Category Theory

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1 Categories

1.1 Definition.

A category $\mathcal{C}$ consists of:

- a class of objects A, B, C, ...
- for each pair (A, B) of objects of $\mathcal{C}$, a set $\mathcal{C}(A, B)$ of morphisms of A on B, denoted by f, g, ...; we will write $f : A \to B$ when f belongs to $\mathcal{C}(A, B)$;
- for each triple of objects (A, B, C) of $\mathcal{C}$, a composition law

$$\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$$

such that:

Axiom 1: for $f : A \to B$, $g : B \to C$, $h : C \to D$,

$$h \circ (g \circ f) = (h \circ g) \circ f;$$

Axiom 2: for each object A, there exists a morphism $1_A : A \to A \in \mathcal{C}(A, A)$, called the identity of A, such that, for each $f : A \to B$ and $g : C \to A$,

$$1_A \circ g = g \;\&\; f \circ 1_A = f.$$

1.2 Examples.

1. The category $\mathcal{S}et$ of sets and maps, with the usual composition law.
2. The category $\mathcal{G}rp$ of groups and group homomorphisms, with the usual composition law.

   The categories $\mathcal{M}on$ and $\mathcal{S}Grp$, of monoids and semigroups, are defined analogously.
3. The category $\mathcal{V}ec_K$ of vector spaces over the field K and their linear maps, with their usual composition law.
4. The category $\mathcal{G}rph$ of (directed) graphs and their homomorphisms, with the usual composition law.
5. The category $\mathcal{M}etr$ of metric spaces and non-expansive maps, with the usual composition law.
6. The category $\mathcal{POS}et$ of partially ordered sets and monotone maps, with the usual composition law.
7. The category $\mathcal{S}et_*$ of pointed sets and maps that preserve the selected point; that is, $\mathcal{S}et_*$ has as objects pairs (X, x_0), where X is a set and $x_0 \in X$, and as morphisms $f : (X, x_0) \to (Y, y_0)$ maps $f : X \to Y$ such that $f(x_0) = y_0$.
8. The category $\mathcal{P}fn$ of sets and partial maps (that is, $f \in \mathcal{P}fn(X, Y)$ if f is a map whose domain of definition, DD_f, is a subset of X, and whose codomain is Y. The composition $g \circ f \in \mathcal{P}fn(X, Z)$ of $f \in \mathcal{P}fn(X, Y)$ and $g \in \mathcal{P}fn(Y, Z)$ has as domain of definition $\{x \in X\,;\, x \in DD_f \;\&\; f(x) \in DD_g\}$, with $(g \circ f)(x) = g(f(x))$.
9. If X is any set, we may consider X as a discrete category $\mathcal{C}$: the objects of $\mathcal{C}$ are the elements of X, $\mathcal{C}(x, y) = \emptyset$ if $x \neq y$ and $\mathcal{C}(x, x) = \{1_x\}$.

10. if $(X, \leq)$ is a preordered set, we may consider the category $\mathcal{C}_{(X,\leq)}$ having as objects the elements of X, and with

$$\mathcal{C}_{(X,\leq)}(x,y) = \begin{cases} \{x \to y\} & \text{if } x \leq y \\ \emptyset & \text{otherwise,} \end{cases}$$

with the obvious composition law.

11. If $(M, \times, e)$ is a monoid, then we may consider the category $\mathcal{C}_M$ having as unique object M, with $\mathcal{C}_M(M, M) = M$, and with the composition law given by the monoid product: $\times : M \times M \to M$.

12. The category $\mathcal{M}at_{\mathbb{R}}$ whose objects are the natural numbers and whose morphisms, from n to m, the real matrices with n rows and m columns. Its composition law is the matrices product.

1.3 Definition.

A category is said to be small if its class of objects (and consequently its class of morphisms) is a set; it is said to be finite if it has a finite number of morphisms.

1.4 Examples.

Examples of small (finite) categories:

1. the category 0, whose class of objects is empty.
2. the category 1, with a unique object and a unique morphism (its identity).
3. the category 2, with two objects, their respective identities, and one non-trivial morphism.
4. the category 1+1, with two objects and their respective identities.

The categories defined by preordered sets and by monoids are also examples of small categories.

1.5 Exercises.

1. Verify whether the following constructions define categories:

   (a) For each category $\mathcal{C}$, the category $\mathcal{C} \downarrow \mathcal{C}$ (also denoted by Mor$\mathcal{C}$ or $\mathcal{C}^2$) of morphisms of $\mathcal{C}$ that has as objects the morphisms of $\mathcal{C}$, and as morphisms from $f : A \to B$ to $g : C \to D$ pairs (h, k) of morphisms of $\mathcal{C}$ for which the diagrams

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"] & D \end{tikzcd}$$

   are commutative, i.e. $g \circ h = k \circ f$.

   (b) If A is an object of $\mathcal{C}$, the category $\mathcal{C} \downarrow A$ (also denoted by $\mathcal{C}/A$) whose objects are the morphisms of $\mathcal{C}$ with codomain A. A morphism from $f : B \to A$ to $g : C \to A$ is a $\mathcal{C}$-morphism $h : B \to C$ such that $g \circ h = f$, that is, for which the following diagram

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

   commutes.

(c) If $(X, \leq)$ is a preordered set considered as a category $C_{(X,\leq)}$ and $x \in X$, interpret $C_{(X,\leq)} \downarrow x$.

2. A partially ordered set $(X, \leq)$ is said to be ω-complete if every monotone sequence (or chain) in X has supremum. A map $f : X \to Y$ between ω-complete partially ordered sets is said to be continuous if, for each supremum s of a chain $C = (c_n)_{n \in \mathbb{N}}$ in X, $f(s)$ is the supremum of the image of C by f.

   (a) Show that every continuous map between ω-complete partially ordered sets is monotone.

   (b) Let X be a set. Show that:

      i. the powerset $\mathcal{P}(X)$ of X, ordered by inclusion, is an ω-complete partially ordered set.

      ii. The set $\mathcal{P}$ of the partial maps from X to X, equipped with the partial order

$$f \leq g \;:\Leftrightarrow\; DD_f \subseteq DD_g \text{ and } (\forall x \in DD_f)\; f(x) = g(x),$$

      is an ω-complete partially ordered set.

   (c) Show that ω-complete partially ordered sets and continuous maps, with the usual composition of maps, form a category.

1.6 Definition.

A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is a subclass Ob$\mathcal{D}$ of the class of objects of $\mathcal{C}$ and a subclass Mor$\mathcal{D}$ of the class of morphisms of C such that:

- if $f \in$ Mor$\mathcal{D}$, then its domain and codomain belong to Ob$\mathcal{D}$;
- if $X \in$ Ob$\mathcal{D}$, then $1_X \in$ Mor$\mathcal{D}$;
- if $f, g \in$ Mor$\mathcal{D}$, then $g \circ f \in$ Mor$\mathcal{D}$ (whenever f and g are composable).

Note that $\mathcal{D}$ is itself a category, that inherits the identities and the composition law from $\mathcal{C}$.

1.7 Examples.

1. The category $\mathcal{F}in$ of finite sets and maps is a subcategory of $\mathcal{S}et$.
2. The category of sets and injective maps is a subcategory of $\mathcal{S}et$.
3. $\mathcal{S}et$ is a subcategory of $\mathcal{P}fn$.
4. The category $\mathcal{A}b\mathcal{G}rp$ of abelian groups and their homomorphisms is a subcategory of $\mathcal{G}rp$, $\mathcal{G}rp$ is a subcategory of $\mathcal{M}on$, and $\mathcal{M}on$ is a subcategory of $\mathcal{SG}rp$.
5. The category of non-directed graphs is a subcategory of $\mathcal{G}rph$.

1.8 Definition.

A subcategory $\mathcal{D}$ of $\mathcal{C}$ is said to be a full subcategory if Mor$\mathcal{D}$ contains all the morphisms of $\mathcal{C}$ with domain and codomain in $\mathcal{D}$.

1.9 Exercises.

1. Which of the subcategories mentioned in the previous examples are full?
2. Show that $\mathcal{C} \downarrow A$ is a non-full subcategory of $\mathcal{C} \downarrow \mathcal{C}$ whenever $\mathcal{C}(A, A)$ has cardinal larger or equal to 2.
3. Verify whether the category $\mathcal{A}$ of ω-complete partially ordered sets (and continuous maps) is a full subcategory of the category $\mathcal{POS}et$.

1.10 Definition.

Given two categories $\mathcal{A}$ and $\mathcal{B}$, the product category $\mathcal{A} \times \mathcal{B}$ is the category with ordered pairs (A, B), $A \in \mathrm{Ob}\mathcal{A}$, $B \in \mathrm{Ob}\mathcal{B}$ as objects, and ordered pairs $(f, g) : (A, B) \to (C, D)$, where $f \in \mathcal{A}(A, C)$ and $g \in \mathcal{B}(B, D)$, as morphisms. Its composition law is defined componentwisely.

1.11 Definition.

Given a category $\mathcal{C}$, its opposite category or dual category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is the category whose class of objects is exactly $\mathrm{Ob}\mathcal{C}$ and with $\mathcal{C}^{\mathrm{op}}(A, B) := \mathcal{C}(B, A)$; its composition law is defined using the composition law of $\mathcal{C}$.

1.12 Exercise.

Describe the dual category of the following algebraic structures (considered as categories):

1. a group,
2. a monoid,
3. a preordered set,

and show that, in each of the cases, the category obtained may be also described by the same type of structure. Verify whether in each of the cases the structure obtained is isomorphic to the initial one.

2 Functors.

2.1 Definition.

A functor $F : \mathcal{A} \to \mathcal{B}$ of a category $\mathcal{A}$ on a category $\mathcal{B}$ consists of:

- a map $\mathrm{Ob}\mathcal{A} \to \mathrm{Ob}\mathcal{B}$ between the classes of objects of $\mathcal{A}$ and $\mathcal{B}$ (the image of $A \in \mathrm{Ob}\mathcal{A}$ is denoted by $F(A)$ or simply by FA),
- for each pair of objects A, A' of $\mathcal{A}$, a map $\mathcal{A}(A, A') \to \mathcal{B}(FA, FA')$, usually denoted by $F_{A,A'}$; the image of $f : A \to A'$ is denoted by $F_{A,A'}(f)$ (or $F(f)$, or simply Ff),

such that:

F1. if $f \in \mathcal{A}(A, A')$ and $g \in \mathcal{A}(A', A'')$, then $F(g \circ f) = F(g) \circ F(f)$;

F2. for each $A \in \mathrm{Ob}\mathcal{A}$, $F(1_A) = 1_{FA}$.

2.2 Examples.

1. For each category $\mathcal{C}$ the functor identity $1_\mathcal{C} : \mathcal{C} \to \mathcal{C}$, with $1_\mathcal{C}(C) = C$ and $1_\mathcal{C}(f) = f$ for each object C and each morphism f of $\mathcal{C}$.

2. If $\mathcal{A}$ is a subcategory of $\mathcal{C}$, the inclusion functor $I_\mathcal{A} : \mathcal{A} \to \mathcal{C}$ with $I_\mathcal{A}(A) = A$ and $I_\mathcal{A}(f) = f$, for each object A and each morphism f of $\mathcal{A}$.

3. The forgetful functors

   $U : \mathcal{G}rp \to \mathcal{S}et$ with $U(G, \times) = G$ and $Uf = f$;

   $U : \mathcal{V}ec_K \to \mathcal{S}et$, that assigns to each vector space its underlying set and to each linear map the underlying map;

   $U : \mathcal{G}rph \to \mathcal{S}et$ with $U(X, K_X) = X$ and $Uf = f$; etc.

4. $\mathcal{P} : \mathcal{S}et \to \mathcal{S}et$, that assigns to each set X the set $\mathcal{P}(X)$ of its subsets and to each map $f : X \to Y$ the map $\mathcal{P}f = f(-) : \mathcal{P}X \to \mathcal{P}Y$ with $\mathcal{P}f(S) = f(S)$ for each subset S of X.

5. For each object C of a category $\mathcal{C}$, $\mathcal{C}(C, -) : \mathcal{C} \to \mathcal{S}et$, assigning to each A of $\mathcal{C}$ the set $\mathcal{C}(C, A)$ and to each morphism $f : A \to B$ the map

$$\begin{array}{rcl} \mathcal{C}(C,f) : \mathcal{C}(C,A) & \longrightarrow & \mathcal{C}(C,B) \\ (g : C \to A) & \longmapsto & f \circ g : C \to B \end{array}$$

2.3

Given two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, the functor $G \circ F : \mathcal{A} \to \mathcal{C}$ is defined by composition. Moreover, for each category $\mathcal{A}$, the identity functor $1_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ is an identity for the composition law. Therefore, although we cannot consider, because of "size" issues, the category of all categories, we can consider the category $\mathcal{C}at$ of all small categories and their functors, with their natural composition law.

2.4 Definitions.

Each functor $F : \mathcal{A} \to \mathcal{B}$ defines, for each pair of objects A, A' of $\mathcal{A}$, a map $F_{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(FA, FA')$.

The functor $F : \mathcal{A} \to \mathcal{B}$:

1. is faithful if the map $F_{A,A'}$ is injective, for all pairs of objects A, A' of $\mathcal{A}$;

2. is full if the map $F_{A,A'}$ is surjective, for all pairs of objects A, A' of $\mathcal{A}$;

3. is injective on objects if the map $\mathrm{Ob}\mathcal{A} \to \mathrm{Ob}\mathcal{B}$ is injective;

4. is an embedding if it is faithful, full, and injective on objects;

5. is an isomorphism if there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that $G \circ F = 1_\mathcal{A}$ and $F \circ G = 1_\mathcal{B}$.

2.5 Exercises.

1. Show that the following are functors:

(a) For each category $\mathcal{C}$ and each object A of a category $\mathcal{A}$,

$$\begin{array}{rcl} C_A : \mathcal{C} & \longrightarrow & \mathcal{A}. \\ C & \longmapsto & A \\ (f : C \to C') & \longmapsto & (1_A : A \to A) \end{array}$$

(C_A is a constant functor.)

(b) The projection of the product of categories $\mathcal{A} \times \mathcal{B}$ into each of its factors.

(c) For each category $\mathcal{C}$ and each object A of $\mathcal{C}$,

$$\begin{array}{rcl} U_A : \mathcal{C} \downarrow A & \longrightarrow & \mathcal{C}. \\ (C \xrightarrow{f} A) & \longmapsto & C \\ ((C, f) \xrightarrow{h} (C', f')) & \longmapsto & h \end{array}$$

(d) If Y is a set, $Y \times - : \mathcal{S}et \to \mathcal{S}et$, with $(Y \times -)X = Y \times X$ e $(Y \times -)f = 1_Y \times f$.

(e) $Q : \mathcal{S}et \to \mathcal{S}et$ with $Q(X) = X \times X$ and $Q(f) = f \times f$.

(f)

$$\begin{array}{rcl} \overline{\mathcal{P}} : \mathcal{S}et^{\mathrm{op}} & \longrightarrow & \mathcal{S}et \\ X & \longmapsto & \mathcal{P}X \\ (f^{\mathrm{op}} : Y \to X) & \longmapsto & \begin{array}{rcl} f^{-1}(-) : \mathcal{P}Y & \to & \mathcal{P}X \\ S & \mapsto & f^{-1}(S). \end{array} \end{array}$$

(g)

$$\begin{array}{rcl} \mathcal{Q} : \mathcal{S}et & \longrightarrow & \mathcal{S}et \\ X & \longmapsto & \mathcal{Q}(X) := \mathcal{P}(X) \\ (f : X \to Y) & \longmapsto & \begin{array}{rcl} \mathcal{Q}(f) : \mathcal{P}(X) & \to & \mathcal{P}(Y) \\ A & \mapsto & \{y \in Y \mid f^{-1}(y) \subseteq A\}. \end{array} \end{array}$$

(h)

$$\begin{array}{rcl} F : \mathcal{G}rf & \longrightarrow & \mathcal{S}et \\ (X, K_X) & \longmapsto & \{x \in X \mid (x, x) \in K_X\} \\ (f : (X, K_X) \to (Y, K_Y)) & \longmapsto & \begin{array}{rcl} Ff : FX & \to & FY \\ x & \mapsto & f(x). \end{array} \end{array}$$

(i) If C is an object of a category $\mathcal{C}$,

$$\begin{array}{rcl} \mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{S}et. \\ A & \longmapsto & \mathcal{C}(A, C) \\ f^{\mathrm{op}} : A \to B & \longmapsto & \begin{array}{rcl} \mathcal{C}(f^{\mathrm{op}}, C) : \mathcal{C}(A, C) & \to & \mathcal{C}(B, C) \\ (g : A \to C) & \mapsto & g \circ f \end{array} \end{array}$$

2. Interpret functor in the following situations

(a) $\mathsf{1} \to \mathcal{C}$, $\mathsf{2} \to \mathcal{C}$,

(b) $F : \mathcal{C}_{(X,\leq)} \to \mathcal{C}_{(Y,\preceq)}$, with $(X, \leq)$ and $(Y, \preceq)$ partially ordered sets,

(c) $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are categories defined by monoids,

and in each case study the meaning of faithful and full.

3. Identify the faithful and the full functors studied in the first exercise.

3 Isomorphisms

3.1 Definition.

A morphism $f : A \to B$ of $\mathcal{C}$ is an isomorphism if there exists a morphism $g : B \to A$ in $\mathcal{C}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

3.2 Proposição.

In a category:

1. Every identity morphism is an isomorphism;
2. The composition of two isomorphisms is an isomorphism.

3.3 Examples.

1. The isomorphisms in $\mathcal{S}et$ are the bijections.
2. The isomorphisms in $\mathcal{G}rph$ are the bijections $f : (X, K_X) \to (Y, K_Y)$ such that $(x, x') \in K_X$ if and only if $(f(x), f(x')) \in K_Y$.
3. The isomorphisms in $\mathcal{G}rp$ ($\mathcal{A}b\mathcal{G}rp$, $\mathcal{M}on$, $\mathcal{S}\mathcal{G}rp$) are the bijective homomorphisms.

3.4 Exercises.

Describe the isomorphisms of the categories:

1. $\mathcal{POS}et$;
2. $\mathcal{C}_{(X,\leq)}$ when $(X, \leq)$ is a preordered set;
3. $\mathcal{C}_{(X,\leq)}$ when $(X, \leq)$ is a partially ordered set;
4. $\mathcal{C}_M$ where M is a monoid;
5. $\mathcal{M}at_{\mathbb{R}}$;
6. $\mathcal{P}fn$.

4 Initial and final objects

4.1 Definitions.

Let $\mathcal{C}$ be a category.

1. An object A of $\mathcal{C}$ ia an initial object if, for each object C of $\mathcal{C}$, the set $\mathcal{C}(A, C)$ is a singleton.
2. An object A of $\mathcal{C}$ is a terminal object if, for each object C of $\mathcal{C}$, the set $\mathcal{C}(C, A)$ is a singleton.

4.2 Proposition.

If C and C' are initial (terminal) objects of the category $\mathcal{C}$, then there exists an isomorphism $h : C \to C'$. (One says then that the initial (terminal) object is unique up to isomorphism.)

4.3

The initial and final objects of a category $\mathcal{C}$ (when they exist) are usually denoted by 0 and 1, respectively. For each $C \in \mathcal{C}$, the unique morphism from 0 into C is denoted by $0_C : 0 \to C$ and the unique morphism from C to 1 by $!_C : C \to 1$.

4.4 Examples.

1. The category $\mathcal{S}et$ as initial object – the empty set – and terminal object – any singleton set.
2. The category $\mathcal{G}rp$ ($\mathcal{A}b\mathcal{G}rp$, $\mathcal{M}on$) as initial and terminal object: $\{e\}$.
3. The category $\mathcal{G}rph$ has initial object – $(\emptyset, \emptyset)$ – and terminal object – $(\{0\}, \{(0,0)\})$.

4.5 Exercises.

1. Identify, when they exist, the initial and terminal objects of the category:
   (a) $\mathcal{C}_{(X,\leq)}$, where $(X, \leq)$ is a partially ordered set;
   (b) i. $\mathcal{C} \downarrow \mathcal{C}$,
       ii. $\mathcal{C} \downarrow A$,
       where $\mathcal{C}$ is a category with initial object 0 and terminal object 1, and A is an object of $\mathcal{C}$;
   (c) $\mathcal{S}et_*$;
   (d) $\mathcal{POS}et$;
   (e) $\mathcal{P}fn$;
   (f) $\mathcal{M}on$.
2. A zero object is an object that is simultaneously initial and terminal.
   (a) Show that the following conditions are equivalent:
       i. $\mathcal{C}$ has zero object;
       ii. $\mathcal{C}$ has initial object 0 and terminal 1, and 0 and 1 are isomorphic;
       iii. $\mathcal{C}$ has initial object 0 and terminal 1, and $\mathcal{C}(1,0) \neq \emptyset$.

5 Monomorphisms, epimorphisms and the Categorical Duality Principle.

5.1 Definition.

A morphism $f : A \to B$ of a category $\mathcal{C}$ is a monomorphism if, for all pairs $u, v : C \to A$ with $f \circ u = f \circ v$, $u = v$.

5.2 Proposition.

In a category $\mathcal{C}$,

1. every isomorphism is a monomorphism; in particular, every identity morphism is a monomorphism;
2. the composition of two monomorphisms is a monomorphism.

5.3 Examples.

1. A morphism $f : X \to Y$ of $\mathcal{S}et$ is a monomorphism if and only if it is an injective map.
2. A morphism $f : (X, K_X) \to (Y, K_Y)$ of $\mathcal{G}rph$ is a monomorphism if and only if $f : X \to Y$ is an injective map.
3. A group homomorphism is a monomorphism in $\mathcal{G}rp$ ($\mathcal{A}b\mathcal{G}rp$) if and only if it is injective.

5.4 Definition.

A morphism $f : A \to B$ of a category $\mathcal{C}$ is said to be an epimorphism if, for any pair $u, v : B \to C$ of morphisms of $\mathcal{C}$ with $u \circ f = v \circ f$, $u = v$.

5.5 Examples.

1. a map $f : X \to Y$ is an epimorphism in $\mathcal{S}et$ if and only if it is surjective.
2. A morphism of graphs $f : (X, K_X) \to (Y, K_Y)$ is an epimorphism in $\mathcal{G}rph$ if and only if the map f is surjective.
3. A homomorphism of groups is an epimorphism in $\mathcal{G}rp$ if and only it is an a surjective map.
4. Consider the category $\mathcal{C}$ of commutative rings with unit and ring homomorphisms. In $\mathcal{C}$ the epimorphisms are not necessarily surjective:
   consider the inclusion i of $\mathbb{Z}$ in $\mathbb{Q}$, that is not obviously surjective; however, given any pair of ring homomorphisms $u, v : \mathbb{Q} \to A$ such that $u \circ i = v \circ i$ (that is, u and v coincide in the integers), then, for any $\frac{p}{q} \in \mathbb{Q}$,
   $$u(\tfrac{p}{q}) = u(p \cdot \tfrac{1}{q}) = u(p) \cdot u(\tfrac{1}{q}) = u(p) \cdot u(q^{-1}) = u(p) \cdot u(q)^{-1} = v(p) \cdot v(q)^{-1} = v(\tfrac{p}{q}).$$

5.6 Remark.

A morphism f of a category $\mathcal{C}$ is an epimorphism if and only if, as a morphism of $\mathcal{C}^{\mathrm{op}}$, it is a monomorphism. Hence we can conclude immediately that the epimorphisms have the "dual" properties of those stated for monomorphisms.

This statement is a particular instance of:

5.7 Principle of the Categorical Duality.

If an assertion is valid in any category, its "dual assertion" will be also valid, that is, the assertion obtained by inverting the direction of the morphisms in the first assertion.

5.8 Exercises.

1. Show that, if $g \circ f$ is a monomorphism, then f is a monomorphism.
2. Show that, if 1 is a terminal object of $\mathcal{C}$, then any morphism in $\mathcal{C}$ with domain 1 is a monomorphism.
3. (a) Show that, if m is a monomorphism in a category $\mathcal{C}$, then m is a monomorphism in any subcategory of $\mathcal{C}$.
   (b) Give an example of a morphism that is not monic in the category but it is a monomorphism in a subcategory.

4. A morphism $f : A \to B$ in a category $\mathcal{C}$ is said to be a split monomorphism (or section) if there exists a morphism $g : B \to A$ in $\mathcal{C}$ such that $g \circ f = 1_A$.

   Show that:

   (a) Every isomorphism is a split monomorphism and every split monomorphism is a monomorphism.

   (b) If f is simultaneously a split monomorphism and an epimorphism, then f is an isomorphism.

5. Dualize the results of the last 4 exercises.

   (Note: The dual notion of split monomorphism is split epimorphism or retraction.)

6. Describe monomorphisms, split monomorphisms, epimorphisms and split epimorphisms in the category:

   (a) $\mathcal{S}et$;

   (b) if X is a set, $\mathcal{S}et \downarrow X$;

   (c) $\mathcal{S}et \downarrow \mathcal{S}et$;

   (d) $\mathcal{C}_{(X,\leq)}$, where $(X, \leq)$ is a preordered set;

   (e) $\mathcal{P}fn$.

7. A functor $F : \mathcal{A} \to \mathcal{B}$ preserves property (P) of morphisms (of objects) if Ff has that property whenever f has it (respectively FA has that property whenever A has it).

   Show that:

   (a) Every functor preserves isomorphisms.

   (b) In general a functor does not preserve monomorphisms (epimorphisms). (And split monomorphisms?)

8. A functor $F : \mathcal{A} \to \mathcal{C}$ reflects one property if f fulfils that property whenever Ff does (analogously for objects).

   Show (giving counter-examples) that the following assertions are in general false:

   (a) Every functor reflects isomorphisms.

   (b) Every functor reflects terminal objects.

6 Products

6.1 Definition.

If A and B are objects of a category $\mathcal{C}$, the product of A and B is a pair $(P, (p_A, p_B))$ where P is an object of $\mathcal{C}$ and $p_A : P \to A$ and $p_B : P \to B$ are morphisms of $\mathcal{C}$ such that, for each pair $(Q, (q_A, q_B))$ where $Q \in \mathrm{Ob}\mathcal{C}$, $q_A \in \mathcal{C}(Q, A)$ and $q_B \in \mathcal{C}(Q, B)$, there exists a unique morphism $t : Q \to P$ verifying the identities $q_A = p_A \circ t$ and $q_B = p_B \circ t$ (that is, t is the unique morphism making the two triangles commutative).

$$\begin{tikzcd} & Q \arrow[dl, "q_A"'] \arrow[d, dashed, "t"] \arrow[dr, "q_B"] & \\ A & P \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd}$$

6.2 Proposition.

In a category, the product of two objects – when it exists – is unique up to isomorphism.

6.3 Definition.

If $(C_i)_{i\in I}$ is a family of objects of a category $\mathcal{C}$, the product of the family $(C_i)_{i\in I}$ is a pair $(P,(p_i : P \to C_i)_{i\in I})$, where $P \in \mathrm{Ob}\mathcal{C}$ and $p_i \in \mathrm{Mor}\mathcal{C}$ for each $i \in I$, such that, for any pair $(Q,(q_i : Q \to C_i)_{i\in I})$ with $Q \in \mathrm{Ob}\mathcal{C}$ and $q_i \in \mathrm{Mor}\mathcal{C}$, there exists a unique morphism $t : Q \to P$ satisfying the equality $q_i = p_i \circ t$ for all $i \in I$.

If $(P,(p_i)_{i\in I})$ is the product of $(C_i)_{i\in I}$, it is usual to denote the object P by $\prod\limits_{i\in I} C_i$; the morphisms $p_i : P \to C_i$ are called projections.

6.4 Exercise.

Show that the product of a family of objects of a category, when it exists, is unique up to isomorphism.

6.5 Examples.

Let I be a set.

1. In $\mathcal{S}et$ the product of a family of sets $(X_i)_{i\in I}$ is exactly the cartesian product
$$\prod_{i\in I} X_i = \{(x_i)_{i\in I} \,|\, x_i \in X_i\},$$
with projections $p_j((x_i)_{i\in I}) = x_j$.

2. In the category $\mathcal{C}at$ of small categories the product of two categories $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{A}\times\mathcal{B}$ as defined in 1.10.

   The product of a family $(\mathcal{A}_i)_{i\in I}$ of categories is defined analogously.

3. In the category $\mathcal{G}rp$ of groups (as well as in $\mathcal{A}b\mathcal{G}rp$) the product of a family $(G_i,+_i)_{i\in I}$ is $((\prod\limits_{i\in I} G_i,+),(p_i : \prod\limits_{i\in I} G_i \to G)_{i\in I})$, where $\prod\limits_{i\in I} G_i$ is the cartesian product of the sets G_i and the operation $+$ is defined componentwise, that is,
$$(x_i)_{i\in I} + (y_i)_{i\in I} := (x_i +_i y_i)_{i\in I}.$$

4. In the category $\mathcal{G}rph$ of directed graphs and their homomorphisms, the product of a family $(X_i,K_i)_{i\in I}$ is the pair $((X,K),(p_i : (X,K) \to (X_i,K_i))_{i\in I})$, where X is the cartesian product of $(X_i)_{i\in I}$ and
$$K = \{((x_i)_{i\in I},(y_i)_{i\in I}) \in X\times X \,|\, \text{for all } i \in I,\ (x_i,y_i) \in K_i\},$$
where p_i is the corresponding projection of the cartesian product.

5. Let $(X,\le)$ be a partially ordered set. If $(x_i)_{i\in I}$ is a family of elements of X, their product in $\mathcal{C}_{(X,\le)}$ is exactly (when it exists) the infimum of the set $\{x_i \,|\, i \in I\}$.

6. In the category of partially ordered sets $\mathcal{POS}et$ the product of a family $(X_i,\le_i)_{i\in I}$ is the pair $((X,\le),(p_i : (X,\le) \to (X_i,\le_i))_{i\in I})$, where X is the cartesian product of the family of sets $(X_i)_{i\in I}$ and the order relation $\le$ is defined by:
$$(x_i)_{i\in I} \le (y_i)_{i\in I} \ :\Leftrightarrow\ (\forall i \in I)\ x_i \le_i y_i,$$
where p_i is the projection of the cartesian product into X_i.

6.6 Remark.

The product of an empty family of objects of $\mathcal{C}$ is, when it exists, the terminal object of $\mathcal{C}$.

6.7 Definitions.

One says that a category $\mathcal{C}$:

1. has binary products if, given any pair of objects A and B, there exists the product of A and B.
2. has finite products if any family of objects of $\mathcal{C}$ indexed by a finite set has product.
3. has products if any set-indexed family of objects of $\mathcal{C}$ has product in $\mathcal{C}$.

6.8 Remarks.

1. If $f : A \to B$ and $g : C \to D$ are morphisms in $\mathcal{C}$ and if the products $(A \times C, (p_A, p_C))$, $(B \times D, (p_B, p_D))$ of A and B and of C and D, respectively, exist, the only morphism of $A \times C$ in $B \times D$ making the diagram commutative

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times C \arrow[l, "p_A"'] \arrow[r, "p_C"] \arrow[d, dashed] & C \arrow[d, "g"] \\ B & B \times D \arrow[l, "p_B"'] \arrow[r, "p_D"] & D \end{tikzcd}$$

   is usually denoted by $f \times g$.

2. If $f : A \to B$ and $g : A \to C$ are morphisms in $\mathcal{C}$ and if the product $(B \times C, (p_B, p_C))$ of B and C exists in $\mathcal{C}$, the only morphism of A in $B \times C$ that makes the diagram commutative

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dashed] \arrow[dr, "g"] & \\ B & B \times C \arrow[l, "p_B"'] \arrow[r, "p_C"] & C \end{tikzcd}$$

   is denoted by $< f, g >$.

6.9 Exercises.

1. Show that a category has finite products if and only if it has a terminal object and binary products.
2. (a) For the cartesian product of sets, show that the following maps are bijections:
      i. $c : X \times Y \longrightarrow Y \times X$, with $c(x, y) = (y, x)$.
      ii. $a : X \times (Y \times Z) \longrightarrow (X \times Y) \times Z$, with $a(x, (y, z)) = ((x, y), z)$.

   (b) Show that, if A, B, C are objects of a category $\mathcal{C}$, then:
      i. there exists an isomorphism from $A \times B$ to $B \times A$;
      ii. $A \times (B \times C)$ and $(A \times B) \times C$ are isomorphic.
3. Let $\mathcal{C}$ be a category with binary products. Show that, for each object A of $\mathcal{C}$,

$$\begin{array}{rcl} - \times A : \mathcal{C} & \longrightarrow & \mathcal{C} \\ C & \longmapsto & C \times A \\ (f : C \to C') & \longmapsto & (f \times 1_A : C \times A \to C' \times A) \end{array}$$

   is a functor.

4. (a) Show that, if $\mathcal{C}$ has terminal object 1, then, for every object C of $\mathcal{C}$, the product of 1 by C exists and it is isomorphic to C.

   (b) Assume now that $\mathcal{C}$ has initial object 0. Verify that, in general, $C \times 0$ is not isomorphic to 0 for an object C of $\mathcal{C}$.

   And if 0 is a zero object?

5. Consider in $\mathcal{S}et$ the sets $\mathbb{Z}$ and $\mathbb{R}$ of integer and real numbers, respectively. Let $\mathbb{Z} \times \mathbb{R}$ be their cartesian product, $z_0 \in \mathbb{Z}$, $r_0 \in \mathbb{R}$,

$$\begin{array}{rcl} p_{z_0} : \mathbb{Z} \times \mathbb{R} & \longrightarrow & \mathbb{Z} \\ (z,r) & \longmapsto & z + z_0 \end{array} \quad \text{and} \quad \begin{array}{rcl} p_{r_0} : \mathbb{Z} \times \mathbb{R} & \longrightarrow & \mathbb{R}. \\ (z,r) & \longmapsto & r + r_0 \end{array}$$

   Show that, for any z_0 and r_0, $(\mathbb{Z} \times \mathbb{R}, (p_{z_0}, p_{r_0}))$ is a product of $\mathbb{Z}$ and $\mathbb{R}$ in $\mathcal{S}et$.

6. Let $(G, +)$ be an abelian group and $(G^2, (\pi_1, \pi_2))$ the cartesian product of G and G. Show that $(G^2, (\pi_1, \pi_2))$, $(G^2, (\pi_1, +))$ and $(G^2, (\pi_2, +))$ are products of (G, G) in the category of abelian groups $\mathcal{A}b\mathcal{G}rp$.

7. Show that the category $\mathcal{P}fn$ has finite products.

   (Suggestion: Verify that the product of X and Y is $(X \prod Y, (p_X, p_Y))$, where

$$X \prod Y = (X \times Y) \overset{+}{\cup} X \overset{+}{\cup} Y,$$

   $p_X : X \prod Y \to X$ is the partial map with domain of definition $(X \times Y) \overset{+}{\cup} X$, where $p_X(x, y) = x$ and $p_X(x) = x$, and p_Y is defined in an analogous way.)

8. Let $\mathcal{C}$ be a category with finite products. Show that $\mathcal{C} \downarrow \mathcal{C}$ has finite products.

6.10 Definition.

If $(C_i)_{i \in I}$ is a family of objects of a category $\mathcal{C}$, the coproduct of $(C_i)_{i \in I}$ is a pair $(C, (c_i : C_i \to C)_{i \in I})$, where $C \in \mathrm{Ob}\mathcal{C}$ and $c_i \in \mathrm{Mor}\mathcal{C}$ for every $i \in I$, such that, if $(D, (d_i : C_i \to D)_{i \in I})$ is a pair with $D \in \mathrm{Ob}\mathcal{C}$ and $d_i \in \mathrm{Mor}\mathcal{C}$ $(i \in I)$, then there exists a unique morphism $t : C \to D$ verifying the equality $d_i = t \circ c_i$ for all $i \in I$.

If $(C, (c_i)_{i \in I})$ is the coproduct of $(C_i)_{i \in I}$, C is usually denoted by $\coprod_{i \in I} C_i$ and the morphisms $c_i : C_i \to C$ are called coprojections.

6.11 Examples.

Let I be a set.

1. In $\mathcal{S}et$ the coproduct of a family of sets $(X_i)_{i \in I}$ is its disjoint union

$$\coprod_{i \in I} X_i = \bigcup_{i \in I} (X_i \times \{i\})$$

   with inclusions $c_j : X_j \to \coprod X_i$, where $c_j(x) = (x, j)$, for all $x \in X_j$.

2. In the category $\mathcal{G}rph$, the coproduct of $(X_i, K_i)_{i \in I}$ is the pair (X, K) where X is the disjoint union of (X_i) and

$$K = \{((x, i), (y, i)) \,|\, (x, y) \in K_i,\ i \in I\}.$$

3. In the category of groups, the coproduct of a family $(G_i, +_i, e_i)_{i\in I}$ is its free product, that is built as follows:

   - consider the disjoint union A of the sets G_i (A is called the alphabet);
   - next consider the set B of all finite sequences (called words) of elements of A;
   - in B define the equivalence relation $\sim$ generated by:
     - for any $i, j \in I$, $e_i \sim e_j \sim \emptyset$ ("empty word");
     - whenever a_m and a_{m+1} belong to the same G_j,

$$a_1 a_2 \cdots a_m a_{m+1} \cdots a_n \;\sim\; a_1 a_2 \cdots (a_m +_j a_{m+1}) \cdots a_n.$$

   $B/\sim$, equipped with the operation of concatenation, is the free product of $(G_i)_{i\in I}$, usually denoted by $\coprod_{i\in I} G_i$, where $c_j(a)$ is the singleton word a, for each $a \in G_j$, $j \in I$.

4. In $\mathcal{A}b\mathcal{G}rp$ the coproduct of a family $(G_i, +_i, e_i)_{i\in I}$ is its direct sum

$$\coprod_{i\in I} G_i = \{(x_i)_{i\in I} \,|\, x_i \in G_i,\ \{i \in I \,|\, x_i \neq e_i\} \text{ is finite}\},$$

   where the group operation is the product operation, and $c_j : G_j \to \coprod G_i$ is defined by $c_j(x) = (x_i)_{i\in I}$ with $x_j = x$ and $x_i = e_i$ for $i \neq j$.

6.12 Definitions.

As for products, we say that a category $\mathcal{C}$:

1. has binary coproducts if, given any pair of objects A and B, there exists the coproduct of A and B.

2. has finite coproducts if any family of objects of $\mathcal{C}$ indexed by a finite set has coproduct.

3. has coproducts if any family of objects of $\mathcal{C}$ indexed by a set has coproduct in $\mathcal{C}$.

6.13 Remark.

If $f : A \to B$ and $g : C \to D$ are morphisms in $\mathcal{C}$ and if the coproducts $(A + C, (c_A, c_C))$, $(B + D, (c_B, c_D))$ of A and B and of C and D, respectively, exist, then the unique morphism of $A + C$ to $B + D$ making the following diagram commutative

$$\begin{tikzcd}
A \arrow[r, "c_A"] \arrow[d, "f"'] & A + C \arrow[d, dashed] & C \arrow[l, "c_C"'] \arrow[d, "g"] \\
B \arrow[r, "c_B"] & B + D & D \arrow[l, "c_D"']
\end{tikzcd}$$

is usually denoted by $f + g$.

6.14 Exercises.

Verify whether the following categories have coproducts and, if it is the case, describe them:

1. $\mathcal{C}_{(X,\leq)}$, where $(X, \leq)$ is a partially ordered set;

2. $\mathcal{P}fn$;

3. $\mathcal{POS}et$.

7 Equalizers and coequalizers

7.1 Definition.

Given a pair of morphisms $f, g : A \to B$ in a category $\mathcal{C}$, an equalizer of f and g is a pair $(M, m : M \to A)$, where $M \in \mathrm{Ob}\mathcal{C}$ and $m \in \mathrm{Mor}\mathcal{C}$, such that:

1. $f \circ m = g \circ m$;
2. if $(D, d : D \to A)$, with $D \in \mathrm{Ob}\mathcal{C}$ and $d \in \mathrm{Mor}\mathcal{C}$, verifies $f \circ d = g \circ d$, then there exists a unique morphism $t : D \to M$ such that $d = m \circ t$.

7.2 Proposition.

When it exists, the equalizer of a pair of morphisms is unique up to isomorphism.

7.3 Proposition.

Every equalizer is a monomorphism.

7.4 Proposition.

If $f \in \mathcal{C}(A, B)$, then the equalizer of (f, f) exists in $\mathcal{C}$ and it is the identity in A.

7.5 Examples.

1. In $\mathcal{S}et$ the equalizer of two maps $f, g : X \to Y$ is a pair (M, m) where
$$M = \{x \in X \mid f(x) = g(x)\},$$
and m is the inclusion.
2. In $\mathcal{G}rp$ (and $\mathcal{A}b\mathcal{G}rp$) the equalizer of two homomorphisms $f, g : (G, +) \to (G', +')$ is $((M, +_M), m)$, where $(M, +_M)$ is the subgroup $\{x \in G \mid f(x) = g(x)\}$ of G and m the inclusion.
3. In $\mathcal{G}rph$ the equalizer of two morphisms $f, g : (X, K_X) \to (Y, K_Y)$ is the pair $((M, K_M), m)$, where $M = \{x \in X \mid f(x) = g(x)\}$, $K_M = K \cap (M \times M)$ and m is the inclusion.

7.6 Definition.

Given a pair of morphisms $f, g : A \to B$ in a category $\mathcal{C}$, a coequalizer of f and g is an equalizer of (f, g) in $\mathcal{C}^{\mathrm{op}}$. That is, a coequalizer of $f, g : A \to B$ in $\mathcal{C}$ is a pair $(Q, q : B \to Q)$, where $Q \in \mathrm{Ob}\mathcal{C}$ and $q \in \mathrm{Mor}\mathcal{C}$, such that:

1. $q \circ f = q \circ g$;
2. if $(R, r : B \to R)$, with $R \in \mathrm{Ob}\mathcal{C}$ and $r \in \mathrm{Mor}\mathcal{C}$, verifies $r \circ f = r \circ g$, then there exists a unique morphism $t : Q \to R$ such that $r = t \circ q$.

7.7 Examples.

1. In the category of sets, to define the coequalizer (Q, q) of a pair of morphisms $f, g : X \to Y$ one considers the equivalence relation $\sim$ generated by the pairs $(f(x), g(x))$ for every element x of X and the set Q of equivalence classes of this relation; the map q is the projection of Y in $Q = Y/\sim$.

2. In the category of directed graphs, given a pair of morphisms $f, g : (X, K_X) \to (Y, K_Y)$, its coequalizer is the pair $((Q, K), q)$, where Q and q are defined as in $\mathcal{S}et$ and
$$K = \{(a, b) \in Q \times Q \mid \text{ there exists } (y, z) \in K_Y \text{ such that } q(y) = a \;\&\; q(z) = b\}.$$

3. In the category of abelian groups, if $f : G \to G'$, the coequalizer of $(f, 0)$ is the quotient $G' \to G'/f(G)$; if $f, g : G \to G'$, then the coequalizer of (f, g) is the coequalizer of $(f - g, 0)$.

7.8 Exercises.

1. Show that in the full subcategory of $\mathcal{S}et$ of non-empty sets there are pairs of morphisms without equalizer.

2. Let $(X, \leq)$ be a partially ordered set. Identify the morphisms in $\mathcal{C}_{(X,\leq)}$ that are equalizers (of some pair of morphisms).

3. A monomorphism $m : M \to X$ is said to be an extremal monomorphism if, whenever $m = g \circ f$ with f an epimorphism, f is necessarily an isomorphism.
   (a) We have shown that every split monomorphism is an equalizer. Show that every equalizer is an extremal monomorphism.
   (b) Show that every morphism that is simultaneously an epimorphism and an extremal monomorphism is an isomorphism.
   (c) Show that, in the category of sets, every monomorphism is an equalizer although not every monomorphism splits.
   (d) Give examples of categories where not every monomorphism is an equalizer.

4. Show that $\mathcal{P}fn$ has equalizers.

5. Show that $\mathcal{P}fn$ has coequalizers.

8 Pullbacks and Pushouts

8.1 Definitions.

1. The pullback of a pair of morphisms $f : A \to B$, $g : C \to B$ in a category $\mathcal{C}$ consists of a pair $(P, (g' : P \to A,\ f' : P \to C))$, where $P \in \mathrm{Ob}\mathcal{C}$ and $f', g' \in \mathrm{Mor}\mathcal{C}$, such that $f \circ g' = g \circ f'$ and, for each pair $(Q, (u : Q \to A,\ v : Q \to C))$ such that $f \circ u = g \circ v$, there exists a unique morphism $t : Q \to P$ making the following diagram

$$\begin{tikzcd}
Q \arrow[dr, dashed, "t"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\
& P \arrow[r, "f'"] \arrow[d, "g'"'] & C \arrow[d, "g"] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

commutative. f' (resp. g') is the pullback of f (resp. g) along g (resp. f).

2. The dual of a pullback is a pushout.

8.2 Exercises.

1. Show that the following categories have pullbacks:

   (a) $\mathcal{S}et$;

   (b) $\mathcal{G}rph$.

2. Consider, in the category of sets, a morphism $f : X \to Y$, a subset $M \subseteq X$ and its inclusion $m : M \to X$. Identify the pullback of m along f.

3. Show that, for every morphism $f : A \to B$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[d, "1_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

   is a pullback.

4. Show that in a pullback diagram

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

   if f is a monomorphism (resp. isomorphism), then f' is also a monomorphism (resp. isomorphism).

5. A kernel pair of a morphism $f : A \to B$ is the pullback (when it exists) of f along f.

   Let $f : A \to B$ be a morphism in the category $\mathcal{C}$. Show that the following conditions are equivalent:

   (a) f is a monomorphism;

   (b) the kernel pair of f exists and it is given by $(A, (1_A, 1_A))$;

   (c) the kernel pair $(P, (\alpha, \beta))$ of f exists and it is such that $\alpha = \beta$.

6. Show that, if the following squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \end{tikzcd}$$

   are pushout diagrams, then the outer diagram is also a pushout.

7. State the duals of the statements of the previous 4 exercises.

9 Limits and colimits

9.1 Definitions.

Let $F : \mathcal{D} \to \mathcal{C}$ be a functor.

1. a cone of F consists of a pair $(C, (p_D : C \to FD)_{D \in \mathrm{Ob}\mathcal{D}})$, where $C \in \mathrm{Ob}\mathcal{C}$ and $p_D \in \mathrm{Mor}\mathcal{C}$ for all $D \in \mathrm{Ob}\mathcal{D}$, such that, for each morphism $f : D \to D'$ in $\mathcal{D}$, the diagram

$$\begin{tikzcd} & FD \arrow[d, "Ff"] \\ C \arrow[ur, "p_D"] \arrow[r, "p_{D'}"'] & FD' \end{tikzcd}$$

   commutates, that is, $Ff \circ p_D = p_{D'}$.

2. A cone $(L, (p_D : L \to FD)_{D \in \mathrm{Ob}\mathcal{D}})$ is a limit cone of F (or simply a limit of F) if, for each cone $(M, (q_D : M \to FD)_{D \in \mathrm{Ob}\mathcal{D}})$ of F, there exists a unique morphism $t : M \to L$ such that $q_D = p_D \circ t$ for every object $D \in \mathcal{D}$:

$$\begin{tikzcd} L \arrow[dr, "p_D"] & \\ & FD \\ M \arrow[uu, dashed, "t"] \arrow[ur, "q_D"'] & \end{tikzcd}$$

9.2 Proposition.

If the functor $F : \mathcal{D} \to \mathcal{C}$ has a limit, this is unique up to isomorphism; that is, if $(L, (p_D))$ and $(M, (q_D))$ are limit cones of F, then there exists an isomorphism $h : M \to L$ such that $p_D \circ h = q_D$ for all $D \in \mathrm{Ob}\mathcal{D}$.

9.3 Proposition.

If $(L, (p_D : L \to FD)_{D \in \mathrm{Ob}\mathcal{D}})$ is a limit of $F : \mathcal{D} \to \mathcal{C}$ and $f, g : M \to L$ are morphisms in $\mathcal{C}$ such that $p_D \circ f = p_D \circ g$ for all $D \in \mathrm{Ob}\mathcal{D}$, then $f = g$.

9.4 Definitions.

Let $F : \mathcal{D} \to \mathcal{C}$ be a functor.

1. A cocone of F consists of a pair $(C, (c_D : FD \to C)_{D \in \mathrm{Ob}\mathcal{D}})$, where C is an object of $\mathcal{C}$ and $c_D : FD \to C$ is a morphism in $\mathcal{C}$ for all $D \in \mathrm{Ob}\mathcal{D}$, such that, for each morphism $f : D \to D'$ in $\mathcal{D}$, the diagram

$$\begin{tikzcd} & FD \arrow[dl, "c_D"'] \arrow[d, "Ff"] \\ C & FD' \arrow[l, "c_{D'}"] \end{tikzcd}$$

   is commutative, that is, $c_{D'} \circ Ff = c_D$.

2. A cocone $(Q, (c_D : FD \to Q)_{D \in \mathrm{Ob}\mathcal{D}})$ is a colimit cocone of F (or simply colimit of F) if, for each cocone $(M, (q_D : FD \to M)_{D \in \mathrm{Ob}\mathcal{D}})$ of F, there exists a unique morphism $t : Q \to M$ such that $q_D = t \circ c_D$ for every object $D \in \mathcal{D}$:

$$\begin{tikzcd} Q \arrow[d, dashed, "t"'] & & FD \arrow[ll, "c_D"'] \arrow[dll, "q_D"] \\ M & & \end{tikzcd}$$

9.5 Examples.

1. If I is a set interpreted as a discrete category $\mathcal{I}$, a functor $F : \mathcal{I} \to \mathcal{C}$ consists exactly of a family $(Fi)_{i \in I}$ of objects of $\mathcal{C}$ indexed by I, and its limit, when it exists, is the product of a family (Fi) in $\mathcal{C}$.

2. Let $\mathcal{D}$ be the category with two objects D_1 and D_2, and two distinct non-trivial morphisms $d, d' : D_1 \to D_2$. A functor $F : \mathcal{D} \to \mathcal{C}$ consists of the choice of two morphisms $f, g : A \to B$ in $\mathcal{C}$, and its limit is exactly the equalizer of (f, g).

3. Let $\mathcal{D}$ be the category with three objects, D_1, D_2 and D_3, and two non-trivial morphisms $d : D_1 \to D_3$ and $d' : D_2 \to D_3$. A functor $F : \mathcal{D} \to \mathcal{C}$ consists of the choice of two morphisms with common codomain, $f : A \to B$ and $g : C \to B$, and its limit, when it exists, is the pullback of f and g.

9.6 Definitions.

Let $\mathcal{C}$ be a category.

1. $\mathcal{C}$ is complete if, for any small category $\mathcal{D}$ and any functor $F : \mathcal{D} \to \mathcal{C}$, there exists a limit of F.

2. $\mathcal{C}$ is finitely complete if, for any finite category $\mathcal{D}$ and any functor $F : \mathcal{D} \to \mathcal{C}$, the limit of F exists.

9.7 Remark.

The following result justifies the restriction to small categories in the definition of complete category.

9.8 Theorem.

If the category $\mathcal{C}$ has limits for any functor $F : \mathcal{D} \to \mathcal{C}$ and any category $\mathcal{D}$, then between each two objects of $\mathcal{C}$ there exists at most one morphism.

(That is, $\mathcal{C} = \mathcal{C}_{(X, \leq)}$ for some preordered class $(X, \leq)$.)

9.9 Theorem of Existence of Limits.

Let $\mathcal{C}$ be a category. The following conditions are equivalent:

(i) $\mathcal{C}$ is complete;

(ii) $\mathcal{C}$ has products and equalizers.

9.10 Theorem of Existence of Finite Limits.

Let $\mathcal{C}$ be a category. The following assertions are equivalent:

(i) $\mathcal{C}$ is finitely complete;

(ii) $\mathcal{C}$ has terminal object, binary products and equalizers;

(iii) $\mathcal{C}$ has terminal object and pullbacks.

9.11 Exercises.

1. Let $\mathcal{C}$ be a category. Verify whether the following functors have limit and colimit in $\mathcal{C}$:

   (a) $F : 1 \to \mathcal{C}$;

   (b) $F : 2 \to \mathcal{C}$.

2. Show that, if $(A, (f_D : A \to HD)_{D \in \mathrm{Ob}\mathcal{D}})$ is a cone of the functor $H : \mathcal{D} \to \mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ is a functor, then $(FA, (Ff_D))$ is a cone of the functor $F \circ H$.

3. Consider the set $\mathbb{N}$ of natural numbers with the usual order relation $\leq$. Verify whether $\mathcal{C}_{(\mathbb{N},\leq)}$ is complete and cocomplete

4. Let X be a set and $\mathcal{P}(X)$ its powerset. Consider the functor

$$\begin{array}{rcl} F : \mathcal{C}_{(\mathcal{P}(X),\subseteq)} & \longrightarrow & \mathcal{S}et \\ S & \longmapsto & S \\ S \to S' & \longmapsto & S \hookrightarrow S' \end{array}$$

   where $S \hookrightarrow S'$ is the inclusion map. Compute the limit and the colimit of F.

5. A par $(\Lambda, \leq)$ is said to be a directed set if $\leq$ is a binary relation on X such that:

   (1) $\leq$ is reflexive;

   (2) $\leq$ is transitive;

   (3) given $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

   Consider a directed set $(\Lambda, \leq)$ as a category $\mathcal{D}$ having as set of objects Λ and

$$\mathcal{D}(\alpha, \beta) = \begin{cases} \{*^\alpha_\beta\} & \text{if } \beta \leq \alpha \\ \emptyset & \text{otherwise.} \end{cases}$$

   Let $F : \mathcal{D} \to \mathcal{S}et$ be a functor. Show that $(P, (p_\alpha : P \to F(\alpha)))$, where

$$P = \{(x_\alpha) \in \prod F(\alpha)\,;\, F(*^\alpha_\beta)(x_\alpha) = x_\beta\}$$

   and each p_α is the obvious projection, is a limit cone of F.

   (Note: One calls inverse system to the image of F and inverse limit to its limit.)

6. Show that the following assertions are equivalent:

   (i) $\mathcal{A}$ has an initial object;

   (ii) the functor $id_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ has limit.

7. A functor $G : \mathcal{A} \to \mathcal{B}$ preserves limits if, for every small category $\mathcal{D}$ and functor $F : \mathcal{D} \to \mathcal{A}$, whenever $(L, (p_L : L \to FD)_{D \in \mathrm{Ob}\mathcal{D}})$ is a limit cone of F, also $(GL, (Gp_D : GL \to GFD)_{D \in \mathrm{Ob}\mathcal{D}})$ is a limit cone of the functor GF. Show that:

   (a) Every functor that preserves limits preserves necessarily monomorphisms.

   (b) If $\mathcal{A}$ is a category and A an object of $\mathcal{A}$, then the functor $\mathcal{A}(A, -)$ preserves limits.

10 Natural transformations

10.1 Definition.

Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors.

1. A natural transformation $\alpha : F \to G$ from F to G is a class of morphisms $(\alpha_A : FA \to GA)_{A \in \mathrm{Ob}\mathcal{A}}$ in $\mathcal{B}$, indexed by the objects of $\mathcal{A}$, and such that, for each morphism $f : A \to B$ in $\mathcal{A}$, the diagram

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"] & GB \end{tikzcd}$$

   commutates, that is $\alpha_B \circ Ff = Gf \circ \alpha_A$.

2. A natural transformation $\alpha : F \to G$ is a natural isomorphism if there exists a natural transformation $\beta : G \to F$ such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$.

10.2 Properties.

1. If $F, G, H : \mathcal{A} \to \mathcal{B}$ are functors and $\alpha : F \to G$ and $\beta : G \to H$ are natural transformations, then $\beta \circ \alpha$, defined by $(\beta \circ \alpha)_A := \beta_A \circ \alpha_A$ $(A \in \mathrm{Ob}\mathcal{A})$, is a natural transformation from F to H.

2. The composition law for natural transformations defined above is associative and has a unit: for each functor $F : \mathcal{A} \to \mathcal{B}$, $1_F : F \to F$ it is defined by $(1_{FA} : FA \to FA)_{A \in \mathrm{Ob}\mathcal{A}}$.

3. If $F : \mathcal{A} \to \mathcal{B}$, $G, H : \mathcal{B} \to \mathcal{C}$ and $K : \mathcal{C} \to \mathcal{D}$ are functors and $\alpha : G \to H$ is a natural transformation, then we can define natural transformations $\alpha F : G \circ F \to H \circ F$ and $K\alpha : K \circ G \to K \circ H$, where $\alpha F_A = \alpha_{FA} : GFA \to HFA$ and $K\alpha_B = K(\alpha_B) : KGB \to KHB$.

10.3 Proposition.

Let $\mathcal{A}$ be a small category and $\mathcal{B}$ an arbitrary category. The functors from $\mathcal{A}$ to $\mathcal{B}$ and the natural transformations between them constitute a category, $\mathcal{F}un(\mathcal{A}, \mathcal{B})$. This category is small whenever $\mathcal{B}$ is.

10.4 Exercise.

Show that the following conditions are equivalent, for a natural transformation $\alpha : F \to G$, where $F, G : \mathcal{A} \to \mathcal{B}$ are functors:

(i) α is a natural isomorphism.

(ii) for every object A of $\mathcal{A}$, $\alpha_A : FA \to GA$ is an isomorphism in $\mathcal{B}$.

10.5 Examples.

1. Let $\mathcal{C} = \mathcal{V}ec_K$ be the category of vector spaces over the field K, and let $(\ )^{**} : \mathcal{C} \to \mathcal{C}$ be the bidual functor; that is, $V^{**} = \mathcal{C}(\mathcal{C}(V,K),K)$ and, if $f \in \mathcal{C}(V,W)$, then

$$\begin{array}{rcl} (f)^{**} : \mathcal{C}(\mathcal{C}(V,K),K) & \longrightarrow & \mathcal{C}(\mathcal{C}(W,K),K) \\ g : \mathcal{C}(V,K) \to K & \longmapsto & \begin{array}{rcl} (f)^{**}(g) : \mathcal{C}(W,K) & \to & K \\ u & \mapsto & g(u \circ f). \end{array} \end{array}$$

The canonical morphisms

$$\begin{array}{rcl} \alpha_V : V & \longrightarrow & V^{**} \\ v & \longmapsto & \begin{array}{rcl} \mathcal{C}(V,K) & \to & K \\ f & \mapsto & f(v) \end{array} \end{array}$$

form a natural transformation $\alpha : 1_{\mathcal{C}} \to (\ )^{**}$.

2. The determinant is a natural transformation:

Consider the category $\mathcal{C}$ of rings with unit, and the functors

$$\begin{array}{rcl} U : \mathcal{C} & \longrightarrow & \mathcal{G}rp \\ A & \longmapsto & UA \text{ (multiplicative group of invertible elements of } A) \end{array}$$

and

$$\begin{array}{rcl} GL_n : \mathcal{C} & \longrightarrow & \mathcal{G}rp \\ A & \longmapsto & GL_n(A) \text{ (group of invertible matrices } n \times n \text{ with entries on } A) \end{array}$$

The "determinants" $\det_A : GL_n(A) \to UA$ define a natural transformation

$$\det : GL_n \to U.$$

3. Let $\mathcal{A}$ be a category. Each morphism $f : B \to A$ on $\mathcal{A}$ defines a natural transformation $\mathcal{A}(f,-) : \mathcal{A}(A,-) \to \mathcal{A}(B,-)$, with $\mathcal{A}(f,-) = (\mathcal{A}(f,C))_{C \in \mathrm{Ob}\mathcal{A}}$, where

$$\begin{array}{rcl} \mathcal{A}(f,C) : \mathcal{A}(A,C) & \longrightarrow & \mathcal{A}(B,C) \\ g : A \to C & \longmapsto & g \circ f \end{array}$$

10.6 Exercises.

1. Let $(X, \leq)$ and $(Y, \preceq)$ be two preordered sets. Given functors $F, G : \mathcal{C}_{(X,\leq)} \to \mathcal{C}_{(Y,\preceq)}$, explain what a natural transformation $\mu : F \to G$ is.

2. If G and G' are groups and $H, K : \mathcal{C}_G \to \mathcal{C}_{G'}$ are functors, show that there exists a natural transformation $\beta : H \to K$ if and only if H and K are conjugated homomorphisms, i.e. if there exists $x \in G'$ such that, for all $g \in G$

$$H(g) = x \cdot K(g) \cdot x^{-1}.$$

3. Consider the functors

$$\begin{array}{rcl} \mathcal{P} : \mathcal{S}et & \longrightarrow & \mathcal{S}et, \\ X & \longmapsto & \mathcal{P}(X) \\ (f : X \to Y) & \longmapsto & \begin{array}{rcl} \mathcal{P}(f) : \mathcal{P}(X) & \to & \mathcal{P}(Y) \\ A & \mapsto & f(A) \end{array} \end{array}$$

$$
\begin{array}{rcl}
\mathcal{Q} : \mathcal{S}et & \longrightarrow & \mathcal{S}et. \\
X & \longmapsto & \mathcal{Q}(X) := \mathcal{P}(X) \\
(f : X \to Y) & \longmapsto & \begin{array}{rcl} \mathcal{Q}(f) : \mathcal{P}(X) & \to & \mathcal{P}(Y) \\ A & \mapsto & \{y \in Y \mid f^{-1}(y) \subseteq A\} \end{array}
\end{array}
$$

(a) Verify that $\lambda = (\lambda_X : X \to \mathcal{P}(X))_{X \in \mathcal{S}et}$, where $\lambda_X(x) = \{x\}$ for each $x \in X$, is a natural transformation from the identity functor into the functor $\mathcal{P}$.

(b) Show that the same $\lambda = (\lambda_X)_{X \in \mathcal{S}et}$ is not a natural transformation from $1_{\mathcal{S}et}$ to $\mathcal{Q}$.

4. Let S be a set. Consider the functors

$$
\begin{array}{rcl}
F = - \times S : \mathcal{S}et & \longrightarrow & \mathcal{S}et \\
X & \longmapsto & X \times S
\end{array}
\quad \text{and} \quad
\begin{array}{rcl}
G = \mathcal{S}et(S, -) : \mathcal{S}et & \longrightarrow & \mathcal{S}et \\
X & \longmapsto & \mathcal{S}et(S, X) =: X^S.
\end{array}
$$

Show that the evaluation maps

$$
\begin{array}{rcl}
\alpha_X : X^S \times S & \longrightarrow & X \\
(f, s) & \longmapsto & f(s)
\end{array}
$$

define a natural transformation $\alpha = (\alpha_X)_{X \in \mathcal{S}et}$ from FG to $1_{\mathcal{S}et}$.

10.7 Theorem.

If $\mathcal{A}$ is a small category, we may consider the functor

$$
\begin{array}{rcl}
Y : \mathcal{A}^{\mathrm{op}} & \longrightarrow & \mathcal{F}un(\mathcal{A}, \mathcal{S}et) \\
A & \longmapsto & \mathcal{A}(A, -) \\
f \in \mathcal{A}^{\mathrm{op}}(A, B) & \longmapsto & \mathcal{A}(f, -) : \mathcal{A}(A, -) \to \mathcal{A}(B, -).
\end{array}
$$

The functor Y is an embedding, called Yoneda Embedding.

10.8 Remarks.

The injectivity of Y on objects is immediate. That it is full and faithful means: every natural transformation of $\mathcal{A}(A, -)$ in $\mathcal{A}(B, -)$ is defined by a unique $\mathcal{A}$-morphism from B to A, that is, by a unique element of $\mathcal{A}(B, -)(A) = \mathcal{A}(B, A)$.

This result is valid when we replace $\mathcal{A}(B, -)$ by an arbitrary functor $F : \mathcal{A} \to \mathcal{S}et$.

10.9 Yoneda Lemma.

Let $\mathcal{A}$ be a category and $F : \mathcal{A} \to \mathcal{S}et$ a functor. Consider an object A of $\mathcal{A}$ and the functor $\mathcal{A}(A, -) : \mathcal{A} \to \mathcal{S}et$.

1. There exists a bijection

$$\theta_{F,A} : \mathrm{Nat}(\mathcal{A}(A, -), F) \longrightarrow FA$$

between the natural transformations from $\mathcal{A}(A, -)$ to F and the elements of the set FA.

2. The bijections $\theta_{F,A}$ constitute a natural transformation on the variable A; that is, $\theta_F = (\theta_{F,A})_{A\in \mathrm{Ob}\mathcal{A}}$ is a natural transformation from the functor

$$\begin{array}{rcl} N:\mathcal{A} & \longrightarrow & \mathcal{S}et \\ A & \longmapsto & \mathrm{Nat}(\mathcal{A}(A,-),F) \end{array} \quad \text{to } F,$$

where, if $f:A\to B$,

$$\begin{array}{rcl} N(f):\mathrm{Nat}(\mathcal{A}(A,-),F) & \to & \mathrm{Nat}(\mathcal{A}(B,-),F) \\ \alpha & \mapsto & \alpha\circ\mathcal{A}(f,-) \end{array}.$$

3. Moreover, if $\mathcal{A}$ is a small category, the bijections $\theta_{F,A}$ constitute a natural transformation on the variable F; that is, $\theta_A = (\theta_{F,A})_{F\in\mathcal{F}un(\mathcal{A},\mathcal{S}et)}$ is a natural transformation from the functor

$$\begin{array}{rcl} M:\mathcal{F}un(\mathcal{A},\mathcal{S}et) & \longrightarrow & \mathcal{S}et \\ F & \longmapsto & \mathrm{Nat}(\mathcal{A}(A,-),F) \end{array} \quad \text{to} \quad \begin{array}{rcl} ev:\mathcal{F}un(\mathcal{A},\mathcal{S}et) & \longrightarrow & \mathcal{S}et \\ F & \longmapsto & FA \end{array},$$

for each natural transformation $\alpha:F\to G$,

$$\begin{array}{rcl} M(\alpha):\mathrm{Nat}(\mathcal{A}(A,-),F) & \to & \mathrm{Nat}(\mathcal{A}(A,-),G) \\ \beta & \mapsto & \alpha\circ\beta \end{array} \quad \text{and } ev(\alpha)=\alpha_A.$$

10.10 Exercises.

1. Proof the assertions 2 and 3 of Yoneda Lemma.

2. Identify the category $\mathcal{S}et^{\mathcal{D}} = \mathcal{F}un(\mathcal{D},\mathcal{S}et)$, when:

   (a) $\mathcal{D}$ is the discrete category with two objects;

   (b) $\mathcal{D} = 2$ (defined in Example 1.4);

   (c) $\mathcal{D}$ is the category defined by a monoid M.

3. Let $\mathcal{C}$ be a small category. Verify whether the category $\mathcal{S}et^{\mathcal{C}}$ has

   (a) terminal object,

   (b) binary products,

   and, if it the case, build them.

11 Representable functors

11.1 Definition.

A functor $F:\mathcal{A}\to\mathcal{S}et$ is said to be representable if there exists an object A of $\mathcal{A}$ and a natural isomorphism of F in $\mathcal{A}(A,-)$.

We say then that the object A represents the functor F.

11.2 Corollary of the Yoneda Lemma.

If the functor $F:\mathcal{A}\to\mathcal{S}et$ is represented by two objects, A and A', of the category $\mathcal{A}$ then there exists an isomorphism $h:A\to A'$ in $\mathcal{A}$.

11.3 Examples.

1. The identity functor $1 : \mathcal{S}et \to \mathcal{S}et$ is representable, being represented by any singleton.
2. The forgetful functor $U : \mathcal{G}rph \to \mathcal{S}et$ is representable, being represented by the graph $1 = (\{*\}, \{(*, *)\})$.
3. (a) The forgetful functor $U : \mathcal{SG}rp \to \mathcal{S}et$ is representable, being represented by the additive semigroup $\mathbb{N}$ (not containing 0).
   (b) The forgetful functor $U : \mathcal{M}on \to \mathcal{S}et$ is representable, being represented by the monoid $\mathbb{N}_0$.
   (c) The forgetful functor $U : \mathcal{G}rp \to \mathcal{S}et$ is representable, being represented by the group $\mathbb{Z}$.

11.4 Exercises.

Verify whether the following functors are representable:

1. The forgetful functor $U : \mathcal{POS}et \to \mathcal{S}et$.
2. The forgetful functor $U : \mathcal{S}et_* \to \mathcal{S}et$.
3. The functor
$$\begin{array}{rcl} F : \mathcal{G}rf & \longrightarrow & \mathcal{S}et \\ (X, K_X) & \longmapsto & \{x \in X \mid (x, x) \in K_X\} \\ (f : (X, K_X) \to (Y, K_Y)) & \longmapsto & \begin{array}{rcl} Ff : FX & \to & FY \\ x & \mapsto & f(x). \end{array} \end{array}$$
4. The functor
$$\begin{array}{rcl} G : \mathcal{G}rf & \longrightarrow & \mathcal{S}et \\ (X, K_X) & \longmapsto & K_X \\ (f : (X, K_X) \to (Y, K_Y)) & \longmapsto & \begin{array}{rcl} Gf : K_X & \to & K_Y \\ (x, x') & \mapsto & (f(x), f(x')). \end{array} \end{array}$$

12 Functors and limits

12.1 Definition.

A functor $G : \mathcal{A} \to \mathcal{B}$ preserves limits if, for every small category $\mathcal{D}$ and functor $F : \mathcal{D} \to \mathcal{A}$, whenever $(L, (p_L : L \to FD)_{D \in \mathrm{Ob}\mathcal{D}})$ is a limit cone of F, also $(GL, (Gp_D : GL \to GFD)_{D \in \mathrm{Ob}\mathcal{D}})$ is a limit cone of the functor GF.

12.2 Lemma.

Every functor preserving limits preserves monomorphisms.

12.3 Proposition.

Let $\mathcal{A}$ be a (finitely) complete category and $\mathcal{B}$ any category. A functor $F : \mathcal{A} \to \mathcal{B}$ preserves (finite) limits if and only if it preserves (finite) products and equalizers.

12.4 Proposition.

Let $\mathcal{A}$ be a category and A an object of $\mathcal{A}$. The functor $\mathcal{A}(A, -)$ preserves limits.

12.5 Corollary 1.

Let $\mathcal{A}$ be a category and A an object of $\mathcal{A}$. The "contravariant functor"

$$\mathcal{A}(-,A) : \mathcal{A} \to \mathcal{S}et$$

maps (existing) colimits to limits.

12.6 Corollary 2.

Every representable functor preserves limits.

12.7 Definition.

Let $G : \mathcal{A} \to \mathcal{B}$ be a functor. One says that G reflects limits when, for every small category $\mathcal{D}$ and functor $F : \mathcal{D} \to \mathcal{A}$, if $(L, (p_D : L \to FD)_{D \in \mathrm{Ob}\mathcal{D}})$ is a cone for F and $(GL, (Gp_D : GL \to GFD)_{D \in \mathrm{Ob}\mathcal{D}})$ is a limit cone limit of GF, then $(L, (p_D))$ is a limit cone of F.

12.8 Lemma.

Every functor that reflects limits reflects necessarily isomorphisms and monomorphisms.

12.9 Proposition.

Let $\mathcal{A}$ be a complete category and $F : \mathcal{A} \to \mathcal{B}$ a functor that preserves limits. The following assertions are equivalent:

(i) F reflects limits;

(ii) F reflects isomorphisms.

12.10 Proposition.

Every full and faithful functor reflects limits.

12.11 Remark.

Although representable functors do not reflect limits in general, they reflect jointly limits, as explained next.

12.12 Proposition.

Representable functors reflect jointly limits; that is, if $F : \mathcal{D} \to \mathcal{A}$ is a functor and $(L, (p_D : L \to FD)_{D \in \mathrm{Ob}\mathcal{D}})$ is a cone for F, then this cone is a limit cone if and only if, for each object A of $\mathcal{A}$,

$$(\mathcal{A}(A,L), (\mathcal{A}(A,p_D) : \mathcal{A}(A,L) \to \mathcal{A}(A,FD))_{D \in \mathrm{Ob}\mathcal{D}})$$

is a limit cone of $\mathcal{A}(A, F-)$ in $\mathcal{S}et$.

12.13 Exercise.

Prove Proposition 12.10.

13 Adjoint Functors

13.1 Definitions.

Let $G : \mathcal{A} \to \mathcal{B}$ be a functor and B an object of $\mathcal{B}$.

1. A universal morphism from B to G is a pair (η_B, A_B), with an object A_B of $\mathcal{A}$ and a morphism $\eta_B : B \to G(A_B)$ in $\mathcal{B}$, such that, for each $f : B \to GA'$, there exists a unique morphism $\overline{f} : A \to A'$ in $\mathcal{A}$ making the diagram

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[dr, "f"'] & G(A_B) \arrow[d, "G\overline{f}"] & A_B \arrow[d, dashed, "\overline{f}"] \\ & GA' & A' \end{tikzcd}$$

commute.

2. A universal morphism universal from G to B is a pair (σ_B, A_B), with an object A_B of $\mathcal{A}$ and a morphism $\sigma_B : G(A_B) \to B$ in $\mathcal{B}$, such that, for each $g : GA' \to B$, there exists a unique morphism $\overline{g} : A' \to A$ in $\mathcal{A}$ making the diagram

$$\begin{tikzcd} A_B & G(A_B) \arrow[r, "\sigma_B"] & B \\ A' \arrow[u, dashed, "\overline{g}"] & GA' \arrow[u, "G\overline{g}"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutative.

13.2 Proposition.

Let $G : \mathcal{A} \to \mathcal{B}$ be a functor such that, for each object B of $\mathcal{B}$, there exists a universal morphism (η_B, A_B) of B in G. Then there exists a functor $F : \mathcal{B} \to \mathcal{A}$ such that $FB = A_B$ for all $B \in \mathrm{Ob}\mathcal{B}$, and $\eta = (\eta_B : B \to GFB)_{B \in \mathrm{Ob}\mathcal{B}}$ is a natural transformation from $1_{\mathcal{B}}$ to $G \circ F$.

13.3 Definition.

One says that a functor $F : \mathcal{B} \to \mathcal{A}$ is left adjoint to the functor $G : \mathcal{A} \to \mathcal{B}$ if there exists a natural transformation $\eta : 1_{\mathcal{B}} \to GF$ such that, for each object B of $\mathcal{B}$, η_B is a universal morphism from B to G.

One says then that G is a right adjoint of F, and one writes $F \dashv G$. $(F, G; \eta)$ is said to be an adjunction.

13.4 Proposition.

Given an adjunction $(F : \mathcal{B} \to \mathcal{A}, G : \mathcal{A} \to \mathcal{B}; \eta)$, there exists a natural transformation $\varepsilon : FG \to 1_{\mathcal{A}}$ such that, for each object A of $\mathcal{A}$, $\varepsilon_A : FGA \to A$ is a universal morphism from F to A. Moreover, for each $A \in \mathrm{Ob}\mathcal{A}$ and each $B \in \mathrm{Ob}\mathcal{B}$,

$$G\varepsilon_A \circ \eta_{GA} = 1_{GA} \quad \text{and} \quad \varepsilon_{FB} \circ F\eta_B = 1_{FB}.$$

13.5 Definition.

In an adjoint situação like the one of the previous proposition, the unit of the adjunction is the natural transformation η, and the counit of the adjunction is the natural transformation ε.

Saying that $(F, G; \eta, \varepsilon)$ is an adjunction means that $F \dashv G$, and that η is the unit and ε is the counit of the adjunction.

13.6 Exercises.

1. Show that, if $F : \mathcal{B} \to \mathcal{A}$ is left adjoint to $G : \mathcal{A} \to \mathcal{B}$, with unit η and counit ε, then $G^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is left adjoint to $F^{\mathrm{op}} : \mathcal{B}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$, with unit $\varepsilon^{\mathrm{op}}$ and counit η^{op}.

2. Show that, if F and $F' : \mathcal{B} \to \mathcal{A}$ are left adjoint to the functor $G : \mathcal{A} \to \mathcal{B}$, then F and F' are naturally isomorphic.

3. Show that, if $F : \mathcal{B} \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{B}$ are functors and $\eta : 1_{\mathcal{B}} \to GF$ and $\varepsilon : FG \to 1_{\mathcal{A}}$ are natural transformations such that, for every object A of $\mathcal{A}$ and every object B of $\mathcal{B}$,

$$G\varepsilon_A \circ \eta_{GA} = 1_{GA} \quad \text{and} \quad \varepsilon_{FB} \circ F\eta_B = 1_{FB},$$

then $(F, G; \eta, \varepsilon)$ is an adjunction.

13.7 Theorem.

Given two categories $\mathcal{A}$ and $\mathcal{B}$ and functors $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$, the following conditions are equivalent:

(i) There exists a natural transformation $\eta : 1_{\mathcal{B}} \to GF$ such that, for each $B \in \mathrm{Ob}\mathcal{B}$, $\eta_B : B \to GFB$ is a universal morphism from B to G.

(ii) There exists a natural transformation $\varepsilon : FG \to 1_{\mathcal{A}}$ such that, for each $A \in \mathrm{Ob}\mathcal{A}$, $\varepsilon_A : FGA \to A$ is a universal morphism from F to A.

(iii) There exist natural transformations $\eta : 1_{\mathcal{B}} \to GF$ and $\varepsilon : FG \to 1_{\mathcal{A}}$ such that

$$G\varepsilon_A \circ \eta_{GA} = 1_{GA} \quad \text{and} \quad \varepsilon_{FB} \circ F\eta_B = 1_{FB}.$$

(iv) The functors $\mathcal{A}(F-, -), \mathcal{B}(-, G-) : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{S}et$ are naturally isomorphic.

13.8 Exercises.

1. Show that, if F, G, H and K are functors,

$$\mathcal{A} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{B} \underset{H}{\overset{K}{\rightleftarrows}} \mathcal{C}\,,$$

and $F \dashv G$ and $H \dashv K$, then $F \circ H \dashv K \circ G$.

2. Describe adjoint situations for functors between partially ordered sets considered as categories.

3. Let X and Y be sets, and $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ the powersets of X and Y respectively, ordered by inclusion, and considered as categories. Let $f : X \to Y$ be a map.

Consider the functor

$$\begin{array}{rcl} f^{-1}(\ ) : \mathcal{P}(Y) & \longrightarrow & \mathcal{P}(X). \\ B & \longmapsto & f^{-1}(B) \end{array}$$

Show that:

(a) the functor

$$\begin{array}{rcl} f(\ ):\mathcal{P}(X) & \longrightarrow & \mathcal{P}(Y) \\ A & \longmapsto & f(A) \end{array}$$

is a left adjoint of $f^{-1}(\ )$;

(b) the functor

$$\begin{array}{rcl} f!(\ ):\mathcal{P}(X) & \longrightarrow & \mathcal{P}(Y) \\ A & \longmapsto & \{y \in Y \mid f^{-1}(y) \subseteq A\} \end{array}$$

is a right adjoint of $f^{-1}(\ )$.

4. Show that:

(a) the functor $\mathcal{C} \to 1$ has a right adjoint if and only if $\mathcal{C}$ has terminal object;

(b) the functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, with $\Delta(C) = (C, C)$ and $\Delta(f) = (f, f)$, has a right adjoint if and only if $\mathcal{C}$ has binary products.

5. Show that, for each set S, the functor $F = - \times S : \mathcal{S}et \longrightarrow \mathcal{S}et$ is left adjoint to $\mathcal{S}et(S, -) : \mathcal{S}et \to \mathcal{S}et$.

13.9 Proposition.

Every right adjoint functor (that is, with a left adjoint) preserves limits.

13.10 Corollary.

Every left adjoint functor preserves colimits.

13.11 Remark.

The preservation of limits by a functor $G : \mathcal{A} \to \mathcal{B}$ doesn't imply the existence of a left adjoint of G, except when the category is small and complete. This assertion follows from the Theorem we state next.

13.12 Freyd's Adjoint Functor Theorem.

Let $\mathcal{A}$ be a complete category and $G : \mathcal{A} \to \mathcal{B}$ a functor. The following assertions are equivalent:

(i) G has a left adjoint;

(ii) G preserves limits and satisfies the following "Solution Set Condition":

For each object B of $\mathcal{B}$ there exists a set C_B of objects of $\mathcal{A}$ such that, for each object A of $\mathcal{A}$ and each morphism $f : B \to GA$ in $\mathcal{B}$, there exist $A' \in C_B$, $f' : A \to A'$ and $h : B \to GA'$ in $\mathcal{B}$ such that $Gf' \circ h = f$:

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[dr, "f"'] & GA' \arrow[d, "Gf'"] \\ & GA \end{tikzcd}$$

13.13 Exercises.

1. Consider the inclusion functor $g : \mathbb{N} \hookrightarrow \mathbb{N}_\infty$ (where the ordered sets $\mathbb{N}$ and $\mathbb{N}_\infty$ are interpreted as categories). Show that:
   (a) g preserves limits;
   (b) g is not a right adjoint.
2. Show that the forgetful functor $U : \mathcal{G}rp \to \mathcal{S}et$ does not have a right adjoint.
3. Let $\mathcal{C}$ be a category and A an object of $\mathcal{C}$. Consider the functor

$$\begin{array}{rcl} U_A : \mathcal{C} \downarrow A & \longrightarrow & \mathcal{C}. \\ (C \xrightarrow{f} A) & \longmapsto & C \\ ((C,f) \xrightarrow{h} (C',f')) & \longmapsto & h \end{array}$$

   (a) Show that, if $\mathcal{C}$ has products, then U_A has a right adjoint.
   (b) Show that, in general, U_A has no left adjoint.
   (c) Characterize the objects A of $\mathcal{C}$ for which U_A has a left adjoint.

13.14 Theorem.

Let $G : \mathcal{A} \to \mathcal{S}et$ be a functor. The following assertion are equivalent:

(i) G has left adjoint;

(ii) G is representable, and the object A of $\mathcal{A}$ that represents G has coproducts in $\mathcal{A}$.

13.15 Exercise.

Verify whether the functor

$$\begin{array}{rcl} G : \mathcal{G}rf & \longrightarrow & \mathcal{S}et \\ (X, K_X) & \longmapsto & \{x \in X \mid (x,x) \in K_X\} \\ (f : (X,K_X) \to (Y,K_Y)) & \longmapsto & \begin{array}{rcl} Gf : FX & \to & FY \\ x & \mapsto & f(x) \end{array} \end{array}$$

has a left adjoint.

13.16 Proposition.

If $(F : \mathcal{B} \to \mathcal{A}, G : \mathcal{A} \to \mathcal{B}; \eta, \varepsilon)$ is an adjunction, then:

(a) G is faithful if and only if, for every object A of $\mathcal{A}$, ε_A is an epimorphism;

(b) G is full if and only if, for every object A of $\mathcal{A}$, ε_A is a split monomorphism.

(c) G is full and faithful if and only if, for every object A of $\mathcal{A}$, ε_A is an isomorphism.

13.17 Corollary.

If $(F : \mathcal{B} \to \mathcal{A}, G : \mathcal{A} \to \mathcal{B}; \eta, \varepsilon)$ is an adjunction, then:

(a) F is faithful if and only if, for every object B of $\mathcal{B}$, η_B is a monomorphism;

(b) F is full if and only if, for every object B of $\mathcal{B}$, η_B is a split epimorphism.

(c) F is full and faithful if and only if, for every object B of $\mathcal{B}$, η_B is an isomorphism.

14 Reflective Subcategories.

14.1 Definitions.

Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$, and $I : \mathcal{A} \to \mathcal{B}$ the inclusion functor.

1. The subcategory $\mathcal{A}$ is said to be replete if it is closed under isomorphisms; that is, if, whenever $h : A \to B$ is an isomorphism with A in $\mathcal{A}$, also B belongs to $\mathcal{A}$.

2. The subcategory $\mathcal{A}$ is said to be reflective (co-reflective) if the inclusion functor has left (right) adjoint. We will call reflector (co-reflector) the left (right) adjoint of I.

14.2 Lemma.

If $\mathcal{A}$ is a full subcategory of $\mathcal{B}$ and the inclusion functor $I : \mathcal{A} \to \mathcal{B}$ is a right adjoint, then we can define a left adjoint $R : \mathcal{B} \to \mathcal{A}$ of I such that R, when restricted to $\mathcal{A}$, is the identity functor.

14.3 Remark.

From now on, whenever I has a left adjoint, we will consider a functor $R \dashv I$ fulfilling the conditions of the Lemma.

If η is the unit of the adjunction $R \dashv I$, one says that $\eta_B : B \to RB (= IRB)$ is the reflexion of B in $\mathcal{A}$.

14.4 Theorem.

Let $\mathcal{A}$ be a reflective, full and replete, subcategory of $\mathcal{B}$.

(a) If $\mathcal{B}$ is a complete category, then $\mathcal{A}$ is complete.

(b) If $\mathcal{B}$ is a cocomplete category, then $\mathcal{A}$ is cocomplete.

14.5 Exercises.

1. Considering the natural order $\leq$ in $\mathbb{N}$ and $\mathbb{Z}$, verify whether or not the category $\mathcal{C}_{(\mathbb{N},\leq)}$ is a (co-)reflective subcategory of $\mathcal{C}_{(\mathbb{Z},\leq)}$.

2. Verify whether the category of finite sets and maps is a (co-)reflective subcategory of $\mathcal{S}et$.

3. * Verify if the category $\mathcal{S}et$ is a (co)-reflective subcategory of $\mathcal{P}fn$.

4. Let $\mathcal{B}$ be the (full) subcategory of $\mathcal{G}rph$ of reflexive graphs (that is, graphs (X, K_X) such that, for all $x \in X$, $(x, x) \in K_X$). Let $I : \mathcal{B} \to \mathcal{G}rf$ be the inclusion functor.

   (a) For each graph (X, K_X), let $X' = \{x \in X \,|\, (x, x) \in K_X\}$ and $K' = \{(x, y) \in K_X \,|\, x, y \in X'\}$. Show that:
   $$\begin{array}{rcl} \varepsilon_{(X,K_X)} : (X', K') & \longrightarrow & (X, K_X) \\ x & \longmapsto & x \end{array}$$
   is a universal morphism from I to (X, K_X).

   (b) $\mathcal{B}$ is a coreflective subcategory of $\mathcal{G}rf$. Justify this assertion.

   (c) Show that $\mathcal{B}$ is also a reflective subcategory of $\mathcal{G}rf$.

5. Let $\mathcal{A}$ be the (full) subcategory of $\mathcal{G}rph$ of symmetric directed graphs (i.e., those graphs (X, K_X) such that, if $(x, y) \in K_X$, also $(y, x) \in K_X$). Show that $\mathcal{A}$ is both a reflective and a coreflective subcategory of $\mathcal{G}rph$.

6. If G is an abelian group, the torsion of G is the subgroup of G

$$\mathcal{T}or(G) = \{g \in G \,|\, g \text{ has finite order}\}.$$

An abelian group is said to be a torsion group if $G = \mathcal{T}or(G)$ and a torsion-free group if $\mathcal{T}or(G) = \{0\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be the (full) subcategories of $\mathcal{A}b$ of torsion and of torsion-free abelian groups. Show that:

   (a) $\mathcal{A}$ is a coreflective subcategory of $\mathcal{A}b$;

   (b) $\mathcal{B}$ is a reflective subcategory of $\mathcal{A}b$.

7. Let $\mathcal{B}$ be a category with two objects A, B, and such that, besides the identities, has morphisms $f : A \to B$ and $g : B \to A$ with $g \cdot f = 1_A$ and $f \cdot g = 1_B$. Show that the subcategory $\mathcal{A}$ of $\mathcal{B}$ with objects A and B and morphisms $1_A, 1_B, f$ is reflective but the reflector functor $R : \mathcal{B} \to \mathcal{A}$, when restricted to $\mathcal{A}$, cannot be the identity.

15 Adjunctions versus equivalences of categories

15.1 Definition.

Two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there exist functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $\alpha : 1_{\mathcal{A}} \to G \circ F$ and $\beta : 1_{\mathcal{B}} \to F \circ G$.

One says then that F (and then also G) is an equivalence of categories.

15.2 Examples.

1. Let $\mathcal{A}$ be a category with one object, A, and one morphism, 1_A, and $\mathcal{B}$ a category with two objects, B_1 and B_2, and two non-trivial morphisms, $f : B_1 \to B_2$ and $g : B_2 \to B_1$, such that $g \circ f = 1_{B_1}$ and $f \circ g = 1_{B_2}$. Then the categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

2. The functors F and G defined below establish an equivalence between the category $\mathcal{P}fn$ and the category $\mathcal{PS}et$ of pointed sets:

$$\begin{array}{rcl} F : \mathcal{PS}et & \longrightarrow & \mathcal{P}fn \\ (X, x_0) & \longmapsto & X \setminus \{x_0\} \\ f & \longmapsto & Ff \end{array} \qquad e \qquad \begin{array}{rcl} G : \mathcal{P}fn & \longrightarrow & \mathcal{PS}et \\ X & \longmapsto & (X \overset{+}{\cup} \{\infty\}, \infty) \\ g & \longmapsto & Gg \end{array}$$

where, for $f \in \mathcal{PS}et((X, x_0), (Y, y_0))$, $DD_{Ff} = X \setminus f^{-1}(y_0)$ and $Ff(x) = f(x)$ for all $x \in DD_{Ff}$, and, for $g \in \mathcal{P}fn(X, Y)$, $Gg(x) = g(x)$ if $x \in DD_g$ and $Gg(x) = \infty$ when $x \notin DD_f$.

15.3 Theorem.

Let $G : \mathcal{A} \to \mathcal{B}$ be a functor. The following conditions are equivalent:

(i) G is full and faithful and it has a full and faithful left adjoint;

(ii) G is an equivalence of categories;

(iii) G is full and faithful, and each object B of $\mathcal{B}$ is isomorphic to an object of the form GA for some $A \in \mathrm{Ob}\mathcal{A}$.

15.4 Corollary.

Let $F : \mathcal{A} \to \mathcal{B}$ be an equivalence of categories. Then, if $\mathcal{A}$ is (finitely) complete, also $\mathcal{B}$ is (finitely) complete.

16 Cartesian closed categories

For each set A, the functor $F : - \times A : \mathcal{S}et \to \mathcal{S}et$ has a right adjoint, $G = \mathcal{S}et(A, -) : \mathcal{S}et \to \mathcal{S}et$, with the counit of the adjunction given by the evaluation maps

$$\begin{array}{rcl} \mathrm{ev}_B : \mathcal{S}et(A,B) \times A & \longrightarrow & B \\ (f,a) & \longmapsto & f(a). \end{array}$$

16.1 Definition.

A category $\mathcal{C}$ is cartesian closed if it has finite products and, for each object A of $\mathcal{C}$, the functor $- \times A : \mathcal{C} \to \mathcal{C}$ has a right adjoint. In this case, if G_A is the right adjoint of $- \times A$, $G_A(B)$ is called the exponential object – or exponential of B with exponent A –, and it is denoted by B^A or $[A \to B]$.

16.2 Proposition.

A category $\mathcal{C}$ is cartesian closed if and only if the following functors have right adjoint

$$\mathcal{C} \longrightarrow 1, \qquad \begin{array}{rcl} \Delta : \mathcal{C} & \longrightarrow & \mathcal{C} \times \mathcal{C} \\ C & \longmapsto & (C, C) \end{array} \quad \text{and} \quad - \times A : \mathcal{C} \longrightarrow \mathcal{C} \quad \text{(for each object } A \text{ of } \mathcal{A}\text{).}$$

16.3 Exercises.

1. Let $\mathcal{C}$ be a cartesian closed category with initial object 0.

   Show that, if A is an object of $\mathcal{C}$,

   (a) $0 \cong 0 \times A$;

   (b) if $\mathcal{C}(A, 0) \neq \emptyset$, then $A \cong 0$;

   (c) if $0 \cong 1$, then the category $\mathcal{C}$ is degenerated, that is, all $\mathcal{C}$-objects are isomorphic;

   (d) every morphism $0 \to A$ is a monomorphism;

   (e) $A^1 \cong A$, $A^0 \cong 1$ and $1^A \cong 1$.

2. Show that, if $\mathcal{C}$ is a cartesian closed category and A and B are objects of $\mathcal{C}$, there exists a bijection between the sets $\mathcal{C}(A, B)$ and $\mathcal{C}(1, B^A)$.

   (The elements of $\mathcal{C}(1, X)$ are called points of X.)

16.4 Examples.

1. The following categories are cartesian closed:

   (a) $\mathcal{S}et$;

   (b) $\mathcal{S}et \times \mathcal{S}et$;

   (c) $\mathcal{S}et \downarrow \mathcal{S}et$;

   (d) $\mathcal{S}et \downarrow I$, for any set I;

   (e) $\mathcal{C}at$;

   (f) $\mathcal{G}rph$.

2. The following categories are not cartesian closed: $\mathcal{SG}rp$, $\mathcal{V}ec_K$, $\mathcal{M}on$, $\mathcal{G}rp$, $\mathcal{A}b$, $\mathcal{T}op$.

16.5 Exercises.

1. Show that the category of finite sets and maps is cartesian closed.

2. A partially ordered set $(X, \leq)$ is a Boolean algebra if:

   - it has finite infima and finite suprema,
   - $\wedge$ is distributive with respect to $\vee$ (that is: $(\forall x, y, z \in X)\ \ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$), and
   - every element x of X has a complement $\neg x$ in X (that is: $x \wedge \neg x = 0$ and $x \vee \neg x = 1$).

   Show that, if $(X, \leq)$ is a Boolean algebra, then the category $\mathcal{C}_{(X,\leq)}$ is cartesian closed.

3. If $(X, \leq)$ is totally ordered with a top element 1, then $\mathcal{C}_{(X,\leq)}$ is a cartesian closed category, with

$$q^p := \begin{cases} 1 & \text{if } p \leq q \\ q & \text{if } q < p. \end{cases}$$

17 Toposes

17.1 Definition.

If $\mathcal{C}$ is a category with terminal object 1, a subobject classifier of $\mathcal{C}$ is a pair $(\Omega, \top : 1 \to \Omega)$, where Ω is an object of $\mathcal{C}$ and $\top : 1 \to \Omega$ is a $\mathcal{C}$-morphism, such that:

> For each monomorphism $f : A \to B$, there exists a unique morphism $\chi_f : B \to \Omega$ such that the diagram
>
> $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "\chi_f"] \\ 1 \arrow[r, "\top"] & \Omega \end{tikzcd}$$
>
> is a pullback.

(χ_f is usually called characteristic morphism – or character – of f.)

17.2 Exercises.

1. Show that a subobject classifier, when it exists, is unique up to isomorphism.

2. Show that, if the category $\mathcal{C}$ has a subobject classifier and $f : A \to C$ and $g : B \to C$ are monomorphisms in $\mathcal{C}$, then: $f \cong g \quad \Leftrightarrow \quad \chi_f = \chi_g$.

17.3 Definition.

A category $\mathcal{C}$ is said to be a (elementary) topos if

(a) $\mathcal{C}$ is finitely complete;

(b) $\mathcal{C}$ is finitely cocomplete;

(c) $\mathcal{C}$ is cartesian closed;

(d) $\mathcal{C}$ has subobject classifier.

17.4 Remark.

In the previous definition condition (b) follows from (a), (c) and (d).

17.5 Examples.

1. $\mathcal{S}et$.
2. The category of finite sets and maps.
3. $\mathcal{S}et \times \mathcal{S}et$.
4. $\mathcal{S}et \downarrow \mathcal{S}et$.
5. $\mathcal{S}et \downarrow I$, for any set I.
6. The category M-$\mathcal{S}et$ of sets equipped with an action of the monoid M and equivariant maps.

17.6 Proposition.

Let $\mathcal{C}$ be a topos.

1. Every monomorphism in $\mathcal{C}$ is an equalizer.
2. In $\mathcal{C}$, a morphism is an isomorphism if and only it it is both a monomorphism and an epimorphism.

17.7 Theorem.

In a topos $\mathcal{C}$ (Epi$\mathcal{C}$,Mono$\mathcal{C}$) is a factorisation system.

17.8 Remark.

In the proof of this theorem we use the following result, that we didn't prove:

Theorem. If $\mathcal{C}$ is a topos and A and object of $\mathcal{C}$, then $\mathcal{C} \downarrow A$ is a topos.

17.9 Corollary.

In a topos every epimorphism is a coequalizer.

17.10 Proposition.

In a non-degenerated topos (that is, with $0 \ncong 1$), with subobject classifier $(\top : 1 \to \Omega)$, the object Ω has at least two points:

- the morphism "true" $\top : 1 \to \Omega$,
- the morphism "false" $\perp : 1 \to \Omega$, that is the characteristic morphism of the monomorphism $0 \to 1$.

17.11 Exercise.

Let $\mathcal{C}$ be a topos. The points of Ω are called truth values of the topos $\mathcal{C}$.

Calculate the truth values of the following toposes:

1. $\mathcal{S}et \times \mathcal{S}et$;
2. $\mathcal{S}et \downarrow \mathcal{S}et$;
3. $\mathcal{S}et \downarrow I$ (where I is a set);
4. M-$\mathcal{S}et$.

17.12 Definition.

A natural numbers object in a topos is a diagram

$$1 \xrightarrow{o} N \xrightarrow{s} N$$

such that, for any other such diagram

$$1 \xrightarrow{x} X \xrightarrow{u} X$$

there exists a unique morphism $f : N \to X$ such that the following diagram

$$\begin{tikzcd} 1 \arrow[r, "o"] \arrow[dr, "x"'] & N \arrow[r, "s"] \arrow[d, "f"] & N \arrow[d, "f"] \\ & X \arrow[r, "u"] & X \end{tikzcd}$$

commutes.

17.13 Proposition.

If $1 \xrightarrow{o} N \xrightarrow{s} N$ is a natural numbers object in a topos, then:

1. $1 \xrightarrow{o} N \xleftarrow{s} N$ is a coproduct diagram.
2. $N \underset{1_N}{\overset{s}{\rightrightarrows}} N \longrightarrow 1$ is a coequalizer diagram.

17.14 Theorem.

Properties 1. and 2. of the previous Proposition characterize the natural numbers object in a topos.

17.15 Theorem.

If there exists an object X in a topos such that $X \cong X \coprod 1$, then X has a subobject with a structure of natural numbers object.

17.16 Exercises.

Verify whether there exist natural numbers objects in the toposes already studied.

18 Abelian Categories

18.1 Definitions.

Ley $\mathcal{C}$ be a category with zero object. For each pair of objects A and B of $\mathcal{C}$, the zero morphism from A to B is the morphism $A \longrightarrow 0 \longrightarrow B$. We define the *kernel* of the morphism $f : A \to B$ as the equalizer of f and the zero morphism, and denote it by $\ker f$. Dually one defines cokernel.

18.2 Examples.

1. In the category $\mathcal{S}et_*$ of pointed sets, and in the category $\mathcal{A}b$ of abelian groups, every monomorphism is a kernel. In $\mathcal{S}et_*$ not every epimorphism is a cokernel.

2. In the category $\mathcal{T}op_*$ of pointed spaces, and in the category $\mathcal{G}rp$ of groups, not every monomorphism is a kernel.

18.3 Remark.

Let $\mathcal{C}$ be a category with a zero object, kernels and cokernels. For each object C of $\mathcal{C}$ let

$$P^C = \{f \mid \mathrm{cod} f = C\}/\sim \ \text{ and } Q_C = \{f \mid \mathrm{dom} f = C\}/\sim,$$

where the equivalence relations $\sim$ are defined by $f \sim g$ if and only if $f \leq g$ and $g \leq f$, and the preorder relation $\leq$ is the following one: given f, g, both with codomain C, or both with domain C, $f \leq g$ if f factors through g. Then $(P^C, \leq)$ and $(Q_C, \leq)$ are partially ordered (possibly proper classes), and the assignments $u \mapsto \ker u$ and $f \mapsto \mathrm{coker} f$ define adjoint functors

$$\begin{tikzcd} Q^C \arrow[r, shift right, "\ker"'] \arrow[r, phantom, "\top"] & P_C^{\mathrm{op}} \arrow[l, shift right, "\mathrm{coker}"'] \end{tikzcd} .$$

(we will not mention the equivalence classes if not necessary). That is,

$$f \leq \ker u \iff u \leq \mathrm{coker} f.$$

18.4 Exercise.

Prove that, for each $u \in Q^C$ and each $f \in P_C$,

$$\ker(\mathrm{coker}(\ker u)) = \ker u \text{ and } \mathrm{coker}(\ker(\mathrm{coker} f)) = \mathrm{coker} f.$$

Conclude that every kernel is the kernel of its cokernel and every cokernel is the cokernel of its kernel.

18.5 Proposition.

If $\mathcal{C}$ has a zero object, kernels and cokernels, then every morphism $f : A \to B$ factors through $m := \ker(\mathrm{coker} f)$; that is, there exists q such that $f = m \circ q$. This factorisation has the following property: if $f = m' \circ q'$, with m' a kernel, then there exists a unique morphism t making the following diagram commute:

$$\begin{tikzcd} \cdot \arrow[r, "q"] \arrow[d, "q'"'] & \cdot \arrow[d, "m=\ker(\mathrm{coker} f)"] \arrow[dl, dotted, "t"'] \\ \cdot \arrow[r, "m'"'] & \cdot \end{tikzcd}$$

If, moreover, $\mathcal{C}$ has equalizers and every monomorphism in $\mathcal{C}$ is a kernel, then q is an epimorphism.

18.6 Definition.

A category $\mathcal{C}$ is enriched in $\mathcal{A}b$ (or $\mathcal{A}b$-category) if every set of morphisms $\mathcal{C}(A, B)$ has an abelian group structure compatible with the composition law, that is, such that the morphisms composition is bilinear:

$$(\forall f, f' \in \mathcal{C}(A, B))\ (\forall g, g' \in \mathcal{C}(B, C))\ (g + g') \circ (f + f') = (g \circ f) + (g \circ f') + (g' \circ f) + (g' \circ f').$$

18.7 Exercise.

Show that, if C is an object in an $\mathcal{A}b$-category, then the following conditions are equivalent:

(i) C is initial;

(ii) C is terminal;

(iii) 1_C is a zero morphism;

(iv) $\mathcal{C}(C, C)$ is the trivial group.

18.8 Definition.

A diagram of biproduct of the objects A, B on the $\mathcal{A}b$-enriched category $\mathcal{C}$ is a diagram

$$\begin{tikzcd} A \arrow[r, shift right, "i_1"'] & C \arrow[l, shift right, "p_1"'] \arrow[r, shift left, "p_2"] & B \arrow[l, shift left, "i_2"] \end{tikzcd} \tag{1}$$

such that $p_1 \circ i_1 = 1_A$, $p_2 \circ i_2 = 1_B$ and $i_1 \circ p_1 + i_2 \circ p_2 = 1_C$.

18.9 Exercise.

Show that, if the diagram (1) is a biproduct, then $p_1 \circ i_2 = 0$ and $p_2 \circ i_1 = 0$.

18.10 Theorem.

In an $\mathcal{A}b$-category two objects have product if and only if they have biproduct. In particular, given the biproduct diagram (1), $A \xleftarrow{p_1} C \xrightarrow{p_2} B$ is the product of A and B, while, dually, $A \xrightarrow{i_1} C \xleftarrow{i_2} B$ is a coproduct. In particular, there exists the product of A and B if and only if there exists their coproduct.

18.11 Definition.

An $\mathcal{A}b$-category is additive if it has zero object and biproducts.

18.12 Definition.

In an additive category we can define a functor (tensor product):

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r] & \mathcal{C} \\ (A, B) \arrow[r, mapsto] \arrow[d, "(f,g)"'] & A \oplus B \arrow[d, "f \oplus g"] \\ (A', B') \arrow[r, mapsto] & A' \oplus B' \end{tikzcd}$$

where $A \oplus B$ is the biproduct object of A and B, and $f \oplus g$ may be defined as $f \times g$ or $f + g$, using the universal property of $A \oplus B$ as product and coproduct of A and B, respectively.

18.13 Exercise.

Show that $f \times g = f + g$ in the definition given above.

18.14 Proposition.

If $f, f' : A \to B$ are morphisms in an additive category $\mathcal{C}$, then:

$$(A \xrightarrow{f+f'} B) = (A \xrightarrow{\Delta_A} A \oplus A \xrightarrow{f \oplus f'} B \oplus B \xrightarrow{\nabla_B} B) \ ,$$

where $\Delta_A : A \to A \times A$ is the morphism $\langle 1_A, 1_A \rangle$ and $\nabla_B : B + B \to B$ is the morphism $[1_B, 1_B] : B + B \to B$.

18.15 Exercise.

Prove the Proposition above.

18.16 Definition.

A functor $T : \mathcal{A} \to \mathcal{B}$, between two categories enriched in $\mathcal{A}b$, is said to be an additive functor if it preserves the morphisms sum, that is, for $f, f' : A \to B$ in $\mathcal{A}$, $T(f + f') = T(f) + T(f')$.

18.17 Proposition.

If $\mathcal{A}$ and $\mathcal{B}$ are additive categories, then a functor $T : \mathcal{A} \to \mathcal{B}$ is additive if and only if it preserves biproducts.

18.18 Definition.

An $\mathcal{A}b$-category is abelian if:

(1) it has a zero object;

(2) it has biproducts;

(3) it has kernels and cokernels;

(4) every monomorphism is a kernel and every epimorphism is a cokernel.

18.19 Remarks.

(a) By Exercise 18.7, in (1) it is enough to assume the existence of terminal object.

(b) By Theorem 18.10, in (2) it is enough to impose the existence of products, or of coproducts.

(c) The notion of abelian category is self-dual; that is, a category $\mathcal{C}$ is abelian if and only if $\mathcal{C}^{\mathrm{op}}$ is.

(d) If we replace (2) by

   (2') it has binary products and binary coproducts;

   we don't need to impose that the category is $\mathcal{A}b$-enriched. One can define the sum of two morphisms as given in Proposition 18.14 [non-trivial proof, that will not be presented in this course].

18.20 Lemma.

Every abelian category is finitely complete.

18.21 Exercise.

Show that, if $\mathcal{A}$ is a small category and $\mathcal{C}$ an abelian category, then the category $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$ is abelian.

18.22 Proposition.

Is $\mathcal{C}$ is an abelian category, then every morphism f has a factorisation $f = m \circ e$, with m a monomorphism and e an epimorphism, being $m = \ker(\mathrm{coker} f)$ and $e = \mathrm{coker}(\ker f)$. Moreover, if $g = m' \circ e'$ is such a factorisation and the diagram

$$\begin{tikzcd} {} \arrow[r, "g"] \arrow[d, "h"'] & {} \arrow[d, "k"] \\ {} \arrow[r, "f"] & {} \end{tikzcd}$$

commutes, then there exists a unique morphism t making the following diagram commute:

$$\begin{tikzcd} {} \arrow[r, "e'"] \arrow[d, "h"'] & {} \arrow[r, "m'"] \arrow[d, "t"] & {} \arrow[d, "k"] \\ {} \arrow[r, "e"] & {} \arrow[r, "m"] & {} \end{tikzcd}$$

We denote this factorisation of f by $f = \mathrm{im} f \circ \mathrm{coim} f$, with $m = \ker(\mathrm{coker} f) = \mathrm{im} f$ and $e = \mathrm{coker}(\ker f) = \mathrm{coim} f$.

18.23 Definition.

A composable pair of morphisms $\xrightarrow{f} B \xrightarrow{g}$ is exact in B if $\mathrm{im} f \cong \ker g$, or, equivalently, $\mathrm{coker} f \cong \mathrm{coim} g$.

18.24 Exercise.

Prove that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact in A, B and C if and only if $f = \ker g$ and $g = \mathrm{coker} f$.

18.25 Definitions.

1. A diagram $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a short exact sequence if it is exact in A, B and C; that is, $f = \ker g$ and $g = \mathrm{coker} f$.

2. One says that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is a left short exact sequence if $f = \ker g$, that is, if it is exact in A and B.

3. One says that $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a right short exact sequence if $g = \mathrm{coker} f$, that is, if it is exact in B and C.

18.26 Definition.

A functor $T : \mathcal{A} \to \mathcal{B}$ between two abelian categories is exact if it preserves finite limits and finite colimits. It is left exact if it preserves finite limits.

18.27 Theorem.

The following conditions are equivalent, for a functor $T : \mathcal{A} \to \mathcal{B}$ between abelian categories:

(i) T is exact;

(ii) T is additive and preserves kernels and cokernels;

(iii) T is additive and preserves left and right short exact sequences.

18.28 Short Five Lemma.

Given a commutative diagram in an abelian category

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact sequences,

(1) if f and h are monomorphisms, then g is a monomorphism;

(2) dually, if f and h are epimorphisms, then g is an epimorphism;

(3) therefore, if f and h are isomorphisms, then g is an isomorphism.

18.29 Proposition.

In an abelian category, let us consider a pullback diagram:

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

1. If f is an epimorphism, then so is f'.
2. Moreover, $\ker f = g' \cdot \ker f'$.