Numerical Approximation of Mean Curvature Flow

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Outline

1 Introduction
   • Parkinson’s Disease
   • Level Sets Method

2 Existence and unicity
   • Viscosity solutions
   • Energy estimate

3 Numerical Analysis
   • Numerical IMEX Method
   • Parallel Splitting Algorithm

4 Segmentation Model
   • Chan and Vese Model
   • Numerical Results

5 Conclusions and Future Work
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1. Introduction
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2. Existence and unicity
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   - Numerical Results

5. Conclusions and Future Work
Parkinson disease

What is it?

- **Degenerative disorder** of the central nervous system that affects the control of muscles and so may affect movement, speech and posture
- Caused by insufficient formation and action of **dopamine**
Parkinson disease

What is it?
- Degenerative disorder of the central nervous system that affects the control of muscles and so may affect movement, speech and posture
- Caused by insufficient formation and action of dopamine

Diagnoses
- There are currently no blood or laboratory tests that have been proven to help in diagnosing the disease
- 75% of clinical diagnoses of Parkinson disease are confirmed at autopsy
The symptoms of Parkinson disease result from the loss of dopaminergic cells and subsequent loss of melanin. The neurons project to the striatum and their loss leads to alteration in the activity of neural circuits within the basal ganglia.
Parkinson disease

- **Decreased dopamine activity** in the basal ganglia, a pattern which aids in diagnosing Parkinson disease
- PET and SPECT images may help

Credits: European Parkinson’s Disease Association
Level Sets Method

Γ(𝑡) is implicitly represented by the zero level set of a higher dimension function 𝜙:

\[ Γ(𝑡) = \{(x, y) ∈ Ω : 𝜙(𝑡, x, y) = 0\} \]

- Notions of interior and exterior of a curve are immediate
- Union and division of curves are automatic
Level Sets Method

- Evolve the curve in the direction of the normal with speed \( v \) solving a PDE

\[
\frac{\partial \phi}{\partial t} = v |\nabla \phi| , \quad \phi(0, x, y) = \phi_0(x, y)
\]

with suitable boundary conditions
Level Sets Method

- Evolve the curve in the direction of the normal with speed $v \iff$ solving a PDE
  \[
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  \]
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- Motion by mean curvature
  \[
  v = \text{div} \left( \frac{\nabla \phi}{|\nabla \phi|} \right)
  \]
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- For $\phi \in C^{2,1}([0, T] \times \Omega)$ and $\nabla \phi \neq 0$ in a neighborhood of $\Gamma(t)$:
  \[
  \begin{cases}
    \phi_t = |\nabla \phi| \text{div} \left( \frac{\nabla \phi}{|\nabla \phi|} \right) & (x, y) \in \Omega, t > 0 \\
    \phi(0, x, y) = \phi_0(x, y) & (x, y) \in \bar{\Omega} \\
    \phi(t, x, y) = 0 & (x, y) \in \partial \Omega , t > 0
  \end{cases}
  \]
  (IBVP)

  with $\Omega$ a bounded open set of $\mathbb{R}^2$ and $\phi_0 \in C(\Omega)$. 
Let $\phi \in C([0, T] \times \Omega) \cap L^\infty([0, T] \times \Omega)$.

- $\phi$ is a viscosity sub-solution (super-solution) of (IBVP) if for all $\nu \in C^2([0, T], \Omega)$, $\phi - \nu$ has a local maximum (minimum) in $(t_0, x_0, y_0)$ then $(\nabla \phi(t_0, x_0, y_0) \neq 0)$

$$\nu_t(t_0, x_0, y_0) \leq (\geq) |\nabla \phi(t_0, x_0, y_0)| \text{div} \left( \frac{\nabla \phi(t_0, x_0, y_0)}{|\nabla \phi(t_0, x_0, y_0)|} \right)$$

- $\phi$ is a viscosity solution of (IBVP) if it simultaneity a viscosity sub and super-solution.
Results

- Under certain conditions, the viscosity solution of (IBVP) exists and its unique (Evans & Spruck, 1991)

- The curves $\Gamma(t)$ are independent of the initial choice $\phi_0$ (Evans & Spruck, 1991)

- The following stability result holds (Caselles et al., 1993)

$$
\sup_{0 \leq s \leq t} \| \phi(s) - \hat{\phi}(s) \|_{L^\infty} \leq \| \phi_0 - \hat{\phi}_0 \|_{L^\infty} \quad \forall t \in [0, T]
$$
Motion by Mean Curvature

Determine $\phi$ from the initial boundary value problem (IBVP):

\[
\begin{aligned}
\frac{\phi_t}{|\nabla \phi|} &= \nabla^T \left( \frac{\nabla \phi}{|\nabla \phi|} \right) & (x, y) \in \Omega, \ t > 0 \\
\phi(0, x, y) &= \phi_0(x, y) & (x, y) \in \bar{\Omega} \\
\phi(t, x, y) &= 0 & (x, y) \in \partial \Omega, \ t > 0
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\phi(t, x, y) = 0 & (x, y) \in \partial \Omega, \ t > 0
\end{cases}$$

Theorem

$$\left\| \phi(t) \right\|_{L^1} + \left\| \nabla \phi(t) \right\|_{L^1} \leq C \left( \left\| \phi_0 \right\|_{L^1} + \left\| \nabla \phi_0 \right\|_{L^1} \right) \left\| \phi(t) \right\|_{W^{1,1}} \left\| \phi_0 \right\|_{W^{1,1}}$$

Proof:

1. $\left\| \nabla \phi(t) \right\|_{L^1} + \int_0^t \int_{\Omega} \frac{\phi^2}{|\nabla \phi|} \, dx \, dy \, ds = \left\| \nabla \phi_0 \right\|_{L^1}$ (Walkington, 1996)
2. $\left\| \nabla \phi(t) \right\|_{L^1} \leq \left\| \nabla \phi_0 \right\|_{L^1}$
3. Poincaré inequality in $L^1$: $\left\| \phi \right\|_{L^1} \leq C^* \left\| \nabla \phi \right\|_{L^1}$
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IMEX Method: time discretization

\[
\frac{1}{|\nabla \phi^n|} \frac{\phi^{n+1} - \phi^n}{\Delta t} = \text{div} \left( \frac{\nabla \phi^{n+1}}{|\nabla \phi^n|} \right)
\]

- \( t^n = n\Delta t, \ n = 0, \ldots, N \), with \( t_0 = 0 \) and \( t_N = T \)
- \( \phi^n \approx \phi(n\Delta t, x, y), \ \forall (x, y) \in \Omega \)
IMEX Method: time discretization

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- \(t^n = n\Delta t, \ n = 0, \ldots, N\), with \(t_0 = 0\) and \(t_N = T\)
- \(\phi^n \approx \phi(n\Delta t, x, y), \ \forall (x, y) \in \Omega\)

Theorem

\[
\|\phi^{n+1}\|_{W^{1,1}} \leq C \|\phi^n\|_{W^{1,1}} \quad \forall n = 0, 1, \ldots, N - 1
\]

Proof:

- Multiply the equation by \(\phi^{n+1} - \phi^n\) with respect to the \(L^2\) inner product and integrate by parts
- \(\|\nabla \phi^{n+1}\|_{L^1} \leq \|\nabla \phi^n\|_{L^1} \quad \forall n = 0, 1, \ldots, N - 1\)
- Poincaré inequality in \(L^1\)
IMEX Method: full discretization

\[ \frac{1}{|\nabla_h \phi_{ij}^n|} \left( \phi_{ij}^{n+1} - \phi_{ij}^n \right) \Delta t = D_x^+ \left( \frac{D_x \phi_{ij}^{n+1}}{|\nabla_h \phi_{ij}^n|} \right) + D_y^+ \left( \frac{D_y \phi_{ij}^{n+1}}{|\nabla_h \phi_{ij}^n|} \right) \]

- \( \tilde{\Omega}_h = \text{grid in } \tilde{\Omega} \text{ with space step } h \)
- Finite differences: \( D_x^+ \), \( D_y^+ \) (forward); \( D_x^- \), \( D_y^- \) (backward)
- \( \phi_{ij}^n \approx \phi(n\Delta t, x_i, y_j), \forall (x_i, y_j) \in \tilde{\Omega}_h \)
- \( |\nabla_h \phi_{ij}^n| = \sqrt{(D_x^- \phi_{ij}^n)^2 + (D_y^- \phi_{ij}^n)^2} \)
- Norm in the discrete \( W^{1,1} \) space

\[ \| \phi \|_{1,1} = \sum_{i,j} h^2 |\phi_{ij}| + \sum_{i,j} h^2 |\nabla_h \phi_{ij}| \]
IMEX Method: full discretization

**Theorem**

\[ \| \phi^{n+1} \|_{1,1} \leq C \| \phi^n \|_{1,1} \quad \forall n = 0, \ldots, N - 1 \]

**Proof:**

- Multiply

\[
\frac{1}{|\nabla h \phi_{ij}^n|} \frac{\phi_{ij}^{n+1} - \phi_{ij}^n}{\Delta t} = D_x^+ \left( \frac{D_x^- \phi_{ij}^{n+1}}{|\nabla h \phi_{ij}^n|} \right) + D_y^+ \left( \frac{D_y^- \phi_{ij}^{n+1}}{|\nabla h \phi_{ij}^n|} \right)
\]

by \( \phi_{ij}^{n+1} - \phi_{ij}^n \) with respect to the discrete \( L^2 \) inner product and use summation by parts

\[ \sum_{ij} h^2 |\nabla h \phi_{ij}^{n+1}| \leq \sum_{ij} h^2 |\nabla h \phi_{ij}^n| \quad \forall n = 0, \ldots, N - 1 \]

- Discrete Poincaré inequality in \( \ell^1 \):

\[ \sum_{ij} h^2 |\phi_{ij}^n| \leq C^* \sum_{ij} h^2 |\nabla h \phi_{ij}^n| \]
Parallel Splitting Algorithm

\[
\frac{\partial \phi}{\partial t} = A\phi + f(t) \quad \text{in} \quad \Omega \times [0, T], \quad \phi(0) = \phi_0
\]

- \( A = A_1 + A_2 + \cdots + A_m \) and \( f = f_1 + f_2 + \cdots + f_m \)
- \( A \) is time independent

**Algorithm (Lu, Neittaanmaki and Tai, 1992)**

At each level time \( n = 0, \ldots, N - 1 \) compute:

1. \[
\frac{\phi^{n+\frac{k}{2m}} - \phi^{n}}{m\Delta t} = A_k \phi^{n+\frac{k}{2m}} + f_k \left( (n + \frac{1}{2})\Delta t \right) \quad k = 1, \ldots, m
\]

2. \[
\phi^{n+1} = \frac{1}{m} \sum_{k=1}^{m} \phi^{n+\frac{k}{2m}}
\]
Parallel Splitting Algorithm

Consider \( A_1 \phi^{n+\frac{1}{4}} = D_x^+ \left( \frac{D_x^+ \phi^{n+\frac{1}{4}}}{|\nabla \phi_{ij}^n|} \right) \) and \( A_2 \phi^{n+\frac{1}{2}} = D_x^+ \left( \frac{D_y^+ \phi^{n+\frac{1}{2}}}{|\nabla \phi_{ij}^n|} \right) \)

**Construction of \( A_1 \)**

\[
\frac{\phi^{n+\frac{1}{4}} - \phi^n}{2\Delta t} = A_1 \phi^{n+\frac{1}{4}} \Leftrightarrow \\
\frac{1}{h^2|\nabla h\phi_{ij}^n|} \frac{\phi^{n+\frac{1}{4}} - \phi_{ij}}{2\Delta t} = \frac{\phi^{n+\frac{1}{4}}_{i-1,j}}{h^2|\nabla h\phi_{i,j}^n|} - \frac{2}{h^2 \phi_{ij}^{n+\frac{1}{4}}} \left( \frac{1}{h^2|\nabla h\phi_{i+1,j}^n|} + \frac{1}{h^2|\nabla h\phi_{i,j}^n|} \right) + \frac{\phi^{n+\frac{1}{4}}_{i+1,j}}{h^2|\nabla h\phi_{i+1,j}^n|}
\]

\( A_1 \) is tridiagonal and diagonally dominant with

\[
a_{i,i-1} = \frac{1}{h^2}, \quad a_{i,i} = -\frac{2}{h^2} \left( \frac{|\nabla h\phi_{i,j}^n|}{|\nabla h\phi_{i+1,j}^n|} + 1 \right), \quad a_{i,i+1} = \frac{|\nabla h\phi_{i,j}^n|}{h^2|\nabla h\phi_{i+1,j}^n|}
\]

A similar construction can be made for \( A_2 \).
Parallel Splitting Algorithm

Algorithm

At each level time \( n = 0, \ldots, N - 1 \) compute:

1. Compute
   \[
   |\nabla_h \phi^n_{ij}| = \sqrt{(D_x \phi^n_{ij})^2 + (D_y \phi^n_{ij})^2}
   \]

2. Construct \( A_1 \) and \( A_2 \)

3. Solve
   \[
   (I - 2\Delta tA_1)\phi^{n + \frac{1}{4}} = \phi^n \quad \text{and} \quad (I - 2\Delta tA_2)\phi^{n + \frac{1}{2}} = \phi^n
   \]

4. \[
   \phi^{n+1} = \frac{\phi^{n + \frac{1}{4}} + \phi^{n + \frac{1}{2}}}{2}
   \]
Theorem (Stability)

*The algorithm is unconditionally stable in the $\| \cdot \|_\infty$ norm.*

Proof:

- $I - 2\Delta tA_1$ and $I - 2\Delta tA_2$ are M-matrices
- $\exists c_1, c_2 \geq 0: \| (I - 2\Delta tA_1)^{-1} \|_\infty \leq c_1$ and $\| (I - 2\Delta tA_2)^{-1} \|_\infty \leq c_2$
- $\| \Phi^{n+1} \|_\infty \leq \frac{1}{2} \left( \| \Phi^{n+\frac{1}{4}} \|_\infty + \| \Phi^{n+\frac{1}{2}} \|_\infty \right) \leq \frac{1}{2} (c_1 + c_2) \| \Phi^n \|_\infty$
Parallel Splitting Algorithm

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- \( \exists c_1, c_2 \geq 0: \| (I - 2\Delta tA_1)^{-1} \|_\infty \leq c_1 \) and \( \| (I - 2\Delta tA_2)^{-1} \|_\infty \leq c_2 \)
- \( \| \Phi^{n+1} \|_\infty \leq \frac{1}{2} \left( \| \Phi^{n+\frac{1}{4}} \|_\infty + \| \Phi^{n+\frac{1}{2}} \|_\infty \right) \leq \frac{1}{2} (c_1 + c_2) \| \Phi^n \|_\infty \)

Theorem (Convergence)

If \( (-A_k) \), \( k = 1, \ldots, m \), are irreducible M-matrices, then the algorithm is convergent of first order in \( \Delta t \).
CV model (Chan and Vese, 2001)

- Find the curve that minimizes:

\[
F(c_1, c_2, \phi) = \mu \int_{\Omega} \delta_0(\phi)|\nabla \phi|dxdy + \nu \int_{\Omega} H(\phi)dxdy \\
+ \lambda_1 \int_{\Omega} |u_0 - c_1|^2 H(\phi)dxdy + \lambda_2 \int_{\Omega} |u_0 - c_2|^2 (1 - H(\phi))dxdy
\]

- It reduces to the resolution of a PDE:

\[
\frac{\partial \phi}{\partial t} = \delta_0(\phi) \left( \mu \text{div} \left( \frac{\nabla \phi}{|\nabla \phi|} \right) - \nu - \lambda_1 (u_0 - c_1)^2 + \lambda_2 (u_0 - c_2)^2 \right)
\]
CV model (Chan and Vese, 2001)

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F(c_1, c_2, \phi) = \mu \int_{\Omega} \delta_0(\phi)|\nabla \phi| \, dx \, dy + \nu \int_{\Omega} H(\phi) \, dx \, dy
\]

\[
+ \lambda_1 \int_{\Omega} |u_0 - c_1|^2 H(\phi) \, dx \, dy + \lambda_2 \int_{\Omega} |u_0 - c_2|^2 (1 - H(\phi)) \, dx \, dy
\]

- It reduces to the resolution of a PDE:

\[
\frac{\partial \phi}{\partial t} = \delta_0(\phi) \left( \mu \text{div} \left( \frac{\nabla \phi}{|\nabla \phi|} \right) - \nu - \lambda_1 (u_0 - c_1)^2 + \lambda_2 (u_0 - c_2)^2 \right)
\]

All the previous results could be generalized for

\[
\phi_t = g(\phi) \text{div} \left( \frac{\nabla \phi}{|\nabla \phi|} \right) \quad (x, y) \in \Omega \ , \ t > 0 , \quad g > 0
\]
SPECT Images (given by IBILI)
Numerical Results

Time = 20 sec

Time = 20 sec
Numerical Results

Evolution of the zero level set in the iteration for $\mu = 0.05$
Numerical Results

Results of segmentation algorithm for $\mu = 0.05$ (left) and $\mu = 0.001$ (right)
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Conclusions and Future Work
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Conclusions

- IMEX method with good stability properties
- A parallel splitting algorithm

Future Work

- Higher order splitting
- Optical Coherence Tomography (OCT) images
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References

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