An algorithm to compute generalised Feng-Rao numbers of a numerical semigroup

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Seminar of the PHD program
Coimbra, February 11th, 2011

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Numerical semigroups

A numerical semigroup $S$ is a submonoid of $\mathbb{N} = \mathbb{Z}_{\geq 0}$ (under addition) such that $\#(\mathbb{N} \setminus S) < \infty$.

Definitions and notation

Let $S$ be a numerical semigroup.
- The elements of $\mathbb{N} \setminus S$ are said to be the gaps of $S$;
- $g(S) := \#(\mathbb{N} \setminus S)$ is the genus of $S$;
- the unique element $c(S) \in S$ such that $c(S) - 1 \notin S$ and $c(S) + \ell \in S$ for all $\ell \in \mathbb{N}$ is the conductor of $S$;
- the Frobenius number of $S$ is the greatest integer $F(S)$ not belonging to $S$ (i.e., $F(S) = c(S) - 1$);
- the multiplicity of $S$ is the least positive integer belonging to $S$.

Sometimes we write $g$ for $g(S)$ and $c$ for $c(S)$.

Fact

Let $S$ be a numerical semigroup. The following inequalities hold: $g(S) + 1 \leq c(S) \leq 2g(S)$.

Proof. The first inequality is obvious.

Let $m \in \mathbb{N}$ and consider the sets

$A = \{(r, m - r) \mid 0 \leq r \leq m\}$ and $B = \{(h, m - h) \mid h \in \mathbb{N} \setminus S\} \cup \{(m - h, h) \mid h \in \mathbb{N} \setminus S\}$.

Note that $\#A = m + 1$, $\#B = 2g$.

If $m \geq 2g$, then $A \not\subseteq B$ and therefore $A$ contains a pair $(r, m - r)$ such that both components belong to $S$, giving that $m = r + (m - r) \in S$.

It follows that all integers $m$ such that $m \geq 2g$ belong to $S$ and therefore $c \leq 2g$. \qed
We say that a numerical semigroup $S$ is generated by a set of elements $G \subseteq S$ if every element $x \in S$ can be written as a linear combination

$$x = \sum_{g \in G} \lambda_g g,$$

where finitely many $\lambda_g \in \mathbb{N}$ are non-zero.

**Remark**

Let $S$ be a numerical semigroup. $x \in S$ is said to be **irreducible** if $x = u + v$ and $u, v \in S$ implies $u \cdot v = 0$.

Every set of generators of $S$ contains the set of irreducible elements, and this set actually generates $S$, so that it is usually called the **generating set** of $S$.

It is finite and its cardinality is called the **embedding dimension** of $S$.

Let $S$ be a numerical semigroup generated by $n_1 < \cdots < n_e$. By definition, if $s$ belongs to $S$, then there exists a tuple of nonnegative integers $(a_1, \ldots, a_e)$ such that $s = a_1 n_1 + \cdots + a_e n_e$.

We say that $(a_1, \ldots, a_e)$ is a **factorization** of $s$ in $S$.

The **length of a factorization** $(a_1, \ldots, a_e)$ is defined as $a_1 + \ldots + a_e$.

**Remark**

Most of the terminology used in numerical semigroups comes from Algebraic Geometry.

For instance, if $S$ is the Weierstrass semigroup of a curve $\chi$ at a point $P$, $g$ is equal to the geometric genus of $\chi$, and the elements of $G(S) := \mathbb{N} \setminus S$ are called the Weierstrass gaps at $P$. 
Details on numerical semigrops may be found in the book


For computations, one can use the following package of the computer algebra system GAP.


The Frobenius problem

Given positive integers $a_1, \ldots, a_n$, with $\gcd(a_1, \ldots, a_n) = 1$, which is the greatest integer that cannot be written as a non negative linear combination of $a_1, \ldots, a_n$?

The Frobenius problem, although apparently too specialised, appears naturally in several areas of mathematics.

So, to find “formulas” or “efficient algorithms” even for some particular cases may have some interest.

Proposition 1 (Sylvester)

Let $p, q$ be coprime positive integers. Then

$$F(<p, q>) = pq - p - q$$
Sylvester solved the case of numerical semigroups of embedding dimension 2, but no (polynomial) formula is known for semigroups of higher embedding dimension.

For much more on the Frobenius problem, one may consult a book by Ramírez Alfonsín.

We can think of the integers as points disposed regularly on a cylindrical helix.

As some of the integers would be hidden, we will consider planifications of the cylinder instead.

They are usually obtained by cutting the cylinder by a vertical line passing through a point previously chosen, and this corresponds to taking a partition of the integers.

We shall use this type of drawings to depict the most relevant parts of the sets considered. For instance, if we want to highlight the elements of a numerical semigroup, we do not add any information by depicting the points below 0 and those above the conductor.

The following parallelograms highlight the elements of the semigroup $S = \langle 9, 13, 15 \rangle$, and the elements of $60 - S$, respectively.

**Figure:** $S = \langle 9, 13, 15 \rangle$ and $60 - S$, respectively
Most times we are interested in finite sets of integers which are non smaller than a given integer \( m \).
Suppose that the cylinder is cutted by a vertical line passing through \( m + b \) and that the length of an arc of the helix is \( a \).
In this case we prefer to draw all the points from \( m \) to \( m + a + b \) at the same level.

**Example**

Suppose \( m = 189 \), \( a = 19 \) and \( b = 4 \)... The following is a possible figure.

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A convenient visualisation of the integers

**Definition**

Given \( x \in S \), we say that \( \alpha \in S \) divides \( x \) if \( x - \alpha \in S \). We denote by \( D(x) = \{ \alpha \in S \mid x - \alpha \in S \} \) the set of divisors of \( x \).

It is clear that \( D(x) \subseteq [0, x] \).
Furthermore, noting that \( x - s \in S \) and \( s - \alpha \in S \) implies \( x - \alpha \in S \), we have that \( s \in D(x) \) implies \( D(s) \subseteq D(x) \).

**Lemma 2**

\[
D(x) = S \cap (x - S).
\]

**Proof.** Let \( \alpha \in D(x) \). By definition, \( \alpha \in S \) and \( x - \alpha \in S \). But then \( \alpha = x - (x - \alpha) \in x - S \).
Conversely, let \( \alpha \in S \) be such that there exists \( \beta \in S \) for which \( x - \beta = \alpha \). But then \( x - \alpha = \beta \in S \), proving that \( \alpha \) divides \( x \).
We observe that elements greater than $x$ need not to be used to compute the divisors of $x$. Denoting $S_x = \{ n \in S \mid n \leq x \}$, we get the following:

**Corollary 3**

$$D(x) = S_x \cap (x - S_x).$$

The computation of the divisors of an element can be easily implemented due to this consequence of Lemma 2.

**Algorithm 1**: Divisors

**Input**: A numerical semigroup $S$, $x \in S$

**Output**: The divisors of $x$

1. $S_x := \{ s \in S \mid s \leq x \}$ /* Compute the elements of $S$ smaller than $x$ */
2. return $\{ s \in S_x \mid x - s \in S_x \}$

Note that, once we compute the elements of $S$ smaller than $x$ (which can easily be done if the conductor is known), the computation of the divisors is immediate.

The highlighted elements in Figure 2 represent the divisors of $60 \in \langle 9, 13, 15 \rangle$. They are obtained intersecting the highlighted elements of the pictures in Figure 1.

![Figure: The divisors of 60 in the semigroup $S = \langle 9, 13, 15 \rangle$](image)
Notation

\[ D(m_1, \ldots, m_r) = D(m_1) \cup \cdots \cup D(m_r) \]

Example

Let \( S = \langle 19, 20, 21, 22, 23 \rangle \). The elements of

\[ D(235, 199, 247, 229) \cap [189, \infty) \]

are the highlighted elements in the following figure.

---

Let \( S \) be a numerical semigroup with conductor \( c \) and let \( x \geq 2c - 1 \). Observe that \( x - S \) contains all the integers not greater than \( x - c \) and that the number of integers smaller than \( x \) not belonging to \( x - S \) is precisely the genus of \( S \).

As the number of non-negative integers not greater than \( x \) is \( x + 1 \), one gets immediately the well known fact:

**Proposition 4**

If \( x \geq 2c - 1 \), then \( \#D(x) = \#S \cap (x - S) = x + 1 - 2g \).
In the framework of the Theory of Error-Correcting Codes, Feng and Rao introduced a notion of distance for the Weirstrass semigroup at a rational point of an algebraic curve, with decoding purposes. It is a purely combinatorial concept that can be defined for any numerical semigroup. Later on, that notion has been generalised and is used not only in the theory of error correcting codes, but also in cryptography.
Generalized Feng-Rao distances

**Definition 5**

Let $S$ be a numerical semigroup. The (classical) **Feng-Rao distance** of $S$ is defined by the function

$$
\delta_{FR} : S \rightarrow \mathbb{N} \\
m \mapsto \delta_{FR}(m) := \min\{\sharp D(m_1) \mid m_1 \geq m, \ m_1 \in S\}.
$$

There are some well-known facts about the function $\delta_{FR}$ for an arbitrary semigroup $S$. An important one follows from Proposition 4:

$$
\delta_{FR}(m) = m + 1 - 2g \text{ for all } m \in S \text{ with } m \geq 2c - 1.
$$

The classical Feng-Rao distance corresponds to $r = 1$ in the following definition.

**Definition 6**

Let $S$ be a numerical semigroup. For any integer $r \geq 1$, the **$r$-th Feng-Rao distance** of $S$ is defined by the function

$$
\delta^r_{FR} : S \rightarrow \mathbb{N} \\
m \mapsto \delta^r_{FR}(m)
$$

where

$$
\delta^r_{FR}(m) = \min\{\sharp D(m_1, \ldots, m_r) \mid m \leq m_1 < \cdots < m_r, \ m_i \in S\}.
$$
Few results are known for the numbers $\delta_{FR}^r$. The following describes the asymptotical behavior for $m >> 0$, and was proved by Farrán and Munuera [2].

**Proposition 7**

There exists a certain constant $E(S, r)$, depending on $r$ and $S$, such that

$$\delta_{FR}^r(m) = m + 1 - 2g + E(S, r)$$

for $m \geq 2c - 1$.

**Definition 8**

The constant $E(S, r)$ is called the $r$th-Feng-Rao number of the semigroup $S$.

Note that, when restricted to $S \cap [2c - 1, \infty)$, $\delta_{FR}^r$ is strictly increasing.

**Definition 9**

Let $S$ be a numerical semigroup and let $m \in S$. A finite subset of $S \cap [m, \infty)$ is called a $(S, m)$-configuration, or, simply, a configuration.

A configuration $M$ of cardinality $r$ is said to be optimal if

$$\delta_{FR}^r(m) = \# D(M)$$

Recall that we already have a quite efficient algorithm to compute the divisors of an element of a numerical semigroup.

Now we will work towards finding optimal configurations.
Amenable sets

Definition 10

Let $S$ be a numerical semigroup with conductor $c$. Let $m, m_1, \ldots, m_r$ be integers such that $2c - 1 \leq m = m_1 < \cdots < m_r$. We say that the set $M = \{m_1, \ldots, m_r\} \subseteq S$ is $(S, m, r)$-amenable if:

$$\text{for all } i \in \{1, \ldots, r\}, D(m_i) \cap [m, \infty) \subseteq M.$$ (1)

We will refer to a set satisfying (1) as being $m$-closed under division. So, a subset of $S \cap [m, \infty)$ with cardinality $r$ is $(S, m, r)$-amenable if and only if it contains $m$ and is $m$-closed under division.

As a convention, the empty set is considered an $(S, m, 0)$-amenable set, for any $m$. When no confusion arises or only the concept is important, we say $(m, r)$-amenable set or, simply, amenable set.

Example 11

Let $S = \langle 19, 20, 21, 22, 23 \rangle$. Its conductor is $c = 95$. Take $m = 2c - 1 = 189$. The set $M$ consisting of the highlighted elements of the following figure is an amenable subset of $S$.

Example 12

Let $S$ be a numerical semigroup with conductor $c$. Let $m \geq 2c - 1$, and $r$ a non negative integer. Then the interval $[m, m + r - 1] \cap \mathbb{N}$ is a $(S, m, r)$-amenable set.
The importance of amenable sets comes from the following result, which states that among the optimal configurations of cardinality \( r \) there is one \((S, m, r)\)-amenable set.

**Proposition 13**

Let \( S \) be a numerical semigroup with conductor \( c \) and let \( m \geq 2c - 1 \). Let \( r \) be a positive integer. Among the optimal configurations of cardinality \( r \) there is one \((S, m, r)\)-amenable set.

**Proof.** Let \( M = \{m_1, \ldots, m_r\} \) be an optimal configuration. As \( \delta_{FR} \) is strictly increasing when restricted to \( S \cap [2c - 1, \infty) \), \( m \) cannot be less than \( m_1 \), which implies that \( m_1 = m \).

If \( M \) is not \( m \)-closed under division, we may assume that for some \( i \in \{1, \ldots, r\} \) there exists \( t \in S \) such that \( m_i - t > m \) (which implies \( m_i - t \in S \)) and \( m_i - t \notin \{m_1, \ldots, m_r\} \). As \( m_i - t \) divides \( m_i \), we have \( D(m_i - t) \subseteq D(m_i) \), and thus

\[
D(m_1, \ldots, m_{i-1}, m_i - t, m_{i+1}, \ldots, m_r) \subseteq D(m_1, \ldots, m_r).
\]

In other words, we can substitute \( m_i \) by \( m_i - t \) and the number of divisors does not increase.

Now we can repeat the process with the set obtained until we reach a \( m \)-closed under division set. Note that this must happen in a finite number of steps (\( \mathbb{N}^r \) has no infinite descending chains). \( \square \)
The definition of amenable set, which seems to be suitable for proofs, does not seem to help very much to do computations unless we can prove some consequences. The following one, proving that the distances between elements is somehow controlled, guarantees that the search of the amenable sets can be done in a bounded subset of $S$, and therefore all $r$-amenable sets can be effectively computed. An algorithm will be presented (Algorithm 5).

**Proposition 14**

Let $S$ be a numerical semigroup with conductor $c$ and let $m \geq 2c - 1$. Let $M = \{m_1 < \cdots < m_r\} \subseteq S$ be an $(S, m, r)$-amenable set and suppose that $S = \{0 = \rho_1 < \rho_2 < \cdots\}$. Then

(a) $m_i \leq m + \rho_i$, for all $i \in \{1, \ldots, r\}$,

(b) $m_{i+1} - m_i \leq \rho_2$, for all $i \in \{1, \ldots, r - 1\}$.

**Proof.** (a) Suppose that there exists $i_0 \in \{1, \ldots, r\}$ such that $m_{i_0} - \rho_{i_0} > m$. Let $D = \{m_{i_0} - \rho_j \mid j \in \{1, \ldots, i_0\}\}$. All the elements of $D$ are bigger than $m$, that is, $D \subseteq (m, \infty)$. On the other hand, using Lemma 2, $D \subseteq D(m_{i_0})$. Thus $D \subseteq D(m_{i_0}) \cap (m, \infty) \subseteq \{m_1, \ldots, m_{i_0}\}$. The containment is strict since $m_1 = m$. But this is absurd, since the two ends of the chain have the same cardinality.

(b) Note that $m_{i+1} - \rho_2$ is a divisor of $m_{i+1}$. This implies that, if $m_{i+1} - \rho_2 \geq m$, then $m_{i+1} - \rho_2 \in M$. As $m_{i+1} - \rho_2 < m_{i+1}$ and there is no element in $M$ strictly between $m_i$ and $m_{i+1}$, $m_{i+1} - \rho_2$ must be non greater than $m_i$. \qed

The following result is important for efficiency reasons.
Proposition 15

A subset $M = \{m = m_1, \ldots, m_r\}$ of a numerical semigroup $S$ is $(S, m, r)$-amenable if and only if

$$\text{for all } i \in \{1, \ldots, r\} \text{ and } g \text{ minimal generator of } S,$$

$$\text{if } m_i - g \geq m, \text{ then } m_i - g \in \{m_1, \ldots, m_r\}. \quad (2)$$

**Proof.** Let $m_i \in M$ and $u \in D(m_i) \cap [m, \infty)$, with $u \neq m_i$. We shall prove that if (2) holds, then $u \in M$, thus concluding that $M$ is $(S, m, r)$-amenable. We can write $u = m_i - \gamma$. Assume as an hypothesis that $m_i - \alpha \in D(m_i) \cap [m, \infty)$ implies $m_i - \alpha \in M$, for all $\alpha$ with factorization length lesser than the factorization length of $\gamma$.

Let $g$ be a minimal generator that divides $\gamma$. As $\gamma - g$ has smaller factorization length than $\gamma$, we have, by hypothesis, that $m_i - \gamma + g \in M$. But then, by (2), $m_i - \gamma = (m_i - \gamma + g) - g \in M$. $\square$

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Algorithm 2: $(S, m, r)$-amenable sets

**Input**: A numerical semigroup $S$, $m \geq 2c - 1$ and $r$ an integer  
**Output**: The set of $(S, m, r)$-amenable sets

$SM := [[m]]/*$ the set of amenable sets */

Compute the generators $gens = \{n_1 < \cdots < n_e\}$ and the elements 

$\{0 = \rho_1 < \rho_2 < \cdots\}$ of $S$;

1. for $i$ in $[2..r]$ do
   1.1. newM := [ ];
   1.2. for $x$ in $SM$ do
      1.2.1. $min := \text{Minimum}(x[\text{Length}(x)] + \rho_2, m + \rho_i$);
      1.2.2. for $m_j$ in $[x[\text{Length}(x)] + 1..min]$ do
         1.2.2.1. $divs := \{d \in m_j - gens \mid d > m\}$;
         1.2.2.2. if $divs \subseteq x$ then
            1.2.2.2.1. Append(newM, [Union(x, [mj])])
      1.2.3. $SM := newM$;
   1.3. return $SM$;
We got an algorithm to compute generalized Feng Rao numbers: compute the amenable sets and then the divisors of each of these sets. Not all the amenable sets are needed as will be shown.

We continue considering $S$ a numerical semigroup minimally generated by $\{n_1 < \cdots < n_e\}$ with conductor $c$. Let $m \geq 2c - 1$. The set $\{m, \ldots, m + n_e - 1\}$ is called the $(S, m)$-ground, or, simply, ground.

The intersection of an $(S, m, r)$-amenable set $M$ with the $(S, m)$-ground is called the shadow of $M$.

Note that the shadow of an amenable set is amenable.
Lemma 16

Let $S$ be a numerical semigroup minimally generated by $\{n_1 < \cdots < n_e\}$ with conductor $c$. Let $m \geq 2c - 1$ and let $M = \{m = m_1 < \cdots < m_r\}$ be an amenable set. Let $L = M \cap [m, m + n_e)$ be the shadow of $M$. Then

$$D(M) = (M \setminus L) \cup D(L),$$

and furthermore

$$\#D(M) = \#(M \setminus L) + \#D(L).$$

Proof. The inclusion $(M \setminus L) \cup D(L) \subseteq D(M)$ is clear. For the other inclusion, let $x \in D(M) \setminus (M \setminus L) = (D(M) \setminus M) \cup L$. We want to prove that $x \in D(L)$.

Since $L \subseteq D(L)$, we may assume that $x \in D(M) \setminus M$. Then $x \in D(m_i)$ for some $i \in \{1, \ldots, r\}$ and $m_i \geq m + n_e$. As $m_i - x \in S \setminus \{0\}$, there exists $j \in \{1, \ldots, e\}$ such that $m_i - x - n_j \in S$. Hence $x \in D(m_i - n_j)$.

By hypothesis $M$ is amenable and thus $m_i - n_j \in M$, since $m_i - n_j \in D(m_i) \cap [m, \infty)$. If needed, we can repeat the process until $m_i - n_j \in L$, that is, $x \in D(L)$.

The second assertion follows easily since the above union is disjoint.
As an easy but useful consequence, we get the following:

**Corollary 17**

Let $M$ and $N$ be $(m, r)$-amenable sets with shadows $L_M$ and $L_N$ respectively. $L_M \subseteq L_N \implies \#D(M) \leq \#D(N)$.

**Proof.** Suppose that $L_N$ is the disjoint union of $L_M$ and a set $K$ of cardinality $k$. Observe that $\#(M \setminus L_M) = \#(N \setminus L_N) + k$.

As $D(L_N) = D(L_M) \cup D(K) \supseteq D(L_M) \cup K$, it follows that $\#D(L_N) \geq \#(L_M) + k$, that is, $\#D(L_M) \leq \#(L_N) - k$.

$\#D(M) = \#(M \setminus L_M) + \#D(L_M) \leq \#(N \setminus L_N) + k + \#D(L_N) - k = \#D(N)$. ⊠

**Corollary 18**

Let $S$ be a numerical semigroup minimally generated by $\{n_1 < \cdots < n_e\}$ with conductor $c$. Let $m \geq 2c - 1$ and let $M \subseteq [m, \infty)$ be an amenable set which is an optimal configuration of cardinality $r$. Let $L = M \cap [m, m + n_e)$ be the shadow of $M$. Then $\delta_{FR}^r(m) = \#D(L) + \#(M \setminus L)$.

**Corollary 19**

In particular, if there exists an optimal configuration $M$ of cardinality $r$ such that $[m, m + n_e) \subseteq M$, then $[m, m + r - 1 + k] \cap \mathbb{N}$ is also an optimal configuration of cardinality $r + k$. 
## Proposition 20

Let $S$ be a numerical semigroup minimally generated by \( \{ n_1 < \cdots < n_e \} \) with conductor $c$. Let $m \geq 2c - 1$ and let $M \subset [m, \infty)$ be a finite amenable set. Choose $s \geq n_e$. Let $L = M \cap [m, m+s)$, $\ell = \#L$, and $k = \#(M \setminus L)$. If $\delta_{FR}^\ell(m) = \#D(L)$, then $\delta_{FR}^{\ell+k}(m) = \delta_{FR}^\ell(m) + k$.

### Proof.

By Lemma 16, we know that $\#D(M) = \#(M \setminus L) + \#D(L)$. Hence $\delta_{FR}^\ell(m) + k = \#D(L) + \#(M \setminus L) = \#D(M) \geq \delta_{FR}^{\ell+k}(m)$. Now assume that $\delta_{FR}^{\ell+k}(m) = \#D(m_1 < \cdots < m_{\ell+k})$ for some $m_1, \ldots, m_{\ell+k} \in S$. Since $\#D(m_1 < \cdots < m_{\ell+k}) \geq \#D(m_1, \ldots, m_{\ell}) + \#\{m_{\ell+1}, \ldots, m_{\ell+k}\} \geq \delta_{FR}^\ell(m) + k$, we conclude that $\delta_{FR}^{\ell+k}(m) = \delta_{FR}^\ell(m) + k$.

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In the cases where computing divisors is “easy”, finding optimal configurations is as difficult as computing generalized Feng Rao numbers. Computing generalized Feng Rao numbers is referred to as difficult in the literature...

We got an algorithm to compute generalized Feng Rao numbers. Note that its efficiency depends on the number of amenable sets. Due to Corollary 17, our algorithm can be sharpened, since we only need to consider one amenable set for each possible shadow.
**Algorithm 3: Generalised Feng Rao numbers**

**Input**: A numerical semigroup $S$, $m \in S$, $r \in \mathbb{N}$

**Output**: $\delta_{FR}(m)$

$SM := \emptyset$;  
$AM := \{ M \subset S \mid M$ is a $(S, m, r)$-amenable set$\}$

/* Compute the $(m, r)$-amenable sets, by making a call to Algorithm 5 */

2 For each possible shadow $s$, add to $SM$ an element of $AM$ with shadow $s$, if it exists;  
$\nu := m + r$; /* an obvious upper bound */

3 for $M$ in $SM$ do  
   $D := \bigcup \{ \text{Divisors}(x) \mid x \in M \}$ /* Compute the divisors of $M$, by using Algorithm 1 */  
   $\nu := \text{minimum}(\#D, \nu)$

4 return $\nu$
This algorithm (even preliminary versions of it) has been extensively used to perform computations which gave the intuition that ultimately led to discovering a formula for the generalised Feng Rao numbers of numerical semigroups generated by intervals.

Finally, if we enumerate the elements of $S$ in increasing order

$$S = \{ \rho_1 = 0 < \rho_2 < \cdots \},$$

we note that every $x \geq c$ is the $(x + 1 - g)$-th element of $S$, that is

$x = \rho_{x+1-g}.$

The last part of this paper will be devoted to semigroups generated by intervals.

Let $a$ be a positive integer and $b$ an integer with $0 < b < a$. Let $S = \langle a, a + 1, \ldots, a + b \rangle$. Then $S$ is a numerical semigroup with multiplicity $a$ and embedding dimension $b + 1$. As usual, let $c$ denote the conductor of $S$ and $m \geq 2c - 1$. Membership problem for these semigroups is trivial as the following known result (and with many different formulations) shows.
Lemma 21

[?; Lemma 10, for \(d = 1\)] Let \(k\) and \(r\) be integers such that \(0 \leq r \leq a - 1\). Then \(ka + r \in S\) if and only if \(r \leq kb\).

Suppose \(S\) is minimally generated by \(\{n_1 < \cdots < n_e\}\). Note that if \(S\) is minimally generated by an interval \([a, a + b]\), then \(n_1\) is \(a\) and \(n_e - n_1\) is \(b\).

Figure: the biggest ordered amenable
Figure: A set coming from line ??
Figure: A set coming from line ??

Figure: A set coming from line ??
References

The GAP Group.
http://www.gap-system.org

M. Delgado, P. A. García-Sánchez and J. Morais,
“numericalsgps”: a GAP package on numerical semigroups.
http://www.gap-system.org/Packages/numericalsgps.html

J.C. Rosales and P. A. García-Sánchez,
“Numerical Semigroups”,
Springer. 2009.

J. I. Farrán, P. A. García-Sánchez and D. Llena,
On the Feng-Rao numbers,
Seventh Conference on Discrete Mathematics and Computer Science,
Castro Urdiales, 2010

J. I. Farrán and C. Munuera,
Goppa-like bounds for the generalized Feng-Rao distances,

M. Delgado, J. I. Farrán, P. A. García-Sánchez and D. Llena,
On the generalized Feng-Rao numbers of numerical semigroups
generated by intervals,
Work in progress