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Spectra of certain Jacobi operators

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Abstract
We discuss the spectra and, in particular, the essential spectra, of some bounded self-adjoint Jacobi operators associated with orthogonal polynomial sequences obtained via polynomial mappings.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction and preliminaries

In this paper, we study spectral properties of Jacobi operators related to certain polynomial mappings. We are particularly interested in the description of the essential spectra of such operators. Let H be a Hilbert space and $\mathbf{T}$ a bounded and self-adjoint linear operator in H. Under such conditions, it is well known that the spectrum of $\mathbf{T}$, $\sigma(\mathbf{T})$, is a subset of $\mathbb{R}$, and it can be decomposed as

$$\sigma(\mathbf{T}) = \sigma_c(\mathbf{T}) \cup \sigma_p(\mathbf{T}),$$

where $\sigma_c(\mathbf{T})$ denotes the continuum spectrum and $\sigma_p(\mathbf{T})$ is the point spectrum. Recall that $\lambda \in \sigma_p(\mathbf{T})$ if and only if $\mathbf{T}_\lambda := \mathbf{T} - \lambda\mathbf{I}$ is not one to one ($\mathbf{I}$ is the identity operator in H), and $\lambda \in \sigma_c(\mathbf{T})$ if and only if $\mathbf{T}_\lambda$ is one to one and $\mathbf{T}_\lambda^{-1}$ is not a bounded operator in H. Moreover, if $\mathbf{T}_\lambda$ is one to one, then the range of $\mathbf{T}_\lambda$, $\mathbf{T}_\lambda(H)$, is a dense subset of H. A limit point of the spectrum of $\mathbf{T}$ is any point in $\sigma_c(\mathbf{T})$, or any accumulation point of $\sigma(\mathbf{T})$, or an eigenvalue of $\mathbf{T}$ with infinite multiplicity. The set of limit points of the spectrum of $\mathbf{T}$ is called the essential spectrum of $\mathbf{T}$ and it is denoted by $\sigma_{\text{ess}}(\mathbf{T})$. Therefore, $\sigma(\mathbf{T})$ also admits the decomposition

$$\sigma(\mathbf{T}) = \sigma_{\text{ess}}(\mathbf{T}) \cup \sigma_p^f(\mathbf{T})$$

(a disjoint union), where $\sigma_p^f(\mathbf{T}) := \sigma(\mathbf{T}) \setminus \sigma_{\text{ess}}(\mathbf{T})$ is a set of real numbers which contains only isolated points of the spectrum which are eigenvalues of finite multiplicity. $\sigma_p^f(\mathbf{T})$ is called the discrete spectrum. It is well known that the essential spectrum of a bounded and self-adjoint linear operator may be described by using the associated spectral family. This fact can be used to prove the following characterization of the essential spectrum.

Theorem 1.1 (Weyl's criterium)**.** *Let* $\mathbf{T}$ *be a bounded and self-adjoint linear operator in a Hilbert space* H*, and let* λ *be a real number. Then* $\lambda \in \sigma_{\text{ess}}(\mathbf{T})$ *if and only if there exists a sequence* $(f_n)_n$ *in* H *such that* $\|f_n\| = 1$ *for all* n *and the following conditions hold as* $n \to +\infty$*:*

$$f_n \rightharpoonup 0 \text{ weakly in } H, \quad (\mathbf{T} - \lambda \mathbf{I}) f_n \to 0 \text{ strongly in } H.$$

Weyl's criterium implies the following two useful propositions.

Theorem 1.2 (Weyl's theorem)**.** *Let* H *be a Hilbert space,* $\mathbf{T}$ *a bounded and self-adjoint linear operator in* H*, and* $\mathbf{K}$ *a compact operator in* H*. Then*

$$\sigma_{\text{ess}}(\mathbf{T} + \mathbf{K}) = \sigma_{\text{ess}}(\mathbf{T}).$$

Theorem 1.3. *If* $\mathbf{T}$ *is a bounded self-adjoint linear operator in a Hilbert space, then*

$$q(\sigma_{\text{ess}}(\mathbf{T})) = \sigma_{\text{ess}}(q(\mathbf{T}))$$

for any nonzero polynomial q*.*

The above results (and some of their extensions to non-self-adjoint linear operators) can be found e.g. in the books by Riesz and Nagy [29, section 133] and Reed and Simon [27, section VII.3], [28, section XIII.4].

Let us consider a Jacobi operator $\mathbf{J}$ acting on the complex Hilbert space of the square summable sequences $\ell^2 \equiv \ell^2(\mathbb{C})$ represented in the canonical basis of ℓ^2 by the infinite and symmetric tridiagonal matrix (which we also denote by $\mathbf{J}$)

$$\mathbf{J} := \begin{pmatrix} b_0^{(0)} & c_0^{(1)} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ c_0^{(1)} & b_0^{(1)} & c_0^{(2)} & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & c_0^{(2)} & b_0^{(2)} & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & b_0^{(k-2)} & c_0^{(k-1)} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & c_0^{(k-1)} & b_0^{(k-1)} & c_1^{(0)} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & c_1^{(0)} & b_1^{(0)} & c_1^{(1)} & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & c_1^{(1)} & b_1^{(1)} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.1}$$

where $b_n^{(i)} \in \mathbb{R}$ and $c_n^{(i)} > 0$ for all $0 \leqslant i \leqslant k-1$ and $n = 0, 1, 2, \ldots,$ with k being a fixed positive integer number, with

$$B_i := \sup_{n\in\mathbb{N}_0} \left|b_n^{(i)}\right| < \infty, \quad C_i := \sup_{n\in\mathbb{N}_0} c_n^{(i)} < \infty, \quad 0 \leqslant i \leqslant k-1. \tag{1.2}$$

Under such conditions, $\mathbf{J}$ is a symmetric bounded linear operator on $\ell^2(\mathbb{C})$ with

$$\|\mathbf{J}\| := \sup_{\|x\|=1} \|\mathbf{J}x\| \leqslant \sup_{0\leqslant i\leqslant k-1} \{B_i + 2C_i\}.$$

If $\mathbf{J}$ is a bounded and self-adjoint Jacobi operator, then the eigenspace associated with any eigenvalue is one dimensional [15, theorem 5.1]; hence, there are no eigenvalues with infinite multiplicity. Furthermore, $\sigma_c(\mathbf{J}) \subset \sigma(\mathbf{J})'$. Therefore, we may write

$$\sigma_{\text{ess}}(\mathbf{J}) = \sigma(\mathbf{J})',$$

so that $\sigma_{\mathrm{ess}}(\mathbf{J})$ becomes the set of accumulation points of $\sigma(\mathbf{J})$, and $\sigma_p^f(\mathbf{J})$ the set of isolated points of $\sigma(\mathbf{J})$. We mention the work [17] by Koelink for background on the spectral theory of Jacobi operators.

Our aim in this paper is to characterize the spectrum and the essential spectrum of $\mathbf{J}$ in a special situation where this operator can be related to a certain polynomial mapping. The study of polynomial mappings in the framework of orthogonal polynomial sequences (OPS) theory has been a subject which attracted several researchers, especially after an important work by Bessis and Moussa [3]. The subject has been treated for general polynomial transformations by Geronimo and Van Assche [10], by Charris, Ismail and Monsalve [5, 6] (in the more general framework of the so-called blocks of orthogonal polynomials), and by Peherstorfer [24]. Applications of this type of polynomials have appeared in quantum chemistry and physics (see e.g. Wheeler [30] and Pettifor and Weaire [25]), as well as in quantum physics in the study of the so-called chain models (Álvarez-Nodarse *et al* [2])—concerning information about the chain model, see the papers by Haydock [13, 14]. More recently [8, 9], characterization results have been stated in order to ensure that a given OPS is obtained via another OPS via a polynomial mapping. In order to describe this mapping, denote by $(\widetilde{p}_n)_n$ the sequence of orthonormal polynomials associated with $\mathbf{J}$, so that

$$\begin{aligned}&\big(x-b_n^{(j)}\big)\widetilde{p}_{nk+j}(x)=c_n^{(j+1)}\widetilde{p}_{nk+j+1}(x)+c_n^{(j)}\widetilde{p}_{nk+j-1}(x),\\&j=0,1,\ldots,k-1;\quad n=0,1,2,\ldots,\end{aligned}\tag{1.3}$$

with the convention $c_n^{(k)}=c_{n+1}^{(0)}$ for all $n=0,1,2,\ldots,$ and satisfying initial conditions $\widetilde{p}_{-1}(x)=0$ and $\widetilde{p}_0(x)=1$. The monic OPS corresponding to this orthonormal sequence $(\widetilde{p}_n)_n$ is the sequence $(p_n)_n$ characterized by

$$\begin{aligned}&\big(x-b_n^{(j)}\big)p_{nk+j}(x)=p_{nk+j+1}(x)+a_n^{(j)}p_{nk+j-1}(x),\\&j=0,1,\ldots,k-1;\quad n=0,1,2,\ldots,\end{aligned}\tag{1.4}$$

where the relation between $a_n^{(j)}$ and $c_n^{(j)}$ is given by

$$a_n^{(j)}:=\big[c_n^{(j)}\big]^2\quad(j=0,1,\ldots,k-1;\quad n=0,1,2,\ldots),\tag{1.5}$$

and satisfying initial conditions $p_{-1}(x)=0$ and $p_0(x)=1$. Without loss of generality, we will take $a_0^{(0)}=1$, and polynomials p_j with degree $j\leqslant-1$ will always be defined as the zero polynomial. With these numbers $b_n^{(j)}$ and $a_n^{(j)}$, we may construct the determinants $\Delta_n(i,j;x)$ as in [5, 6], so that

$$\Delta_n(i,j;x):=\begin{cases}0 & \text{if}\quad j<i-2\\ 1 & \text{if}\quad j=i-2\\ x-b_n^{(i-1)} & \text{if}\quad j=i-1\end{cases}\tag{1.6}$$

and, if $j\geqslant i\geqslant 1$,

$$\Delta_n(i,j;x):=\begin{vmatrix}x-b_n^{(i-1)} & 1 & 0 & \ldots & 0 & 0\\ a_n^{(i)} & x-b_n^{(i)} & 1 & \ldots & 0 & 0\\ 0 & a_n^{(i+1)} & x-b_n^{(i+1)} & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & x-b_n^{(j-1)} & 1\\ 0 & 0 & 0 & \ldots & a_n^{(j)} & x-b_n^{(j)}\end{vmatrix}\tag{1.7}$$

for every $n=0,1,2,\ldots,$ where the order of this last determinant is $j-i+2$. Taking into account that $\Delta_n(i,j;\cdot)$ is a polynomial whose degree may exceed k, and since in (1.4) the $a_n^{(j)}$s and $b_n^{(j)}$s were defined only for $0\leqslant j\leqslant k-1$, we adopt the convention

$$b_n^{(k+j)}:=b_{n+1}^{(j)},\quad a_n^{(k+j)}:=a_{n+1}^{(j)}\tag{1.8}$$

for all $i, j = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$, and so the useful equality

$$\Delta_n(k+i, k+j; x) = \Delta_{n+1}(i, j; x) \tag{1.9}$$

holds for every $i, j = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$.

Theorem 1.4 ([9, theorem 2.1])**.** *Let $k \in \mathbb{N}$ and $(p_n)_n$ be a monic OPS characterized by the general block of recurrence relations* (1.4). *Fix an integer number m such that $0 \leqslant m \leqslant k-1$, and put*

$$\theta_m(x) := \Delta_0(1, m-1; x) \equiv p_m(x).$$

Assume that, for all $n = 0, 1, 2, \ldots$, the following four conditions hold:

(i) $b_n^{(m)}$ *is independent of n;*
(ii) $\Delta_n(m+2, m+k-1; x)$ *is independent of n for every x;*
(iii) $\Delta_0(m+2, m+k-1; \cdot)$ *is divisible by θ_m, i.e. there exists a polynomial η_{k-1-m} with degree $k-1-m$ such that*

$$\Delta_0(m+2, m+k-1; x) = \theta_m(x)\eta_{k-1-m}(x);$$

(iv) the expression

$$\begin{aligned} r_n := a_n^{(m+1)}\Delta_n(m+3, m+k-1; x) - a_0^{(m+1)}\Delta_0(m+3, m+k-1; x) \\ + a_n^{(m)}\Delta_{n-1}(m+2, m+k-2; x) - a_0^{(m)}\Delta_0(1, m-2; x)\eta_{k-1-m}(x) \end{aligned}$$

is a constant with respect to x for every $n = 1, 2, \ldots$.

Consider the polynomial π_k of degree k defined as

$$\pi_k(x) = \Delta_0(1, m; x)\,\eta_{k-1-m}(x) - a_0^{(m+1)}\,\Delta_0(m+3, m+k-1; x), \tag{1.10}$$

and let $(q_n)_n$ be the monic OPS generated by the recurrence relation

$$q_{n+1}(x) = (x - r_n)\,q_n(x) - s_n q_{n-1}(x), \quad n = 0, 1, 2, \ldots, \tag{1.11}$$

with initial conditions $q_{-1}(x) = 0$ and $q_0(x) = 1$, where

$$r_0 := 0, \quad s_n := a_n^{(m)} a_{n-1}^{(m+1)} \cdots a_{n-1}^{(m+k-1)}, \quad n = 1, 2, \ldots. \tag{1.12}$$

Then, for each $j = 0, 1, 2, \ldots, k-1$ and all $n = 0, 1, 2, \ldots$,

$$\begin{aligned} p_{kn+m+j+1}(x) = \frac{1}{\eta_{k-1-m}(x)} \Big\{ \Delta_n(m+2, m+j; x)\, q_{n+1}(\pi_k(x)) \\ + \left(\prod_{i=1}^{j+1} a_n^{(m+i)}\right) \Delta_n(m+j+3, m+k-1; x)\, q_n(\pi_k(x)) \Big\}. \end{aligned} \tag{1.13}$$

In particular, for $j = k-1$,

$$p_{kn+m}(x) = \theta_m(x)\, q_n(\pi_k(x)), \quad n = 0, 1, 2, \ldots. \tag{1.14}$$

Remark 1.5. In [9], we have indeed stated a more general result, by showing that the four conditions (i)–(iv) appearing in theorem 1.4 are also necessary for the existence of a polynomial mapping in the sense of (1.14), provided that we assign $q_1(0) = 0$. The result remains true without imposing the conditions

$$b_n^{(j)} \in \mathbb{R}, \quad a_n^{(j)} > 0 \quad (j = 0, 1, \ldots, k-1; \quad n = 0, 1, 2, \ldots),$$

i.e. without assuming that $(p_n)_n$ is orthogonal in the positive-definite sense (by simply assuming that the $a_n^{(j)}$s and the $b_n^{(j)}$s are complex numbers with $a_n^{(j)} \neq 0$ for all n and j).

Theorem 1.6 ([9, theorem 3.4])**.** *Under the conditions of theorem 1.4, with the monic OPS $(p_n)_n$ being orthogonal in the positive-definite sense with respect to some positive measure $\mathrm{d}\mu$, $(q_n)_n$ is also a monic OPS in the positive-definite sense, orthogonal with respect to a measure $\mathrm{d}\tau$. Further, assume that the following four conditions hold:*

(i) $[\xi,\eta] := \mathrm{co}\,(\mathrm{supp}(\mathrm{d}\tau))$ *is a compact set;*

(ii) if $m \geqslant 1$,

$$\int_\xi^\eta \frac{\mathrm{d}\tau(x)}{|x-\pi_k(z_i)|} < \infty, \quad i = 1, 2, \ldots, m,$$

where $z_1 < z_2 < \cdots < z_m$ *are the zeros of* θ_m*;*

(iii) either $\pi_k(y_{2i-1}) \geqslant \eta$ *and* $\pi_k(y_{2i}) \leqslant \xi$ *(for all possible i) if k is odd, or* $\pi_k(y_{2i-1}) \leqslant \xi$ *and* $\pi_k(y_{2i}) \geqslant \eta$ *if k is even, where* $y_1 < \cdots < y_{k-1}$ *denote the zeros of* π_k'*;*

(iv) $\theta_m\eta_{k-1-m}$ *and* π_k' *have the same sign at each point of the set* $\pi_k^{-1}([\xi,\eta])$.

Then the Stieltjes transforms $F(\cdot;\mathrm{d}\mu)$ *and* $F(\cdot;\mathrm{d}\tau)$ *are related by*

$$F(z;\mathrm{d}\mu) = \frac{-v_0\Delta_0(2, m-1; z) + \left(\prod_{j=1}^m a_0^{(j)}\right)\eta_{k-1-m}(z)F(\pi_k(z);\mathrm{d}\tau)}{\theta_m(z)},$$

$$z \in \mathbb{C}\setminus\left(\pi_k^{-1}([\xi,\eta]) \cup \{z_1,\ldots,z_m\}\right),$$

where the normalization condition $v_0 := \int_\xi^\eta \mathrm{d}\tau = \int_{\mathrm{supp}(\mathrm{d}\sigma)} \mathrm{d}\mu =: u_0$ *is assumed. Further, the measure* $\mathrm{d}\mu$ *can be obtained from* $\mathrm{d}\tau$ *by*

$$\mathrm{d}\mu(x) = \sum_{i=1}^m M_i\,\delta(x-z_i)\,\mathrm{d}x + \left|\frac{\eta_{k-1-m}(x)}{\theta_m(x)}\right| \frac{\mathrm{d}\tau(\pi_k(x))}{\pi_k'(x)} \tag{1.15}$$

(up to overall constant factors), where, if $m \geqslant 1$,

$$M_i := \frac{v_0\,\Delta_0(2, m-1; z_i)/\left(\prod_{j=1}^m a_0^{(j)}\right) - \eta_{k-1-m}(z_i)\,F(\pi_k(z_i);\mathrm{d}\tau)}{\theta_m'(z_i)} \geqslant 0 \tag{1.16}$$

for all $i = 1,\ldots,m$*. The support of* $\mathrm{d}\mu$ *is contained in the set*

$$\pi_k^{-1}([\xi,\eta]) \cup \{z_1,\ldots,z_m\},$$

a union of k intervals and m possible mass points.

Remark 1.7. In statement (i), $\mathrm{co}(A)$ means the convex hull of a set A. Under the conditions of theorem 1.6, if $\mathrm{d}\tau$ is an absolutely continuous measure with density w_τ (weight function), then the absolutely continuous part of $\mathrm{d}\mu$ has the density

$$w_\mu(x) := \left|\frac{\eta_{k-1-m}(x)}{\theta_m(x)}\right| w_\tau(\pi_k(x))$$

with support contained in a union of at most k closed intervals, and it may have mass points at the zeros of θ_m.

In the following sections, we will study the spectral properties of the operator $\mathbf{J}$ in (1.1), assuming the hypothesis of theorem 1.4.

2. Spectra of Jacobi operators via polynomial mappings

We begin with the following useful

Lemma 2.1. *Let k be a fixed positive integer number and $(r_n)_n$ and $(s_n)_n$ two bounded sequences of real numbers with $s_n > 0$ for all n. Denote by $\mathbf{J}_Q$ and $\mathbf{J}_{0,k}$ the linear operators defined in $\ell^2(\mathbb{C})$ by the infinite band matrices*

$$\mathbf{J}_Q = \begin{pmatrix} r_0 & \sqrt{s_1} & & & & \\ \sqrt{s_1} & r_1 & \sqrt{s_2} & & & \\ & \sqrt{s_2} & r_2 & \sqrt{s_3} & & \\ & & \sqrt{s_3} & r_3 & \sqrt{s_4} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

and

$$\mathbf{J}_{0,k} = \begin{pmatrix} r_0 I_k & \sqrt{s_1} I_k & & & & \\ \sqrt{s_1} I_k & r_1 I_k & \sqrt{s_2} I_k & & & \\ & \sqrt{s_2} I_k & r_2 I_k & \sqrt{s_3} I_k & & \\ & & \sqrt{s_3} I_k & r_3 I_k & \sqrt{s_4} I_k & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where I_k is the identity matrix of order k. Then $\mathbf{J}_Q$ and $\mathbf{J}_{0,k}$ are bounded self-adjoint operators and their spectra, point spectra, essential spectra and discrete spectra coincide respectively:

$$\text{(i)}\quad \sigma(\mathbf{J}_{0,k}) = \sigma(\mathbf{J}_Q) \qquad \text{(ii)}\quad \sigma_p(\mathbf{J}_{0,k}) = \sigma_p(\mathbf{J}_Q)$$

$$\text{(iii)}\quad \sigma_{\text{ess}}(\mathbf{J}_{0,k}) = \sigma_{\text{ess}}(\mathbf{J}_Q) \qquad \text{(iv)}\quad \sigma_p^f(\mathbf{J}_{0,k}) = \sigma_p^f(\mathbf{J}_Q).$$

Proof. It is clear that $\mathbf{J}_Q$ and $\mathbf{J}_{0,k}$ are bounded self-adjoint operators in $\ell^2(\mathbb{C})$. For the proof of (i)–(iv), we make use of the notion of tensor product of operators (see e.g. Reed and Simon [27, section II.4 and VIII.10], [28, section XIII.9] and Prugovečki [26]). A different proof is presented in the appendix, without using tensor products. We begin by noting that $\mathbf{J}_{0,k}$ is equivalent (via a unitary operator) to the tensor product $\mathbf{J}_Q \otimes \mathbf{I}_k$, where $\mathbf{I}_k$ is the identity operator in $\mathbb{C}^k$ (regarded as a vector space over $\mathbb{C}$) which is represented by the matrix I_k with respect to the canonical basis of $\mathbb{C}^k$. Clearly, $\sigma(\mathbf{I}_k) = \sigma_p(\mathbf{I}_k) = \{1\}$ and, since $\mathbb{C}^k$ has finite dimension, $\sigma_{\text{ess}}(\mathbf{I}_k) = \emptyset$. Therefore, on the first hand, to prove (i) we just need to take into account that the spectrum of the tensor product of two operators acting in the tensor product of two separable Hilbert spaces is equal to the product of the spectra of the two involved operators (this is a classical result due to Brown and Pearcy [4]; see also Reed and Simon [28, theorem XIII.34]), so that

$$\sigma(\mathbf{J}_{0,k}) = \sigma(\mathbf{J}_Q \otimes \mathbf{I}_k) = \sigma(\mathbf{J}_Q)\sigma(\mathbf{I}_k) = \sigma(\mathbf{J}_Q).$$

On the other hand, (iii) may be proved as follows:

$$\sigma_{\text{ess}}(\mathbf{J}_{0,k}) = \sigma_{\text{ess}}(\mathbf{J}_Q \otimes \mathbf{I}_k) = \sigma_{\text{ess}}(\mathbf{J}_Q)\sigma(\mathbf{I}_k) \cup \sigma(\mathbf{J}_Q)\sigma_{\text{ess}}(\mathbf{I}_k) = \sigma_{\text{ess}}(\mathbf{J}_Q),$$

where the second equality may be justified by theorem 4.2 in [16] by Ichinose (note that our operators are self-adjoint; hence all the different notions of essential spectra considered in [16] coincide—cf [16, p 79]). Statement (iv) is an immediate consequence of (i) and (iii), so it remains to prove (ii). Indeed, from the general theory of tensor products, we have (see e.g. Kubrusly and Duggal [18, proposition 0])

$$\sigma_p(\mathbf{J}_{0,k}) = \sigma_p(\mathbf{J}_Q \otimes \mathbf{I}_k) \supseteq \sigma_p(\mathbf{J}_Q)\sigma_p(\mathbf{I}_k) = \sigma_p(\mathbf{J}_Q).$$

The reverse inclusion also holds, because $\mathbf{I}_k$ is the identity matrix of finite size k and so it may be readily verified that $\sigma_p(\mathbf{J}_Q \otimes \mathbf{I}_k) \subseteq \sigma_p(\mathbf{J}_Q)$. □

Returning to the Jacobi operator $\mathbf{J}$ in (1.1), it is clear that we can write

$$\pi_k(\mathbf{J}) = \mathbf{J}_{0,k} + \begin{pmatrix} A_0 & B_1 & & & \\ B_1^t & A_1 & B_2 & & \\ & B_2^t & A_2 & B_3 & \\ & & B_3^t & A_3 & B_4 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.1}$$

where the last matrix is an infinite tridiagonal block matrix with $(A_n)_n$ and $(B_n)_n$ uniquely determined sequences of matrices of order k, each B_n being a lower triangular matrix. Note that the explicit expressions for the entries of the matrices A_n and B_n are achieved by computing the difference $\pi_k(\mathbf{J}) - \mathbf{J}_{0,k}$. We are ready to state our main result.

Theorem 2.2. *Let* $\mathbf{J}$ *be the Jacobi operator in* (1.1) *and assume that the conditions*

$$b_n^{(j)} \in \mathbb{R}, \quad c_n^{(j)} > 0 \quad (j = 0, 1, \ldots, k-1; n = 0, 1, 2 \ldots)$$

as well as (1.2) *hold, so that* $\mathbf{J}$ *is bounded and self-adjoint in* $\ell^2(\mathbb{C})$. *Assume that the hypothesis of theorem 1.4 holds and suppose that*

$$A_n \to 0, \quad B_n \to 0 \quad (\text{as } n \to +\infty), \tag{2.2}$$

where A_n *and* B_n *are the matrices of order* k *defined by* (2.1). *Let* $\mathbf{J}_Q$ *be the infinite Jacobi matrix defined in lemma 2.1 with the numbers* r_n *and* s_n *appearing in the entries of* $\mathbf{J}_Q$ *defined by* (1.12). *Then the following holds.*

(i) The essential spectra of $\mathbf{J}$ *and* $\mathbf{J}_Q$ *satisfy the relation*

$$\pi_k(\sigma_{\text{ess}}(\mathbf{J})) = \sigma_{\text{ess}}(\mathbf{J}_Q). \tag{2.3}$$

(ii) In addition, if we also assume that the hypothesis of theorem 1.6 holds together with the condition

$$C := \min_{x \in \pi_k^{-1}(\sigma(\mathbf{J}_Q))} \frac{\eta_{k-1-m}(x)}{\theta_m(x)\pi_k'(x)} > 0, \tag{2.4}$$

then the following holds:

$$\sigma_p(\mathbf{J}) = \pi_k^{-1}(\sigma_p(\mathbf{J}_Q)) \cup \Xi, \tag{2.5}$$

$$\sigma_c(\mathbf{J}) = \pi_k^{-1}(\sigma_c(\mathbf{J}_Q)) \cap (\mathbb{R} \setminus \Xi), \tag{2.6}$$

$$\sigma(\mathbf{J}) = \pi_k^{-1}(\sigma(\mathbf{J}_Q)) \cup \Xi, \tag{2.7}$$

where Ξ *is the subset of* $\{z_1, \ldots, z_m\}$ *(set of the zeros of* θ_m*) characterized by the condition*

$$z_i \in \Xi \quad \text{if and only if} \quad M_i > 0 \quad (i = 1, \ldots, m),$$

with M_i *being defined by (1.16).*

(iii) Under the conditions of (ii),

$$\sigma_{\text{ess}}(\mathbf{J}) = \pi_k^{-1}(\sigma_{\text{ess}}(\mathbf{J}_Q)). \tag{2.8}$$

Proof. Note that hypothesis (2.2) ensures that the infinite banded matrix appearing in the second term on the right-hand side of equality (2.1) is a compact operator (see e.g. [12, theorem 16.4]). Thus, it follows from theorem 1.3, lemma 2.1 and Weyl's theorem that the equalities

$$\pi_k(\sigma_{\rm ess}(\mathbf{J})) = \sigma_{\rm ess}(\pi_k(\mathbf{J})) = \sigma_{\rm ess}(\mathbf{J}_{0,k}) = \sigma_{\rm ess}(\mathbf{J}_Q)$$

hold. This gives (2.3). Note also that (2.3) implies

$$\sigma_{\rm ess}(\mathbf{J}) \subset \pi_k^{-1}(\sigma_{\rm ess}(\mathbf{J}_Q)). \tag{2.9}$$

Next we prove (2.5). Let $x \in \sigma_p(\mathbf{J})$. Then

$$\sum_{n=0}^{\infty} p_n^2(x) < \infty.$$

Therefore, if x is a zero of θ_m, say $x = z_i$ for some $i \in \{1, \dots, m\}$, then the orthogonality measure for the sequence $(p_n)_n$ has a mass point at $x = z_i$; hence, the corresponding mass M_i is a positive number, and so $x \in \Xi$; otherwise, if x is not a zero of θ_m, we may write (using theorem 1.4)

$$\sum_{n=0}^{\infty} q_n^2(\pi_k(x)) = \frac{1}{\theta_m^2(x)} \sum_{n=0}^{\infty} \theta_m^2(x) q_n^2(\pi_k(x)) = \frac{1}{\theta_m^2(x)} \sum_{n=0}^{\infty} p_{kn+m}^2(x) < \infty;$$

hence, $\pi_k(x) \in \sigma_p(\mathbf{J}_Q)$, i.e. $x \in \pi_k^{-1}(\sigma_p(\mathbf{J}_Q))$. Therefore, $\sigma_p(\mathbf{J}) \subset \pi_k^{-1}(\sigma_p(\mathbf{J}_Q)) \cup \Xi$. Further, it is clear that $\Xi \subset \sigma_p(\mathbf{J})$. Hence, to prove (2.5) it remains to demonstrate that $\pi_k^{-1}(\sigma_p(\mathbf{J}_Q)) \subset \sigma_p(\mathbf{J})$. Let $x \in \pi_k^{-1}(\sigma_p(\mathbf{J}_Q))$. Then

$$\sum_{n=0}^{\infty} q_n^2(\pi_k(x)) < \infty. \tag{2.10}$$

We need to prove that $\sum_{n=0}^{\infty} p_n^2(x) < \infty$. Note that

$$\sum_{n=0}^{\infty} p_n^2(x) = \sum_{n=0}^{m} p_n^2(x) + \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} p_{kn+m+j+1}^2(x);$$

hence, we have to show that

$$\sum_{n=0}^{\infty} p_{kn+m+j+1}^2(x) < \infty, \quad j = 0, 1, \dots, k-1. \tag{2.11}$$

Indeed, according to theorem 1.4, we may write

$$p_{kn+m+j+1}(x) = K_{1,j}(x; n) q_{n+1}(\pi_k(x)) + K_{2,j}(x; n) q_n(\pi_k(x))$$

for all $j = 0, 1, \dots, k-1$, where

$$K_{1,j}(x, n) := \frac{\Delta_n(m+2, m+j; x)}{\eta_{k-1-m}(x)},$$
$$K_{2,j}(x, n) := \left(\prod_{i=1}^{j+1} a_n^{(m+i)} \right) \frac{\Delta_n(m+j+3, m+k-1; x)}{\eta_{k-1-m}(x)}.$$

Since $\pi_k(x) \in \sigma_p(\mathbf{J}_Q)$, (2.4) implies that $\eta_{k-1-m}(x) \neq 0$. Moreover, since the sequences $(a_n^{(j)})_n$ and $(b_n^{(j)})_n$ are bounded, we conclude from the definition of the determinants $\Delta_n(i, j; x)$s that there exist continuous functions $\Delta_{1,j}(x)$ and $\Delta_{2,j}(x)$, independent of n, such that

$$|K_{i,j}(x, n)| \leqslant \left| \frac{\Delta_{i,j}(x)}{\eta_{k-1-m}(x)} \right| \quad (i = 1, 2; \quad j = 0, 1, \dots, k-1).$$

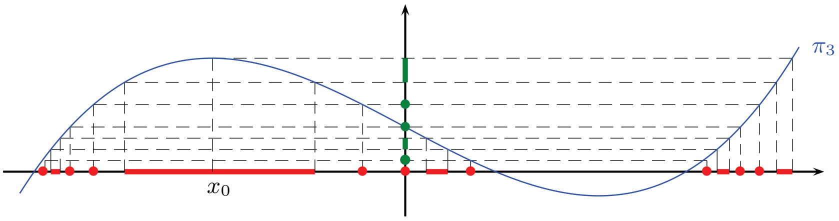

Figure 1. A situation s.t. $\pi_k(x_0) = \max \pi_k(I)$; $\sigma(\mathbf{J}_Q)$ is represented in the y-axis.

It follows that

$$p^2_{kn+m+j+1}(x) \leqslant 2K_j^2(x)\left\{ q^2_{n+1}(\pi_k(x)) + q_n^2(\pi_k(x))\right\} \tag{2.12}$$

for all $j = 0, 1, \ldots, k-1$, being $K_j(x) := \max\{|\Delta_{1,j}(x)|, |\Delta_{2,j}(x)|\}/\eta_{k-1-m}(x)$. (Of course, if needed, we may redefine $K_j(x)$ at each zero x of η_{k-1-m} in such a way that (2.12) holds for all such values of x.) Thus, (2.11) follows, taking into account (2.10). This completes the proof of (2.5).

Next we prove (2.6). Note first that, taking into account (2.9), we may write

$$\sigma_c(\mathbf{J}) \subset \sigma_{\text{ess}}(\mathbf{J}) \subset \pi_k^{-1}(\sigma_{\text{ess}}(\mathbf{J}_Q)) \subset \pi_k^{-1}(\sigma(\mathbf{J}_Q)). \tag{2.13}$$

Let $x \in \sigma_c(\mathbf{J})$. Then $x \notin \sigma_p(\mathbf{J})$ and so, by (2.5), $\pi_k(x) \notin \sigma_p(\mathbf{J}_Q)$ and $x \notin \Xi$. Since $\pi_k(x) \notin \sigma_p(\mathbf{J}_Q)$ and, by (2.13), $\pi_k(x) \in \sigma(\mathbf{J}_Q)$, it follows that $\pi_k(x) \in \sigma_c(\mathbf{J}_Q)$, i.e. $x \in \pi_k^{-1}(\sigma_c(\mathbf{J}_Q))$. Therefore, we conclude that $\sigma_c(\mathbf{J}) \subset \pi_k^{-1}(\sigma_c(\mathbf{J}_Q)) \cap (\mathbb{R}\setminus \Xi)$. To prove that the reverse inclusion also holds, let $x \in \pi_k^{-1}(\sigma_c(\mathbf{J}_Q)) \cap (\mathbb{R}\setminus\Xi)$. Then $x \notin \Xi$ and $\pi_k(x) \in \sigma_c(\mathbf{J}_Q)$; hence, $\pi_k(x) \notin \sigma_p(\mathbf{J}_Q)$. It follows from (2.5) that $x \notin \sigma_p(\mathbf{J})$. As a consequence, if we can prove that

$$\pi_k^{-1}(\sigma(\mathbf{J}_Q)) \subset \sigma(\mathbf{J}), \tag{2.14}$$

then from $\pi_k(x) \in \sigma_c(\mathbf{J}_Q)$ we may write $x \in \pi_k^{-1}(\sigma_c(\mathbf{J}_Q)) \subset \pi_k^{-1}(\sigma(\mathbf{J}_Q)) \subset \sigma(\mathbf{J})$, and so $x \in \sigma(\mathbf{J}) \setminus \sigma_p(\mathbf{J}) = \sigma_c(\mathbf{J})$, concluding the proof of (2.6).

To prove (2.14), take $x_0 \in \pi_k^{-1}(\sigma(\mathbf{J}_Q))$ and let us prove that $x_0 \in \sigma(\mathbf{J})$. Since we are assuming the hypothesis of theorem 1.6, then

$$\sigma(\mathbf{J}) = \text{supp}\,(\mathrm{d}\mu)\,, \quad \sigma(\mathbf{J}_Q) = \text{supp}\,(\mathrm{d}\tau)\,.$$

Thus, we need to prove that $x_0 \in \text{supp}\,(\mathrm{d}\mu)$, i.e.

$$\mu\,(]x_0-\delta, x_0+\delta[) > 0, \quad \forall \delta > 0.$$

Fix $\delta > 0$ and set $I :=]x_0-\delta, x_0+\delta[$. Then, according to (1.15),

$$\mu\,(I) = \int_I \mathrm{d}\mu(x) \geqslant \int_I \left|\frac{\eta_{k-1-m}(x)}{\theta_m(x)}\right| \frac{\mathrm{d}\tau\,(\pi_k(x))}{\pi_k'(x)} \geqslant C\int_I \text{sgn}(\pi_k'(x))\,\mathrm{d}\tau\,(\pi_k(x)),$$

where the last inequality is justified by (2.4) and by hypothesis (iv) in theorem 1.6. Therefore, making the change of variables $y = \pi_k(x)$, we obtain

$$\mu\,(I) \geqslant C\int_{\pi_k(I)} \mathrm{d}\tau\,(y) = C\tau\,(\pi_k(I)) > 0.$$

Note that the last inequality holds since $\pi_k(x_0) \in \sigma(\mathbf{J}_Q) = \text{supp}\,(\mathrm{d}\tau)$ and we have only two possibilities (figure 1 illustrates a situation such that $\pi_k(x_0) = \max \pi_k(I)$, with the set $\sigma(\mathbf{J}_Q)$ being represented in the y-axis):

(i) if $\pi_k'(x_0) \neq 0$, then we choose $\delta_1 > 0$ such that $\pi_k(I) \supset]\pi_k(x_0) - \delta_1, \pi_k(x_0) + \delta_1[$; hence $\tau(\pi_k(I)) > 0$;

(ii) if $\pi_k'(x_0) = 0$, then either $\pi_k(x_0) = \max \pi_k(I)$ or $\pi_k(x_0) = \min \pi_k(I)$; hence, either there exists $\delta_1 > 0$ such that $\pi_k(I) \supset]\pi_k(x_0) - \delta_1, \pi_k(x_0)]$ (in this case, $]\pi_k(x_0), \pi_k(x_0) + \delta_1[\cap \pi_k(I) = \emptyset$) or there exists $\delta_1 > 0$ such that $\pi_k(I) \supset [\pi_k(x_0), \pi_k(x_0) + \delta_1[$ (in this case, $\pi_k(I) \cap]\pi_k(x_0) - \delta_1, \pi_k(x_0)[= \emptyset$) and so, assuming without loss of generality that the first of these situations holds, we deduce $\tau(\pi_k(I)) \geqslant \tau(]\pi_k(x_0) - \delta_1, \pi_k(x_0)]) = \tau(]\pi_k(x_0) - \delta_1, \pi_k(x_0) + \delta_1[) > 0$.

Next, we note that (2.7) is an immediate consequence of (2.5) and (2.6):

$$\sigma(\mathbf{J}) = \sigma_p(\mathbf{J}) \cup \sigma_c(\mathbf{J}) = \pi_k^{-1}(\sigma_p(\mathbf{J}_Q)) \cup \pi_k^{-1}(\sigma_c(\mathbf{J}_Q)) \cup \Xi = \pi_k^{-1}(\sigma(\mathbf{J}_Q)) \cup \Xi.$$

Finally, we will prove (2.8). Taking into account (2.9), we need only to show that $\pi_k^{-1}(\sigma_{\text{ess}}(\mathbf{J}_Q)) \subset \sigma_{\text{ess}}(\mathbf{J})$. Indeed, to prove this relation, we first show that

$$\pi_k^{-1}(\sigma(\mathbf{J}_Q)') \subset \sigma(\mathbf{J})'. \tag{2.15}$$

To prove this relation, let $x \in \pi_k^{-1}(\sigma(\mathbf{J}_Q)')$. Then $\pi_k(x) \in \sigma(\mathbf{J}_Q)'$ and so there exists a sequence $(z_n)_n \subset \sigma(\mathbf{J}_Q)$ such that $z_n \to \pi_k(x)$ as $n \to +\infty$. Let $\{\varphi_j^{-1}\}_{j=1}^k$ be a complete set of inverse branches of π_k (this set exists, according to the hypothesis of theorem 1.6—cf [10, 21]). Then there exists $j \in \{1, \ldots, k\}$ such that $\varphi_j^{-1}(\pi_k(x)) = x$. Set $y_n := \varphi_j^{-1}(z_n)$ for each $n = 1, 2, \ldots$. Note that $y_n \in \pi_k^{-1}(\{z_1, z_2, z_3, \ldots\}) \subset \pi_k^{-1}(\sigma(\mathbf{J}_Q)) \subset \sigma(\mathbf{J})$ for all $n = 1, 2, \ldots$. Therefore, as $n \to +\infty$, we have $y_n := \varphi_j^{-1}(z_n) \to \varphi_j^{-1}(\pi_k(x)) = x$; hence $x \in \sigma(\mathbf{J})'$. This proves (2.15). Therefore, we may write

$$\pi_k^{-1}(\sigma_{\text{ess}}(\mathbf{J}_Q)) = \pi_k^{-1}(\sigma(\mathbf{J}_Q)') \subset \sigma(\mathbf{J})' = \sigma_{\text{ess}}(\mathbf{J}).$$

This completes the proof of the theorem. □

In the next sections, we will analyze some special cases, involving a periodic Jacobi operator, as well as Jacobi operators related to quadratic and cubic polynomial mappings. For convenience, we set

$$\widetilde{A}_n := r_n I_k + A_n, \qquad \widetilde{B}_n := \sqrt{s_n}\, I_k + B_n, \tag{2.16}$$

so that (2.1) can be rewritten as

$$\pi_k(\mathbf{J}) = \begin{pmatrix} \widetilde{A}_0 & \widetilde{B}_1 & & & \\ \widetilde{B}_1^t & \widetilde{A}_1 & \widetilde{B}_2 & & \\ & \widetilde{B}_2^t & \widetilde{A}_2 & \widetilde{B}_3 & \\ & & \widetilde{B}_3^t & \widetilde{A}_3 & \widetilde{B}_4 \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.17}$$

We also need the following useful result due to Chihara [7] (see also Last and Simon [20, theorem 7.2]).

Theorem 2.3 ([7, 20])**.** *Under the conditions of lemma 2.1, if $s_n \to 0$ and R is the set of limit points of $(r_n)_n$, then*

$$\sigma_{\text{ess}}(\mathbf{J}_Q) = R.$$

Remark 2.4. We recall that a real number x is called a limit point of a real sequence $(x_n)_n$ if there exists a subsequence of $(x_n)_n$ which converges to x.

3. Essential spectrum of a periodic Jacobi operator revisited

In this section, we consider a periodic Jacobi operator. Such an operator may be represented by the infinite tridiagonal matrix

$$\mathbf{J}_k := \begin{pmatrix} b_0 & c_1 & 0 & \cdots \\ c_1 & b_1 & c_2 & \cdots \\ 0 & c_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.1}$$

where $(b_n)_{n\geqslant 0}$ and $(c_n)_{n\geqslant 1}$ are sequences of real numbers such that

$$b_0 = b_k, \quad b_{nk+j} = b_j, \quad c_{nk+j} = c_j \quad (j = 1, \ldots, k; \quad n = 0, 1, 2, \ldots). \tag{3.2}$$

It is assumed that

$$b_j \in \mathbb{R}, \quad c_j > 0 \quad (j = 1, \ldots, k). \tag{3.3}$$

The spectrum of $\mathbf{J}_k$ is well known (cf [11, 23, 22, 19, 1, 9]). In what follows we will characterize it using results in the framework of polynomial mappings.

Theorem 3.1. *Let $\mathbf{J}_k$ be the periodic Jacobi operator defined by* (3.1) *and* (3.2) *such that conditions* (3.3) *hold (hence $\mathbf{J}_k$ defines a bounded self-adjoint linear operator in $\ell^2(\mathbb{C})$). Set $D_{i,j}(x) = 1$ if $i > j$ and, for $i \leqslant j$,*

$$D_{i,j}(x) := \begin{vmatrix} x-b_i & c_{i+1} & 0 & \cdots & 0 & 0 \\ c_{i+1} & x-b_{i+1} & c_{i+2} & \cdots & 0 & 0 \\ 0 & c_{i+2} & x-b_{i+2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x-b_{j-1} & c_j \\ 0 & 0 & 0 & \cdots & c_j & x-b_j \end{vmatrix}.$$

Let π_k be the polynomial of degree k defined by

$$\pi_k(x) := D_{0,k-1}(x) - c_k^2 D_{1,k-2}(x)$$

and let Σ and Ξ be the subsets of $\mathbb{R}$ defined as

$$\Sigma := \pi_k^{-1}([-2c, 2c]), \quad c := \textstyle\prod_{j=1}^k c_j,$$

$$\Xi := \left\{x \in \mathbb{R} \mid D_{0,k-2}(x) = 0,\ D_{0,k-1}(x)/D_{1,k-2}(x) > -c_k^2\right\}.$$

Then the following holds:

$$\sigma(\mathbf{J}_k) = \Sigma \cup \Xi, \quad \sigma_c(\mathbf{J}_k) = \sigma_{\rm ess}(\mathbf{J}_k) = \Sigma, \quad \sigma_p(\mathbf{J}_k) = \sigma_p^f(\mathbf{J}_k) = \Xi.$$

Remark 3.2. The set Σ is a union of at most k disjoint intervals, and Ξ is a finite set (possibly the empty set) containing at most $k-1$ points taken from the set of the zeros of the polynomial $\theta_{k-1}(x) := D_{0,k-2}(x)$.

Proof. Note first that in the present situation, we have

$$r_n = 0, \quad s_n = c^2 \quad (n = 0, 1, 2, \ldots);$$

hence, $q_n(x) = c^n U_n\left(\frac{x}{2c}\right)$ for all $n = 0, 1, 2, \dots$, where $(U_n)_n$ is the sequence of the Chebyshev polynomials of the second kind. Moreover,

$$\mathbf{J}_Q = \begin{pmatrix} 0 & c & & & \\ c & 0 & c & & \\ & c & 0 & c & \\ & & c & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \qquad \pi_k(\mathbf{J}_k) = \begin{pmatrix} G_k & cI_k & & & \\ cI_k & 0_k & cI_k & & \\ & cI_k & 0_k & cI_k & \\ & & cI_k & 0_k & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

with the last equality being justified by [1, theorem 1], where G_k is a matrix of order k, I_k is the identity matrix of order k and 0_k is the null matrix of order k. Therefore, we see that the matrices A_n and B_n in (2.16) are given by

$$A_0 = G_k, \quad A_n = B_n = 0_k, \quad n = 1, 2, 3, \dots.$$

Moreover, the polynomials θ_m and η_{k-1-m} appearing in theorems 1.4 and 1.6 are given by (being $m = k - 1$)

$$\theta_{k-1}(x) = D_{0,k-2}(x), \quad \eta_0(x) = 1.$$

Thus, taking into account the results in [9] (cf theorems 5.1 and 7.1 therein, together with their proofs), we see that all the hypotheses of theorem 2.2 are fulfilled. It is well known that

$$\sigma(\mathbf{J}_Q) = \sigma_c(\mathbf{J}_Q) = \sigma_{\mathrm{ess}}(\mathbf{J}_Q) = [-2c, 2c], \quad \sigma_p(\mathbf{J}_Q) = \emptyset.$$

Moreover, according to [9, theorems 5.1 and 7.1], in the present situation the set Ξ defined in theorem 2.2 coincides with the set Ξ defined above, and

$$\Sigma \cap \Xi = \emptyset.$$

To see that this last equality holds, take $x \in \Sigma$ and let us prove that $x \notin \Xi$. Indeed, if $x \in \Sigma$, then $\pi_k(x) \in [-2c, 2c]$. Suppose that $D_{0,k-2}(x) = 0$. Then $\pi_k(x) = \pm 2c$. Without lost of generality, assume that $\pi_k(x) = 2c$. Then, taking into account that $\pi_k^2(x) - 4c^2 = \left\{2c_k^2 D_{1,k-2}(x) + \pi_k(x)\right\}^2$ (see the proof of theorem 5.1 in [9]), we obtain $D_{1,k-2}(x) = -c/c_k^2$ and so, by the definition of $\pi_k(x)$, we obtain $D_{0,k-1}(x) = c$; hence $D_{0,k-1}(x)/D_{1,k-2}(x) = -c_k^2$, and so $x \notin \Xi$. The required equalities for the spectra follow now immediately from theorem 2.2. □

Example 3.3. Let $\mathbf{J}_4$ be the 4-periodic Jacobi operator (3.1)–(3.2), with

$$b_0 = -1, \quad b_1 = 0, \quad b_2 = 1, \quad b_3 = -a, \quad c_1 = c_2 = 2, \quad c_3 = a, \quad c_4 = 4,$$

where a is a fixed positive number. In this case, $(k, m) = (4, 3)$ and

$$\sigma(\mathbf{J}_Q) = [\xi, \eta] = [-32a, 32a],$$

$$\pi_4(x) = x^4 + ax^3 - (a^2 + 25)x^2 - (a^2 + 9a - 16)x + 4(a^2 + 16),$$

$$\theta_3(x) := D_{0,2}(x) = x^3 - 9x,$$

$$R(x) := \frac{D_{0,3}(x)}{D_{1,2}(x)} = \frac{x^4 + ax^3 - (a^2 + 9)x^2 - a(a + 9)x + 4a^2}{x^2 - x - 4}.$$

We have $\theta_3(x) = 0$ iff $x \in \{0, \pm 3\}$, $R(-3) = -\frac{a^2}{4}$, $R(0) = -a^2$ and $R(3) = -4a^2$; hence,

$$\Xi = \{-3, 0, 3\} \quad \text{if} \quad 0 < a < 2; \quad \Xi = \{-3, 0\} \quad \text{if} \quad 2 \leqslant a < 4;$$

$$\Xi = \{-3\} \quad \text{if} \quad 4 \leqslant a < 8; \quad \Xi = \emptyset \quad \text{if} \quad a \geqslant 8.$$

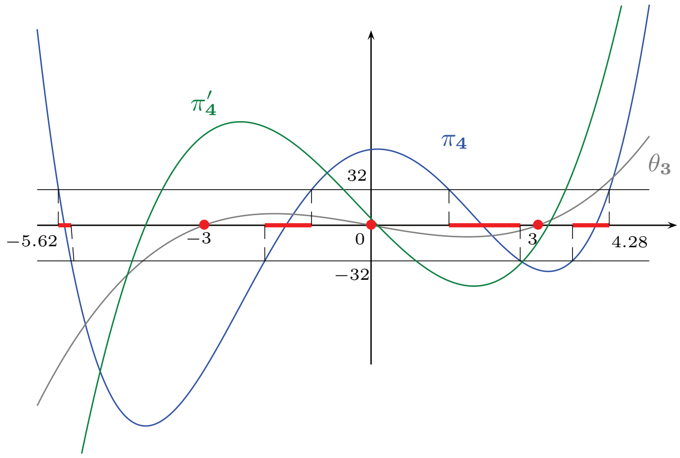

Figure 2. Spectra of $\mathbf{J}_4$ for $a = 1$.

Moreover, from theorem 3.1,

$$\sigma_p(\mathbf{J}_4) = \Xi, \quad \sigma_c(\mathbf{J}_4) = \sigma_{\mathrm{ess}}(\mathbf{J}_4) = \Sigma := \pi_4^{-1}([-32a, 32a]), \quad \sigma(\mathbf{J}_4) = \Sigma \cup \Xi.$$

For instance, letting $a = 1$, we obtain

$$\Xi = \{-3, 0, 3\},$$

$$\Sigma \approx [-5.62, -5.39] \cup [-1.91, -1.07] \cup [1.4, 2.68] \cup [3.62, 4.28].$$

This last example is illustrated in figure 2.

4. Essential spectra via quadratic polynomial mappings

Let $\mathbf{J}$ be the Jacobi operator in $\ell^2(\mathbb{C})$ given by

$$\mathbf{J} := \begin{pmatrix} b_0 & c_1 & 0 & 0 & 0 & \cdots \\ c_1 & b_1 & c_2 & 0 & 0 & \cdots \\ 0 & c_2 & b_0 & c_3 & 0 & \cdots \\ 0 & 0 & c_3 & b_1 & c_4 & \cdots \\ 0 & 0 & 0 & c_4 & b_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.1}$$

where we suppose

$$b_0, b_1 \in \mathbb{R}, \quad c_n > 0 \quad (n = 1, 2, 3, \ldots), \quad \sup_{n\in\mathbb{N}} c_n < \infty. \tag{4.2}$$

Under such conditions, $\mathbf{J}$ is a bounded and self-adjoint Jacobi operator in $\ell^2(\mathbb{C})$. This operator can be studied using a polynomial mapping with $k = 2$ and $m = 1$. In fact, we have (take into account (1.5))

$$b_n^{(0)} = b_0, \quad b_n^{(1)} = b_1, \quad a_n^{(0)} = c_{2n}^2, \quad a_n^{(1)} = c_{2n+1}^2$$

for all $n = 0, 1, 2, \ldots$ and so we see that all the hypotheses of theorem 1.4 are satisfied, being

$$\begin{aligned} r_n &:= c_{2n+2}^2 + c_{2n+1}^2 - c_2^2 - c_1^2, \quad s_n := c_{2n}^2 c_{2n+1}^2 \\ \theta_1(x) &:= x - b_0, \quad \pi_2(x) := x^2 - (b_0 + b_1)x + b_0 b_1 - c_1^2 - c_2^2. \end{aligned} \tag{4.3}$$

Making the convenable choice $c_0 = 0$ and computing $\pi_2(\mathbf{J})$, after straightforward computations, we may identify the matrices $\widetilde{A}_n$ and $\widetilde{B}_n$ in (2.17) as

$$\widetilde{A}_n = \begin{pmatrix} c_{2n}^2 - c_2^2 + c_{2n+1}^2 - c_1^2 & 0 \\ 0 & c_{2n+2}^2 - c_2^2 + c_{2n+1}^2 - c_1^2 \end{pmatrix}, \qquad n = 0, 1, 2, \ldots,$$

and

$$\widetilde{B}_n = \begin{pmatrix} c_{2n-1}c_{2n} & 0 \\ 0 & c_{2n}c_{2n+1} \end{pmatrix}, \qquad n = 1, 2, \ldots.$$

Consequently, according to (2.16), we find

$$A_n = \begin{pmatrix} c_{2n}^2 - c_{2n+2}^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_n = c_{2n} \begin{pmatrix} c_{2n-1} - c_{2n+1} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.4}$$

for all n. Hence by theorem 2.2, we may state the following proposition.

Theorem 4.1. *Let* $\mathbf{J}$ *be the Jacobi operator defined by (4.1) and (4.2). Let* π_2 *be the quadratic polynomial in* (4.3) *and* Y *the set of limit points of* $(c_{2n-1})_n$. *If* $c_{2n} \to 0$ *as* $n \to +\infty$, *then*

$$\pi_2(\sigma_{\mathrm{ess}}(\mathbf{J})) = S := \left\{x = y^2 - \left(c_1^2 + c_2^2\right) \middle| y \in Y\right\}.$$

Example 4.2. Let $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ be the set of rational numbers in the interval $]0, 1[$, and set in (4.1)

$$b_0 = b_1 = 0, \quad c_{2n} = 1/n, \quad c_{2n-1} = \alpha_n \quad (n = 1, 2, \ldots).$$

Then we have $\pi_2(x) = x^2 - 1 - \alpha_1^2$, $Y = [0, 1]$ and $S = \left[-1 - \alpha_1^2, -\alpha_1^2\right]$; hence,

$$\pi_2\left(\sigma_{\mathrm{ess}}(\mathbf{J})\right) = \left[-1 - \alpha_1^2, -\alpha_1^2\right], \quad \sigma_{\mathrm{ess}}(\mathbf{J}) \subset [-1, 1].$$

5. Essential spectra via cubic polynomial mappings

Let $\mathbf{J}$ be the Jacobi operator in $\ell^2(\mathbb{C})$ given by

$$\mathbf{J} := \begin{pmatrix} b_0 & c_1 & 0 & 0 & 0 & \cdots \\ c_1 & b_1 & c_2 & 0 & 0 & \cdots \\ 0 & c_2 & b_2 & c_3 & 0 & \cdots \\ 0 & 0 & c_3 & b_3 & c_4 & \cdots \\ 0 & 0 & 0 & c_4 & b_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.1}$$

where we suppose

$$b_n \in \mathbb{R}, \quad c_n > 0, \quad \sup_{n\in\mathbb{N}}\{c_n + |b_n|\} < \infty. \tag{5.2}$$

Assume further that the conditions

$$\begin{array}{llll} \text{(i)} & b_{3n+2} = b_2 & \text{(iii)} & c_{3n+1}^2 - b_{3n}b_{3n+1} = c_1^2 - b_0 b_1 \\ \text{(ii)} & b_{3n} + b_{3n+1} = b_0 + b_1 & \text{(iv)} & c_{3n}^2 + c_{3n-1}^2 = c_3^2 + c_2^2 \end{array} \tag{5.3}$$

hold for all $n = 1, 2, \ldots$. Under these conditions, $\mathbf{J}$ can be related to a polynomial mapping with $k = 3$ and $m = 2$. In fact, we have

$$b_n^{(j)} = b_{3n+j}, \quad a_n^{(j)} = c_{3n+j}^2 \quad (j = 0, 1, 2)$$

for every n, and conditions (5.3) ensure that all the hypotheses of theorem 1.4 are fulfilled, being

$$\begin{aligned}
r_n &:= c_3^2 b_4 + c_2^2 b_0 - c_{3n+3}^2 b_{3n+4} - c_{3n+2}^2 b_{3n},\\
s_n &:= c_{3n}^2 c_{3n+1}^2 c_{3n+2}^2,\\
\theta_2(x) &:= x^2 - (b_0 + b_1)x + b_0 b_1 - c_1^2, \qquad (5.4)\\
\pi_3(x) &:= x^3 - (b_0 + b_1 + b_2)x^2 + \big(b_0 b_1 + b_0 b_2 + b_1 b_2 - c_1^2 - c_2^2 - c_3^2\big)x\\
&\quad + c_2^2 b_0 + c_1^2 b_2 + c_3^2 b_4 - b_0 b_1 b_2.
\end{aligned}$$

We now compute $\pi_3(\mathbf{J})$, so that after straightforward computations, we can identify the matrices $\widetilde{A}_n$ and $\widetilde{B}_n$ in (2.17) and then from (2.16) we compute

$$A_n = \begin{pmatrix} c_{3n}^2(b_{3n} - b_{3n+1}) + c_{3n+3}^2(b_{3n+4} - b_{3n}) & c_{3n+1}\big(c_{3n}^2 - c_{3n+3}^2\big) & 0\\ c_{3n+1}\big(c_{3n}^2 - c_{3n+3}^2\big) & c_{3n+3}^2(b_{3n+4} - b_{3n+1}) & 0\\ 0 & 0 & 0 \end{pmatrix}$$

for all $n = 0, 1, 2, \ldots$ (with the convention $c_0 = 0$) and

$$B_n = c_{3n} \begin{pmatrix} c_{3n-2}c_{3n-1} - c_{3n+1}c_{3n+2} & 0 & 0\\ c_{3n-1}\,(b_{3n} - b_{3n-3}) & c_{3n+1}\,(c_{3n-1} - c_{3n+2}) & 0\\ 0 & 0 & 0 \end{pmatrix}$$

for all $n = 1, 2, \ldots$. Therefore, we may state the following

Theorem 5.1. *Let $\mathbf{J}$ be a Jacobi operator satisfying (5.1)–(5.3). Let π_3 be the cubic polynomial in* (5.4) *and Y the set of limit points of $(b_{3n})_n$. If $c_{3n} \to 0$ as $n \to +\infty$, then*

$$\pi_3\left(\sigma_{\text{ess}}(\mathbf{J})\right) = S := \left\{x = c_2^2 b_0 + c_3^2 b_4 - \big(c_2^2 + c_3^2\big)y \,\big|\, y \in Y\right\}.$$

Example 5.2. For the choice

$$b_n = 0, \quad c_{3n+1} = 1, \quad c_{3n} = 1/\sqrt{n+1}, \quad c_{3n-1} = \sqrt{n/(n+1)},$$

we find $\pi_3(x) = x(x^2 - 2)$, $\theta_2(x) = x^2 - 1$, and $S = \{0\}$; hence,

$$\pi_3\left(\sigma_{\text{ess}}(\mathbf{J})\right) = \{0\}, \quad \sigma_{\text{ess}}(\mathbf{J}) \subset \pi_3^{-1}(\{0\}) = \{-\sqrt{2}, 0, \sqrt{2}\}.$$

Example 5.3. Fix arbitrarily $b \in \mathbb{R}$ and $c > 0$. Choosing

$$\begin{aligned}
&b_{3n} = c\sin^2(n+1), \quad b_{3n+1} = c\cos^2(n+1), \quad b_{3n+2} = b,\\
&c_{3n+1} = \tfrac{c}{2}\left|\sin(2n+2)\right|, \quad c_{3n} = 1/\sqrt{n+1}, \quad c_{3n-1} = \sqrt{n/(n+1)},
\end{aligned}$$

we find

$$\begin{aligned}
&\theta_2(x) = x(x-c),\\
&\pi_3(x) = x^3 - (b+c)x^2 + (bc-1)x + \frac{c}{2}(\sin^2 1 + \cos^2 2).
\end{aligned}$$

Therefore, taking into account that $\{\sin n \mid n \in \mathbb{N}\}$ is a dense subset of the interval $[-1, 1]$, we derive $Y = [0, c]$, so that

$$\pi_3\left(\sigma_{\text{ess}}(\mathbf{J})\right) = S := \left[\frac{c}{2}(\sin^2 1 + \cos^2 2) - c, \frac{c}{2}(\sin^2 1 + \cos^2 2)\right];$$

hence

$$\sigma_{\text{ess}}(\mathbf{J}) \subset \pi_3^{-1}(S) = I_1 \cup I_2 \cup I_3,$$

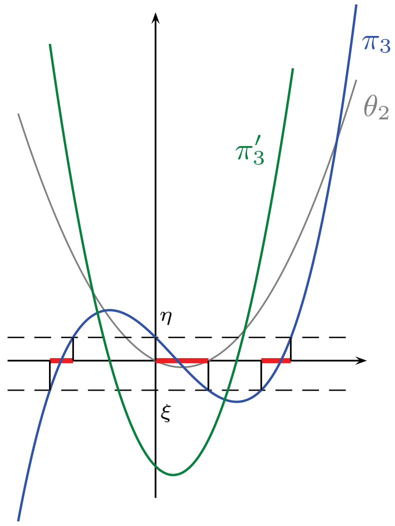

Figure 3. $b = 0$, $c = \frac{1}{2}$.

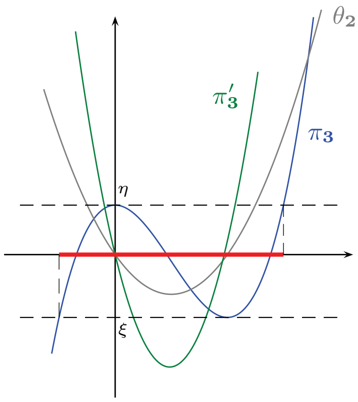

Figure 4. $b = \frac{1}{\sqrt{2}}$, $c = \sqrt{2}$.

where

$$I_1 := \left[\frac{b - \sqrt{b^2+4}}{2}, \min\left\{0, \frac{c+b-\sqrt{(c-b)^2+4}}{2}\right\}\right],$$
$$I_2 := \left[\max\left\{0, \frac{c+b-\sqrt{(c-b)^2+4}}{2}\right\}, \min\left\{c, \frac{b+\sqrt{b^2+4}}{2}\right\}\right],$$
$$I_3 := \left[\max\left\{c, \frac{b+\sqrt{b^2+4}}{2}\right\}, \frac{c+b+\sqrt{(c-b)^2+4}}{2}\right].$$

In this example, we see that $\pi_3^{-1}(S)$ is

(i) a union of three disjoint intervals if $bc \neq 1$ and $c(b-c) \neq 1$;

(ii) a union of two disjoint intervals if $bc \neq 1$ and $c(c-b) = 1$, or if $bc = 1$ and $c(c-b) \neq 1$;
(iii) the single interval $\left[-\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right]$ if $bc = c(c-b) = 1$, i.e. $c = 2b = \sqrt{2}$.

Figures 3 and 4 illustrate the set $\pi_3^{-1}(S)$ for two particular choices of the parameters b and c.

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Appendix

For the sake of completeness, we include an alternative proof of the basic lemma 2.1 for readers not familiar with tensor products. Taking into account proposition VII.12 in [27], statement (i) is a consequence of the proof of (iii) given below, and (iv) follows from (i) and (iii). Therefore, we have to prove only (ii) and (iii).

First we shall prove (ii). Let any $\lambda \in \sigma_p(\mathbf{J}_{0,k})$. Then there exists $\widetilde{x} = (\widetilde{\xi}_j)_{j\geqslant 0} \in \ell^2(\mathbb{C})\backslash\{0\}$ such that $(\mathbf{J}_{0,k} - \lambda\mathbf{I})\widetilde{x} = 0$, i.e.

$$\begin{aligned} &\sqrt{s_n}\,\widetilde{\xi}_{(n-1)k+j} + (r_n - \lambda)\widetilde{\xi}_{nk+j} + \sqrt{s_{n+1}}\,\widetilde{\xi}_{(n+1)k+j} = 0, \\ &n = 0, 1, 2, \ldots; \qquad j = 0, 1, \ldots, k-1. \end{aligned} \tag{A.1}$$

Since $\widetilde{x} \neq 0$, there exist $n_0 \in \mathbb{N}_0$ and $j_0 \in \{0, 1, \ldots, k-1\}$ such that $\widetilde{\xi}_{n_0k+j_0} \neq 0$. Define $\xi_n := \widetilde{\xi}_{nk+j_0}$ for every $n \in \mathbb{N}_0$. Since $\xi_{n_0} \neq 0$, $x := (\xi_n)_n \in \ell^2(\mathbb{C})\backslash\{0\}$, and from (A.1) for $j = j_0$, we obtain

$$\sqrt{s_n}\xi_{n-1} + (r_n - \lambda)\xi_n + \sqrt{s_{n+1}}\xi_{n+1} = 0, \quad n = 0, 1, 2, \ldots,$$

i.e. $(\mathbf{J}_Q - \lambda\mathbf{I})x = 0$; hence $\lambda \in \sigma_p(\mathbf{J}_Q)$. Therefore, $\sigma_p(\mathbf{J}_{0,k}) \subset \sigma_p(\mathbf{J}_Q)$. Next we prove the reverse inclusion. Let $\lambda \in \sigma_p(\mathbf{J}_Q)$. Then there exists $x = (\xi_n)_n \in \ell^2(\mathbb{C})\backslash\{0\}$ such that $(\mathbf{J}_Q - \lambda\mathbf{I})x = 0$, i.e.

$$\sqrt{s_n}\,\xi_{n-1} + (r_n - \lambda)\xi_n + \sqrt{s_{n+1}}\,\xi_{n+1} = 0, \quad n = 0, 1, 2, \ldots. \tag{A.2}$$

Define $\widetilde{x} := (\widetilde{\xi}_n)_n$ by

$$\widetilde{\xi}_{nk+j} := \begin{cases} 0 & \text{if} \quad j \in \{1, \ldots, k-1\} \\ \xi_n & \text{if} \quad j = 0 \end{cases} \qquad (n = 0, 1, 2, \ldots).$$

Then of course $\widetilde{x} \in \ell^2(\mathbb{C})\backslash\{0\}$ and from (A.2), we obtain

$$\begin{aligned} &\sqrt{s_n}\widetilde{\xi}_{(n-1)k+j} + (r_n - \lambda)\widetilde{\xi}_{nk+j} + \sqrt{s_{n+1}}\widetilde{\xi}_{(n+1)k+j} = 0, \\ &n = 0, 1, 2, \ldots; \qquad j = 0, 1, \ldots, k-1, \end{aligned}$$

i.e. $(\mathbf{J}_{0,k} - \lambda\mathbf{I})\widetilde{x} = 0$, or $\lambda \in \sigma_p(\mathbf{J}_{0,k})$. Hence, $\sigma_p(\mathbf{J}_Q) \subseteq \sigma_p(\mathbf{J}_{0,k})$. Thus (ii) is proved.

Now we shall prove (iii). We begin by showing that $\sigma_{\text{ess}}(\mathbf{J}_{0,k}) \subseteq \sigma_{\text{ess}}(\mathbf{J}_Q)$. Let $\lambda \in \sigma_{\text{ess}}(\mathbf{J}_{0,k})$. Then by Weyl's criterium, there exists $\widetilde{f}_n \equiv \left(\widetilde{f}_j^{(n)}\right)_{j\geqslant 0} \in \ell^2(\mathbb{C})$ such that $\|\widetilde{f}_n\| = 1$ for all $n = 0, 1, 2, \ldots$ and

$$\widetilde{f}_n \rightharpoonup 0 \text{ weakly in } \ell^2(\mathbb{C}), \quad \|(\mathbf{J}_{0,k} - \lambda\mathbf{I})\widetilde{f}_n\| \to 0 \quad (\text{as } n \to +\infty).$$

Since $1 = \|\widetilde{f}_n\|^2 = \sum_{i=0}^{k-1}\left(\sum_{j=0}^{\infty}\left|\widetilde{f}_{jk+i}^{(n)}\right|^2\right)$, there exists $i_0 \in \{0,1,\ldots,k-1\}$ such that $\sum_{j=0}^{\infty}\left|\widetilde{f}_{jk+i_0}^{(n)}\right|^2 \geqslant \frac{1}{k}$ for all $n = 0,1,2,\ldots$. Define $\widehat{f}_n \equiv \left(\widehat{f}_j^{(n)}\right)_j \in \ell^2(\mathbb{C})$ by $\widehat{f}_j^{(n)} := \widetilde{f}_{jk+i_0}^{(n)}$. Note that $\|\widehat{f}_n\|^2 = \sum_{j=0}^{\infty}|\widehat{f}_j^{(n)}|^2 \geqslant \frac{1}{k}$ for all n. Now, set

$$f_n := \frac{\widehat{f}_n}{\|\widehat{f}_n\|} \quad (n = 0,1,2,\ldots).$$

Clearly, $f_n \in \ell^2(\mathbb{C})$ and $\|f_n\| = 1$ for all n. To prove that $f_n \rightharpoonup 0$ weakly in $\ell^2(\mathbb{C})$, we need to show only that

$$\langle f_n, h\rangle \to 0 \quad \text{as } n \to +\infty \tag{A.3}$$

for all $h \in \ell^2(\mathbb{C})$, where $\langle\cdot,\cdot\rangle$ means the usual inner product in $\ell^2(\mathbb{C})$. In fact, take arbitrarily $h \equiv (h_n)_n \in \ell^2(\mathbb{C})$. Define $g \equiv (g_n)_n$ by

$$g_{jk+i} := \begin{cases} h_j & \text{if} \quad i = i_0 \\ 0 & \text{if} \quad i \in \{0,1,\ldots,k-1\}\backslash\{i_0\} \end{cases} \qquad (j = 0,1,2,\ldots).$$

Then $g \in \ell^2(\mathbb{C})$ and

$$\langle f_n, h\rangle = \frac{1}{\|\widehat{f}_n\|}\sum_{j=0}^{\infty}\widetilde{f}_{jk+i_0}^{(n)} g_{jk+i_0} = \frac{1}{\|\widehat{f}_n\|}\sum_{\nu=0}^{\infty}\widetilde{f}_{\nu}^{(n)} g_{\nu} = \frac{1}{\|\widehat{f}_n\|}\langle\widetilde{f}_n, g\rangle;$$

hence

$$|\langle f_n, h\rangle| \leqslant \sqrt{k}\,|\langle\widetilde{f}_n, g\rangle|, \quad n = 0,1,2,\ldots.$$

This proves (A.3), taking into account that $\widetilde{f}_n \rightharpoonup 0$, which implies $\langle\widetilde{f}_n, g\rangle \to 0$. To conclude that $\lambda \in \sigma_{\text{ess}}(\mathbf{J}_Q)$, it remains to show that $\|(\mathbf{J}_Q - \lambda\mathbf{I})f_n\| \to 0$. Indeed,

$$\begin{aligned}\|\widehat{f}_n\|^2\|(\mathbf{J}_Q - \lambda\mathbf{I})f_n\|^2 &= \sum_{j=0}^{\infty}\left|\sqrt{s_j}\widehat{f}_{j-1}^{(n)} + (r_j-\lambda)\widehat{f}_j^{(n)} + \sqrt{s_{j+1}}\widehat{f}_{j+1}^{(n)}\right|^2 \\ &= \sum_{j=0}^{\infty}\left|\sqrt{s_j}\widetilde{f}_{(j-1)k+i_0}^{(n)} + (r_j-\lambda)\widetilde{f}_{jk+i_0}^{(n)} + \sqrt{s_{j+1}}\widetilde{f}_{(j+1)k+i_0}^{(n)}\right|^2 \\ &\leqslant \|(\mathbf{J}_{0,k} - \lambda\mathbf{I})\widetilde{f}_n\|^2;\end{aligned}$$

hence, $\|(\mathbf{J}_Q - \lambda\mathbf{I})f_n\| \leqslant \sqrt{k}\,\|(\mathbf{J}_{0,k} - \lambda\mathbf{I})\widetilde{f}_n\| \to 0$ as $n \to +\infty$. Therefore, $\lambda \in \sigma_{\text{ess}}(\mathbf{J}_Q)$ and so $\sigma_{\text{ess}}(\mathbf{J}_{0,k}) \subseteq \sigma_{\text{ess}}(\mathbf{J}_Q)$.

Conversely, let us prove that $\sigma_{\text{ess}}(\mathbf{J}_Q) \subseteq \sigma_{\text{ess}}(\mathbf{J}_{0,k})$. Let $\lambda \in \sigma_{\text{ess}}(\mathbf{J}_Q)$. Then there exists $f_n \equiv \left(f_j^{(n)}\right)_j \in \ell^2(\mathbb{C})$ such that $\|f_n\| = 1$ for all n and

$$f_n \rightharpoonup 0, \quad \|(\mathbf{J}_Q - \lambda\mathbf{I})f_n\| \to 0 \quad (\text{as } n \to +\infty).$$

Define $\widetilde{f}_n \equiv \left(\widetilde{f}_j^{(n)}\right)_j \in \ell^2(\mathbb{C})$ by

$$\widetilde{f}_{jk+i}^{(n)} := \begin{cases} f_j^{(n)} & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \in \{1,\ldots,k-1\} \end{cases} \qquad (j = 0,1,2,\ldots).$$

Then $\|\widetilde{f}_n\|^2 = \sum_{j=0}^{\infty}\left|\widetilde{f}_j^{(n)}\right|^2 = \sum_{i=0}^{k-1}\sum_{j=0}^{\infty}\left|\widetilde{f}_{jk+i}^{(n)}\right|^2 = \sum_{j=0}^{\infty}\left|f_j^{(n)}\right|^2 = \|f_n\|^2$; hence, $\|\widetilde{f}_n\| = \|f_n\| = 1$ for all n. Next take arbitrarily $f \equiv (f_j)_j \in \ell^2(\mathbb{C})$ and set $\widehat{f} \equiv (\widehat{f}_j)_j$ with $\widehat{f}_j := f_{jk}$ for all $j = 0,1,2,\ldots$. Then $\widehat{f} \in \ell^2(\mathbb{C})$ and

$$\langle\widetilde{f}_n, f\rangle = \sum_{j=0}^{\infty}\widetilde{f}_j^{(n)} f_j = \sum_{i=0}^{k-1}\sum_{j=0}^{\infty}\widetilde{f}_{jk+i}^{(n)} f_{jk+i} = \sum_{j=0}^{\infty} f_j^{(n)} f_{jk} = \langle f_n, \widehat{f}\rangle \to 0;$$

hence, $\widetilde{f}_n \rightharpoonup 0$ weakly in $\ell^2(\mathbb{C})$. Finally, we prove that $\|(\mathbf{J}_{0,k} - \lambda\mathbf{I})\widetilde{f}_n\| \to 0$. In fact,

$$\begin{aligned}
\|(\mathbf{J}_{0,k} - \lambda\mathbf{I})\widetilde{f}_n\|^2 &= \sum_{i=0}^{k-1}\sum_{j=0}^{\infty}\left|\sqrt{s_j}\widetilde{f}^{(n)}_{(j-1)k+i} + (r_j - \lambda)\widetilde{f}^{(n)}_{jk+i} + \sqrt{s_{j+1}}\widetilde{f}^{(n)}_{(j+1)k+i}\right|^2 \\
&= \sum_{j=0}^{\infty}\left|\sqrt{s_j}f^{(n)}_{j-1} + (r_j - \lambda)f^{(n)}_j + \sqrt{s_{j+1}}f^{(n)}_{j+1}\right|^2 \\
&= \|(\mathbf{J}_Q - \lambda\mathbf{I})f_n\|^2 \to 0.
\end{aligned}$$

We may conclude that $\lambda \in \sigma_{\text{ess}}(\mathbf{J}_{0,k})$ and so $\sigma_{\text{ess}}(\mathbf{J}_Q) \subseteq \sigma_{\text{ess}}(\mathbf{J}_{0,k})$.

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