Numerical Approximation of Mean Curvature Flow

Adérito Araújo, Marta Cavaleiro

CMUC
Centre for Mathematics, University of Coimbra

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Outline

1. Introduction
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   - Level Sets Method
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   - Viscosity solutions
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Outline

Parkinson disease

What is it?

- Degenerative disorder of the central nervous system that affects the control of muscles and so may affect movement, speech and posture
- Caused by insufficient formation and action of dopamine

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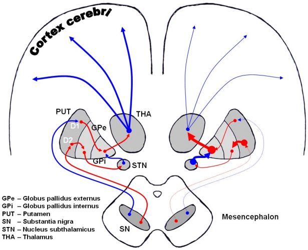
Diagnoses

- There are currently no blood or laboratory tests that have been proven to help in diagnosing the disease
- 75% of clinical diagnoses of Parkinson disease are confirmed at autopsy

Parkinson disease

Pathology

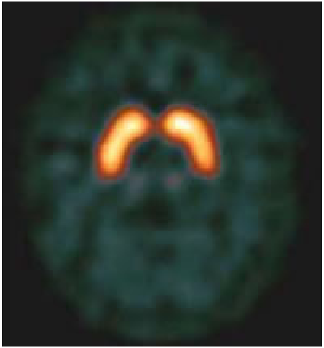
- The symptoms of Parkinson disease result from the loss of dopaminergic cells and subsequent loss of melanin
- The neurons project to the striatum and their loss leads to alteration in the activity of neural circuits within the basal ganglia

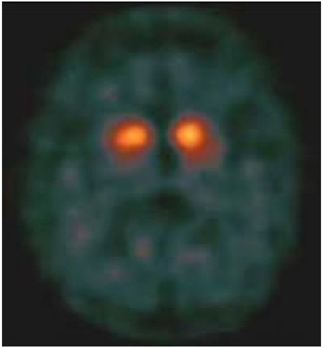


Credits: Wikipedia, the free encyclopedia

Parkinson disease

- Decreased dopamine activity in the basal ganglia, a pattern which aids in diagnosing Parkinson disease
- PET and SPECT images may help



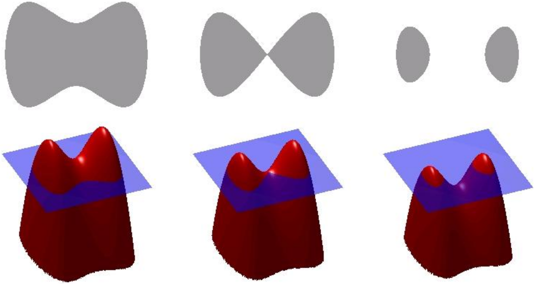


Credits: European Parkinson's Disease Association

Level Sets Method

$\Gamma(t)$ is implicitly represented by the zero level set of a higher dimension function ϕ:

$$\Gamma(t) = \{(x, y) \in \Omega : \phi(t, x, y) = 0\}$$



Credits: Oleg Alenxandrov, en.wikipedia.org

- Notions of interior and exterior of a curve are immediate
- Union and division of curves are automatic

Level Sets Method

- Evolve the curve in the direction of the normal with speed $v \Leftrightarrow$ solving a PDE

$$\frac{\partial \phi}{\partial t} = v \left| \nabla \phi \right|, \qquad \phi(0, x, y) = \phi_0(x, y)$$

with suitable boundary conditions

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- Motion by mean curvature

$$v = \text{ div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right)$$

- For $\phi \in C^{2,1}(]0,T] \times \Omega)$ and $\nabla \phi \neq 0$ in a neighborhood of $\Gamma(t)$:

$$\text{(IBVP)} \qquad \begin{cases} \phi_t = |\nabla \phi| \,\text{div}\left(\dfrac{\nabla \phi}{|\nabla \phi|}\right) & (x,y) \in \Omega,\ t > 0 \\ \phi(0,x,y) = \phi_0(x,y) & (x,y) \in \bar{\Omega} \\ \phi(t,x,y) = 0 & (x,y) \in \partial\Omega\ ,\ t > 0 \end{cases}$$

with Ω a bounded open set of $\mathbb{R}^2$ and $\phi_0 \in C(\Omega)$.

Outline

Viscosity solution (Evans & Spruck, 1991)

Let $\phi \in C(]0,T] \times \Omega) \cap L^\infty(]0,T] \times \Omega)$.

- ϕ is a viscosity sub-solution (super-solution) of (IBVP) if for all $v \in C^2(]0,T],\Omega)$, $\phi - v$ has a local maximum (minimum) in (t_0, x_0, y_0) then $(\nabla\phi(t_0,x_0,y_0) \neq 0)$

$$v_t(t_0,x_0,y_0) \leq (\geq) |\nabla\phi(t_0,x_0,y_0)| \operatorname{div}\left(\frac{\nabla\phi(t_0,x_0,y_0)}{|\nabla\phi(t_0,x_0,y_0)|}\right)$$

- ϕ is a viscosity solution of (IBVP) if it simultaneity a viscosity sub and super-solution.

Results

- Under certain conditions, the viscosity solution of (IBVP) exists and its unique (Evans & Spruck, 1991)
- The curves $\Gamma(t)$ are independent of the initial choice ϕ_0 (Evans & Spruck, 1991)
- The following stability result holds (Caselles *et al.*, 1993)

$$\sup_{0 \le s \le t} \|\phi(s) - \hat{\phi}(s)\|_{L^\infty} \le \|\phi_0 - \hat{\phi}_0\|_{L^\infty} \quad \forall t \in [0, T]$$

Motion by Mean Curvature

Determine ϕ from the initial boundary value problem (IBVP):

$$\begin{cases} \dfrac{\phi_t}{|\nabla\phi|} = \nabla^T \left(\dfrac{\nabla\phi}{|\nabla\phi|}\right) & (x,y) \in \Omega \text{ , } t > 0 \\ \phi(0,x,y) = \phi_0(x,y) & (x,y) \in \bar{\Omega} \\ \phi(t,x,y) = 0 & (x,y) \in \partial\Omega \text{ , } t > 0 \end{cases}$$

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Theorem

$$\underbrace{\|\phi(t)\|_{L^1} + \|\nabla\phi(t)\|_{L^1}}_{\|\phi(t)\|_{W^{1,1}}} \le C(\underbrace{\|\phi_0\|_{L^1} + \|\nabla\phi_0\|_{L^1}}_{\|\phi_0\|_{W^{1,1}}})$$

Proof:

- $\|\nabla\phi(t)\|_{L^1} + \int_0^t\int_\Omega \frac{\phi_t^2}{|\nabla\phi|}dxdyds = \|\nabla\phi_0\|_{L^1}$ (Walkington, 1996)
- $\|\nabla\phi(t)\|_{L^1} \le \|\nabla\phi_0\|_{L^1}$
- Poincaré inequality in L^1: $\|\phi\|_{L^1} \le C^*\|\nabla\phi\|_{L^1}$

Outline

IMEX Method: time discretization

$$\frac{1}{|\nabla \phi^n|} \frac{\phi^{n+1} - \phi^n}{\Delta t} = \text{div}\left(\frac{\nabla \phi^{n+1}}{|\nabla \phi^n|}\right)$$

- $t^n = n\Delta t$, $n = 0, ..., N$, with $t_0 = 0$ and $t_N = T$
- $\phi^n \approx \phi(n\Delta t, x, y),\ \forall (x, y) \in \Omega$

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Theorem

$$\|\phi^{n+1}\|_{W^{1,1}} \leq C\|\phi^n\|_{W^{1,1}} \quad \forall n = 0, 1, ..., N-1$$

Proof:

- Multiply the equation by $\phi^{n+1} - \phi^n$ with respect to the L^2 inner product and integrate by parts
- $\|\nabla \phi^{n+1}\|_{L^1} \leq \|\nabla \phi^n\|_{L^1} \quad \forall n = 0, 1, ..., N-1$
- Poincaré inequality in L^1

IMEX Method: full discretization

$$\frac{1}{|\nabla_h \phi_{ij}^n|} \frac{\phi_{ij}^{n+1} - \phi_{ij}^n}{\Delta t} = D_x^+ \left(\frac{D_x^- \phi_{ij}^{n+1}}{|\nabla_h \phi_{ij}^n|} \right) + D_y^+ \left(\frac{D_y^- \phi_{ij}^{n+1}}{|\nabla_h \phi_{ij}^n|} \right)$$

- $\bar{\Omega}_h =$ grid in $\bar{\Omega}$ with space step h
- Finite differences: D_x^+, D_y^+ (forward); D_x^-, D_y^- (backward)
- $\phi_{ij}^n \approx \phi(n\Delta t, x_i, y_j), \ \forall (x_i, y_j) \in \bar{\Omega}_h$
- $|\nabla_h \phi_{ij}^n| = \sqrt{(D_x^- \phi_{ij}^n)^2 + (D_y^- \phi_{ij}^n)^2}$
- Norm in the discrete $W^{1,1}$ space

$$\|\phi\|_{1,1} = \sum_{i,j} h^2 |\phi_{ij}| + \sum_{i,j} h^2 |\nabla_h \phi_{ij}|$$

IMEX Method: full discretization

Theorem

$$\|\phi^{n+1}\|_{1,1} \leq C\|\phi^n\|_{1,1} \quad \forall n = 0, ..., N-1$$

Proof:

- Multiply

$$\frac{1}{|\nabla_h \phi_{ij}^n|} \frac{\phi_{ij}^{n+1} - \phi_{ij}^n}{\Delta t} = D_x^+ \left(\frac{D_x^- \phi_{ij}^{n+1}}{|\nabla_h \phi_{ij}^n|} \right) + D_y^+ \left(\frac{D_y^- \phi_{ij}^{n+1}}{|\nabla_h \phi_{ij}^n|} \right)$$

  by $\phi_{ij}^{n+1} - \phi_{ij}^n$ with respect to the discrete L^2 inner product and use summation by parts

- $\sum_{ij} h^2 |\nabla_h \phi_{ij}^{n+1}| \leq \sum_{ij} h^2 |\nabla_h \phi_{ij}^n| \quad \forall n = 0, ..., N-1$
- Discrete Poincaré inequality in ℓ^1: $\quad \sum_{ij} h^2 |\phi_{ij}^n| \leq C^* \sum_{ij} h^2 |\nabla_h \phi_{ij}^n|$

Parallel Splitting Algorithm

$$\frac{\partial \phi}{\partial t} = A\phi + f(t) \quad \text{in} \quad \Omega \times [0, T], \qquad \phi(0) = \phi_0$$

- $A = A_1 + A_2 + \cdots + A_m$ and $f = f_1 + f_2 + \cdots + f_m$
- A is time independent

Algorithm (Lu, Neittaanmaki and Tai, 1992)

At each level time $n = 0, ..., N-1$ compute:

1. $$\frac{\phi^{n+\frac{k}{2m}} - \phi^n}{m\Delta t} = A_k \phi^{n+\frac{k}{2m}} + f_k\left((n+\frac{1}{2})\Delta t\right) \quad k = 1, \ldots, m$$
2. $$\phi^{n+1} = \frac{1}{m}\sum_{k=1}^{m} \phi^{n+\frac{k}{2m}}$$

Parallel Splitting Algorithm

Consider $A_1\phi^{n+\frac{1}{4}} = D_x^+\left(\frac{D_x^-\phi_{ij}^{n+\frac{1}{4}}}{|\nabla\phi_{ij}^n|}\right)$ and $A_2\phi^{n+\frac{1}{2}} = D_x^+\left(\frac{D_y^-\phi_{ij}^{n+\frac{1}{2}}}{|\nabla\phi_{ij}^n|}\right)$

- Construction of A_1

$$\frac{\phi^{n+\frac{1}{4}} - \phi^n}{2\Delta t} = A_1\phi^{n+\frac{1}{4}} \Leftrightarrow$$

$$\frac{1}{|\nabla_h\phi_{ij}^n|}\frac{\phi_{ij}^{n+\frac{1}{4}} - \phi_{ij}^n}{2\Delta t} = \frac{\phi_{i-1,j}^{n+\frac{1}{4}}}{h^2|\nabla_h\phi_{i,j}^n|} - \frac{2}{h^2}\phi_{ij}^{n+\frac{1}{4}}\left(\frac{1}{|\nabla_h\phi_{i+1,j}^n|} + \frac{1}{|\nabla_h\phi_{i,j}^n|}\right) + \frac{\phi_{i+1,j}^{n+\frac{1}{4}}}{h^2|\nabla_h\phi_{i+1,j}^n|}$$

- A_1 is tridiagonal and diagonally dominant with

$$a_{i,i-1} = \frac{1}{h^2}, \quad a_{i,i} = -\frac{2}{h^2}\left(\frac{|\nabla_h\phi_{i,j}^n|}{|\nabla_h\phi_{i+1,j}^n|} + 1\right), \quad a_{i,i+1} = \frac{|\nabla_h\phi_{i,j}^n|}{h^2|\nabla_h\phi_{i+1,j}^n|}$$

- A similar construction can be made for A_2

Parallel Splitting Algorithm

Algorithm

At each level time $n = 0, ..., N-1$ compute:

1. Compute $|\nabla_h \phi_{ij}^n| = \sqrt{(D_x^- \phi_{ij}^n)^2 + (D_y^- \phi_{ij}^n)^2}$
2. Construct A_1 and A_2
3. Solve

$$(I - 2\Delta t A_1)\phi^{n+\frac{1}{4}} = \phi^n \quad \text{and} \quad (I - 2\Delta t A_2)\phi^{n+\frac{1}{2}} = \phi^n$$

4. $\phi^{n+1} = \dfrac{\phi^{n+\frac{1}{4}} + \phi^{n+\frac{1}{2}}}{2}$

Parallel Splitting Algorithm

Theorem (Stability)

The algorithm is unconditionally stable in the $\|.\|_\infty$ norm.

Proof:

- $I - 2\Delta t A_1$ and $I - 2\Delta t A_2$ are M-matrices
- $\exists c_1, c_2 \geq 0$: $\|(I - 2\Delta t A_1)^{-1}\|_\infty \leq c_1$ and $\|(I - 2\Delta t A_2)^{-1}\|_\infty \leq c_2$
- $\|\Phi^{n+1}\|_\infty \leq \frac{1}{2}\left(\|\Phi^{n+\frac{1}{4}}\|_\infty + \|\Phi^{n+\frac{1}{2}}\|_\infty\right) \leq \frac{1}{2}(c_1 + c_2)\|\Phi^n\|_\infty$

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Theorem (Convergence)

If $(-A_k)$, $k = 1, ..., m$, are irreducible M-matrices, then the algorithm is convergent of first order in Δt.

Outline

CV model (Chan and Vese, 2001)

- Find the curve that minimizes:

$$F(c_1, c_2, \phi) = \mu \int_\Omega \delta_0(\phi)|\nabla\phi| dxdy \ + \ \nu \int_\Omega H(\phi) dxdy$$

$$+\lambda_1 \int_\Omega |u_0 - c_1|^2 H(\phi) dxdy + \ \lambda_2 \int_\Omega |u_0 - c_2|^2 (1 - H(\phi)) dxdy$$

- It reduces to the resolution of a PDE:

$$\frac{\partial\phi}{\partial t} = \delta_0(\phi) \left(\mu \, \mathrm{div} \left(\frac{\nabla\phi}{|\nabla\phi|} \right) - \nu - \lambda_1 (u_0 - c_1)^2 + \lambda_2 (u_0 - c_2)^2 \right)$$

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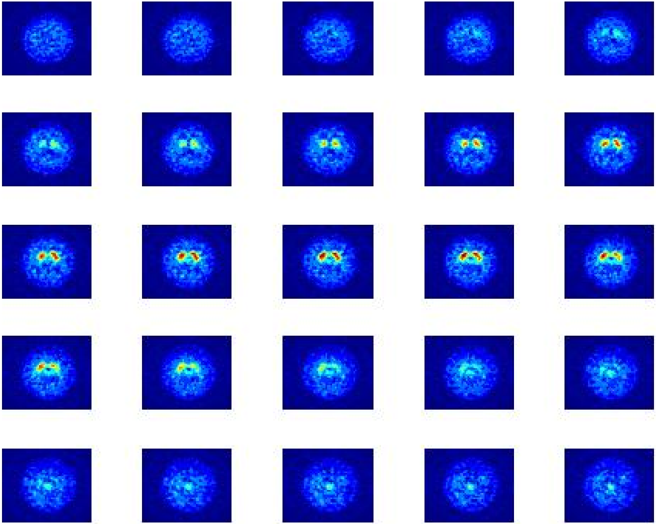
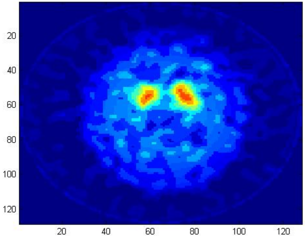
All the previous results could be generalized for

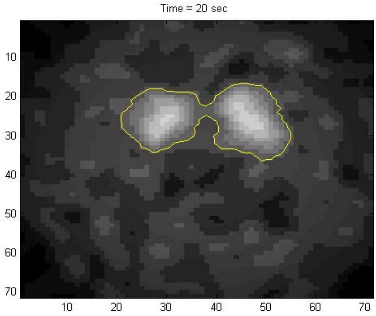
$$\phi_t = g(\phi) \, \mathrm{div} \left(\frac{\nabla\phi}{|\nabla\phi|} \right) \quad (x, y) \in \Omega \, , \, t > 0, \quad g > 0$$

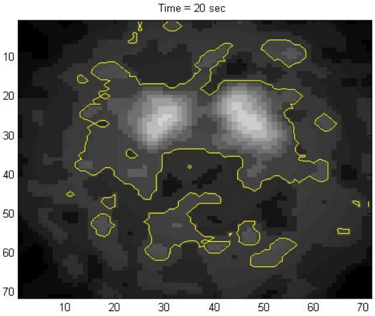
SPECT Images (given by IBILI)



Numerical Results







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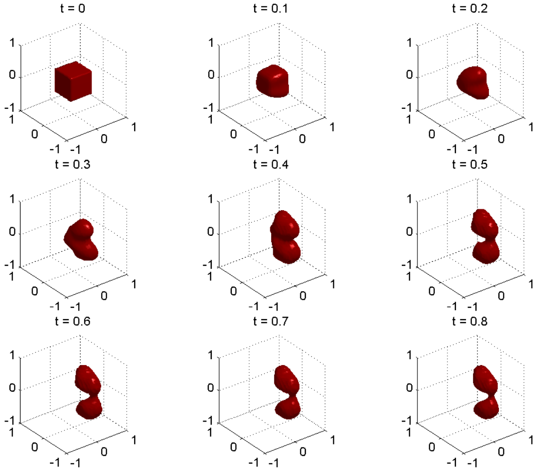
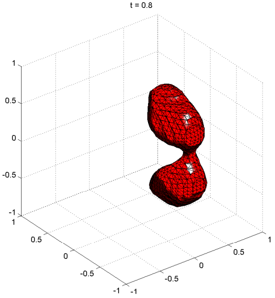
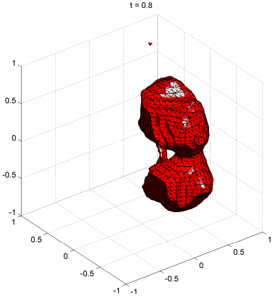


Evolution of the zero level set in the iteration for $\mu = 0.05$

Numerical Results





Results of segmentation algorithm for $\mu = 0.05$ (left) and $\mu = 0.001$ (right)

Outline

Conclusions and Future Work

Conclusions

- IMEX method with good stability properties
- A parallel splitting algorithm

Conclusions and Future Work

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- IMEX method with good stability properties
- A parallel splitting algorithm

Future Work

- Higher order splitting
- Optical Coherence Tomography (OCT) images

References

T.F. Chan and L.A. Vese
Active Contour Without Edges.
IEEE Trans. Image Processing 10, 266-277, 2001.

L.C. Evans and J. Spruck
Motion of level sets by mean curvature I.
J. Differential Geometry 33, 635-681, 1991.

V. Caselles, F. Catté, T.Coll and F. Dibos
A geometric model for active contours in image processing.
Numerische Mathematik 66, 1-31, 1993.

T. Lu, P. Neittaanmaki and X.C. Tai
A parallel splitting-up method for PDEs and its applications to Navier-Stokes equations.
RAIRO Math. Model Numer. Anal. 26(6), 673-708, 1992.

N.J. Walkington
Algorithms for computing motion by mean curvature.
SIAM J. Numer. Anal. 33(6), 2215-2238, 1996.