Splitting methods in the design of coupled flow and mechanics simulators

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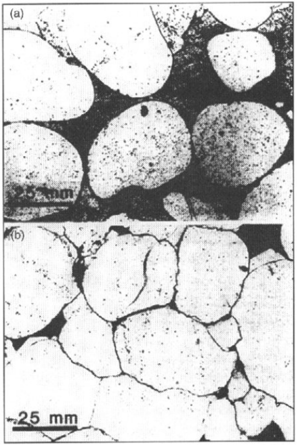
Introduction



Modeling porous media systems

aims to describe the changes in a permeable media (made of somewhat rigid material containing open spaces) due to changes of the fluid it held.

Introduction

The changes in pressure caused by motion, removal or addition of liquid or gas may cause deformations in the structure holding the fluid.

In the past, modelers tended to ignore the geomechanics in their calculations.

Side effects of drilling:

consolidation - reduction in volume due to fluid extraction
compaction - reduction in volume due to air removal
subsidence - collapse

Motivation

Subsidence of Ekofisk Oil Field

was a side effect of drilling.

Motivation

"They subsided so much they had to go in and raise the platforms, costing them several billion dollars. If they'd known ahead of time, they could have built their platforms taller"

Rick Dean, in "Modeling complex, multiphase porous media systems", Siam News, April 2002.

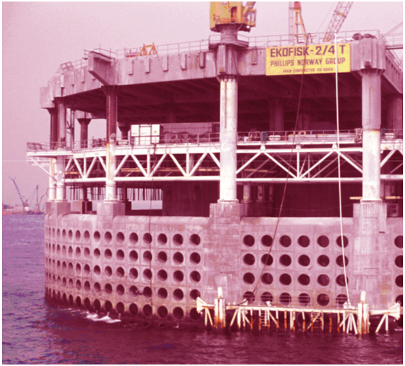


Photo: Norwegian Petroleum Museum/ConocoPhillips

Motivation

Subsidence of Venice

cased by the extraction of the aquifer.

Modeling

Poroelasticity refers to fluid flow within a deformable porous medium under the assumption of relatively small deformations.

Examples of poroelastic structures

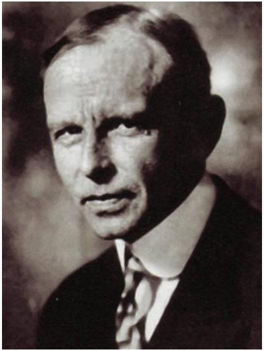
soil, rock, cartilage, brain, heart, bone

Various environmental, energy industry and biomechanics applications

- Subsidence, reservoir compaction
- Well stability, sand production
- Waste disposal
- Sequestration of carbon in saline aquifers
- Estimate tumor-induced stress levels in the brain
- Development of prosthetic devices for cartilage, bone, heart valves

Biot's consolidation model

Karl von Terzaghi (October 2, 1883 - October 25, 1963) Austrian civil engineer and geologist. Frequently called the father of soil mechanics.
Maurice Anthony Biot (May 25, 1905 - September 12, 1985) Belgian-American physicist Founder of the theory of poroelasticity.



Karl von Terzaghi

Maurice Anthony Biot

Biot's consolidation model

primary variables: displacement $\mathbf{u} = (u_1, u_2, u_3)$ and fluid pressure p

Balance of linear momentum

$$-\nabla \cdot (\sigma(\mathbf{u}) - \alpha p I) = \mathbf{f}$$

$\sigma(\mathbf{u})$ - effective stress tensor, linear elastic, $\mathbf{f}$ - body force, α - Biot-Willis constant

$$\sigma(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda \mathrm{tr}(\epsilon(\mathbf{u}))I,$$

where

$$\epsilon(\mathbf{u}) = \frac{1}{2}\Big(\mathrm{grad}\,\mathbf{u} + (\mathrm{grad}\,\mathbf{u})^t\Big),$$

λ, μ - Lamé constants

The momentum conservation equation is very similar to the equation governing linear elasticity, the exception is the addition of the term involving pressure.

Biot's consolidation model

Mass conservation

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot \mathbf{v_f} + s_f$$

η - fluid content , $\mathbf{v_f}$ - flux of fluid, s_f - volumetric fluid source term

Equation for fluid content η in terms of fluid pressure p and material volume $\nabla \cdot \mathbf{u}$

$$\eta = c_0 p + \alpha \nabla \cdot \mathbf{u}$$

c_0 - constrained specific storage coefficient

Darcy's law for the flux of fluid

$$\mathbf{v_f} = -\frac{1}{\mu_f} K(\nabla p - \rho_f \mathbf{g})$$

K - permeability tensor, μ_f - fluid viscosity, $\mathbf{g}$ - gravity

Summary of equations

Coupling equations

- In the domain, at any time t

$$-\nabla \cdot (\sigma(\mathbf{u}) - \alpha p I) = \mathbf{f}$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha \nabla \cdot \mathbf{u}) - \frac{1}{\mu_f} \nabla \cdot K(\nabla p - \rho_f \mathbf{g}) = s_f$$

- Initial conditions at time $t = 0$

$$p(0) = p_0, \quad \mathbf{u}(0) = \mathbf{u}_0$$

- Plus adequate boundary conditions

For the mixed formulation for the flow, we introduce the variable

$$\mathbf{z} = -\frac{1}{\mu_f} K(\nabla p - \rho_f \mathbf{g}).$$

Variational problem

Integrating by parts over Ω, we obtain the variational problem

Find $\mathbf{u}$, p and $\mathbf{z}$ such that

$$a_{\mathbf{u}}(\mathbf{u}, \mathbf{v}) - \alpha(p, \nabla \cdot \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{r}_N \cdot \mathbf{v},$$

$$\left(c_0 \frac{\partial p}{\partial t}, w \right) + \alpha \left(\frac{\partial}{\partial t} \nabla \cdot \mathbf{u}, w \right) + (\nabla \cdot \mathbf{z}, w) = \int_\Omega s_f w,$$

$$(\mu_f K^{-1} \mathbf{z}, \mathbf{s}) - (p, \nabla \cdot \mathbf{s}) = \int_\Omega \rho_f \mathbf{g} \cdot \mathbf{s} - \int_{\Gamma_p} p_D \mathbf{s} \cdot \eta$$

holds for all $(\mathbf{v}, w, \mathbf{s})$ and $t \in [0, T]$, where

$$a_{\mathbf{u}}(\mathbf{u}, \mathbf{v}) = \int_\Omega (2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v})) + \lambda(\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v}))\, d\mathbf{x}.$$

Time discretization

Using the backward Euler method.

Let n denote the time step, and Δt the time increment

$$a_{\mathbf{u}}(\mathbf{u}^{n+1}, \mathbf{v}) - \alpha(p^{n+1}, \nabla \cdot \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{r}_N \cdot \mathbf{v},$$

$$\Big(c_0(p^{n+1} - p^n), w\Big) + \alpha\Big(\nabla \cdot (\mathbf{u}^{n+1} - \mathbf{u}^n), w\Big) + \Delta t(\nabla \cdot \mathbf{z}^{n+1}, w) = \Delta t \int_\Omega s_f w,$$

$$(\mu_f K^{-1} \mathbf{z}^{n+1}, \mathbf{s}) - (p^{n+1}, \nabla \cdot \mathbf{s}) = \int_\Omega \rho_f \mathbf{g} \cdot \mathbf{s} - \int_{\Gamma_p} p_D \mathbf{s} \cdot \eta$$

holds for all $(\mathbf{v}, w, \mathbf{s})$.

Time discretization

Operator splitting

$$\alpha \nabla \cdot \mathbf{u} = c_r p + \frac{c_r}{\alpha} \tilde{\sigma}$$

FLOW

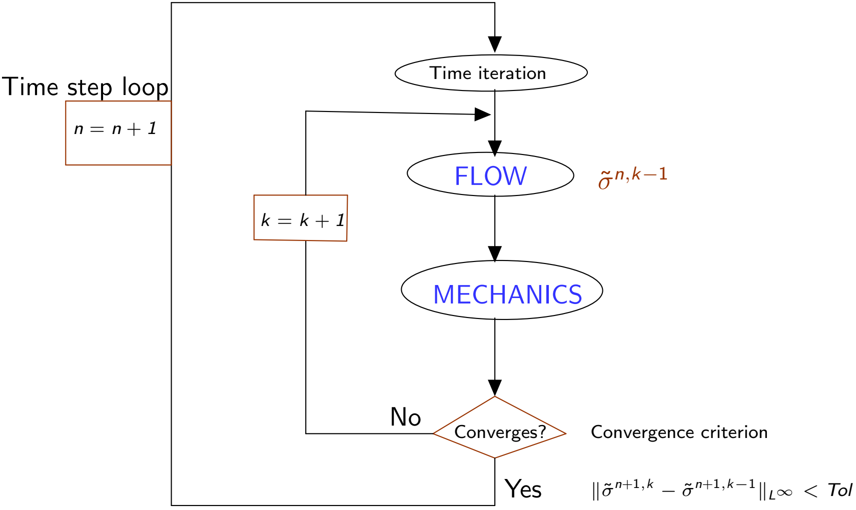
$$\Big((c_0 + c_r)(p^{n+1,k+1} - p^n), w \Big) + \Delta t (\nabla \cdot \mathbf{z}^{n+1,k+1}, w) = \Delta t \int_\Omega s_f w \\ - \Big(\frac{c_r}{\alpha} (\tilde{\sigma}^{n+1,k} - \tilde{\sigma}^n), w \Big),$$

$$(\mu_f K^{-1} \mathbf{z}^{n+1,k+1}, \mathbf{s}) - (p^{n+1,k+1}, \nabla \cdot \mathbf{s}) = - \int_{\Gamma_p} p_D \mathbf{s} \cdot \eta + \int_\Omega \rho_f \mathbf{g} \cdot \mathbf{s}$$

MECHANICS

$$a_{\mathbf{u}}(\mathbf{u}^{n+1,k+1}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{r}_N \cdot \mathbf{v} + \alpha (p^{n+1,k+1}, \nabla \cdot \mathbf{v})$$

Splitting: iterative coupling



Iterative coupling is stable and accurate. [Wheeler and Gai, Numer. Meth. PDEs, 2007]

Fully, iteratively, explicit and loosely coupled

Fully coupled The coupled governing equations of flow and geomechanics are solved simultaneously at every time step.

Iteratively coupled Either the flow, or mechanical, problem is solved first, then the other problem is solved using the intermediate solution information. This sequential procedure is iterated at each time step until the solution converges to within an acceptable tolerance. The converged solution is identical to that obtained using the fully coupled approach.

Explicitly coupled This is a special case of the iteratively coupled method, where only one iteration is taken.

Loosely coupled The coupling between the two problems is resolved only after a certain number of flow time steps. This method can save computational cost compared to the other strategies, but it is less accurate and requires reliable estimates of when to update the mechanical response.

[Kim, Tchelepi, Juanes, SPE, 2009]

Space discretization

A very simple example

$$-u''(x) + u(x) = (1+\pi^2)\sin(\pi x), \quad x \in (0,1), \quad u(0)=0,\ u(1)=0$$

Variational formulation

Multiplying the equation by any arbitrary weight function $v \in H_0^1(0,1)$ and integrating over the interval $(0,1)$

$$\int_0^1 -u''(x)v(x)\,dx + \int_0^1 u(x)v(x)\,dx = \int_0^1 (1+\pi^2)\sin(\pi x)v(x)\,dx.$$

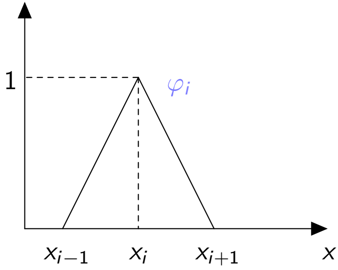
Integrating by parts

$$\begin{aligned}\int_0^1 u'(x)v'(x)\,dx + \int_0^1 u(x)v(x)\,dx \quad &= \quad \int_0^1 (1+\pi^2)\sin(\pi x)v(x)\,dx \\ &\quad\ + u'(1)v(1) - u'(0)v(0).\end{aligned}$$

Finite element method

The finite element method supplies an approximation to the analytical solution in the form of a piecewise polynomial function, defined over the entire computational domain.

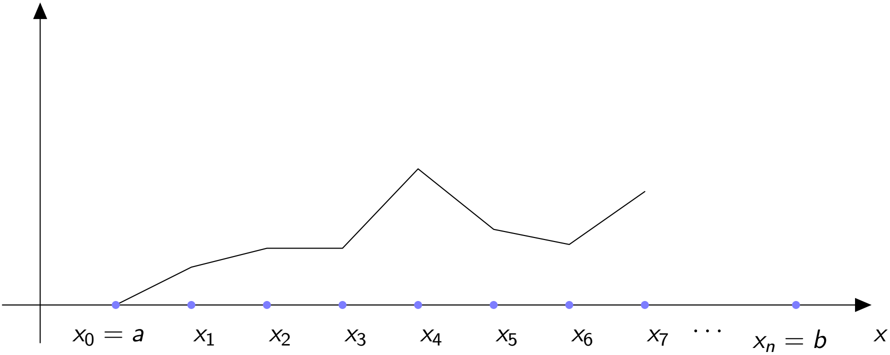
Example: The simplest case of linear splines.



$$\varphi_i(x) = \begin{cases} (x - x_{i-1})/h_i, & x_{i-1} \le x \le x_i, \\ (x_{i+1} - x)/h_{i+1}, & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

A piecewise linear finite element basis function φ_i (hat functions).

Finite element method



A piecewise linear function.

Matrix form

Since the aim of Finite Element Method method is the production of a linear system of equations, we build its matrix form, which can be used to compute the solution by a computer program.

We expand u_n in respect to this basis, $u_n = \sum_{j=1}^{n} c_j \varphi_j$ to obtain

$$\sum_{j=1}^{n} c_j a(\varphi_j, \varphi_i) = f(\varphi_i) \quad i = 1, \ldots, n,$$

which is a linear system of equations $AU = F$, where

$$a_{ij} = a(\varphi_j, \varphi_i), \quad F_i = f(\varphi_i).$$

Finite element method

For the finite element method the important property of the basis functions φ_i, $1 \leq i \leq n$ is that they have *local support*, being nonzero only in one pair of adjacent intervals $(x_{i-1}, x_i]$ and $[x_{i+1}, x_i)$.

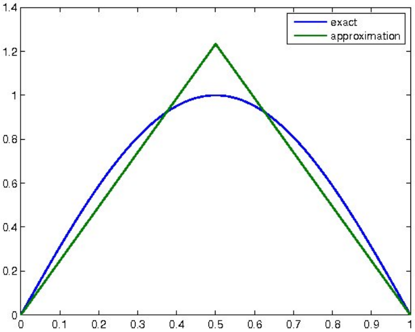
This means that, $A_{ij} = 0$ if $|i - j| > 0$.

$$A = \begin{bmatrix} a_{11} & a_{21} & 0 & \cdot & \cdot & \cdot & & 0 \\ a_{21} & a_{22} & a_{32} & 0 & & & & \cdot \\ 0 & a_{32} & a_{33} & a_{43} & 0 & & & \cdot \\ \cdot & 0 & a_{43} & a_{44} & \cdot & \cdot & & \cdot \\ \cdot & & 0 & \cdot & \cdot & \cdot & \cdot & \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & & \cdot & \cdot & a_{nn} \end{bmatrix}$$

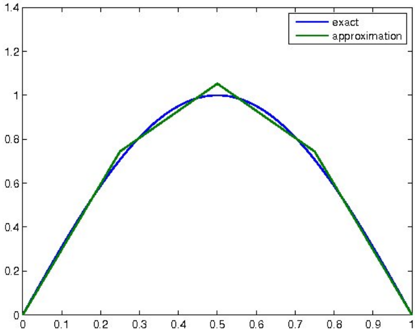
$\Longrightarrow$ The matrix A is symmetric, positive definite and tridiagonal, and the associated system of linear equations can be solved very efficiently.

Finite element method

Back to the very simple example (Exact solution: $u(x) = \sin(\pi x)$)
Uniform subdivision of $[0, 1]$ of spacing $h = 1/n$.



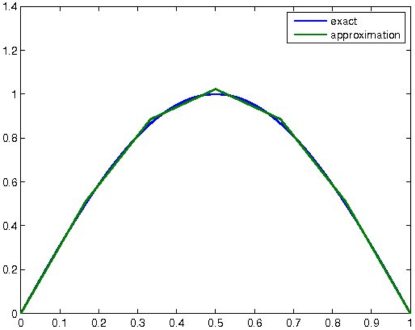
$n = 2$



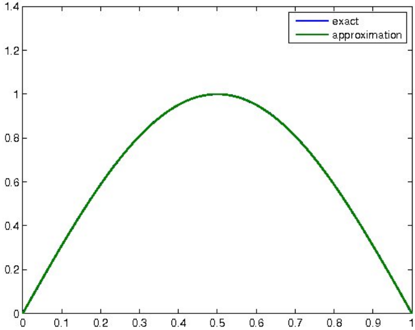
$n = 4$

Finite element method

Uniform subdivision of $[0, 1]$ of spacing $h = 1/n$.



$n = 6$

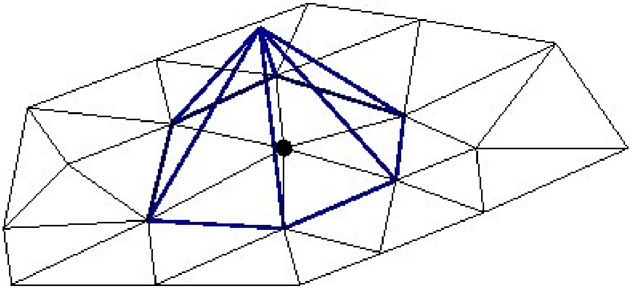


$n = 100$

In the last figure, the approximation error is so small that u and u_n are indistinguishable.

Finite element method

Basis function in 2D



Finite element method

Find $u_n \in V_n$ such that $\forall v_n \in V_n, \quad a(u_n, v_n) = f(v_n)$.

Galerkin orthogonality

The key property of the Galerkin approach is that the error is orthogonal to the chosen subspaces.

Since $V_n \subset V$, we can use v_n as a test vector in the original equation. Subtracting the two, we get the Galerkin orthogonality relation for the error, $e_n = u - u_n$

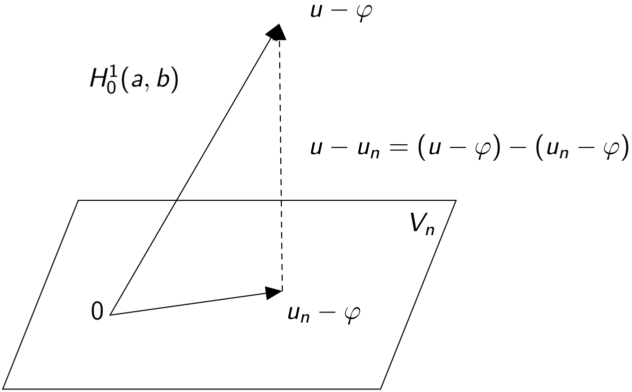
$$a(e_n, v_n) = a(u, v_n) - a(u_n, v_n) = f(v_n) - f(v_n) = 0$$

and

$$a(e_n, e_n) = a(u - u_n, u - u_n) = \min_{v_n \in V_n} a(u - v_n, u - v_n).$$

Finite element method

Galerkin orthogonality



Energy norm: $\|v\|_E = |a(v,v)|^{1/2}$

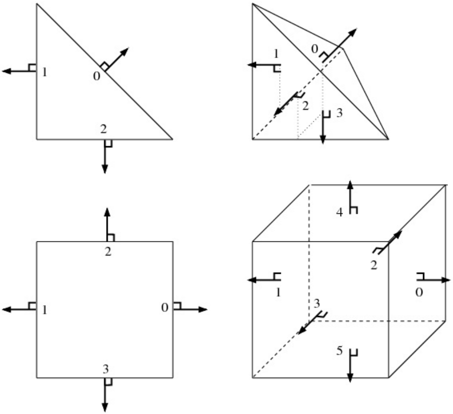
u_n is the *best approximation* from V_n to the weak solution $u \in H_0^1(a,b)$, when we measure the error of the approximation in the energy norm.

Spatial discretization

$\mathcal{E}_h$ and $\mathcal{E}_H$ be two nondegenerate partitions of the polyhedral domain Ω with maximal element diameter h and H, respectively.

Mixed spaces for flow variables

Examples of mixed spaces with the needed properties are the Raviart-Thomas-Nedelec spaces.



Example:
Lowest order Raviart-Thomas

2D

triangles: in each element
$\bar{p}$ =const, $\bar{\mathbf{z}} = (a + bx, c + by)^t$

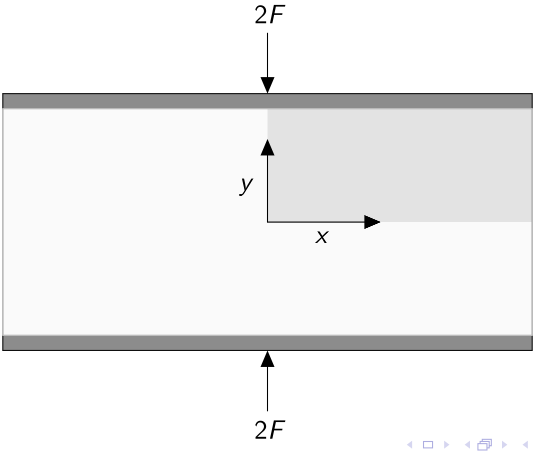
quadrilaterals: in each element
$\bar{p}$ =const, $\bar{\mathbf{z}} = (a + bx, c + dy)^t$

Mandel's problem

Mandel's solution has been used as a benchmark problem for testing the validity of numerical codes of poroelasticity.

[Mandel, 1953] - analytical solution for pressure

[Abousleiman et al., 1996] - analytical solution for displacement and stress



Mandel's problem

$$\begin{aligned} -(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla^2\mathbf{u} + \alpha\nabla p &= 0 \text{ in } \Omega\times(0,T] \\ \frac{\partial}{\partial t}(c_0 p + \alpha\nabla\cdot\mathbf{u}) - \frac{1}{\mu_f}\nabla\cdot K\nabla p &= 0 \text{ in } \Omega\times(0,T] \end{aligned}$$

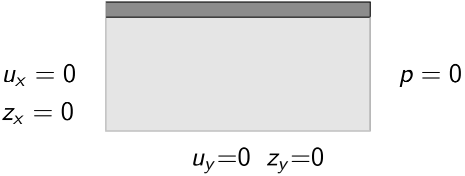
boundary conditions

$$p=0,\ x=a, \quad -\frac{1}{\mu_f}K\nabla p\cdot\eta = 0,\ x=0,\ y=0,\ y=b,$$

$$u_1=0,\ x=0, \quad u_2=0,\ y=0,$$

$$\frac{\partial u_y}{\partial x}=0,\ y=b,$$

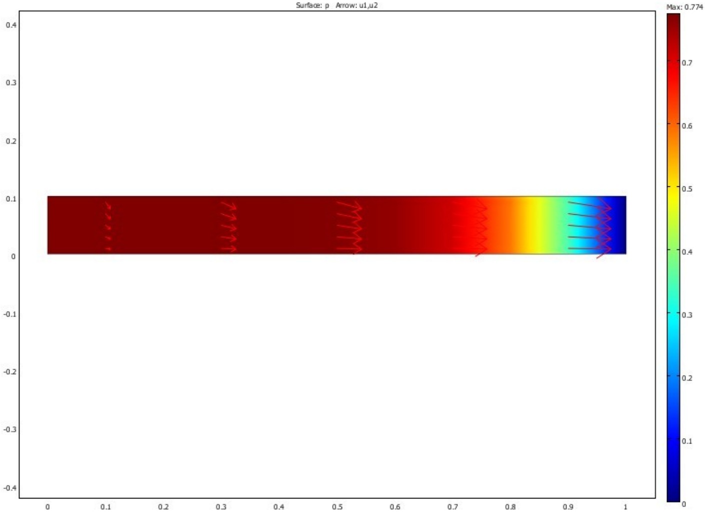
$$\tilde{\sigma}\eta = (-F/a)\eta,\ y=b, \quad \tilde{\sigma}\eta=0,\ x=0,\ x=a,\ y=0,$$



computational domain

Mandel's problem

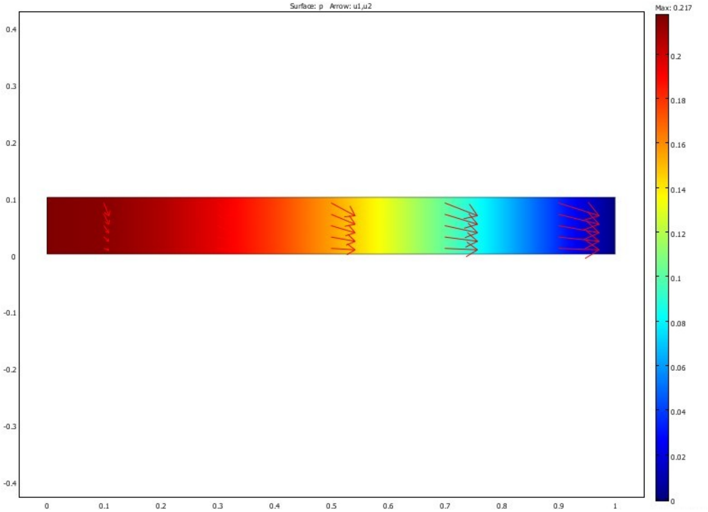
surface: p, arrows: u_1 and u_2



$T = 0.001$

Mandel's problem

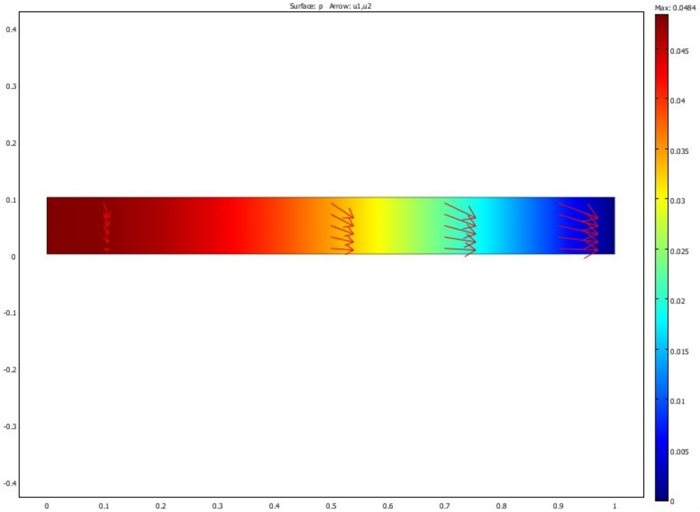
surface: p, arrows: u_1 and u_2



$T = 0.1$

Mandel's problem

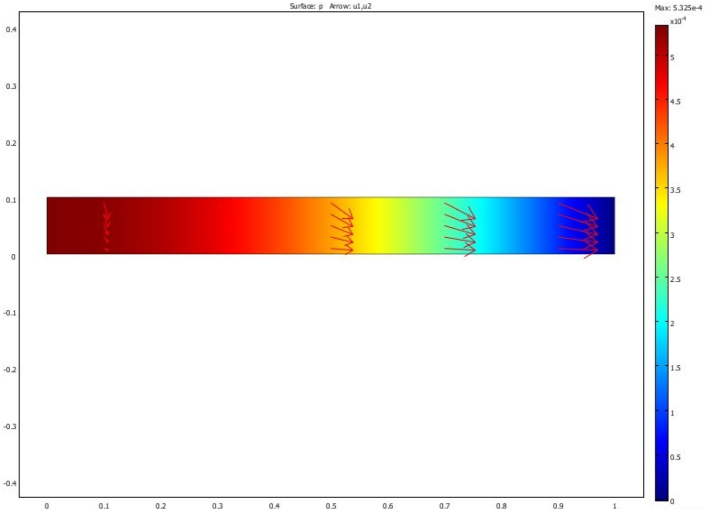
surface: p, arrows: u_1 and u_2



$T = 0.2$

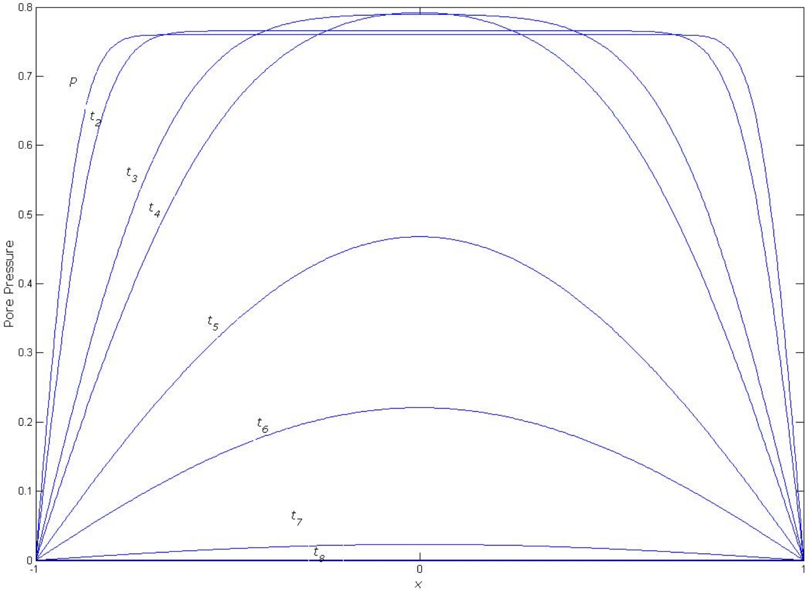
Mandel's problem

surface: p, arrows: u_1 and u_2



$T = 0.5$

Mandel's problem



Mandel's problem

Numerical results

H	$\|\|\mathbf{u}-\bar{\mathbf{u}}\|\|_{H^1}$	rate	$\|\|p-\bar{p}\|\|_{L^2}$	rate	$\|\|\mathbf{z}-\bar{\mathbf{z}}\|\|_{L^2}$	rate
5.000e-2	1.222e-3	1.39	1.389e-2	1.53	2.416e-1	1.35
2.500e-2	4.653e-4	1.19	4.798e-3	1.21	9.452e-2	0.94
1.667e-2	2.878e-4	1.05	2.933e-3	1.03	6.453e-2	1.14
1.250e-2	2.130e-4	1.10	2.179e-3	1.08	4.654e-2	0.82
1.000e-2	1.665e-4	0.89	1.711e-3	0.92	3.875e-2	1.14
8.333e-3	1.415e-4	-	1.446e-3	-	3.149e-2	-

Convergence rates: bilinear elements for $\mathbf{u}$, lowest order Raviart-Thomas space for p and $\mathbf{z}$

Important Questions

Is the method stable?

Is the method convergente?

Next seminar...

"I really enjoy developing efficient and accurate solutions to real-world problems, while maintaining a solid theoretical base." MFW