R}} d\tau = \int_{\mathbb{R}} d\sigma$ *hold, as well as the following three conditions:*

(i) $\mathrm{supp}(d\tau)$ *is a compact set, so*

$$-\infty < \xi := \min \mathrm{supp}(d\tau) < \eta := \max \mathrm{supp}(d\tau) < +\infty;$$

(ii) *either* $T(y_{2i-1}) \ge \eta$ *and* $T(y_{2i}) \le \xi$ *(for all possible* i*) if* k *is odd, or* $T(y_{2i-1}) \le \xi$ *and* $T(y_{2i}) \ge \eta$ *if* k *is even, where* $y_1 < \dots < y_{k-1}$ *denote the zeros of* T';

(iii) $W := \Delta_0(2, k-1; \cdot)$ *and* T' *have the same sign at each point of* $T^{-1}([\xi, \eta])$.

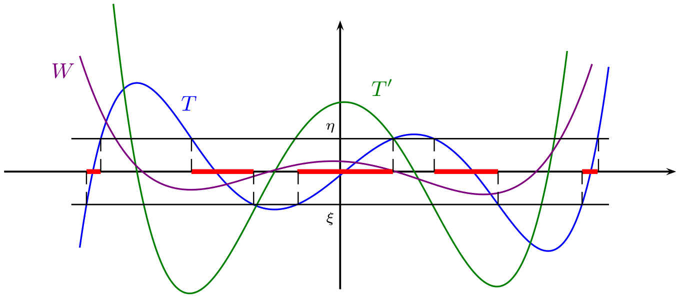

Fig. 1. A polynomial mapping with $(k, m) = (5, 0)$.

Then

$$F(z; \mathrm{d}\sigma) = W(z)F\left(T(z); \mathrm{d}\tau\right), \quad z \in \mathbb{C} \setminus T^{-1}([\xi, \eta])$$

and, up to a constant factor,

$$\mathrm{d}\sigma(x) = |W(x)| \frac{\mathrm{d}\tau(T(x))}{T'(x)}, \tag{4.2}$$

the support of $\mathrm{d}\sigma$ *being contained in* $T^{-1}([\xi, \eta])$ *(a union of* k *intervals).*

Remark 4.3. Fig. 1 illustrates Theorem 4.2 in a situation where $k = 5$ and $m = 0$.

4.2. *Generalized sieved OPUC*

Let $(\Phi_n)_n$ be a sequence of monic orthogonal polynomials on the unit circle (OPUC) with respect to some positive Borel measure $\mathrm{d}\mu$ on $[0, 2\pi]$ and let $-1, \alpha_0, \alpha_1, \alpha_2, \ldots,$ be the corresponding sequence of Verblunsky coefficients such that $\alpha_n \in (-1, 1)$ for all $n = 0, 1, \ldots$ (for background on OPUC see the monographs [29,30] and the survey article [31] by Simon). Fix a vector $(b_1, b_2, \ldots, b_{k-1}) \in (-1, 1)^{k-1}$ and let $(\widetilde{\Phi}_n)_n$ be the monic OPUC characterized by the sequence of Verblunsky coefficients $-1, \widetilde{\alpha}_0, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots,$ defined by

$$\widetilde{\alpha}_{nk-1} = \alpha_{n-1} \qquad \widetilde{\alpha}_{2nk+j-1} = -b_j \qquad \widetilde{\alpha}_{(2n+1)k+j-1} = b_{k-j}$$

for all $n = 0, 1, 2, \ldots$ and $j = 1, \ldots, k-1$. Notice that from an algebraic point of view we start with a sequence of Verblunsky coefficients, $-1, \alpha_0, \alpha_1, \alpha_2, \ldots,$ and then we make a perturbation of this sequence by inserting repeatedly blocks of $k-1$ real parameters in the following way:

$$\begin{aligned} &-1, -b_1, \ldots, -b_{k-1}, \alpha_0, b_{k-1}, \ldots, b_1, \\ &\alpha_1, -b_1, \ldots, -b_{k-1}, \alpha_2, b_{k-1}, \ldots, b_1, \alpha_3, \ldots. \end{aligned} \tag{4.3}$$

When $b_1 = b_2 = \cdots = b_{k-1} = 0$ then (4.3) yields the so-called sieved OPUC, studied by several authors (Badkov [6], Marcellán and Sansigre [19,20], and Ismail and Li [18]). For arbitrary $(b_1, b_2, \ldots, b_{k-1})$, the natural question is how to describe, in terms of the original sequence $(\Phi_n)_n$ and the vector $(b_1, b_2, \ldots, b_{k-1})$, the sequence $(\widetilde{\Phi}_n)_n$ of monic OPUC whose Verblunsky coefficients are given by (4.3) and, in particular, to give explicitly the corresponding orthogonality measure, $\mathrm{d}\widetilde{\mu}$. This problem has been solved in [28] by using an appropriate polynomial mapping linking the sequences $(P_n)_n$ and $(\widetilde{P}_n)_n$ of monic orthogonal polynomials on the real line (OPRL) associated with $(\Phi_n)_n$ and $(\widetilde{\Phi}_n)_n$ given by Szegö's transformation [33]. It has been proved in [28] that

$$\widetilde{P}_n(x) = \mathrm{d}^n \widehat{P}_n\left(\frac{x}{\mathrm{d}}\right), \quad n = 0, 1, 2, \ldots,$$

where $\mathrm{d} := (\prod_{j=1}^{k-1}((1-b_j^2)/2))^{1/k}$ and $(\widehat{P}_n)_n$ is a monic OPRL characterized by a three-term recurrence relation

$$\widehat{P}_{n+1} = (x - \widehat{\beta}_n)\widehat{P}_n - \widehat{\gamma}_n \widehat{P}_{n-1} \quad (n = 0, 1, 2, \ldots),$$

the sequences $(\widehat{\beta}_n)_n$ and $(\widehat{\gamma}_n)_n$ being defined in terms of the given data, i.e., the numbers $b_1, \ldots, b_{k-1}$ and the Verblunsky coefficients $(\alpha_n)_n$ of the given sequence of OPUC $(\Phi_n)_n$ (cf. formulas (3.3) and (3.4) in [28]). Defining $\widehat{\Delta}_n(i, j; \cdot)$ and $\widehat{\Delta}_n$ as in (1.4), (1.5) and (4.1), but with $a_n^{(\nu)}$ and $b_n^{(\nu)}$ replaced by $\widehat{\gamma}_{nk+\nu}$ and $\widehat{\beta}_{nk+\nu}$ (resp.), it follows from the results in [28] (see (3.1), (3.8) and Lemma 3.1 in [28]) that $(P_n)_n$ satisfies the recurrence relation

$$P_{n+1}(x) = (x - \widehat{\Delta}_n(0) + \Phi_1(0))P_n(x) - \left(\prod_{j=0}^{k-1} \widehat{\gamma}_{nk-j}\right) P_{n-1}(x)$$

for all $n = 0, 1, 2, \ldots$. According to [28, Lemma 3.1], $\widehat{\beta}_{nk}$ and $\widehat{\Delta}_n(2, k-1; x)$ are independent of n, and $\widehat{\Delta}_n(x)$ is independent of x. Therefore, setting

$$T(x) := \widehat{\Delta}_0(1, k-1; x) - \Phi_1(0), \qquad W(x) := \widehat{\Delta}_0(2, k-1; x),$$

from Theorem 4.1 we obtain

$$\widehat{P}_{kn+j}(x) = \frac{1}{W(x)} \left\{ \widehat{\Delta}_n(2, j-1; x) P_{n+1}(T(x)) + \left(\prod_{i=1}^{j} \widehat{\gamma}_{nk+i}\right) \widehat{\Delta}_n(j+2, k-1; x) P_n(T(x)) \right\}$$

for all $n = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots, k-1$. In particular

$$\widehat{P}_{nk}(x) = P_n(T(x)), \quad n = 0, 1, 2, \ldots.$$

Furthermore, from [28, Lemma 3.6] we see that the hypotheses of Theorem 4.2 are fulfilled, and so we may conclude that

$$F(z, \mathrm{d}\widetilde{\sigma}) = \frac{1}{d} W\left(\frac{z}{d}\right) F\left(T\left(\frac{z}{d}\right), \mathrm{d}\sigma\right), \quad \frac{z}{d} \in \mathbb{C} \setminus T^{-1}([\xi, \eta])$$

and (up to constant factors)

$$\mathrm{d}\widetilde{\sigma}(x) = \left|W\left(\frac{x}{d}\right)\right| \frac{\mathrm{d}\sigma\left(T\left(\frac{x}{d}\right)\right)}{T'\left(\frac{x}{d}\right)}, \quad \frac{x}{d} \in T^{-1}([\xi, \eta]),$$

the support of $\mathrm{d}\widetilde{\sigma}$ being contained in a union of k intervals on the real line. Finally, by Szegö's transformation, we obtain (see [28])

$$\mathrm{d}\widetilde{\mu}(\theta) = |\mathrm{d}\widetilde{\sigma}(\cos\theta)| = -\mathrm{sgn}\{\sin\theta\} \left|W\left(\frac{\cos\theta}{d}\right)\right| \frac{\mathrm{d}\sigma\left(T\left(\frac{\cos\theta}{d}\right)\right)}{T'\left(\frac{\cos\theta}{d}\right)}, \quad 0 \le \theta < 2\pi.$$

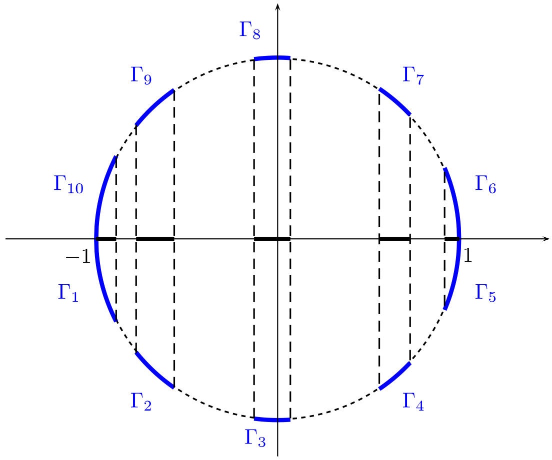

Fig. 2. $2k$ arcs defined by a polynomial mapping with $(k, m) = (5, 0)$.

We notice that the support of $d\widetilde{\mu}$ is contained in a union of $2k$ arcs on the unit circle, $\Gamma_1, \ldots, \Gamma_{2k}$, pairwise symmetric with respect to the real axis, determined by projecting over the unit circle the k intervals on the real line defined by the set $dT^{-1}([\xi, \eta])$, all of which are contained in the interval $[-1, 1]$ (see Fig. 2). All of these sets can be defined explicitly. For details, see [28].

Remark 4.4. If $b_1 = \cdots = b_{k-1} = 0$, then T and W become Chebyshev polynomials of the first and second kind (resp.) and so the above formula gives [6,18]

$$d\widetilde{\mu}(\theta) = \frac{1}{k} d\mu(k\theta), \quad 0 \le \theta < 2\pi,$$

which is the orthogonality measure for the sieved monic OPUC. In this case the above union of arcs covers the entire unit circle. In this situation, as is well known,

$$\widetilde{\Phi}_{nk+j}(z) = z^j \Phi_n(z^k), \quad j = 0, 1, \ldots, k-1; n = 0, 1, 2, \ldots.$$

5. The case $m = k - 1$

In this section we assume $k \ge 3$ and take $m = k - 1$. Fix $2k$ complex numbers $b_0, \ldots, b_{k-1}, c_0, \ldots, c_{k-1}$ with $c_j \neq 0$ for all $j = 0, 1, \ldots, k-1$ and let $(p_n)_n$ be a monic OPS generated by (1.2) with $a_n^{(j)}$ and $b_n^{(j)}$ satisfying

$$\begin{aligned} & b_n^{(j)} := b_j \quad (0 \le j \le k-1) \\ & a_n^{(j)} := c_j \quad (1 \le j \le k-2) \\ & a_1^{(0)} := c_0, \qquad a_0^{(k-1)} := c_{k-1} \\ & a_{n+1}^{(0)} + a_n^{(k-1)} = c_0 + c_{k-1} \end{aligned} \tag{5.1}$$

for all $n = 0, 1, 2, \ldots$. Here $a_{n+1}^{(0)}$ and $a_n^{(k-1)}$ may be arbitrary nonzero complex numbers, which may depend on n, but since we want to obtain $(p_n)_n$ by a polynomial mapping as described in

Theorem 2.1 (with $m = k - 1$) then it is natural to impose the last condition in (5.1), according to Remark 2.4. Define

$$\varphi_k(x) := \begin{vmatrix} x-b_0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ c_1 & x-b_1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & x-b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x-b_{k-3} & 1 & 0 \\ 0 & 0 & 0 & \cdots & c_{k-2} & x-b_{k-2} & 1 \\ c_0 & 0 & 0 & \cdots & 0 & c_{k-1} & x-b_{k-1} \end{vmatrix},$$

φ_k being a monic polynomial of degree exactly k in x, and let $\Delta_{r,s}(x)$ be a monic polynomial of degree $s - r + 1$ in x defined as follows: if $0 \le r < s \le k - 1$ then

$$\Delta_{r,s}(x) := \begin{vmatrix} x-b_r & 1 & 0 & \cdots & 0 & 0 \\ c_{r+1} & x-b_{r+1} & 1 & \cdots & 0 & 0 \\ 0 & c_{r+2} & x-b_{r+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x-b_{s-1} & 1 \\ 0 & 0 & 0 & \cdots & c_s & x-b_s \end{vmatrix},$$

and if $r \le s$ we adopt the convention

$$\Delta_{r,s}(x) := \begin{cases} 0 & \text{if } s < r-1 \\ 1 & \text{if } s = r-1 \\ x-b_r & \text{if } s = r. \end{cases}$$

Notice that when $0 \le r < s \le k - 1$ one observes that $\Delta_{r,s}(x)$ is obtained from the determinant which defines $\varphi_k(x)$ by deleting the first r rows and columns, as well as the last $k - r - s$ rows and columns, provided that $r = 0$ and $s = k - 1$ do not hold simultaneously.

Then we see that $\Delta_n(1, k - 2; x) = \Delta_{0,k-2}(x)$ for all $n = 0, 1, 2, \ldots$, so hypotheses (i) and (ii) of Theorem 2.1 are fulfilled, and

$$\theta_{k-1}(x) = \Delta_{0,k-2}(x).$$

Further, by straightforward computations on the determinant which defines $\varphi_k(x)$, one finds that the polynomial π_k in Theorem 2.1 is given by

$$\pi_k(x) = \Delta_{0,k-1}(x) - c_0\Delta_{1,k-2}(x) = \varphi_k(x) + (-1)^k \left(c_0 + \prod_{i=1}^{k-1} c_i \right). \tag{5.2}$$

Finally, we also have $\Delta_n(2, k - 2; x) = \Delta_{1,k-2}(x)$ and $\Delta_n(1, k - 3; x) = \Delta_{0,k-3}(x)$ for all $n = 0, 1, 2, \ldots$, so taking into account the last equation in (5.1) we see that the polynomial $r_n(x)$ in Theorem 2.1 satisfies

$$r_n(x) = (a_{n+1}^{(0)} - c_0)(\Delta_{1,k-2}(x) - \Delta_{0,k-3}(x)) \tag{5.3}$$

for all $n = 0, 1, 2 \ldots$. This enables us to find conditions ensuring that hypothesis (iv) in Theorem 2.1 holds, i.e., such that $r_n(x)$ is independent of x for every n. For example, if $k = 3$

then we find

$$r_n(x) = (a_{n+1}^{(0)} - c_0)(b_0 - b_1)$$

for all $n = 0, 1, 2, \ldots$, so it is always independent of x. But if $k = 4$ we derive

$$r_n(x) = (a_{n+1}^{(0)} - c_0)((b_0 - b_2)(x - b_1) + c_1 - c_2)$$

for all $n = 0, 1, 2, \ldots$. Hence $r_n(x)$ is independent of x if and only if $b_0 = b_2$ or $a_{n+1}^{(0)} = c_0$ for every $n = 0, 1, 2 \ldots$. Next we analyze in detail some general examples where $r_n(x)$ becomes independent of x. As usual, we denote by $(T_n)_n$ and $(U_n)_n$ the sequences of the Chebyshev polynomials of the first and second kind, which may be defined by the relations

$$T_n(x) = \cos(n\theta), \qquad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta \ (0 < \theta < \pi)$$

for all $n = 0, 1, 2 \ldots$. The corresponding monic polynomials are given by

$$\widehat{T}_n(x) = 2^{1-n} T_n(x), \qquad \widehat{U}_n(x) = 2^{-n} U_n(x), \quad n = 1, 2, \ldots. \tag{5.4}$$

5.1. OPS with periodic recurrence coefficients

If in (5.1) we choose both $a_{n+1}^{(0)}$ and $a_n^{(k-1)}$ to be independent of n for all n, so

$$a_{n+1}^{(0)} = c_0, \qquad a_n^{(k-1)} = c_{k-1} \quad (n = 0, 1, 2, \ldots).$$

Then from (5.3) we obtain $r_n(x) \equiv 0$ (independent of x) and all the hypotheses (i)–(iv) of Theorem 2.1 are fulfilled. Further, since $r_0 = r_n = 0$ and $s_n = \text{const.} = c_0 c_1 \cdots c_{k-1} (n = 1, 2, \ldots)$, then q_n is essentially a Chebyshev polynomial of the second kind,

$$q_n(x) = s_1^{n/2} U_n(x/(2\sqrt{s_1})), \quad n = 0, 1, 2 \ldots.$$

Notice that in this case $(p_n)_n$ is a monic OPS whose recurrence coefficients are k-periodic, characterized by the three-term recurrence relation

$$x p_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \quad n = 0, 1, 2, \ldots, \tag{5.5}$$

with initial conditions $p_{-1}(x) = 0$ and $p_0(x) = 1$, where

$$\beta_{nk+i} := b_i, \qquad \gamma_{nk+i} := c_i, \quad 0 \le i \le k-1; n = 0, 1, 2, \ldots. \tag{5.6}$$

Put

$$c^2 := \prod_{i=0}^{k-1} c_i, \qquad d := (-1)^{k+1} \frac{c_0^2 + c^2}{c_0}$$

(we choose c to be one square root of $\prod_{i=0}^{k-1} c_i$), and set

$$\widetilde{U}_n(x) := c^n U_n\left(\frac{x-d}{2c}\right), \quad n = 0, 1, 2, \ldots. \tag{5.7}$$

The next proposition recovers a classical result due to Geronimus [16,17] stated using continued fractions. Another proof was given by Peherstorfer [26] using the concept of

Chebyshev or the T-polynomial. Other approaches appear in [5,37,14]. We give another proof based on the results in the previous sections. The algebraic properties (5.8)–(5.9) were also given in [12], where the "if" part of Theorem 2.1 was stated in the particular situation $m = k - 1$. It seems that the explicit expression (5.11) for the masses is a new result.

Theorem 5.1. *The monic OPS $(p_n)_n$ characterized by (5.5)–(5.6), with the b_j complex numbers and the c_j nonzero complex numbers for all j, can be expressed in terms of a polynomial mapping over the Chebyshev polynomials of the second kind (suitably shifted and rescaled) as*

$$p_{nk+j}(x) = \Delta_{0,j-1}(x)\widetilde{U}_n(\varphi_k(x)) + \left(\prod_{i=0}^{j} c_i\right) \Delta_{j+1,k-2}(x)\widetilde{U}_{n-1}(\varphi_k(x)) \tag{5.8}$$

for all $n = 0, 1, 2, \ldots$ and $0 \le j \le k-1$. In particular,

$$p_{nk+k-1}(x) = \Delta_{0,k-2}(x)\widetilde{U}_n(\varphi_k(x)), \quad n = 0, 1, 2, \ldots. \tag{5.9}$$

Further, in the positive-definite case (i.e., the b_j's are real and the c_j's are positive for all j) the orthogonality measure for this sequence $(p_n)_n$ is

$$\mathrm{d}\sigma(x) = \sum_{j=1}^{k-1} M_j \delta(x - z_j)\mathrm{d}x + \frac{1}{2\pi c_0} \frac{\sqrt{4c^2 - (\varphi_k(x) - d)^2}}{|\Delta_{0,k-2}(x)|} \chi_{\varphi_k^{-1}(]d-2c,d+2c[)}(x)\mathrm{d}x \tag{5.10}$$

(choosing $c = \prod_{j=0}^{k-1} \sqrt{c_j}$), where $z_1 < \cdots < z_{k-1}$ are the zeros of $\Delta_{0,k-2}(x)$ and the masses M_j are explicitly given by

$$M_j := \frac{\Delta_{1,k-2}(z_j)}{\Delta'_{0,k-2}(z_j)} \left[1 - \min\left\{1, -\frac{\Delta_{0,k-1}(z_j)}{c_0 \Delta_{1,k-2}(z_j)}\right\}\right] \tag{5.11}$$

for all $j = 1, 2, \ldots, k-1$.

Proof. Relations (5.8)–(5.9) follow immediately from (2.1) and (2.5) in Theorem 2.1 and the considerations preceding Theorem 5.1. Let us consider now the positive-definite case. Since $q_n(x) = c^n U_n\left(\frac{x}{2c}\right)$ then $(q_n)_n$ is orthogonal with respect to

$$\mathrm{d}\tau(x) = \frac{1}{2\pi c^2} \chi_{]-2c,2c[}(x)\sqrt{4c^2 - x^2}\mathrm{d}x, \tag{5.12}$$

the corresponding Stieltjes transform being

$$F(z; \mathrm{d}\tau) = \frac{1}{2c} F_U\left(\frac{z}{2c}\right), \quad z \in \mathbb{C} \setminus [-2c, 2c],$$

where $F_U(z) := -2(z-(z^2-1)^{1/2})$ for $z \in \mathbb{C}\setminus[-1, 1]$ (in fact, F_U is the Stieltjes transform of the sequence $(U_n)_n$) and the complex function $(z^2-1)^{1/2}$ is defined taking that branch which satisfies $\left|z + (z^2 - 1)^{1/2}\right| > 1$ whenever $z \notin [-1, 1]$. Notice also that the true interval of orthogonality of this OPS $(q_n)_n$ is

$$[\xi, \eta] = [-2c, 2c] = \mathrm{supp}(\mathrm{d}\tau).$$

Let us show that all the hypotheses (i)–(iv) in Theorem 3.4 are fulfilled. Obviously, hypothesis (i) holds. In order to prove (ii)–(iv) we first state the equalities

$$\pi_k^2(z_j) - 4c^2 = (2c_0\Delta_{1,k-2}(z_j) + \pi_k(z_j))^2, \quad j = 1, \ldots, k-1. \tag{5.13}$$

In fact, from (5.2) we have

$$\pi_k(x) = \Delta_{0,k-1}(x) - c_0\Delta_{1,k-2}(x) = p_k(x) - c_0 p^{(1)}_{k-2}(x),$$

and so, using also the fact that z_j is a zero of $\theta_{k-1} \equiv p_{k-1}$, one can write

$$\begin{aligned}(2c_0\Delta_{1,k-2}(z_j) + \pi_k(z_j))^2 &= \pi_k^2(z_j) + 4c_0\Delta_{1,k-2}(z_j)(\pi_k(z_j) + c_0\Delta_{1,k-2}(z_j)) \\ &= \pi_k^2(z_j) + 4c_0\Delta_{1,k-2}(z_j)\Delta_{0,k-1}(z_j) \\ &= \pi_k^2(z_j) + 4c_0(p^{(1)}_{k-2}(z_j)p_k(z_j) - p_{k-1}(z_j)p^{(1)}_{k-1}(z_j)) \\ &= \pi_k^2(z_j) - 4c_0\prod_{i=1}^{k-1} c_i = \pi_k^2(z_j) - 4c^2,\end{aligned}$$

which proves (5.13). It follows that

$$|\pi_k(z_j)| \geq 2c > 0, \quad j = 1, \ldots, k-1. \tag{5.14}$$

In order to show that hypothesis (ii) holds we have to prove that

$$\int_{-2c}^{2c} \frac{\mathrm{d}\tau(y)}{|y - \pi_k(z_j)|} < +\infty, \quad j = 1, \ldots, k-1.$$

This holds for all z_j such that $|\pi_k(z_j)| > 2c$, so it remains to prove that the same is true for those z_j such that $\pi_k(z_j) = \pm 2c$. For the case $\pi_k(z_j) = 2c$, according to (5.12) we have

$$\begin{aligned}\int_{-2c}^{2c} \frac{\mathrm{d}\tau(y)}{|y - 2c|} &= \frac{1}{2\pi c^2}\int_{-2c}^{2c} \frac{\sqrt{4c^2 - y^2}}{2c - y}\mathrm{d}y = \frac{1}{\pi c}\int_{-1}^{1}(1-t)^{-1/2}(1+t)^{1/2}\mathrm{d}t \\ &= \frac{2}{\pi c}B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{1}{c} < +\infty,\end{aligned}$$

where $B(\cdot,\cdot)$ denotes the beta function. The proof for the case $\pi_k(z_j) = -2c$ is similar. Let us now prove (iii). Taking into account the considerations before the statement of Theorem 3.4, we know that π_k has k real and distinct zeros $x_1 < \cdots < x_k$, and π_k' possesses $k-1$ real and distinct zeros, $y_1 < \cdots < y_{k-1}$, which interlace with those of π_k, i.e., $x_1 < y_1 < x_2 < y_2 < \cdots < y_{k-1} < x_k$. Recall that $\theta_{k-1}(x) = p_{k-1}(x)$ and $\pi_k(x) = p_k(x) - c_0 p^{(1)}_{k-2}(x)$. Therefore

$$\pi_k(z_j) = p_k(z_j) - c_0 p^{(1)}_{k-2}(z_j), \quad j = 1, 2, \ldots, k-1.$$

We know that (cf. [11, p. 86, Theorem 4.1])

$$z_j < z^{(1)}_{k-2,j} < z_{j+1}, \quad j = 1, 2, \ldots, k-2 \tag{5.15}$$

where $z^{(1)}_{k-2,1} < z^{(1)}_{k-2,2} < \cdots < z^{(1)}_{k-2,k-2}$ denote the zeros of $p^{(1)}_{k-2}$. Furthermore (cf. [11, p. 28, Theorem 5.3])

$$z_{k,j} < z_j < z_{k,j+1}, \quad j = 1, 2, \ldots, k-1 \tag{5.16}$$

where $z_{k,1} < z_{k,2} < \cdots < z_{k,k}$ denote the zeros of p_k. Using (5.15) and (5.16) one sees that

$$\operatorname{sgn}\{\pi_k(z_j)\} = (-1)^{k-j}, \quad j = 1, \ldots, k-1 \tag{5.17}$$

and so we have $x_1 < z_1 < x_2 < z_2 < \cdots < z_{k-1} < x_k$. It follows that

$$x_1 < y_1, \quad z_1 < x_2 < y_2, \quad z_2 < x_3 < \cdots < x_{k-1} < y_{k-1}, \quad z_{k-1} < x_k. \tag{5.18}$$

Assume that k is an odd number (the case when k is an even number can be treated in a similar way). Then $\operatorname{sgn}\{\pi_k(x)\} = (-1)^{i-1}$ if $x \in (x_i, x_{i+1})$ for every $i = 1, \ldots, k-1$. Hence $\pi_k(y_{2j-1}) > 0$ and $\pi_k(y_{2j}) < 0$ and so from (5.14) and (5.18) we derive $\pi_k(y_{2j-1}) \geq 2c$ and $\pi_k(y_{2j}) \leq -2c$ for all $j = 1, 2, \ldots, (k-1)/2$ (since the extreme values of π_k occur at the zeros of π_k'). Thus hypothesis (iii) holds, and it is clear that (iv) also holds according to (5.18).

It follows from Theorem 3.4 and taking into account (5.12) and the relation $\pi_k(x) = \varphi_k(x) - d$ that the measure $\mathrm{d}\sigma$ is given (up to the factor c_0/c^2) by (5.10) where

$$M_j := \frac{\Delta_{1,k-2}(z_j) - (c^2/c_0) F(\pi_k(z_j); \mathrm{d}\tau)}{\Delta'_{0,k-2}(z_j)} \tag{5.19}$$

for all $j = 1, \ldots, k-1$ (notice that $v_0 = \int_{-2c}^{2c} \mathrm{d}\tau = 1$). In order to complete the proof we need to show that this expression for M_j coincides with (5.11). Notice first that $\frac{\pi_k(z_j)}{2c} \geq 1$ if $k-j$ is even, and $\frac{\pi_k(z_j)}{2c} \leq -1$ if $k-j$ is odd, as follows from (5.14) and (5.17). As a consequence, according to the choice of the branch of the complex function $(z^2-1)^{1/2}$, we have (in what follows $\sqrt{\ }$ denotes the usual real square root)

$$\begin{aligned} \left(\left(\frac{\pi_k(z_j)}{2c} \right)^2 - 1 \right)^{1/2} &= (-1)^{k-j} \sqrt{\left(\frac{\pi_k(z_j)}{2c} \right)^2 - 1} \\ &= \frac{(-1)^{k-j}}{2c} \left| 2c_0 \Delta_{1,k-2}(z_j) + \pi_k(z_j) \right|, \end{aligned}$$

the last equality being justified by (5.13). Therefore

$$\begin{aligned} F(\pi_k(z_j); \mathrm{d}\tau) &= \frac{1}{2c} F_U \left(\frac{\pi_k(z_j)}{2c} \right) = -\frac{1}{c} \left\{ \frac{\pi_k(z_j)}{2c} - \left(\left(\frac{\pi_k(z_j)}{2c} \right)^2 - 1 \right)^{1/2} \right\} \\ &= -\frac{1}{2c^2} \{ \pi_k(z_j) - (-1)^{k-j} |2c_0 \Delta_{1,k-2}(z_j) + \pi_k(z_j)| \} \\ &= -\frac{1}{2c^2} \{ \Delta_{0,k-1}(z_j) - c_0 \Delta_{1,k-2}(z_j) \\ &\quad - (-1)^{k-j} |\Delta_{0,k-1}(z_j) + c_0 \Delta_{1,k-2}(z_j)| \}. \end{aligned}$$

Substituting this expression in (5.19) we arrive at (5.11), taking into account the relations

$$\min\{g, h\} = \frac{g+h-|g-h|}{2}, \qquad \max\{g, h\} = \frac{g+h+|g-h|}{2} \quad (g, h \in \mathbb{R}).$$

We remark that $M_j \geq 0$ for all $j = 1, \ldots, k-1$, as follows from Theorem 3.4. □

Remark 5.2. For each fixed $j \in \{0, 1, \ldots, k-1\}$, the zeros of the polynomial $p_{nk+j}(x)$ satisfy the equation

$$c \Delta_{0,j-1}(x) \sin(n+1)\theta + \left(\prod_{i=0}^{j} c_i \right) \Delta_{j+1,k-2}(x) \sin(n\theta) = 0,$$

with

$$\cos\theta = \frac{\pi_k(x)}{2c}.$$

In particular, the zeros of the polynomial $p_{nk+k-1}(x)$ are the $k-1$ zeros of the polynomial $\theta_{k-1}(x)$ together with the nk points $\{x^{(n)}_{j,q}\}_{\substack{j=0,\ldots,k-1\\ q=1,\ldots,n}}$ which satisfy

$$\pi_k(x^{(n)}_{j,q}) = 2c\cos\left(\frac{q\pi}{n+1}\right).$$

This result has been stated by Simon in [32, Theorem 2.1]. (We notice that the polynomial $\Delta(x)$ appearing in [32] is related to our polynomial π_k by the relation $\Delta(x) = \pi_k(x)/c$.)

5.2. *Sieved ultraspherical polynomials of the second kind*

Taking

$$b_j := 0, \qquad c_j := \frac{1}{4} \quad (j = 0, 1, \ldots, k-1)$$

in (5.1) then the corresponding monic OPS $(p_n)_n$ is a system of sieved random walk polynomials of the second kind (see [10]). Then

$$\Delta_{r,s}(x) = \widehat{U}_{s-r+1}(x)$$

for all $r, s \in \{0, 1, \ldots, k-1\}$. From these and (5.3) we find again $r_n(x) \equiv 0$; hence all hypotheses (i)–(iv) in Theorem 2.1 are satisfied, $(q_n)_n$ being characterized by

$$r_n = 0, \qquad s_n = 4^{2-k} a_n^{(0)} a_n^{(k-1)}, \quad n = 0, 1, 2, \ldots.$$

Further,

$$\theta_{k-1}(x) = \widehat{U}_{k-1}(x), \qquad \pi_k(x) = \widehat{U}_k(x) - \frac{1}{4}\widehat{U}_{k-2}(x) = \widehat{T}_k(x),$$

and, for every $n = 0, 1, 2, \ldots$ and $0 \le j \le k-1$, $\Delta_n(1, j-1; x) = \widehat{U}_j(x)$ and $\Delta_n(j+2, k-2; x) = \widehat{U}_{k-j-2}(x)$. It follows from Theorem 2.1 that

$$p_{nk+j}(x) = \widehat{U}_j(x) q_n(\widehat{T}_k(x)) + 4^{-j} a_n^{(0)} \widehat{U}_{k-j-2}(x) q_{n-1}(\widehat{T}_k(x)) \tag{5.20}$$

for all $n = 0, 1, 2, \ldots$ and $0 \le j \le k-1$. In particular,

$$p_{nk+k-1}(x) = \widehat{U}_{k-1}(x) q_n(\widehat{T}_k(x)), \quad n = 0, 1, 2, \ldots.$$

The special case when we fix $\lambda > -1/2$ and take

$$a_{n+1}^{(0)} := \frac{n+1}{4(n+1+\lambda)}, \qquad a_n^{(k-1)} := \frac{n+1+2\lambda}{4(n+1+\lambda)} \quad (n = 0, 1, 2, \ldots)$$

is of historical importance since then q_n is (up to an affine change in the variable) an ultraspherical polynomial with parameter λ,

$$q_n(x) = \frac{n!}{2^{kn}(\lambda+1)_n} C_n^{\lambda+1}(2^{k-1}x), \quad n = 0, 1, 2, \ldots,$$

and p_n becomes the monic polynomial corresponding to the sieved ultraspherical polynomial of the second kind $B_n^{\lambda}(\cdot; k)$ introduced by Al-Salam et al. [1]. In fact, from (5.20) we find that, for all $j = 0, \ldots, k-1$ and $n = 0, 1, 2, \ldots,$

$$\begin{aligned} p_{kn+j}(x) &= \widehat{U}_j(x) q_n(\widehat{T}_k(x)) + \frac{n}{4^{j+1}(n+\lambda)} \widehat{U}_{k-2-j}(x) q_{n-1}(\widehat{T}_k(x)) \\ &= \frac{n!}{2^{kn+j}(\lambda+1)_n} B^{\lambda}_{kn+j}(x; k), \end{aligned}$$

the last equality being justified by (see [9])

$$B^{\lambda}_{kn+j}(x; k) = U_j(x) C_n^{\lambda+1}(T_k(x)) + U_{k-j-2}(x) C_{n-1}^{\lambda+1}(T_k(x))$$

and the above relations (5.4). Further, since the OPS $(q_n)_n$ is orthogonal with respect to the measure

$$\mathrm{d}\tau(x) := w_\tau(x) \chi_{]-2^{1-k}, 2^{1-k}[} \mathrm{d}x, \qquad w_\tau(x) := 2^{k-1}(1 - 4^{k-1}x^2)^{\lambda+\frac{1}{2}},$$

then from Theorem 3.4 (we notice that here all the hypotheses in this theorem are very easy to verify, using well-known properties of the Chebyshev polynomials of the first and of the second kind) we obtain, up to a constant factor, the orthogonality measure for the sieved ultraspherical polynomials of the second kind:

$$\mathrm{d}\sigma(x) := (1 - x^2)^{\lambda+\frac{1}{2}} |U_{k-1}(x)|^{2\lambda} \chi_{(-1,1)}(x) \mathrm{d}x.$$

Notice that in this case all the masses M_i in the expression (3.5) for the measure vanish. In fact, since the zeros of $\theta_{k-1} \equiv \widehat{U}_{k-1}$ are $z_j = -\cos(j\pi/k)$ for every $j = 1, \ldots, k-1$, then $\pi_k(z_j) = \widehat{T}_k(z_j) = 2^{1-k}\cos(k\pi - j\pi) = 2^{1-k}(-1)^{k-j}$; hence, using well-known properties of the Gamma and Beta functions (see e.g. [2]), on one hand we compute

$$\begin{aligned} F(\pi_k(z_j); \mathrm{d}\tau) &= F((-1)^{k-j}2^{1-k}; \mathrm{d}\tau) = 2^{k-1}\int_{-1}^{1} \frac{(1-x)^{\lambda+\frac{1}{2}}(1+x)^{\lambda+\frac{1}{2}}}{x - (-1)^{k-j}} \mathrm{d}x \\ &= (-1)^{k-1-j} 2^{k+2\lambda} B\left(\lambda + \frac{1}{2}, \lambda + \frac{3}{2}\right), \end{aligned}$$

and on the other hand, since

$$\begin{aligned} v_0 &= \int_{\mathbb{R}} \mathrm{d}\tau = \int_{-1}^{1} (1 - x^2)^{\lambda+1/2} \mathrm{d}x = 2^{2\lambda+2} B\left(\lambda + \frac{3}{2}, \lambda + \frac{3}{2}\right) \\ &= 2^{2\lambda} B\left(\lambda + \frac{1}{2}, \lambda + \frac{3}{2}\right) \frac{1+2\lambda}{1+\lambda}, \end{aligned}$$

$\Delta_0(2, k-2; z_j) = \widehat{U}_{k-2}(z_j) = 2^{2-k}(-1)^{k-j-1}$ and $\prod_{\nu=1}^{k-1} a_0^{(\nu)} = 4^{1-k}\frac{1+2\lambda}{1+\lambda}$, we see that

$$v_0 \Delta_0(2, k-2; z_j) / \left(\prod_{\nu=1}^{k-1} a_0^{(\nu)}\right) = (-1)^{k-1-j} 2^{k+2\lambda} B\left(\lambda + \frac{1}{2}, \lambda + \frac{3}{2}\right)$$

and, therefore, from (3.6) we obtain $M_j = 0$ for all $j = 1, \ldots, k-1$.

Remark 5.3. The connection between sieved orthogonal polynomials and orthogonal polynomials defined via polynomial mappings has been observed by Geronimo and Van Assche in [15], where these authors used appropriate polynomial mappings to give a new approach to sieved orthogonal polynomials.

6. An example with $k = 5$ and $m = 1$

The examples considered in the previous sections correspond to extremal choices of m (for a given k), i.e., m assumed either its minimum possible value ($m = 0$) or its maximum value ($m = k - 1$). Our next example corresponds to an intermediate situation, with $k = 5$ and $m = 1$. Let us consider the monic OPS $(p_n)_n$ characterized by the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \quad n = 0, 1, 2, \ldots \tag{6.1}$$

(with initial conditions $p_{-1} = 0$ and $p_0 = 1$), where

$$\begin{aligned} &\beta_0 = \beta_{5n+1} = \beta_{5n+3} = \beta_{5n+4} = 0, \qquad \beta_{5n+2} = 1, \qquad \beta_{5n+5} = \frac{1}{2} \quad (n \ge 0), \\ &\gamma_1 = \frac{1}{2}, \qquad \gamma_2 = \frac{3}{2}, \qquad \gamma_{5n+1} = \gamma_{5n+2} = 1 \quad (n \ge 1), \\ &\gamma_{5n+3} = 1, \qquad \gamma_{5n+4} = \frac{3}{2}, \qquad \gamma_{5n+5} = \frac{3}{4} \quad (n \ge 0). \end{aligned} \tag{6.2}$$

Rewriting this recurrence in terms of blocks as in (1.2), we see that the sequences $\{a_n^{(j)}\}_{n\ge0}$ and $\{b_n^{(j)}\}_{n\ge0}$ $(0 \le j \le 4)$ are defined for all $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} &b_n^{(0)} = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{2} & \text{if } n \ge 1, \end{cases} \qquad b_n^{(1)} = b_n^{(3)} = b_n^{(4)} = 0, \qquad b_n^{(2)} = 1, \\ &a_n^{(0)} = \begin{cases} 1 & \text{if } n = 0 \\ \frac{3}{4} & \text{if } n \ge 1, \end{cases} \qquad a_n^{(1)} = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 1 & \text{if } n \ge 1, \end{cases} \qquad a_n^{(2)} = \begin{cases} \frac{3}{2} & \text{if } n = 0 \\ 1 & \text{if } n \ge 1, \end{cases} \\ &a_n^{(3)} = 1, \qquad a_n^{(4)} = \frac{3}{2}. \end{aligned}$$

Then it is straightforward to verify that the hypotheses (i)–(iv) in Theorem 2.1 are fulfilled (with $k = 5$ and $m = 1$), as

$$\begin{aligned} &\Delta_n(3, 5; x) = x^4 - \frac{3}{2}x^3 - \frac{11}{4}x^2 + \frac{7}{2}x \quad \text{(independent of } n\text{)}, \\ &r_n(x) = \begin{cases} 0 & \text{if } n = 0 \\ -\frac{5}{8} & \text{if } n \ge 1 \end{cases} \quad \text{(independent of } x\text{)}, \\ &\theta_1(x) = x, \qquad \eta_3(x) = x^3 - \frac{3}{2}x^2 - \frac{11}{4}x + \frac{7}{2}, \\ &\pi_5(x) = x^5 - \frac{3}{2}x^4 - \frac{19}{4}x^3 + 5x^2 + \frac{19}{4}x - \frac{23}{8}. \end{aligned}$$

Further, the monic OPS $(q_n)_n$ in (2.3) is characterized by the recurrence coefficients

$$r_n = \begin{cases} 0 & \text{if } n = 0 \\ -\frac{5}{8} & \text{if } n \ge 1, \end{cases} \qquad s_n = \begin{cases} \frac{27}{16} & \text{if } n = 1 \\ \frac{9}{8} & \text{if } n \ge 2 \end{cases}$$

and thus one concludes that each q_n can be expressed in terms of Chebyshev polynomials of the second kind as

$$q_n(x)=\left(\frac{3\sqrt{2}}{4}\right)^n\left\{U_n\left(\frac{8x+5}{12\sqrt{2}}\right)-\frac{5\sqrt{2}}{12}\,U_{n-1}\left(\frac{8x+5}{12\sqrt{2}}\right)-\frac{1}{2}\,U_{n-2}\left(\frac{8x+5}{12\sqrt{2}}\right)\right\} \tag{6.3}$$

for all $n=0,1,2,\ldots$. It follows from Theorem 2.1 that $(p_n)_n$ can be obtained from $(q_n)_n$ via a polynomial mapping, with

$$p_{5n+1}(x)=xq_n(\pi_5(x)),\quad n=0,1,2,\ldots$$

and the remaining polynomials $p_{5n+j}(x)$ $(2\le j\le 5;\ n=0,1,2,\ldots)$ can also be given by applying Theorem 2.1. In fact, we readily see that all the polynomials p_n can be expressed by means of a polynomial mapping over the Chebyshev polynomials of the second kind. In order to find the orthogonality measure for the sequence $(p_n)_n$ we will apply Theorem 3.4. Notice first that $(q_n)_n$ is orthogonal with respect to the measure

$$\mathrm{d}\tau(x)=\frac{1}{13}\delta\left(x-\frac{3}{2}\right)+w_\tau(x)\chi_{]-\frac{5}{8}-\frac{3}{2}\sqrt{2},-\frac{5}{8}+\frac{3}{2}\sqrt{2}[}(x)\mathrm{d}x,$$

where

$$w_\tau(x):=-\frac{36\sqrt{2}}{\pi(2x-3)(8x+27)}\sqrt{1-\left(\frac{8x+5}{12\sqrt{2}}\right)^2}.$$

We omit the details here but we mention that these facts can be proved by using results due to Ya. L. Geronimus, also proved by Dombrowski and Nevai (see [13, Section 2, Example A]). Now we can easily see that all the hypotheses (i)–(iv) in Theorem 3.4 are fulfilled. In fact, we have

$$\mathrm{supp}(\mathrm{d}\tau)=\left[-\frac{5}{8}-\frac{3}{2}\sqrt{2},-\frac{5}{8}+\frac{3}{2}\sqrt{2}\right]\cup\left\{\frac{3}{2}\right\},$$

and hence (i) in Theorem 3.4 holds trivially, with $\xi=\frac{5}{8}-\frac{3}{2}\sqrt{2}$ and $\eta=\frac{3}{2}$. It is clear also that hypothesis (ii) holds, since the only zero of $\theta_1(x)$ is $z_1=0$ and $\pi_5(0)=-23/8$, so the poles of the function $w_\tau(x)/|(x-\pi_5(0))|$ are $-23/8=-2.875$, $-27/8=-3.375$ and $3/2=\eta$, all living off the interval

$$\left[-\frac{5}{8}-\frac{3}{2}\sqrt{2},-\frac{5}{8}+\frac{3}{2}\sqrt{2}\right]\approx[-2.74632,1.49632]$$

(which is the support of the absolutely continuous part of the measure $\mathrm{d}\tau$); and the remaining hypotheses (iii) and (iv) can be confirmed geometrically by plotting all the polynomials involved (see Fig. 3). Moreover, since

$$v_0=\int_{\mathbb{R}}\mathrm{d}\tau(x)=\frac{1}{13}-\frac{36\sqrt{2}}{\pi}\int_{-\frac{5}{8}-\frac{3}{2}\sqrt{2}}^{-\frac{5}{8}+\frac{3}{2}\sqrt{2}}\frac{\sqrt{1-\left(\frac{8x+5}{12\sqrt{2}}\right)^2}}{(2x-3)(8x+27)}\mathrm{d}x=1$$

and

$$\begin{aligned}F(\pi_5(0);\mathrm{d}\tau)&=\int_{\mathbb{R}}\frac{\mathrm{d}\tau(x)}{x+\frac{23}{8}}\\&=\frac{8}{455}-\frac{288\sqrt{2}}{\pi}\int_{-\frac{5}{8}-\frac{3}{2}\sqrt{2}}^{-\frac{5}{8}+\frac{3}{2}\sqrt{2}}\frac{\sqrt{1-\left(\frac{8x+5}{12\sqrt{2}}\right)^2}}{(8x+23)(2x-3)(8x+27)}\mathrm{d}x=\frac{4}{7},\end{aligned}$$

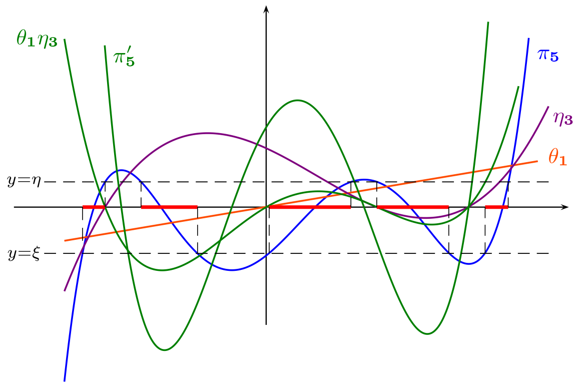

Fig. 3. Polynomial mapping for the OPS $(p_n)_n$ given by (6.1)–(6.2).

then we compute the mass M_1 given by (3.6) getting

$$M_1 = v_0/a_0^{(1)} - \eta_3(0)F(\pi_5(0); \mathrm{d}\tau) = 0.$$

Therefore, the monic OPS $(p_n)_n$ is orthogonal with respect to the measure

$$\mathrm{d}\sigma(x) = \sum_{i=1}^{5} N_i \delta(x - a_i)\mathrm{d}x + w_\sigma(x)\chi_{\pi_5^{-1}(]-\frac{5}{8}-\frac{3}{2}\sqrt{2},-\frac{5}{8}+\frac{3}{2}\sqrt{2}[)}(x)\mathrm{d}x,$$

where (cf. Remark 3.5)

$$w_\sigma(x) := \frac{|4x^3 - 6x^2 - 11x + 14|}{4|x|} w_\tau(\pi_5(x)),$$

$a_1, \ldots, a_5$ are the solutions of the algebraic equation $\pi_5(x) = \frac{3}{2}$, and $N_1, \ldots, N_5$ are nonnegative real numbers. Notice that $w_\sigma(x)$ can be explicitly written as

$$w_\sigma(x) = \frac{-3\sqrt{2}\,\mathrm{sgn}\{4x^3 - 6x^2 - 11x + 14\}\sqrt{-18 - 2x(x-1)(4x^4 - 2x^3 - 21x^2 - x + 18)(4x^4 - 6x^3 - 19x^2 + 20x + 19)}}{2\pi|x|(2x^3 - 4x^2 - 4x + 5)(4x^4 + 2x^3 - 15x^2 - 10x - 1)}.$$

Furthermore, since $\pi_5(x) - \frac{3}{2} = (x^2 + \frac{x}{2} - \frac{7}{4})(x^3 - 2x^2 - 2x + \frac{5}{2})$, then one readily sees that the a_i's are given (in increasing order) by

$$a_1 = \frac{-1 - \sqrt{29}}{4} \approx -1.5963,$$

$$a_2 = \frac{2}{3}\left(1 - \sqrt{10}\cos\frac{\zeta}{3}\right) \approx -1.2398, \qquad a_3 = \frac{2}{3}\left(1 + \sqrt{10}\cos\frac{\pi + \zeta}{3}\right) \approx 0.8405,$$

$$a_4 = \frac{-1 + \sqrt{29}}{4} \approx 1.0963, \qquad a_5 = \frac{2}{3}\left(1 + \sqrt{10}\cos\frac{\pi - \zeta}{3}\right) \approx 2.3993,$$

where $\zeta := \arccos\frac{31}{40\sqrt{10}}$. The corresponding N_i's (masses) are given by

$$N_i = \frac{(a_i - 2)^2}{27a_i^4 - 108a_i^3 + 160a_i^2 - 130a_i + 130}, \quad i = 1, \ldots, 5, \tag{6.4}$$

and hence we compute

$$N_1 \approx 0.0095, \quad N_2 \approx 0.0130, \quad N_3 \approx 0.0162, \quad N_4 \approx 0.0107, \quad N_5 \approx 0.0011.$$

Before proving (6.4), let us point out that

$$\operatorname{supp}(\mathrm{d}\sigma) = E \cup \{a_1, a_2, a_3, a_4, a_5\},$$

where $E := \pi_5^{-1}([-\frac{5}{8} - \frac{3}{2}\sqrt{2}, -\frac{5}{8} + \frac{3}{2}\sqrt{2}])$ is a union of five disjoint intervals:

$$E \approx [-1.8199, -1.5967] \cup [-1.2392, -0.6834] \cup [0.0264, 0.8387] \cup [1.0980, 1.8118] \cup [2.1651, 2.3992].$$

In order to prove (6.4) consider the orthonormal sequence $(\widetilde{p}_n)_n$ corresponding to the monic OPS (p_n), so $\widetilde{p}_n(x) = (u_0 \prod_{i=1}^n \gamma_i)^{-1/2} p_n(x)$ for all $n = 0, 1, 2, \ldots$. Then, we know that

$$N_i = \frac{1}{\sum_{n=0}^{+\infty} \widetilde{p}_n^2(a_i)} \quad (i = 1, \ldots, 5).$$

Since $u_0 = 1$ and taking into account (6.2), we obtain

$$\sum_{n=0}^{+\infty} \widetilde{p}_n^2(x) = 1 + \sum_{n=0}^{+\infty} \sum_{j=1}^{5} \widetilde{p}_{5n+j}^2(x) = 1 + \sum_{n=0}^{+\infty} \frac{4}{3} \left(\frac{8}{9}\right)^n V_n(x), \tag{6.5}$$

where

$$V_n(x) := p_{5n+1}^2(x) + p_{5n+2}^2(x) + p_{5n+3}^2(x) + \frac{2}{3}\, p_{5n+4}^2(x) + \frac{8}{9} p_{5n+5}^2(x).$$

But, according to Theorem 2.1, for each $j \in \{1, \ldots, 5\}$ one has

$$p_{5n+j}(x) = \frac{\Delta_n(3, j-1; x) q_{n+1}(\pi_5(x)) + \left(\prod_{s=1}^{j-1} a_n^{(s+1)}\right) \Delta_n(j+2, 5; x) q_n(\pi_5(x))}{\eta_3(x)}$$

for all $n = 0, 1, 2, \ldots$. Therefore, taking $x = a_i$ $(i = 1, \ldots, 5)$ we see that we have to compute $q_n(\pi_5(a_i)) = q_n(\frac{3}{2})$ for all $n = 0, 1, 2, \ldots$. According to (6.3) we see that

$$q_n\left(\frac{3}{2}\right) = \left(\frac{3\sqrt{2}}{4}\right)^n \left\{ U_n\left(\frac{17}{12\sqrt{2}}\right) - \frac{5\sqrt{2}}{12}\, U_{n-1}\left(\frac{17}{12\sqrt{2}}\right) - \frac{1}{2}\, U_{n-2}\left(\frac{17}{12\sqrt{2}}\right) \right\}$$

for all $n = 0, 1, 2 \ldots$. Now, recall that the Chebyshev polynomials of the second kind satisfy

$$U_n(x) = \frac{\sinh\left((n+1)\operatorname{argcosh} x\right)}{\sinh(\operatorname{argcosh} x)} \quad \text{if } x > 1.$$

Therefore, for $x = \frac{17}{12\sqrt{2}}$ we derive

$$U_n\left(\frac{17}{12\sqrt{2}}\right) = \frac{\sinh\left((n+1)z\right)}{\sinh(z)} = \frac{\mathrm{e}^{(n+1)z} - \mathrm{e}^{-(n+1)z}}{\mathrm{e}^{z} - \mathrm{e}^{-z}}, \qquad \cosh z = \frac{17}{12\sqrt{2}}.$$

The positive solution of the last equation is $z = \ln(3\sqrt{2}/4)$; hence, after straightforward computations (using MAPLE), we obtain

$$V_n(a_i) = \frac{9a_i^4 - 36a_i^3 + 53a_i^2 - 42a_i + 42}{4(a_i - 2)^2}, \quad i = 1, \ldots, 5.$$

Notice that each $V_n(a_i)$ is independent of n. Substituting into (6.5) for $x = a_i$, we finally arrive at the required expressions (6.4).

7. The spectrum of a periodic Jacobi operator revisited

In this section we will discuss the spectrum of a periodic Jacobi operator (and so the spectrum of an asymptotically periodic Jacobi operator), recovering known results stated by Maté et al. [25]. Such an operator can be defined by the infinite tridiagonal matrix

$$\mathbf{J}_k := \begin{pmatrix} b_0 & c_1 & 0 & \cdots \\ c_1 & b_1 & c_2 & \cdots \\ 0 & c_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{7.1}$$

where $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 1}$ are k-periodic sequences of real or complex numbers:

$$b_0 = b_k, \qquad b_{nk+j} = b_j, \qquad c_{nk+j} = c_j \quad (j = 1, \ldots, k) \tag{7.2}$$

for all $n = 0, 1, 2, \ldots$. By the usual operation of multiplication of a matrix by a vector, $\mathbf{J}_k$ defines a bounded linear operator acting on the complex Hilbert space $\ell^2(\mathbb{C})$ of the square summable complex sequences. The next proposition characterizes the spectrum of $\mathbf{J}_k$ in the bounded and self-adjoint case, and it improves Theorem 13 in [25]. The proof in [25] makes use of the Hardy class H^2. Our proof is based on the results presented in the previous sections.

Theorem 7.1. *Consider the k-periodic Jacobi operator $\mathbf{J}_k$ defined by (7.1)–(7.2), with $b_j \in \mathbb{R}$ and $c_j > 0$ for every $j = 1, \ldots, k$, so that $\mathbf{J}_k$ defines a bounded and self-adjoint linear operator in $\ell^2(\mathbb{C})$. Define $D_{i,j}(x) = 1$ if $i > j$ and*

$$D_{i,j}(x) := \begin{vmatrix} x - b_i & c_{i+1} & 0 & \cdots & 0 & 0 \\ c_{i+1} & x - b_{i+1} & c_{i+2} & \cdots & 0 & 0 \\ 0 & c_{i+2} & x - b_{i+2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x - b_{j-1} & c_j \\ 0 & 0 & 0 & \cdots & c_j & x - b_j \end{vmatrix}, \quad i \leq j.$$

Set

$$\pi_k(x) := D_{0,k-1}(x) - c_k^2 D_{1,k-2}(x), \qquad \theta_{k-1}(x) := D_{0,k-2}(x)$$

and define

$$\Sigma := \pi_k^{-1}([-2c, 2c]), \quad c := \prod_{j=1}^{k} c_j.$$

Then:

(i) *π_k has k real and distinct zeros $x_1 < \cdots < x_k$ and θ_{k-1} has $k-1$ real and distinct zeros $z_1 < \cdots < z_{k-1}$ such that $x_j < z_j < x_{j+1}$ and $|\pi_k(z_j)| \geq 2c$ for every $j = 1, \ldots, k-1$.*

(ii) *Σ is a union of k closed intervals such that between two of these intervals there exists a zero of θ_{k-1} and any two of these intervals may touch each other at most at a single point which must be necessarily a common zero of the polynomials θ_{k-1} and π_k'.*

(iii) *The spectrum of $\mathbf{J}_k$ is*

$$\sigma(\mathbf{J}_k) = \Sigma \cup Z,$$

where Z is a subset of $\{z_1, \ldots, z_{k-1}\}$ (the zeros of θ_{k-1}), where

$$z_j \in Z \quad \text{if and only if} \quad D_{0,k-1}(z_j)/D_{1,k-2}(z_j) > -c_k^2$$

for each $j = 1, 2, \ldots, k-1$. Furthermore, Z is the set of eigenvalues of $\mathbf{J}_k$ (i.e., Z is the point spectrum of $\mathbf{J}_k$).

Proof. Notice that $D_{i,j}(x)$ is obtained from $\Delta_{i,j}(x)$ in Section 5, by changing each c_s into c_s^2 in the definition of $\Delta_{i,j}(x)$. Thus we see that Theorem 7.1 is a consequence of Theorem 5.1 and its proof, since under the above hypothesis the spectrum of $\mathbf{J}_k$ coincides with the support of the orthogonality measure of the corresponding OPS $(p_n)_n$. We notice that the condition $D_{0,k-1}(z_j)/D_{1,k-2}(z_j) > -c_k^2$ is equivalent to the condition $M_j > 0$, where M_j is defined in Theorem 5.1. $\square$

Remark 7.2. In [25, Theorem 13] the authors described $\sigma(\mathbf{J}_k)$ by stating that it coincides with the set $\pi_k^{-1}(]-2c, 2c[)$ up to a finite set of possible additional elements. From Theorem 7.1 we see that this additional set of points is precisely the set Z described above together with the reals λ such that $\pi_k(\lambda) = \pm 2c$. Moreover, in [25, p. 520] the authors made a statement according to which it is plausible to admit the existence of real numbers λ which are not spectral points in $\sigma(\mathbf{J}_k)$, but $|\pi_k(\lambda)| = 2c$, and this is why they considered $\pi_k^{-1}(]-2c, 2c[)$ instead of $\Sigma := \pi_k^{-1}([-2c, 2c])$ for the description of the spectrum (up to a finite set of additional points), but as regards such a possibility, they also made the following statement (written here in our notation): "We do not have an example showing that there are reals λ satisfying $|\pi_k(\lambda)| \le 2c$ but not $|\pi_k(\lambda)| < 2c$ that do not belong to the spectrum". The above Theorem 7.1 leads to the conclusion that such an example does not exist. (The main reasons are statements (i) and (ii) in the theorem, which imply that the set Σ does not contain isolated points.)

Remark 7.3. In [25, p. 520] the authors gave an example showing that, in fact, the spectrum of $\mathbf{J}_k$ may have additional points out of the set Σ. (According to Theorem 7.1 these points must necessarily belong to the set Z.) In their example they consider a certain 2-periodic Jacobi matrix, $\mathbf{D}$, and state that 0 is an eigenvalue of $\mathbf{D}$, and they gave an associated eigenvector, namely

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \left\langle 1, 0, -\frac{1}{2}, 0, \frac{1}{4}, 0, -\frac{1}{8}, 0, \ldots \right\rangle^*,$$

where the asterisk indicates the transpose, so it is a column vector. In [25] there is a misprint in the definition of $\mathbf{D}$ as well as in the expression for the given eigenvector, which should be as above. In fact, more generally, when $k = 2$, and so $\mathbf{J}_k$ is the 2-periodic Jacobi matrix

$$\mathbf{J}_2 = \begin{pmatrix} b_0 & c_1 & 0 & 0 & 0 & \cdots \\ c_1 & b_1 & c_2 & 0 & 0 & \cdots \\ 0 & c_2 & b_0 & c_1 & 0 & \cdots \\ 0 & 0 & c_1 & b_1 & c_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$(b_0, b_1 \in \mathbb{R};\ c_1, c_2 > 0)$, then one sees that $\theta_1(x) = x - b_0$; hence from Theorem 7.1 we may conclude that b_0 is the only possible eigenvalue of $\mathbf{J}_2$ and, moreover, b_0 is an eigenvalue of $\mathbf{J}_2$ if and only if $c_1 < c_2$ (see also [35,22,4]).

In the same way, when $k = 3$, we see that the only possible eigenvalues of the 3-periodic Jacobi operator

$$\mathbf{J}_3 = \begin{pmatrix} b_0 & c_1 & 0 & 0 & 0 & \cdots \\ c_1 & b_1 & c_2 & 0 & 0 & \cdots \\ 0 & c_2 & b_2 & c_3 & 0 & \cdots \\ 0 & 0 & c_3 & b_0 & c_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$(b_0, b_1, b_2 \in \mathbb{R};\ c_1, c_2, c_3 > 0)$ are

$$\lambda_\pm := \frac{1}{2}\left(b_0 + b_1 \pm \sqrt{(b_0 - b_1)^2 + 4c_1^2}\right),$$

with $\lambda_\pm$ being an eigenvalue of $\mathbf{J}_3$ if and only if the following condition holds:

$$\frac{c_3^2}{c_2^2} > \frac{b_1 - b_0 \pm \sqrt{(b_0 - b_1)^2 + 4c_1^2}}{b_0 - b_1 \pm \sqrt{(b_0 - b_1)^2 + 4c_1^2}}$$

(see also [4]). In particular, if $b_1 = b_0$, we see that the only possible eigenvalues of $\mathbf{J}_3$ are $b_0 \pm c_1$, and $b_0 + c_1$ is an eigenvalue of $\mathbf{J}_3$ if and only if $b_0 - c_1$ is an eigenvalue of $\mathbf{J}_3$ if and only if $c_2 < c_3$.

Remark 7.4. A description of the spectrum and the essential spectrum of a complex periodic Jacobi operator was given by Almendral Vázquez in [3].

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