Quadratic Malcev superalgebras

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(joint work with H. Albuquerque and S. Benayadi)

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Algebras, Representation and Applications
Lie and Jordan Algebras, their Representation and Applications. IV
Manaus, July 2009

A. Medina and Ph. Revoy, 1985

- Inductive description of quadratic Lie algebras (using double extension of quadratic Lie algebras)

A. Elduque, 1996

- structure of Lie superalgebras with reductive even part and completely reducible action of the even part on the odd part

Quadratic Lie superalgebras

S. Benayadi, 2000

- Inductive description of quadratic Lie superalgebras with reductive even part and completely reducible action of the even part on the odd part
  (using double extension of quadratic Lie superalgebras presented by H. Benamor and S. Benayadi, 1999)

H. Albuquerque, E. B., and S. Benayadi, 2007

- Inductive description of quadratic Lie superalgebras with reductive even part
  (using double extension and also generalized double extension of quadratic Lie superalgebras, introduced by I. Bajo, S. Benayadi and M. Bordemann to give the inductive description of solvable quadratic Lie superalgebras)

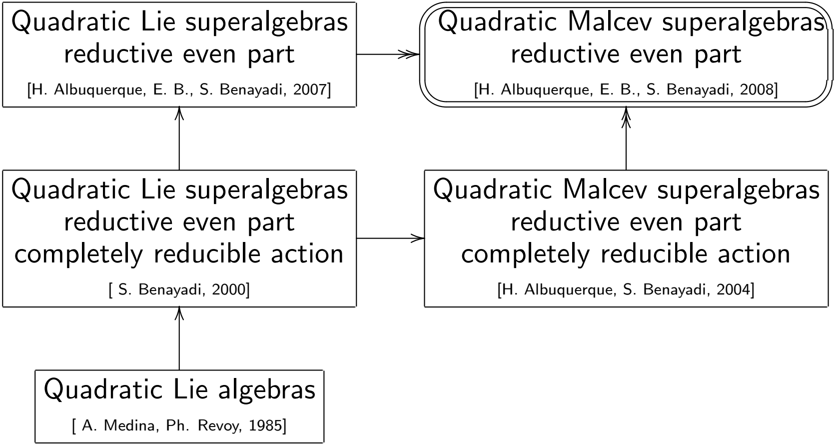
Quadratic Malcev superalgebras

H. Albuquerque and S. Benayadi, 2004

- Inductive description of quadratic Malcev superalgebras with reductive even part and completely reducible action of the even part on the odd part
(double extension of quadratic Malcev superalgebras)

H. Albuquerque, E. B. and S. Benayadi, 2008

- Inductive description of quadratic Malcev superalgebras with reductive even part (using double extension and also generalized double extension of quadratic Malcev superalgebras)


Quadratic Lie superalgebras
reductive even part
[H. Albuquerque, E. B., S. Benayadi, 2007]
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Quadratic Lie algebras
[A. Medina, Ph. Revoy, 1985]

Malcev superalgebras

A superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$
($\mathbb{Z}_2$-graded algebra such that $M_\alpha M_\beta \subseteq M_{\alpha+\beta}, \forall\, \alpha, \beta \in \{\bar{0}, \bar{1}\}$)
is Malcev superalgebra if multiplication satisfies

- graded skew-symmetry property: $\forall_{X \in M_x, Y \in M_y}$
  $XY = -(-1)^{xy} YX$
- graded identity: $\forall_{X \in M_x, Y \in M_y, Z \in M_z, T \in M_t}$
  $$(-1)^{yz}(XZ)(YT) = ((XY)Z)T + (-1)^{x(y+z+t)}((YZ)T)X + (-1)^{(x+y)(z+t)}((ZT)X)Y + (-1)^{t(x+y+z)}((TX)Y)Z$$

- ▷ Finite dimensional Malcev superalgebras over a commutative field $\mathbb{K}$ of zero characteristic
- ▷ subalgebra $M_{\bar{0}}$ is a Malcev algebra
- ▷ vector subspace $M_{\bar{1}}$ is a $M_{\bar{0}}$-module

Malcev algebra M

- center: $\mathfrak{z}(M) = \{X \in M : XY = 0,\ \forall_{Y \in M}\}$
- M is reductive if $M = \mathfrak{z}(M) \oplus M^2$,
  with M^2 the greatest semisimple ideal of M.
- A M-module V is
  - irreducible if it has only two M-submodules (zero and itself).
  - completely reducible if every M-submodule W of V has a *complement* W'
    (W' is a M-submodule of V: $V = W \oplus W'$)

Quadratic Malcev superalgebras

Malcev superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$

A bilinear form $B : M \times M \longrightarrow \mathbb{K}$ is:

- supersymmetric if $B(X,Y) = (-1)^{xy} B(Y,X), \ \forall_{X \in M_x, Y \in M_y}$
- non-degenerate if $X \in M$ satisfies $B(X,Y) = 0, \ \forall_{Y \in M}$, then $X = 0$.
- invariant if $B(XY,Z) = B(X,YZ), \ \forall_{X,Y,Z \in M}$.
- even if $B(M_{\bar{0}}, M_{\bar{1}}) = 0$.

- M is quadratic Malcev superalgebra if there exists a bilinear form B on M such that B is even , supersymmetric, non-degenerate, and invariant.

- B is an invariant scalar product on M

Malcev superalgebra M

- $V = V_{\bar{0}} \oplus V_{\bar{1}}$ $\mathbb{Z}_2$-graded vector space
  $\phi : M \longrightarrow \mathrm{End}(V)$ even linear map
  ϕ is Malcev representation of M in V if: $\forall_{X \in M_x, Y \in M_y, Z \in M_z}$

$$\begin{aligned} \phi((XY)Z) &= \phi(X)\phi(Y)\phi(Z) - (-1)^{x(y+z)}\phi(YZ)\phi(X) \\ &\quad -(-1)^{z(x+y)}\phi(Z)\phi(X)\phi(Y) + (-1)^{x(y+z)}\phi(Y)\phi(ZX) \end{aligned}$$

- V, W $\mathbb{Z}_2$-graded vector spaces
  $\phi : M \longrightarrow \mathrm{End}(V)$, $\psi : M \longrightarrow \mathrm{End}(W)$ Malcev representations of M
  ϕ and ψ are equivalent if there exists $\delta : V \longrightarrow W$ bijective even linear map such that

$$\delta \circ \phi(X) = \psi(X) \circ \delta, \ \forall_{X \in M}$$

Characterization of quadratic Malcev superalgebras

- M Malcev superalgebra, M^* dual of vector space M

- Adjoint representation: $\pi : M \longrightarrow \text{End}(M)$

$$\pi(X)(Y) \;=\; XY, \qquad \forall_{X,Y \in M}$$

- Coadjoint representation: $\pi^* : M \longrightarrow \text{End}(M^*)$

$$\pi^*(X)(f)(Y) \;=\; -(-1)^{x\alpha} f(XY), \qquad \forall_{X \in M_x, Y \in M, f \in (M^*)_\alpha}$$

Proposition [H. Albuquerque and S. Benayadi, 2004]

M is quadratic $\Longleftrightarrow$ adjoint and coadjoint representation of M are equivalent.

- dimension of $M_{\bar{1}}$ is even.

reduction to the B-irreducible case

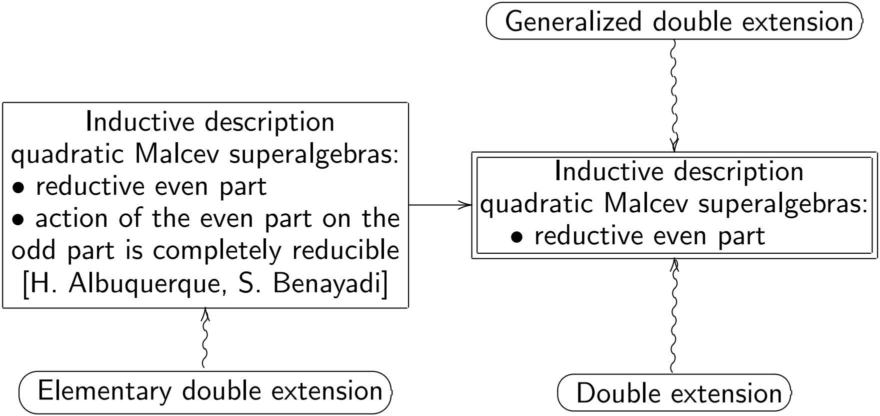
▷ (M, B) quadratic Malcev superalgebra

M is B-irreducible if M contains no non-trivial non-degenerate graded ideal.

Proposition

Then $M = \bigoplus_{k=1}^{m} M_k$, where

- M_k is non-degenerate graded ideal of M
- $(M_k, B_k = B\mid_{(M_k \times M_k)})$ is B_k-irreducible graded ideal of M
- $B(M_k, M_{k'}) = \{0\}$, $\forall k, k' \in \{1, \ldots, m\}$ $(k \neq k')$


Generalized double extension
Inductive description
quadratic Malcev superalgebras:
• reductive even part
• action of the even part on the
odd part is completely reducible
[H. Albuquerque, S. Benayadi]
Inductive description
quadratic Malcev superalgebras:
• reductive even part
Elementary double extension
Double extension

M Malcev superalgebra, V $\mathbb{Z}_2$-graded vector space

- $\psi : M \longrightarrow M$ endomorphism of M
  ψ Malcev operator of M ($\psi \in Op(M)$) if:
  $\forall_{X \in M_x, Y \in M_y, Z \in M_z}$
  $\psi((XY)Z) = (\psi(X)Y)Z - (-1)^{xy}\psi(Y)(XZ) - (-1)^{z(x+y)}(\psi(Z)X)Y - (-1)^{x(y+z)}\psi(YZ)X$
- $\phi : M \times M \longrightarrow V$ homogeneous bilinear map
  ϕ Malcev 2-cocycle on M with values in V if:
  $\forall_{X \in M_x, Y \in M_y, Z \in M_z, T \in M_t}$
  - $\phi(X, Y) = -(-1)^{xy}\phi(Y, X)$
  - $(-1)^{yz}\phi(XZ, YT) = \phi((XY)Z, T) + (-1)^{x(y+z+t)}\phi((YZ)T, X) + (-1)^{(x+y)(z+t)}\phi((ZT)X, Y) + (-1)^{t(x+y+z)}\phi((TX)Y, Z)$

(M, B) quadratic Malcev superalgebra

▷ $\psi : M \longrightarrow M$ endomorphism of M of degree $\alpha \in \mathbb{Z}_2$
ψ is superantisymmetric if

$$B(\psi(X), Y) = -(-1)^{\alpha x} B(X, \psi(Y)), \qquad \forall_{X \in M_x, Y \in M}$$

Proposition

$\phi : M \times M \longrightarrow \mathbb{K}$ bilinear form of degree $\alpha \in \mathbb{Z}_2$.

- There exists $\psi \in \mathrm{End}(M)$ homogeneous map of degree α

$$\phi(X, Y) = B(\psi(X), Y), \qquad \forall_{X, Y \in M}.$$

- ϕ is a Malcev 2-cocycle on $M \iff \psi$ superantisymmetric Malcev operator of M $\left(\psi \in (Op_a(M))_\alpha\right)$

Semi-direct product

M, V Malcev superalgebras

$\rho : M \longrightarrow \text{End}(V)$ Malcev representation of M in V such that:

$\forall_{X \in M_x, Y \in M_y, Z \in M_z, h \in V_\alpha, g \in V_\beta, i \in V_\gamma}$:

(i) $\big(\rho(XY)(h)\big)i - \rho(X)\big(\rho(Y)(h)i\big) - (-1)^{y\alpha}\big(\rho(X)(h)\big)\big(\rho(Y)(i)\big) + (-1)^{xy}\rho(Y)\big(\rho(X)(hi)\big) + (-1)^{\gamma\alpha+xy}\rho(Y)\big(\rho(X)(i)\big)h = 0$;

(ii) $(-1)^{\beta z}\rho(XZ)(gi) = -(-1)^{z(x+\beta)}\rho(Z)\big(\rho(X)(g)\big)i + (-1)^{\beta z}\rho(X)\big(\rho(Z)(g)i\big) - (-1)^{\beta(z+t)}\rho(X)\big(\rho(Z)(i)\big)g + (-1)^{t\beta+(x+\beta)z}\rho(Z)\big(\rho(X)(i)g\big)$;

(iii) $\rho(X)$ is a Malcev operator of V.

Semi-direct product

Proposition

$M \oplus V$

$$(X+f)(Y+h) = XY + fh + \rho(X)(h) - (-1)^{xy}\rho(Y)(f),$$

$\forall_{(X+f)\in(M\oplus V)_x,(Y+h)\in(M\oplus V)_y}$,is Malcev superalgebra.
$M \oplus V$ semi-direct product of M and V (by means of ρ)
ρ Malcev admissible representation of M in V

Central extension

- (M_1, B_1) quadratic Malcev superalgebra
- M_2 Malcev superalgebra
- $\psi : M_2 \longrightarrow \mathrm{End}(M_1)$ Malcev admissible representation of M_2:
  $\forall_{X\in M_2}, \psi(X) \in \mathrm{Op}_a(M_1)$ and $\forall_{X\in M_{2x}, Y\in M_{1y}, Z\in M_{1z}, S\in M_{2s}}$
  $$\psi(SX)(YZ) = \psi(S)((\psi(x)Y)Z) - (-1)^{yz}(\psi(S)(\psi(X)Z))Y + (-1)^{sx+yz}\psi(X)((\psi(S)Z)Y) + (-1)^{sx}(\psi(X)(\psi(S)Y))Z$$

- $\phi : M_1 \times M_1 \longrightarrow {M_2}^*$ Malcev 2-cocycle on M_1
  $\phi(X,Y)(Z) = (-1)^{(x+y)z} B_1(\psi(Z)(X), Y), \ \forall_{X\in M_{1x}, Y\in M_{1y}, Z\in M_{2z}}$

Proposition

$M_1 \oplus M_2^*$

$(X_1 + f)(Y_1 + h) = X_1Y_1 + \phi(X_1, Y_1), \ \forall_{(X_1+f),(Y_1+h)\in(M_1\oplus {M_2}^*)}$

is central extension of ${M_2}^*$ by M_1 (by means of ϕ)

Theorem

- $M = M_2 \oplus M_1 \oplus {M_2}^*$

$$\overbrace{(X_2 + X_1 + f)}^{x}\overbrace{(Y_2 + Y_1 + h)}^{y} = (X_2Y_2) + \psi(X_2)Y_1 + \pi^*(X_2)h$$
$$+(X_1Y_1) + \phi(X_1, Y_1) - (-1)^{xy}\psi(Y_2)(X_1) - (-1)^{xy}\pi^*(Y_2)f$$

  M is semi-direct product of $M_1 \oplus {M_2}^*$ by M_2 by means of $\rho : M_2 \longrightarrow \mathrm{End}(M_1 \oplus {M_2}^*)$, $\rho(X_2) = \psi(X_2) + \pi^*(X_2)$ (ρ Malcev admissible representation of M_2 in $M_1 \oplus {M_2}^*$)

▷ If γ is even supersymmetric invariant bilinear form on M_2

- $B : M \times M \longrightarrow \mathbb{K}$

$$B(X_2 + X_1 + f, Y_2 + Y_1 + h) = B_1(X_1, Y_1) + \gamma(X_2, Y_2)$$
$$+f(Y_2) + (-1)^{xy}h(X_2), \ \forall_{(X_2+X_1+f)\in\mathfrak{M}_x,(Y_2+Y_1+h)\in M_y}$$

  is invariant scalar product on M.

(M, B) is double extension of (M_1, B_1) by M_2 (by means of ψ).

Conversely

- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ irreducible quadratic Malcev superalgebra
- $\mathfrak{z}(M) \cap M_{\bar{0}} \neq \{0\}$
- $\dim M \geq 2$

Proposition

Then (M, B) is a double extension of a quadratic Malcev superalgebra $(A, \widetilde{B})$ $(\dim A = \dim M - 2)$ by a one-dimensional Lie algebra.

Sufficient condition

- (M, B) irreducible quadratic Malcev superalgebra
- Sufficient condition: I maximal graded ideal of M, $M = I \oplus A$, A is a subsuperalgebra of M

Proposition

Then (M, B) is a double extension of the quadratic Malcev superalgebra $(I/I^{\perp}, \widetilde{B})$ by A, $\widetilde{B}$ invariant scalar product on $I/I^{\perp}$ induced by B.

Central extension

- $(\mathbb{K}e)_{\bar{1}}$ one-dimensional Lie superalgebra with even part zero
- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ quadratic Malcev superalgebra
- D odd superantisymmetric Malcev operator of (M, B)
- $\varphi : M \times M \longrightarrow \mathbb{K}e^*$ Malcev 2-cocycle on M

$$\varphi(X, Y) = -B(D(X), Y)e^*, \ \forall_{X,Y \in M}$$

Proposition

- $M \oplus \mathbb{K}e^*$

$$(X + \alpha e^*)(Y + \beta e^*) = XY + \varphi(X, Y), \quad \forall_{(X+\alpha e^*),(Y+\beta e^*)\in(M\oplus\mathbb{K}e^*)}$$

is Malcev superalgebra.

$M \oplus \mathbb{K}e^*$ central extension of M by $\mathbb{K}e^*$ (by means of φ)

Generalized semi-direct product

- M, V Malcev superalgebras
- $\Omega : M \longrightarrow \text{End}(V)$ even linear map
- $\zeta : M \times M \longrightarrow V$ even superantisymmetric bilinear map such that, $\forall_{X\in M_x, Y\in M_y, Z\in M_z, h\in V_\alpha, g\in V_\beta, i\in V_\gamma}$:

(i) $\big(\Omega(XY)(h)\big)i - \Omega(X)\big(\Omega(Y)(h)i\big) - (-1)^{y\alpha}\big(\Omega(X)(h)\big)\big(\Omega(Y)(i)\big) + (-1)^{xy}\Omega(Y)\big(\Omega(X)(hi)\big) + (-1)^{\gamma\alpha+xy}\Omega(Y)\big(\Omega(X)(i)\big)h + \big(\zeta(X,Y)h\big)i = 0;$

(ii) $(-1)^{\beta z}\big\{\zeta(X,Z)(gi) + \Omega(XZ)(gi)\big\} = -(-1)^{z(x+\beta)}\Omega(Z)\big(\Omega(X)(g)\big)i + (-1)^{\beta z}\Omega(X)\big(\Omega(Z)(g)i\big) - (-1)^{\beta(z+\gamma)}\Omega(X)\big(\Omega(Z)(i)\big)g + (-1)^{\gamma\beta+(x+\beta)z}\Omega(Z)\big(\Omega(X)(i)g\big);$

Generalized semi-direct product

$\forall_{X\in M_x, Y\in M_y, Z\in M_z, T\in M_t}$:

(iii) $(-1)^{yz}(\zeta(X,Z)\zeta(Y,T))_V + (-1)^{yz}\Omega(XZ)\big(\zeta(Y,T)\big) - (-1)^{zt+x(y+t)}\Omega(YT)\big(\zeta(X,Z)\big) + (-1)^{yz}\zeta(XZ,YT) = \zeta((XY)Z,T) - (-1)^{t(x+y+z)}\Omega(T)\big(\zeta(XY,Z)\big) + (-1)^{z(x+y)+t(x+y+z)}\Omega(T)\big(\Omega(Z)(\zeta(X,Y))\big) + (-1)^{x(y+z+t)}\zeta((YZ)T,X) - \Omega(X)\big(\zeta(YZ,T)\big) + (-1)^{t(y+z)}\Omega(X)\big(\Omega(T)(\zeta(Y,Z))\big) + (-1)^{(x+y)(z+t)}\zeta((ZT)X,Y) - (-1)^{x(y+z+t)}\Omega(Y)\big(\zeta(ZT,X)\big) + (-1)^{xy}\Omega(Y)\big(\Omega(X)(\zeta(Z,T))\big) + (-1)^{t(x+y+z)}\zeta((TX)Y,Z) - (-1)^{(x+y)(t+z)}\Omega(Z)\big(\zeta(TX,Y)\big) + (-1)^{yz+x(y+z+t)}\Omega(Z)\big(\Omega(Y)(\zeta(T,X))\big)$;

(iv) $\Omega\big((XY)Z\big) - \Omega(X)\Omega(Y)\Omega(Z) + (-1)^{x(y+z)}\Omega(YZ)\Omega(X) + (-1)^{z(x+y)}\Omega(Z)\Omega(X)\Omega(Y) - (-1)^{x(y+z)}\Omega(Y)\Omega(ZX) = -(-1)^{x(y+z)}\pi_V\big(\zeta(Y,Z)\big)\Omega(X) + (-1)^{z(x+y)}\pi_V\big(\Omega(Z)(\zeta(X,Y))\big) - (-1)^{xy}\Omega(Y)\pi_V\big(\zeta(X,Z)\big) - \pi_V\big(\zeta(XY,Z)\big)$;

(v) $\Omega(X)$ is a Malcev operator of V.

Generalized semi-direct product

Proposition

$M \oplus V$

$$(X+f)(Y+h) = XY + fh + \Omega(X)(h) - (-1)^{xy}\Omega(Y)(f) + \zeta(X,Y),$$

$\forall_{(X+f)\in(M\oplus V)_x,(Y+h)\in(M\oplus V)_y}$,is Malcev superalgebra.
$M \oplus V$ generalized semi-direct product of M and V (by means of Ω and ζ)

If $\zeta = 0$

$M \oplus V$ semi-direct product of M and V (by means of Ω)

- (iii) is trivial
- (iv) means that $\Omega : M \longrightarrow End(V)$ is a Malcev representation

Generalized double extension by one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ with even part zero

- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ quadratic Malcev superalgebra
- D odd superantisymmetric Malcev operator of (M, B)
  $A_0 \in M_{\bar{0}}$ such that: for arbitrary $X \in M_x, Y \in M_y$

$$D(A_0X) = A_0D(X) - D(A_0)X,$$
$$A_0(XY) = D(D(X)Y) + D^2(X)Y - (-1)^{xy}\{D^2(Y)X + D(D(Y)X)\}$$

- $$\begin{aligned} \Omega : \mathbb{K}e &\longrightarrow Op(M \oplus \mathbb{K}e^*) \\ e &\longrightarrow \widetilde{D} \end{aligned}$$

  $\widetilde{D} : M \oplus \mathbb{K}e^* \longrightarrow M \oplus \mathbb{K}e^*$ such that $\widetilde{D}(e^*) = 0$

$$\widetilde{D}(X) = D(X) - (-1)^x B(X, A_0)e^*, \ \ \forall_{X \in M_x}$$

- $\zeta : \mathbb{K}e \times \mathbb{K}e \longrightarrow M \oplus \mathbb{K}e^*$ bilinear map $\zeta(e, e) = A_0$

Generalized double extension

Proposition

- $N = \mathbb{K}e \oplus M \oplus \mathbb{K}e^*$
  $(,) : N \times N \longrightarrow N$ even skew-symmetric bilinear map such that

$$\begin{aligned} (ee) &= A_0,\ (e^*N) = \{0\} \\ (eX) &= D(X) - B(-1)^x(X, A_0)e^*, \qquad \forall_{X \in M_x} \\ (XY) &= (XY)_M - B(D(X), Y)e^*, \qquad \forall_{X,Y \in M} \end{aligned}$$

  is generalized semi-direct product of $M \oplus \mathbb{K}e^*$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of Ω, ζ)

- supersymmetric bilinear form $\widetilde{B} : N \times N \longrightarrow \mathbb{K}$ such that

$$\begin{aligned} \widetilde{B}\,|_{M \times M} = B,\ \widetilde{B}(e^*, e) &= 1 \\ \widetilde{B}(M, e) = \widetilde{B}(M, e^*) &= \{0\},\ \widetilde{B}(e, e) = \widetilde{B}(e^*, e^*) = 0 \end{aligned}$$

  is invariant scalar product on N

$(N, \widetilde{B})$ generalized double extension of (M, B) by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of D and A_0)

Conversely

- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ irreducible quadratic Malcev superalgebra
- $\mathfrak{z}(M) \cap M_{\bar{1}} \neq \{0\}$
- $\dim M \geq 2$

Proposition

Then (M, B) is a generalized double extension of a quadratic Malcev superalgebra $(N, \widetilde{B})$ $(\dim N = \dim M - 2)$ by a one-dimensional Lie superalgebra with zero even part.

Quadratic Malcev superalgebras with reductive even part

M Malcev superalgebra
$soc(M)$:= sum of all minimal graded ideals of M

- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ quadratic Malcev superalgebra
- $M_{\bar{0}}$ reductive Malcev algebra

Proposition

- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ B-irreducible quadratic Malcev superalgebra
$\neq \begin{cases} \text{one-dimensional Lie algebra} \\ \text{simple Malcev superalgebra} \end{cases}$
then $soc(M) = \mathfrak{z}(M)$.
- M is semisimple if and only if $\mathfrak{z}(M) = \{0\}$.

Inductive description of quadratic Malcev superalgebra

reductive even part

$$W := \left\{ \begin{array}{l} \{0\}, \text{basic classical Lie superalgebras} \\ \text{one-dimensional Lie algebra} \\ \text{simple (non-Lie) Malcev algebra} \end{array} \right\}$$

- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ quadratic Malcev superalgebra
- $M_{\bar{0}}$ reductive Malcev algebra

Theorem

Then M is either an element of W or M is obtained from a finite number of elements of W by a sequence of double extensions by the one-dimensional Lie algebra and/or a sequence of generalized double extensions by the one-dimensional Lie superalgebra with even part zero and/or by orthogonal direct sums of quadratic Malcev superalgebras from a finite number of elements of W.

H. Albuquerque, 1993

- Classification of Malcev superalgebras with reductive even part and completely reducible action of the even part on the odd part

- Classification of quadratic Malcev superalgebras with reductive even part and completely reducible action of the even part on the odd part

- $L = L_{\bar{0}} \oplus L_{\bar{1}}$ non-trivial quadratic Lie superalgebra
  $L_{\bar{0}} = sl(2,\mathbb{K})$, $L_{\bar{1}}$ irreducible $L_{\bar{0}}$-module

Theorem

Then $L = L(1) = L_{\bar{0}} \oplus L_{\bar{1}}$ Lie superalgebra (isomorphic to $osp(1,2)$), $L_{\bar{0}} = \langle X, Y, H \rangle$, $L_{\bar{1}} = \langle V, W \rangle$
Multiplication:

$$\begin{array}{llll} XY = L & HX = 2X & HY = -2Y & \\ XW = V & YV = W & HV = V & HW = -W \\ V^2 = 2X & W^2 = 2Y & VW = H & \end{array}$$

Invariant scalar product $B : L(1) \times L(1) \longrightarrow \mathbb{K}$

	X	Y	H	V	W
X	0	1	0	0	0
Y	1	0	0	0	0
H	0	0	2	0	0
V	0	0	0	0	2
W	0	0	0	-2	0

- $L = L_{\bar{0}} \oplus L_{\bar{1}}$ non-trivial Lie superalgebra, $L_{\bar{0}} = sl(2,\mathbb{K})$
  $L_{\bar{1}}$ completely reducible $L_{\bar{0}}$-module

Theorem

Then L admits an invariant scalar product $\Longleftrightarrow$ L is isomorphic to $L(1) \oplus Z$, where $Z \subset L_{\bar{1}}$, $ZL = 0$ and $\dim Z$ is even.

- $M = M_{\bar{0}} \oplus M_{\bar{1}}$ Malcev superalgebra
  $M_{\bar{0}}$ semisimple Lie algebra

Theorem

If M admits a quadratic structure then M is Lie superalgebra.
In particular, if $M_{\bar{0}} = sl(2,\mathbb{K})$ then M is Lie superalgebra.

M Malcev superalgebra

- $SJ : M \times M \times M \longrightarrow M$
  superjacobian of $X \in M_x, Y \in M_y, Z \in M_z$:
  $$SJ(X,Y,Z) \quad = (XY)Z + (-1)^{x(y+z)}(YZ)X + (-1)^{z(x+y)}(ZX)Y$$
- superjacobian of M
  $SJ(M,M,M) =$ graded vector subspace of M spanned by all superjacobians
- supernucleus of M
  $SN(M) =$ graded vector subspace of M formed by all $X \in M$ such that $SJ(X,M,M) = \{0\}$.

$SJ(M,M,M)$ and $SN(M)$ are graded ideals of M.

Result

M Lie superalgebra $\Longleftrightarrow SJ(M,M,M) = \{0\} \Longleftrightarrow SN(M) = M$.

Classification of quadratic Malcev superalgebras reductive even part, completely reducible action of the even part on the odd part

- $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ quadratic Malcev superalgebra
- $M_{\bar{0}}$ reductive Malcev algebra
- $M_{\bar{0}}$-module $M_{\bar{1}}$ completely reducible

$Q_{\bar{0}} = J(M_{\bar{0}}, M_{\bar{0}}, M_{\bar{0}})$ sum of simple non-Lie ideals of $M_{\bar{0}}$
$P_{\bar{0}} = N(M_{\bar{0}}) \cap M_{\bar{0}}^2$
$D = \mathfrak{z}(M_{\bar{0}})M_{\bar{1}}$
$V = SN(M) \cap \underbrace{\{X \in M_{\bar{1}} : \mathfrak{z}(M_{\bar{0}})X = 0\}}_{C}$ maximal Lie submodule of C.

Theorem

Then $M = X \oplus Y$, X and Y ideals of M:

$$X = (\mathfrak{z}(M_{\bar{0}}) \oplus P_{\bar{0}}) \oplus (V \oplus D) \text{ and } Y = Q_{\bar{0}}$$