Quadratic non-associative (super)algebras and quasialgebras

Helena Albuquerque
and
Elisabete Barreiro

Centre for Mathematics - University of Coimbra
Portugal

FIRST JOINT MEETING
American Mathematical Society - Sociedad de Matemática de Chile
December 2010, Pucón, Chile

Quadratic Lie (super)algebras

A. Medina and Ph. Revoy, 1985

- Inductive description of quadratic Lie algebras
  (using double extension of quadratic Lie algebras)

Reference: A. Medina and P. Revoy, Algèbres de Lie et produit scalaire invariant, *Ann. Sci. École. Norm. Sup.* (4) 18 (1985), 553–561.

S. Benayadi, 2000

- Inductive description of quadratic Lie superalgebras with reductive even part and completely reducible action of the even part on the odd part
  (using elementary double extension of quadratic Lie superalgebras)

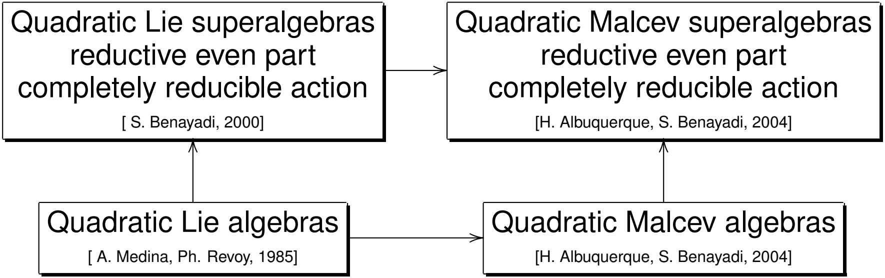
Reference: S. Benayadi, Quadratic Lie superalgebras with the completely reducible action of the even part on the odd part, *J. Algebra* 223 (2000), 344–366.

Quadratic Malcev (super)algebras

H. Albuquerque and S. Benayadi, 2004

- Inductive description of quadratic Malcev algebras (using double extension of quadratic Malcev algebras)
- Inductive description of quadratic Malcev superalgebras with reductive even part and completely reducible action of the even part on the odd part

Reference: H. Albuquerque and S. Benayadi, Quadratic Malcev superalgebras, *J. Pure Appl. Algebra* 187 (2004), 19–45.


Quadratic Lie superalgebras
reductive even part
completely reducible action
[S. Benayadi, 2000]
Quadratic Malcev superalgebras
reductive even part
completely reducible action
[H. Albuquerque, S. Benayadi, 2004]
Quadratic Lie algebras
[A. Medina, Ph. Revoy, 1985]
Quadratic Malcev algebras
[H. Albuquerque, S. Benayadi, 2004]

Lie superalgebras

Superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ ($\mathbb{Z}_2$-graded algebra $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$, $\forall\, \alpha, \beta \in \{\bar{0}, \bar{1}\}$) is Lie superalgebra if multiplication $[,] : L \times L \longrightarrow L$ satisfies

- **L1** $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$ (graded skew-symmetry)
- **L2** $(-1)^{\bar{x}\bar{z}}[[x, y], z] + (-1)^{\bar{x}\bar{y}}[[y, z], x] + (-1)^{\bar{y}\bar{z}}[[z, x], y] = 0$ (graded Jacobi identity), $\forall x \in L_{\bar{x}}, y \in L_{\bar{y}}, z \in L_{\bar{z}}$

- ▷ Finite dimensional Lie superalgebras over an algebraically closed commutative field $\mathbb{K}$ of zero characteristic
- ▷ subalgebra $L_{\bar{0}}$ is Lie algebra
- ▷ vector subspace $L_{\bar{1}}$ is Lie $L_{\bar{0}}$-module

Quadratic Lie superalgebras

Bilinear form $B : L \times L \longrightarrow \mathbb{K}$ is:

- supersymmetric if $B(x, y) = (-1)^{\bar{x}\bar{y}} B(y, x), \ \ \forall x \in L_{\bar{x}}, y \in L_{\bar{y}}$.
- non-degenerate if $x \in L$ satisfies $B(x, y) = 0, \ \ \forall y \in L$, then $x = 0$.
- invariant if $B([x, y], z) = B(x, [y, z]), \ \ \forall x, y, z \in L$.
- even if $B(L_{\bar{0}}, L_{\bar{1}}) = 0$.

- L is quadratic Lie superalgebra if there exists a bilinear form B on L such that B is even, supersymmetric, non-degenerate and invariant.
- B is an invariant scalar product on L

Characterization of quadratic Lie superalgebras

A bilinear form B on a Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is an invariant scalar product on L if and only if the Lie algebra $(L_{\bar{0}}, B_0 = B \mid L_{\bar{0}} \times L_{\bar{0}})$ is quadratic and on the $L_{\bar{0}}$-module $L_{\bar{1}}$ there exists a skew-symmetric non-degenerate $L_{\bar{0}}$-invariant bilinear form B_1 such that

$$B_0([x,y],z) \quad = \quad B_1(x,[y,z]), \qquad \forall x, y \in L_{\bar{1}}, z \in L_{\bar{0}}.$$

Dimension of $L_{\bar{1}}$ is even.

Reference: H. Benamor and S. Benayadi, Double extension of quadratic Lie superalgebras , *Comm. Algebra* 27 (1999), 67–88.

Reduction to the B-irreducible case

Let (L, B) be quadratic Lie superalgebra. Then $L = \bigoplus_{k=1}^{m} L_k$, where

- L_k are non-degenerate graded ideals of L
- $(L_k, B_k = B\mid_{(L_k \times L_k)})$ are B_k-irreducible graded ideals of L
- $B(L_k, L_{k'}) = \{0\}, \forall k, k' \in \{1, \ldots, m\}$ $(k \neq k')$

Double extension

- (L, B) quadratic Lie superalgebra, H Lie superalgebra
- $\psi : H \longrightarrow \mathrm{Der}_a(L, B)$ homomorphism of Lie superalgebras
- $\varphi : L \times L \longrightarrow H^*$

$$\varphi(x, y)(z) = (-1)^{(\bar{x}+\bar{y})\bar{z}} B(\psi(z)(x), y), \quad \forall x \in L_{\bar{x}}, y \in L_{\bar{y}}, z \in H_{\bar{z}}$$

Central extension

Vector space $L \oplus H^*$ endowed with multiplication:

$$[x + f, y + h] \quad = \quad [x, y]_L + \varphi(x, y), \quad \forall (x + f), (y + h) \in (L \oplus H^*)$$

is Lie superalgebra called the central extension of L by H^* (by means of φ)

Double extension

- Vector space $K = H \oplus L \oplus H^*$ with multiplication below is Lie superalgebra: $\forall (z+x+f) \in K_{\bar{x}}, (w+y+g) \in K_{\bar{y}}$,

$$\begin{aligned}[z+x+f, w+y+h] &= [z,w]_H + [x,y]_L + \psi(z)(y) - (-1)^{\bar{x}\bar{y}}\psi(w)(x) \\ &\quad + \pi(z)(h) - (-1)^{\bar{x}\bar{y}}\pi(w)(f) + \varphi(x,y)\end{aligned}$$

- Bilinear form $\widetilde{B} : K \times K \longrightarrow \mathbb{K}$ defined below is invariant scalar product on K: $\forall (z+x+f) \in K_{\bar{x}}, (w+y+h) \in K_{\bar{y}}$,

$$\widetilde{B}(z+x+f, w+y+h) = B(x,y) + \gamma(z,w) + f(w) + (-1)^{\bar{x}\bar{y}} h(z),$$

  where γ is any supersymmetric invariant bilinear form on H.

The quadratic Lie superalgebra $(K, \widetilde{B})$ is called double extension of (L, B) by H (by means of ψ).

Reference: H. Benamor and S. Benayadi, Double extension of quadratic Lie superalgebras , *Comm. Algebra* 27 (1999), 67–88.

Conversely

- $(L = L_{\bar{0}} \oplus L_{\bar{1}}, B)$ is B-irreducible quadratic Lie superalgebra, $\mathfrak{z}(L) \cap L_{\bar{0}} \neq \{0\}$, $\dim L > 1$.

Then (L, B) is a double extension of a quadratic Lie superalgebra $(\mathfrak{h}, \widetilde{B})$ $(\dim \mathfrak{h} = \dim L - 2)$ by a one-dimensional Lie algebra.

Sufficient condition

- (L, B) is B-irreducible quadratic Lie superalgebra,
- I maximal graded ideal of L, $L = I \oplus V$, V is a Lie subsuperalgebra of L.

Then (L, B) is a double extension of the quadratic Lie superalgebra $(I/I^{\perp}, \widetilde{B})$ by V, $\widetilde{B}$ invariant scalar product on $I/I^{\perp}$ induced by B.

Inductive description of quadratic Lie algebras

Theorem

- (L, B) irreducible quadratic Lie algebra
- L not simple Lie algebra

Then (L, B) is a double extension of a quadratic Lie algebra $(A, \widetilde{B})$ by a one-dimensional Lie algebra or by simple Lie algebra.

$$\mathfrak{U} := \{ \text{ abelian Lie algebra, simple Lie algebra } \}$$

Inductive description

- (L, B) quadratic Lie algebra

Then L is an element of $\mathfrak{U}$ or L is obtained in the following way: we take $L_1, L_2, \ldots, L_n$ elements of $\mathfrak{U}$ and we complete by double extensions by the one-dimensional Lie algebra or by a simple Lie algebra and/or orthogonal direct sums of quadratic Lie algebras.

Elementary double extension

Lemma

▷ $(L = L_{\bar{0}} \oplus L_{\bar{1}}, B)$ quadratic Lie superalgebra, $L_{\bar{1}}$ is $L_{\bar{0}}$-module completely reducible, $L_{\bar{1}} \neq \{0\}$.

Then $L_{\bar{1}} = \bigoplus_{i=1}^{n} U_i$, where

- U_i is a $L_{\bar{0}}$-submodule of $L_{\bar{1}}$ such that $B\mid_{(U_i \times U_i)}$ is non-degenerate, $\forall i \in \{1, \ldots, n\}$
- U_i is irreducible or $U_i = U_{i1} \oplus U_{i2}$, where U_{i1} and U_{i2} are irreducible $L_{\bar{0}}$-submodule of $L_{\bar{1}}$ such that $B(U_{i1}, U_{i1}) = \{0\}$, $B(U_{i2}, U_{i2}) = \{0\}$, $\forall i \in \{1, \ldots, n\}$
- $B(U_i, U_j) = \{0\}$, $\forall i, j \in \{1, \ldots, n\}$ $(i \neq j)$

Elementary double extension (by the one-dimensional Lie algebra $\mathbb{K}e$)

- $(L = L_{\bar{0}} \oplus L_{\bar{1}}, B)$ quadratic Lie superalgebra, $L_{\bar{1}}$ is $L_{\bar{0}}$-module completely reducible
- $D : L \longrightarrow L$ superantisymmetric superderivation of degree 0 of L

  (i) $D(L_{\bar{0}}) = D(\bigoplus_{i=1}^{m} U_i) = 0$
  (U_i irreducible $L_{\bar{0}}$-submodule of $L_{\bar{1}}$, $\forall i \in \{1, \dots, m\}$)

  (ii) $D\,|_{U_{i1}} = k_i Id_{U_{i1}}$, $D\,|_{U_{i2}} = -k_i Id_{U_{i2}}$, $k_i \in \mathbb{K}$, $i \in \{m+1, \dots, n\}$
- $\psi : \mathbb{K}e \longrightarrow \mathrm{Der}_a(L, B)$ homomorphism of Lie superalgebras: $\psi(ke) = kD$, $k \in \mathbb{K}$

The double extension $(K, \widetilde{B})$ of (L, B) by $\mathbb{K}e$ (by means of ψ) is called the elementary double extension of (L, B) by the one-dimensional Lie algebra.

Quadratic Lie superalgebras reductive even part and completely reducible action of the even part on the odd part

Examples

- Classical simple Lie superalgebras.
- $\mathfrak{M} = \langle e_1, e_2 \rangle$ two-dimensional abelian Lie superalgebra with zero even part.
  Invariant scalar product on $\mathfrak{M}$: bilinear form $B : \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathbb{K}$
  $B(e_1, e_1) = B(e_2, e_2) = 0$ and $B(e_1, e_2) = -B(e_2, e_1) = 1$.

Lemma

▷ $(L = L_{\bar{0}} \oplus L_{\bar{1}}, B)$ is B-irreducible quadratic Lie superalgebra, $L_{\bar{1}}$ is $L_{\bar{0}}$-module completely reducible, $L_{\bar{0}} \neq 0$.

Then $[L_{\bar{0}}, L_{\bar{1}}] = L_{\bar{1}}$ and $\mathfrak{z}(L) = \mathfrak{z}(L) \cap L_{\bar{0}}$.

Theorem

- $(L = L_{\bar{0}} \oplus L_{\bar{1}}, B)$ is B-irreducible quadratic Lie superalgebra, $L_{\bar{0}}$ reductive Lie algebra, $L_{\bar{1}}$ is $L_{\bar{0}}$-module completely reducible.

Then L is simple if and only if $\mathfrak{z}(L) = \{0\}$.

Inductive description of quadratic Lie superalgebra

reductive even part and completely reducible action of the even part on the odd part

$$\mathfrak{V} := \left\{ \begin{array}{l} \{0\}, \text{basic classical Lie superalgebras} \\ \text{one-dimensional Lie algebra}, \mathfrak{M} \end{array} \right\}$$

- $(L = L_{\bar{0}} \oplus L_{\bar{1}}, B)$ quadratic Lie superalgebra, $L_{\bar{0}}$ reductive Lie algebra, $L_{\bar{1}}$ is $L_{\bar{0}}$-module completely reducible

Inductive description

Then L is either an element of $\mathfrak{V}$ or L is obtained from a finite number of elements of $\mathfrak{V}$ by a sequence of elementary double extensions by the one-dimensional Lie algebra and/or orthogonal direct sums of quadratic Lie superalgebras from a finite number of elements of $\mathfrak{V}$.

Malcev superalgebras

A superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$ ($\mathbb{Z}_2$-graded algebra such that $M_\alpha M_\beta \subseteq M_{\alpha+\beta}, \forall\, \alpha, \beta \in \{\bar{0}, \bar{1}\}$) is Malcev superalgebra if multiplication satisfies

- graded skew-symmetry property: $\forall x \in M_{\bar{x}}, y \in M_{\bar{y}}$
  $xy = -(-1)^{\bar{x}\bar{y}} yx$
- graded identity: $\forall x \in M_{\bar{x}}, y \in M_{\bar{y}}, z \in M_{\bar{z}}, t \in M_{\bar{t}}$

$$\begin{aligned}(-1)^{\bar{y}\bar{z}}(xz)(yt) &= ((xy)z)t + (-1)^{\bar{x}(\bar{y}+\bar{z}+\bar{t})}((yz)t)x \\ &\quad +(-1)^{(\bar{x}+\bar{y})(\bar{z}+\bar{t})}((zt)x)y + (-1)^{\bar{t}(\bar{x}+\bar{y}+\bar{z})}((tx)y)z\end{aligned}$$

- ▷ Finite dimensional Malcev superalgebras over an algebraically closed commutative field $\mathbb{K}$ of zero characteristic
- ▷ subalgebra $M_{\bar{0}}$ is a Malcev algebra
- ▷ vector subspace $M_{\bar{1}}$ is a $M_{\bar{0}}$-module

Malcev superalgebra M

- $V = V_{\bar{0}} \oplus V_{\bar{1}}$ a $\mathbb{Z}_2$-graded vector space, an even linear map $\phi : M \longrightarrow \mathrm{End}(V)$ is Malcev representation of M in V if: $\forall x \in M_{\bar{x}}, y \in M_{\bar{y}}, z \in M_{\bar{z}}$

$$\begin{aligned}\phi((xy)z) &= \phi(x)\phi(y)\phi(z) - (-1)^{\bar{x}(\bar{y}+\bar{z})}\phi(yz)\phi(x) \\ &\quad -(-1)^{\bar{z}(\bar{x}+\bar{y})}\phi(z)\phi(x)\phi(y) + (-1)^{\bar{x}(\bar{y}+\bar{z})}\phi(y)\phi(zx)\end{aligned}$$

- V, W two $\mathbb{Z}_2$-graded vector spaces, $\phi : M \longrightarrow \mathrm{End}(V)$, $\psi : M \longrightarrow \mathrm{End}(W)$ two Malcev representations of M ϕ and ψ are equivalent if there exists $\delta : V \longrightarrow W$ bijective even linear map such that

$$\delta \circ \phi(x) = \psi(x) \circ \delta, \quad \forall x \in M$$

Characterization of quadratic Malcev superalgebras

- M Malcev superalgebra, M^* dual vector space of M

$\pi : M \longrightarrow \text{End}(M)$ is the adjoint representation if
$\pi(x)(y) = xy, \qquad \forall X, Y \in M$
$\pi^* : M \longrightarrow \text{End}(M^*)$ is the coadjoint representation if
$\pi^*(x)(f)(y) \ = \ -(-1)^{\bar{x}\bar{\alpha}} f(xy), \qquad \forall x \in M_{\bar{x}}, y \in M, f \in (M^*)_{\bar{\alpha}}$

Proposition

M is quadratic $\Longleftrightarrow$ adjoint and coadjoint representation of M are equivalent.

- dimension of $M_{\bar{1}}$ is even.

Double extension

M Malcev superalgebra, V $\mathbb{Z}_2$-graded vector space

- An endomorphism $\psi : M \longrightarrow M$ is Malcev operator of M ($\psi \in Op(M)$) if: $\forall x \in M_{\bar{x}}, y \in M_{\bar{y}}, z \in M_{\bar{z}}$

$$\psi((xy)z) = (\psi(x)y)z - (-1)^{\bar{x}\bar{y}}\psi(y)(xz) - (-1)^{\bar{z}(\bar{x}+\bar{y})}(\psi(z)x)y - (-1)^{\bar{x}(\bar{y}+\bar{z})}\psi(yz)x$$

- A homogeneous bilinear map $\phi : M \times M \longrightarrow V$ is Malcev 2-cocycle on M with values in V if: $\forall x \in M_{\bar{x}}, y \in M_{\bar{y}}, z \in M_{\bar{z}}, t \in M_{\bar{t}}$

$$\phi(x, y) = -(-1)^{\bar{x}\bar{y}}\phi(y, x)$$

$$(-1)^{\bar{y}\bar{z}}\phi(xz, yt) = \phi((xy)z, t) + (-1)^{\bar{x}(\bar{y}+\bar{z}+\bar{t})}\phi((yz)t, x) + (-1)^{(\bar{x}+\bar{y})(\bar{z}+\bar{t})}\phi((zt)x, y) + (-1)^{\bar{t}(\bar{x}+\bar{y}+\bar{z})}\phi((tx)y, z)$$

(M, B) quadratic Malcev superalgebra

▷ $\psi : M \longrightarrow M$ endomorphism of M of degree $\alpha \in \mathbb{Z}_2$
ψ is superantisymmetric if

$$B(\psi(x), y) \quad = \quad -(-1)^{\alpha x} B(x, \psi(y)), \qquad \forall x \in M_{\bar{x}}, y \in M$$

Proposition

$\phi : M \times M \longrightarrow \mathbb{K}$ bilinear form of degree $\alpha \in \mathbb{Z}_2$.

- There exists $\psi \in \text{End}(M)$ homogeneous map of degree α

$$\phi(x, y) \quad = \quad B(\psi(x), y), \qquad \forall x, y \in M.$$

- ϕ is a Malcev 2-cocycle on $M \Longleftrightarrow \psi$ superantisymmetric Malcev operator of M $\left(\psi \in (Op_a(M))_\alpha\right)$

Semi-direct product

M, V Malcev superalgebras

$\rho : M \longrightarrow \mathrm{End}(V)$ Malcev representation of M in V such that:

$\forall x \in M_{\bar{x}}, y \in M_{\bar{y}}, z \in M_{\bar{z}}, h \in V_{\bar{\alpha}}, g \in V_{\bar{\beta}}, i \in V_{\bar{\gamma}}$

(i) $\big(\rho(xy)(h)\big)i - \rho(x)\big(\rho(y)(h)i\big) - (-1)^{\bar{y}\bar{\alpha}}\big(\rho(x)(h)\big)\big(\rho(y)(i)\big) + (-1)^{\bar{x}\bar{y}}\rho(y)\big(\rho(x)(hi)\big) + (-1)^{\bar{\gamma}\bar{\alpha}+\bar{x}\bar{y}}\rho(y)\big(\rho(x)(i)\big)h = 0;$

(ii) $(-1)^{\bar{\beta}\bar{z}}\rho(xz)(gi) = -(-1)^{\bar{z}(\bar{x}+\bar{\beta})}\rho(z)\big(\rho(x)(g)\big)i + (-1)^{\bar{\beta}\bar{z}}\rho(x)\big(\rho(z)(g)i\big) - (-1)^{\bar{\beta}(\bar{z}+\bar{\gamma})}\rho(x)\big(\rho(z)(i)\big)g + (-1)^{\bar{\gamma}\bar{\beta}+(\bar{x}+\bar{\beta})\bar{z}}\rho(z)\big(\rho(x)(i)g\big);$

(iii) $\rho(x)$ is a Malcev operator of V.

The $\mathbb{Z}_2$-graded vector space $M \oplus V$ is Malcev superalgebra endowed with multiplication: $\forall (x+f) \in (M \oplus V)_{\bar{x}}, (y+h) \in (M \oplus V)_{\bar{y}}$

$$(x+f)(y+h) = xy + fh + \rho(x)(h) - (-1)^{\bar{x}\bar{y}}\rho(y)(f).$$

The Malcev superalgebra $M \oplus V$ is semi-direct product of M and V (by means of ρ) and ρ Malcev admissible representation of M in V

Central extension

- (M_1, B_1) quadratic Malcev superalgebra, M_2 Malcev superalgebra
- $\psi : M_2 \longrightarrow \mathrm{End}(M_1)$ Malcev admissible representation of M_2 in M_1 such that, $\forall x \in M_2, \psi(x) \in \mathrm{Op}_a(M_1)$.

- $\phi : M_1 \times M_1 \longrightarrow {M_2}^*$ Malcev 2-cocycle on M_1

$$\phi(x,y)(z) = (-1)^{(\bar{x}+\bar{y})\bar{z}} B_1(\psi(z)(x), y), \quad \forall x \in (M_1)_{\bar{x}}, y \in (M_1)_{\bar{y}}, z \in (M_2)_{\bar{z}}$$

The $\mathbb{Z}_2$-graded vector space $M_1 \oplus M_2^*$ is Malcev superalgebra endowed with multiplication:

$$(x_1 + f)(y_1 + h) = x_1 y_1 + \phi(x_1, y_1), \quad \forall (x_1 + f), (y_1 + h) \in (M_1 \oplus {M_2}^*)$$

$M_1 \oplus M_2^*$ is called central extension of ${M_2}^*$ by M_1 (by means of ϕ)

- $\mathbb{Z}_2$-graded vector space $M = M_2 \oplus M_1 \oplus {M_2}^*$ with multiplication:

$$\overbrace{(x_2 + x_1 + f)}^{\bar{x}}\overbrace{(y_2 + y_1 + h)}^{\bar{y}} = (x_2y_2) + \psi(x_2)Y_1 + \pi^*(x_2)h$$
$$+(x_1y_1) + \phi(x_1, y_1) - (-1)^{\bar{x}\bar{y}}\psi(y_2)(x_1) - (-1)^{\bar{x}\bar{y}}\pi^*(y_2)f$$

  M is semi-direct product of $M_1 \oplus {M_2}^*$ by M_2 by means of $\rho : M_2 \longrightarrow \mathrm{End}(M_1 \oplus {M_2}^*)$, $\rho(x_2) = \psi(x_2) + \pi^*(x_2)$
  (ρ Malcev admissible representation of M_2 in $M_1 \oplus {M_2}^*$)
- $B : M \times M \longrightarrow \mathbb{K}$ is invariant scalar product on M:

$$B(x_2 + x_1 + f, y_2 + y_1 + h) = B_1(x_1, y_1) + \gamma(x_2, y_2)$$
$$+f(y_2) + (-1)^{\bar{x}\bar{y}}h(x_2), \quad \forall(x_2 + x_1 + f) \in \mathfrak{M}_{\bar{x}}, (y_2 + y_1 + h) \in M_{\bar{y}}$$

  where γ is even supersymmetric invariant bilinear form on M_2.

(M, B) is double extension of (M_1, B_1) by M_2 (by means of ψ).

Conversely

- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ irreducible quadratic Malcev superalgebra, $\mathfrak{z}(M) \cap M_{\bar{0}} \neq \{0\}$, $\dim M \geq 2$

Then (M, B) is a double extension of a quadratic Malcev superalgebra $(A, \widetilde{B})$ $(\dim A = \dim M - 2)$ by a one-dimensional Lie algebra.

Sufficient condition

- (M, B) irreducible quadratic Malcev superalgebra
- I maximal graded ideal of M,
  $M = I \oplus A$, A is a subsuperalgebra of M

Then (M, B) is a double extension of the quadratic Malcev superalgebra $(I/I^{\perp}, \widetilde{B})$ by A, $\widetilde{B}$ invariant scalar product on $I/I^{\perp}$ induced by B.

Inductive description of quadratic Malcev algebras

Theorem

- (M, B) irreducible quadratic Malcev algebra, $\dim M \geq 2$

Then (M, B) is a double extension of a quadratic Malcev algebra $(A, \widetilde{B})$ $(\dim A = \dim M - 2)$ by a one-dimensional Lie algebra or by the simple non-Lie Malcev algebra.

$$\mathfrak{U} := \{\ \{0\}, \text{one-dimensional Lie algebra}, \text{simple Malcev algebra } C\ \}$$

Inductive description

- (M, B) quadratic Malcev algebra

Then M is an element of $\mathfrak{U}$ or M is obtained in the following way: we take $M_1, M_2, \ldots, M_n$ elements of $\mathfrak{U}$ and we complete by double extensions by the one-dimensional Lie algebra or by a simple Malcev algebra and/or orthogonal direct sums of quadratic Malcev algebras.

Quadratic Malcev superalgebras reductive even part and completely reducible action of the even part on the odd part

Lemma

- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ B-irreducible quadratic Malcev superalgebra, completely reducible action of $M_{\bar{0}}$ over $M_{\bar{1}}$, $M_{\bar{0}} \neq \{0\}$

  Then $M_{\bar{0}} M_{\bar{1}} = M_{\bar{1}}$ and $\mathfrak{z}(M) = \mathfrak{z}(M) \cap M_{\bar{0}}$.

- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ B-irreducible quadratic Malcev superalgebra, $M_{\bar{0}}$ reductive Malcev algebra, completely reducible action of $M_{\bar{0}}$ over $M_{\bar{1}}$, $M_{\bar{0}} \neq \{0\}$

Proposition

Then M is simple if and only if $\mathfrak{z}(M) = \{0\}$.

Theorem

Then M is either a simple Malcev algebra or a Lie superalgebra.

Inductive description of quadratic Malcev superalgebra

reductive even part and completely reducible action of the even part on the odd part

$$\mathfrak{V} := \left\{ \begin{array}{l} \{0\}, \text{basic classical Lie superalgebras} \\ \text{one-dimensional Lie algebra}, \mathfrak{M} \end{array} \right\}$$

▷ $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ quadratic Malcev superalgebra, $M_{\bar{0}}$ reductive Malcev algebra, completely reducible action of $M_{\bar{0}}$ over $M_{\bar{1}}$

Then $M = A \oplus L$, A and L non-degenerate graded ideals of M:

- A is a direct sum of copies of C, where C is the non-Lie Malcev simple algebra;
- L is a Lie superalgebra with $L_{\bar{0}}$ reductive Lie algebra and completely reducible action of $L_{\bar{0}}$ over $L_{\bar{1}}$ such that $L \in \mathfrak{V}$ or L is obtained from a finite number of elements of $\mathfrak{V}$ by a sequence of elementary double extensions by the one-dimensional Lie algebra and/or orthogonal direct sums of quadratic Lie superalgebras from a finite number of elements of $\mathfrak{V}$.

Work in progress:

Classification of quadratic Lie superalgebras with reductive even part and completely reducible action of the even part on the odd part.
Not inductive process!

Reference: A. Elduque, Lie superalgebras with semisimple even part, *J. Algebra* 183 (1996), 649–663.