Rolling Pseudo-Riemannian Manifolds

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May 2011
Rolling Maps for Euclidean Manifolds

Notions of Pseudo-Riemannian Manifolds

Rolling Maps for Pseudo-Riemannian Manifolds

Rolling the Hyperbolic \( n \)-sphere over its affine tangent space at a point, both embedded in the generalized Minkowski space \( \mathbb{R}^{n+1}_1 \):

- Geodesics
- The Kinematic Equations
- Parallel Transport
Some Notations

- $\text{SO}_N$ (Special Orthogonal Group) $\rightarrow$ connected component of the orthogonal group $O_N$ containing the identity matrix.

- $SE_N = \text{SO}_N \rtimes \mathbb{R}^N$ (Special Euclidean Group) $\rightarrow$ group of isometries preserving orientations, also called rigid motions (these include rotations, translations and combinations of them).

Action of $SE_N$ on $\mathbb{R}^N$

$(R, s) \in SE_N$:

- $(R, s) \circ p = Rp + s \rightarrow$ action of $(R, s)$ on points $p \in \mathbb{R}^N$

- The action of $SE_N$ on $\mathbb{R}^N$ induces a linear map between tangent spaces, sending $\eta$ to $R\eta$
$M_1$ and $M_0 \rightsquigarrow$ manifolds of the same dimension embedded in $(\mathbb{R}^N, \langle ., . \rangle)$

A rolling map of $M_1$ on $M_0$, without slipping or twisting over a curve $\alpha : [0, \tau] \to M_1 \ (\tau > 0)$ is a smooth map:

$$h : [0, \tau] \to SE_N = SO_N \ltimes \mathbb{R}^N$$
$$t \mapsto h(t) = (R(t), s(t))$$

satisfying, for each $t \in [0, \tau]$, the following (RMC) conditions:

1. Rolling conditions
2. No-slip condition
3. No-twist conditions
Rolling conditions

- $h(t) \circ \alpha(t) =: \alpha_{\text{dev}}(t) \in M_0$

- $T_{h(t) \circ \alpha(t)}(h(t) \circ M_1) = T_{h(t) \circ \alpha(t)} M_0$

- $\alpha$ is called **rolling curve on** $M_1$

- $\alpha_{\text{dev}}$ is called **development of** $\alpha$ on $M_0$

**Example:** $S^2$
No-slip condition

\( h(t) \circ M_1 \) and \( M_0 \) have the same velocity at the contact point, that is:

\[
h(t) \circ \dot{\alpha}(t) = \dot{\alpha}_{\text{dev}}(t)
\]

\[
\dot{R}(t) R^\top(t)(\alpha_{\text{dev}}(t) - s(t)) + \dot{s}(t) = 0
\]

Example: \( S^2 \)
No-twist conditions

Tangential Part

Any tangent vector field $X(t)$ along $\alpha(t)$ is parallel along $\alpha(t)$ iff $h(t) \circ X(t)$ is parallel along $\alpha_{\text{dev}}(t)$.

\[ \iff \]

\[ \dot{R}(t) R^\top (t) T_{\alpha_{\text{dev}}(t)} M_0 \subset (T_{\alpha_{\text{dev}}(t)} M_0)^\perp \]

Normal Part

Any normal vector field $Z(t)$ along $\alpha(t)$ is normal parallel along $\alpha(t)$ iff $h(t) \circ Z(t)$ is normal parallel vector field along $\alpha_{\text{dev}}(t)$.

\[ \iff \]

\[ \dot{R}(t) R^\top (t)(T_{\alpha_{\text{dev}}(t)} M_0)^\perp \subset T_{\alpha_{\text{dev}}(t)} M_0 \]
No-twist conditions

Remarks:

- Tangential Part is always satisfied for manifolds of dimension 1
- Normal Part is always satisfied for manifolds of codimension 1 (such as Euclidean sphere $S^n$, hyperbolic sphere $H^n$)

Example: $S^2$
Our Goal Now

Is to generalize the concept of rolling for pseudo-Riemannian manifolds

\[ \Downarrow \]

the metric fails to be positive definite
Smooth manifold $\overline{M}$ furnished with a metric tensor $\overline{g}$ (a symmetric nondegenerate $(0,2)$ tensor field on $\overline{M}$ of constant index.)

If $(\overline{M}, \overline{g})$ is a pseudo-Riemannian manifold and $v \in T_p\overline{M}$, then:

- $v$ is ** spacelike if $\overline{g}(v, v) > 0$ or $v = 0$
- $v$ is ** timelike if $\overline{g}(v, v) < 0$
- $v$ is ** lightlike if $\overline{g}(v, v) = 0$ and $v \neq 0$
$M_1$ and $M_0 \rightsquigarrow$ manifolds of the same dimension embedded in a pseudo-Riemannian space $(\overline{M}, \overline{g})$

A rolling motion of $M_1$ over $M_0$, without slipping or twisting is described by a smooth mapping:

$$h : [0, \tau] \rightarrow \overline{G}$$

$$t \quad \mapsto \quad h(t)$$

that satisfies the (RMC) conditions, where:

- $\overline{G}$ is now the connected group of orientation preserving isometries of $\overline{M}$
- orthogonality being taken with respect to the pseudo-Riem. metric $\overline{g}$
- $\circ$ being the action of $\overline{G}$ on our manifolds
Rolling the Hyperbolic \( n \)-sphere

\[ N = n + 1 \]

\[ \overline{M} = \mathbb{R}^{n+1}_1 \hookrightarrow \text{denotes } \mathbb{R}^{n+1} \text{ equipped with the pseudo-Riemannian metric:} \]

\[ \overline{g}(x, y) := \langle x, y \rangle_J = \langle x, Jy \rangle = x^\top Jy, \text{ with } J = \text{diag}(I_n, -1) \]

\[ M_1 := H^n \hookrightarrow n \text{-dimensional hyperbolic sphere (connected component)} \]

\[ H^n = \{ p \in \mathbb{R}^{n+1} : \langle p, p \rangle_J = -1 \text{ and } p_{n+1} > 0 \} \]

\[ T_{p_0}H^n = \{ v \in \mathbb{R}^{n+1} : v = \Omega p_0, \quad \Omega \in \mathfrak{so}(n, 1) \} \]

\[ \mathfrak{so}(n, 1) = \{ A \in \mathfrak{gl}(n+1) : A^\top J = -JA \} \]

\[ \text{SO}(n, 1) = \{ X \in \text{GL}(n + 1) : X^\top JX = J \quad \text{and} \quad \det(X) = 1 \} \]
Rolling the Hyperbolic $n$-sphere

\[ M_0 := T^\text{aff}_{p_0} H^n = \{ x \in \mathbb{R}^{n+1} : x = p_0 + \Omega p_0, \quad \Omega \in so(n, 1) \} \]

\((T_{p_0} H^n)^\perp = \mathbb{R}p_0 \) (codimension 1)

Example: \(n = 2\)

\[ M_1 = H^2 \]

\[ M_0 = T^\text{aff}_{p_0} H^2 \]
SE($n + 1$) $\rightsquigarrow$ is the connected group of orientation preserving isometries of the Euclidean space $\mathbb{R}^{n+1}$

**Question**

What is the connected group of orientation preserving isometries of the Minkowski space $\mathbb{R}_{1}^{n+1}$?

**Answer:**

$$\overline{G} = \text{SO}_{o}(n, 1) \ltimes \mathbb{R}^{n+1}$$

where

$$\text{SO}_{o}(n, 1) = \{ R \in \text{GL}(n + 1) : R^\top J R = J, \ R_{n+1,n+1} > 0 \}$$
Elements in $\overline{G}$ are pairs $(R, s)$, $R \in \text{SO}_o(n, 1)$, $s \in \mathbb{R}^{n+1}_1$

**Group Operations**

$(I, 0)$ is the identity and:

- $(R_1, s_1)(R_2, s_2) := (R_1R_2, R_1s_2 + s_1)$
- $(R, s)^{-1} := (R^{-1}, -R^{-1}s)$

**Action of $\overline{G}$ on $\mathbb{R}^{n+1}$**

$(R, s) \circ x = Rx + s$

\[\downarrow\]

this induces a linear map between $T_x\mathbb{R}^{n+1}$ and $T_{Rx+s}\mathbb{R}^{n+1}$, sending every $\eta$ to $R\eta$
The Lie algebra of $\text{SO}_o(n, 1)$ is $\mathfrak{so}(n, 1)$

$\text{SO}_o(n, 1)$ is a Lie subgroup of $\text{SO}(n, 1)$ and

$$R \in \text{SO}_o(n, 1) \text{ and } p \in H^n \Rightarrow Rp \in H^n$$

↓

the "rotational" part of the rolling map maintains $H^n$ invariant

The restriction of $\langle ., . \rangle_J$ to $T_pH^n$ at any point $p \in H^n$ is positive definite

↓

Although $H^n$ is embedded in a pseudo-Riemannian manifold $\mathbb{R}^{n+1}_1$, it is indeed a Riemannian manifold (all tangent vectors are spacelike)
Geodesics on $H^n$, with respect to the Riemannian metric $\langle ., . \rangle_J$, can be written explicitly:

Let $p \in H^n$ and $v \in T_p H^n$ with $\langle v, v \rangle_J = 1$. Then,

$$t \mapsto \gamma(t) = p \cosh t + v \sinh t$$

is the geodesic in $H^n$ satisfying $\gamma(0) = p$, $\dot{\gamma}(0) = v$.

Let $p, q \in H^n$. Then,

$$t \mapsto \gamma(t) = p \left( \cosh t - \frac{\cosh \theta}{\sinh \theta} \sinh t \right) + q \frac{\sinh t}{\sinh \theta},$$

where $\theta$ is defined by $\cosh \theta = -\langle p, q \rangle_J$ is the geodesic in $H^n$ that joins the point $p$ (at $t = 0$) to the point $q$ (at $t = \theta$).
Rolling the Hyperbolic $n$-sphere - Kinematic Equations

\[ M_1 := H^n \quad M_0 := T_{p_0}^{\text{aff}} H^n = p_0 + T_{p_0} H^n \]

rolling curve $\alpha$ s.t.
\[ \alpha(0) = p_0 \]

\[ H^n \cap T_{p_0}^{\text{aff}} H^n = \{ p_0 \} \]

\[ t \mapsto u(t) \in \mathbb{R}^{n+1} \text{ a piecewise smooth function s.t. } \langle u(t), p_0 \rangle_J = 0 \]

Kinematic Equations (KE)

\[ \begin{cases} \dot{s}(t) = u(t) \\ \dot{R}(t) = R(t) (-u(t)p_0^\top + p_0 u^\top(t)) J \end{cases} \]

If $(R, s) \in \overline{G}$ is the solution of (KE) satisfying $s(0) = 0$, $R(0) = I$, then:

- $t \mapsto h(t) = (R^{-1}(t), s(t)) \in \overline{G} \rightsquigarrow$ rolling map of $H^n$ over $T_{p_0}^{\text{aff}} H^n$
- $t \mapsto \alpha(t) = R(t)p_0 \rightsquigarrow$ rolling curve
- $t \mapsto \alpha_{\text{dev}}(t) = p_0 + s(t) \in T_{p_0}^{\text{aff}} H^n \rightsquigarrow$ development curve
### Definition

\[ A : \mathbb{R} \rightarrow \mathfrak{so}(n, 1) \]

\[ t \mapsto A(t) = (-u(t)p_0^\top + p_0u^\top(t)) J \]

**Assumptions:**

\[ u^\top(t)Jp_0 = 0 \quad p_0^\top Jp_0 = -1 \]

\[ A^{2j-1} = (u^\top(t)Ju(t))^{j-1} A(t); \quad A^{2j} = (u^\top(t)Ju(t))^{j-1} A^2(t) \]

If \( u(t) = u \) (constant), then

\[ e^{At} = I + \frac{\cosh \rho t}{\rho^2} A^2 + \frac{\sinh \rho t}{\rho} A, \]

where \( A = (-up_0^\top + p_0u^\top) J \in \mathfrak{so}(n, 1) \) and \( \rho := (u^\top Ju)^{\frac{1}{2}} \).
**Example 1**

\[ u(t) = u \text{ (constant)} \text{ s.t. } u^\top Jp_0 = 0 \text{ and } (u^\top Ju)^{\frac{1}{2}} = 1 \]

**Solution of KE, with initial conditions** \( s(0) = 0, \ R(0) = I \)

\[
\begin{cases}
  s(t) = ut \\
  R(t) = e^{At}
\end{cases}
\]

- \( \alpha(t) = e^{At} p_0 = p_0 \cosh t + u \sinh t \)
  
  (geodesic in \( \mathbb{H}^n \) passing through \( p_0 \) at \( t = 0 \) with initial velocity \( u \))

- \( \alpha_{\text{dev}}(t) = p_0 + ut \)
  
  (geodesic in \( T_{p_0} \mathbb{H}^n \))
Example 2

\[ p_0 = e_{n+1} = [ 0 \ 0 \ \ldots \ 0 \ 1 ]^\top, \text{ then we must have} \]
\[ u = [ u_1 \ u_2 \ \ldots \ u_n \ 0 ]^\top \]

Solution of KE

\[
\begin{align*}
\dot{s}(t) &= u(t) \\
\dot{R}(t) &= R(t) \left( \sum_{i=1}^{n} u_i(t)(E_{i,n+1} + E_{n+1,i}) \right)
\end{align*}
\]

where the matrices \( E_{i,j} \) have all entries equal to zero except the entry \((i,j)\) which is equal to 1.

- These (KE) can be rewritten as a right-invariant control system evolving on \( \text{SO}_o(n, 1) \ltimes \mathbb{R}^{n+1} \).
- For \( n \geq 2 \), this system is controllable.
Idea:

The tangent (resp. normal) parallel transport of a vector $Y_0$, tangent (resp. normal) to a manifold at a point $p_0$, along a curve $\alpha$, s.t. $\alpha(0) = p_0$, can be accomplished by rolling (without slip or twist) along that curve.
Let $h(t) = (R^{-1}(t), s(t)) \in \overline{G}$ be a rolling map for $H^n$, with rolling curve $\alpha$ satisfying $\alpha(0) = p_0$.

### Tangent Parallel Transport

If $\Omega p_0 \in T_{p_0} H^n$, then

$$Y(t) = R(t)\Omega p_0$$

defines the unique tangent parallel vector field along $\alpha$, satisfying $Y(0) = \Omega p_0$.

### Normal Parallel Transport

If $Z_0 \in T_{p_0}^\perp H^n$, then

$$Z(t) = R(t)Z_0$$

defines the unique normal parallel vector field along $\alpha$, satisfying $Z(0) = Z_0$. 

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Final Remarks

- Lorentzian manifolds - their applications to theory of general relativity
- Stiefel Manifolds
- $S^n$, $SO_n$, and Grassmann Manifolds
- Quadratic Lie groups
- Riemannian manifolds
- Controllability
References

Principal:


Others: