# ABSTRACTS

## 4th Workshop on Categorical Methods in Non-Abelian Algebra, Coimbra, May 30-June 1, 2016

### **Igor Baković** - *Semi-(op)lax pullbacks and bipullbacks*

There are several (equivalent) notions of internal fibrations in a bicategory $\mathcal{B}$. Perhaps the oldest one is the representable definition which can be traced back in the work of Gray [4]. This notion agrees with the original definition of Grothendieck when interpreted in the 2-category Cat of small categories. The work of Gray already captured an intrinsic definition for fibrations in Cat by Chevalley, in terms of the existence of an adjoint one-sided inverse to a certain functor between comma categories. This led Street to an intrinsic definition of fibrations in general 2-categories, and subsequently in general bicategories. Finally, Johnstone [5] gave an alternative way of defining fibrations in 2-categories [6] and bicategories [7], which is closer than Street's to Grothendieck's original definition. Johnstone's definition essentially makes use of semi-(op)lax adjunctions, which are generalized adjunctions first studied by Gray [5]. By a closer look into Johnstone's definition, we isolate a particular universal property of a semi-(op)lax pullback as a component of a counit of the semioplax adjunction. We show that semi-oplax pullbacks of fibrations are (co)reflexive subcategories of bipullbacks, extending the result of Joyal and Street [3]. This enables us to derive a crucial result that the collection of all internal fibrations *in* the bicategory $\mathcal{B}$ is a fibration *of* bicategories [1] *over* $\mathcal{B}$.

[1] I. Baković, Fibrations of bicategories, preprint

[2] P.T. Johnstone, Fibrations and partial products in a 2-category, Applied Categorical Structures, Vol. 1, Nr. 2 (1993) , p. 141-179.

[3] A. Joyal, R. Street, Pullbacks equivalent to pseudopullbacks, Cahiers Topologie Géom. Différentielle Catégoriques XXXIV (1993), no. 2, 153-156.

[4] J. W. Gray, Fibred and cofibred categories. Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965) Springer, New York, 1966, pp. 21-83.

[5] J. W. Gray, Formal category theory: adjointness for 2-categories, Lecture Notes in Mathematics, Vol. 391. Springer-Verlag, Berlin-New York, 1974. xii+282 pp.

[6] R. Street, Fibrations and Yoneda's lemma in a 2-category, Category Seminar (Proc. Sem., Sydney, 1972/1973), pp. 104-133. Lecture Notes in Mathematics, Vol. 420, Springer, Berlin, 1974.

[7] R. Street, Fibrations in bicategories, Cahiers Topologie Géom. Différentielle 21 (1980), no. 2, 111-160.

### **Dominique Bourn** - *Goursat condition in non-regular context*

We introduce a notion of Goursat category valid in the non-regular context and such that: 1) of course, in the regular context, it coincides with the notion introduced by Carboni-Kelly-Pedicchio, 2) any Mal'tsev category is a Goursat category, 3) this notion is characterized by a property of the fibration of points as it is the case for Mal'tsev categories.

### **Alan S. Cigoli** - *General obstruction theory II*

In this talk, I will further investigate the obstruction problem for a fibrewise cofibration of fibrations introduced in the talk by G. Metere.

More precisely, I will explain how, under suitable conditions on the starting fibrewise cofibration $\Pi$, one can get a new fibrewise cofibration $\widehat{\Pi}$ where the fibres are groupoids. This will allow then to state a classification theorem for the hom-sets in the domain of $\widehat{\Pi}$.

Finally I will focus on the main example of

$$\Pi\colon \mathsf{XMod}(\mathcal{C}) \to \mathsf{Mod}(\mathcal{C}),$$

where $\mathcal{C}$ is a semi-abelian category satisfying the condition (SH) and $\Pi$ the functor sending every internal crossed module morphism in $\mathcal{C}$ to the corresponding $\pi_0$-module morphism between the $\pi_1$'s.

We will see how a classification theorem for weak morphisms between internal crossed modules follows from the purely formal theory developed above.

### **Mathieu Duckerts-Antoine** - *A Seifert-van Kampen theorem in non-abelian algebra*

It is a well-known fact that the first fundamental group functor $\pi_1\colon \mathsf{Top}_* \to \mathsf{Gp}$ preserves every pushout of the form

$$\begin{tikzcd}
(X_1 \cap X_2, x) \arrow[r] \arrow[d] & (X_2, x) \arrow[d] \\
(X_1, x) \arrow[r] & (X, x)
\end{tikzcd}$$

where $X_1, X_2$ are open subspaces of $X$ and $X_1 \cap X_2$ is path-connected. This is known as the Seifert - van Kampen theorem.

During the talk, I will explain how it is possible to obtain a version of this result in a non-abelian algebraic context using the concept of algebraic coherence newly introduced in [1].

This is a joint work with Tim Van der Linden.

[1] A. S. Cigoli, J. R. A. Gray and T. Van der Linden, Algebraically coherent categories, Theory Appl. Categ. 30 (2015), no 54, 1864–1905.

**Ramón González** - *The fundamental theorem of Hopf modules in a non-associative setting*

The main motivation of this talk is to extend the fundamental theorem of Hopf modules to the weak non-associative setting. More concretely, we introduce the notion of weak Hopf quasigroup, as a new Hopf algebra generalization that encompasses weak Hopf algebras and Hopf quasigroups, and we prove the fundamental theorem of Hopf modules for these new algebraic structures. As particular instances, we have the classical fundamental theorem of Hopf modules for Hopf algebras, proved by Larson and Sweedler in 1969, the one obtained by Böhm, Nill and Szlachányi in 1999 for weak Hopf algebras, and finally, the similar result established by Brzeziński in 2010 for Hopf quasigroups.

This talk is based on a joint work with José Nicanor Alonso Álvarez and José Manuel Fernández Vilaboa.

**Dirk Hofmann** - *Stone-type dualities beyond lattice structures II*

In this talk we continue the path initiated in the presentation of Pedro Nora and extend the classical Stone and Halmos dualities

$$\mathsf{Stone} \simeq \mathsf{Boole}^{\mathrm{op}} \qquad\qquad \text{and} \qquad\qquad \mathsf{Stone}_{\mathbb{V}} \simeq \mathsf{Boole}^{\mathrm{op}}_{\perp,\vee}$$

to the context of quantale-enriched categories. Besides moving from ordered sets to metric structures on the right hand side, here we also substitute the classical Vietoris monad $\mathbb{V}$ by an appropriate enriched version. Surprisingly or not, parts of the theory work better in this more general setting.

**Fosco Logerian** - t-*structures on stable infinity categories*

Stable $(\infty, 1)$-category theory constitutes the foundation of stable homotopy theory and modern homological algebra. After having introduced the notion of (orthogonal) factorization system on a $(\infty, 1)$-category, I will prove that suitable such factorization systems on a stable $(\infty, 1)$-category C correspond bijectively to t-structures on its homotopy category $\mathbf{ho}(\mathrm{C})$. The notion of factorization system is capable to capture the intrinsic meaning of several classical constructions in algebraic topology and (derived) algebraic geometry.

**Fernando Lucatelli** - *Bilimits Commutativity and Descent Theory*

There are two main constructions in classical descent theory: the category of algebras and the descent category (see [1, 3, 5]). These constructions are known to be examples of bilimits (see [6, 7, 2, 5]). This work aims to investigate whether pure formal methods and commuting properties of bilimits are useful in proving classical and new theorems of descent theory in the classical context of [3, 4].

Willing to give such formal approach, we employ the concept of pointwise pseudo-Kan extensions [5]. In this presentation, we shall give a sketch of the proof of the most basic commutativity result of bilimits, via pseudomonad theory. This result, concisely, says that "bilimits of effective/exact diagrams are effective/exact". Then, we give applications of this basic result to descent theory, proving classical results and giving new results on effective descent morphisms of some categories.

This talk is based on [5] which is part of my PhD work under supervision of Maria Manuel Clementino at the University of Coimbra.

[1] J. Bénabou and J. Roubaud, Monades et descente, *C. R. Acad. Sci. Paris Sér.* A-B 270 (1970) A96-A98.
[2] G. Janelidze, Descent and Galois Theory, *Lecture Notes of the Summer School in Categorical Methods in Algebra and Topology, Haute Bodeux, Belgium* (2007).
[3] G. Janelidze and W. Tholen, Facets of descent I, *Applied Categ. Structures* 2 (1994), no. 3, 245-281.
[4] G. Janelidze and W. Tholen, Facets of descent II, *Applied Categ. Structures* 5 (1997), no. 3, 229-248.
[5] F. Lucatelli, Pseudo-Kan Extensions and Descent Theory (in preparation).
[6] R. Street, Categorical and combinatorial aspects of descent theory, *Applied Categ. Structures* 12 (2004), no. 5-6, 537-576.
[7] R. Street, Limits indexed by category-valued 2-functors, *J. Pure Appl. Algebra* 8 (1976), no. 2, 149-181.
[8] R. Street, Fibrations in bicategories, *Cahiers Topologie Géom. Différentielle* 21 (1980), no. 2, 111-160.

**Pierre-Alain Jacqmin** - *An embedding theorem for regular Mal'tsev categories.*

One can construct an essentially algebraic category M which 'represents' regular Mal'tsev categories. This means that M is itself regular Mal'tsev and that every small regular Mal'tsev category has a faithful embedding into a functor category $[\mathrm{P}, \mathrm{M}]$ which preserves and reflects finite limits, isomorphisms and regular epimorphisms. This embedding theorem gives then a way to make proofs using elements. Similar constructions and theorems hold also in the unital, strongly unital, subtractive, n-permutable and protomodular cases.

This is a join work with Zurab Janelidze.

**George Janelidze** - *Frattini theory in semi-abelian and general categories*

Classically, the Frattini subgroup of a group G is defined as the intersection of all maximal proper subgroups of G. This definition has obvious counterparts for other algebraic structures, and such counterparts have been studied by various authors, especially for Lie algebras, starting from 1960s. The general-categorical counterpart is equally obvious, and the purpose of the talk is to suggest a categorical approach to (at least) the first results of Frattini theory.

**Zurab Janelidze** - *Functorial duality in non-abelian algebra*

The context of an abelian category is suitable for a self-dual treatment of homomorphisms theorems for abelian groups, and more generally, modules over a ring (by homomorphism theorems we mean theorems such as Noether isomorphism theorems, homological diagram lemmas, etc.). This goes back to [9], where a similar development for non-abelian groups and other related structures is left for a future investigation. Let us recall the following passage from [9] to this effect: "A further development giving the first and second isomorphism theorems, and so on, can be made by introducing additional carefully chosen dual axioms. This will be done below only in the more symmetrical abelian case."

In the first half of this talk we report on the main results from "duality in non-abelian algebra" which appear in [6, 7, 8]. The aim of this direction of research is to revisit structures and results in modern categorical algebra, which deal with the categorical study of non-abelian group-like structures, by replacing the context of a category with that of a faithful amnestic functor to the category. We call such functors "forms". When suitably transporting results and structures from the context of a category to the context of a form, it turns out that they are invariant under the *functorial duality*, although they were not invariant under the *categorical duality*. Thus, for instance, the axioms of a semi-abelian category [4] can be reformulated in the language of a form in such a way that on one hand, these axioms are functorially self-dual (they hold for the functor if and only if they hold for the dual functor), and on the other hand, a category having finite products and coproducts is a semi-abelian category if and only if the form canonically associated to it satisfies these axioms. In most cases, the canonically associated form is the *form of subobjects*, i.e., the functor whose fibres are posets of subobjects of a given object, and one of the typical axioms is that this functor is a Grothendieck bifibration [3]. In the second half of this talk we show that the context of a form satisfying the self-dual axioms corresponding to the notion of a semi-abelian category is suitable for a self-dual treatment of homomorphism theorems for non-abelian group-like structures. One of the subtle aspects of this work is how to deal with the construction of a *connecting homomorphism* for a zigzag of homomorphisms, which both in the abelian and in the non-abelian categorical settings are typically obtained via composition of relations. These connecting homomorphisms arise not only in the homological diagram lemmas, but also as the canonical isomorphisms of the isomorphism theorems.

The context in which we carry out the self-dual analysis of homomorphism theorems is free from any assumptions about the existence of products or coproducts in the ground category. This brings us close to the projective approach to homological algebra due to M. Grandis (see [2] and the references there). In fact, the theory of forms evolved from an attempt to compare the notion of a homological category in the sense of F. Borceux and D. Bourn [1] with the notion of a homological category in the sense of M. Grandis (introduced in a preprint in 1991, see the list of references in [2]). The very first results in this direction were reported at the workshop in category theory in Coimbra in 2012 and appear in [5]. The theory of forms in its present state can be considered as a synthesis of categorical algebra of semi-abelian and related categories, which is applicable to non-abelian group-like structures, but is a non-dual theory, and the projective theory developed by M. Grandis, which is not applicable to non-abelian group-like structures, but is a self-dual theory.

[1] F. Borceux and D. Bourn, Mal'cev, protomodular, homological and semi-abelian categories, Mathematics and its Applications 566, Kluwer, 2004.
[2] M. Grandis, Homological algebra in strongly non-abelian settings, World Scientific Publishing Co., Singapore, 2013.
[3] A. Grothendieck, Technique de descente et théorèmes d'existence en géométrie algébrique, I. Généralités. Descente par morphismes fidélement plats. Sémin. Bourbaki 190, 1959, 299?327.
[4] G. Janelidze, L. Márki and W. Tholen, Semi-abelian categories, Journal of Pure and Applied Algebra 168, 2002, 367-386.
[5] Z. Janelidze, On the form of subobjects in semi-abelian and regular protomodular categories, Applied Categorical Structures 22, 2014, 755-766.
[6] Z. Janelidze and T. Weighill, Duality in non-abelian algebra I. From cover relations to Grandis ex2-categories, Theory and Applications of Categories 29, 2014, 315-331.
[7] Z. Janelidze and T. Weighill, Duality in non-abelian algebra II. From Isbell bicategories to Grandis exact categories, Journal of Homotopy and Related Structures, 2016, published online.
[8] Z. Janelidze and T. Weighill, Duality in non-abelian algebra III. Normal categories and 0-regular varieties, Algebra Universalis, 2016, to appear.
[9] S. Mac Lane, Duality for groups, Bulletin of the American Mathematical Society 56, 1950, 485-516.

**Nelson Martins Ferreira** - *Normalized bicategories internal to weakly Mal'tsev categories endowed with a V-Mal'tsev operation*

In this talk we will consider weakly Mal'tsev categories endowed with a V-Mal'tsev operation in the sense of Pedicchio and show that an internal (normalized) bicategory in that context, as a structure, is equivalent to a reflexive graph in which the object of arrows is equipped with a structure of a $\mathtt{cat}-1-\mathtt{group}$ (in the sense of Loday), together with two extra endomorphisms. The two extra endomorphisms correspond to the left and right identity constrains of the tensor-composition in the bicategory and they uniquely determine the associative structure on it. In particular we may easily interpret the results in any Mal'tsev variety (in the sense of universal algebra). As an example, we explicitly describe, using commutators, the structure in the case of groups.

**Giuseppe Metere** - *General obstruction theory I*

Schreier obstruction theory gives existence and structure theorems for group extensions in terms of cohomology invariants determined by so-called *abstract kernels* (see [2]). More specifically, the obstruction part of the theorem deals with the existence of extensions inducing a given abstract kernel, and, in this case, a structure theorem describes the isomorphism classes of such extensions.

A more recent approach investigates this problem from a two dimensional perspective, by considering group extensions as a specific kind of monoidal functor between 2-groups (see [1]). Indeed, with any 2-group $\mathbb{G}$ it is possible to associate two *homotopy* invariants: the connected component group $\pi_0\mathbb{G}$ and the (abelian) group of loops $\pi_1\mathbb{G}$, so that the cohomology classification of group-extensions becomes an instance of the homotopy classification of monoidal functors between 2-groups (see [3]).

In fact, the monoidal structure of $\mathbb{G}$ induces on $\pi_1\mathbb{G}$ a natural structure of $\pi_0\mathbb{G}$-module, so that one can define a functor $\Pi = (\pi_0, \pi_1)$ from 2-groups to *Grp*-modules. The obstruction problem and the corresponding structure theorem can then be stated for the functor $\Pi$: given two 2-groups $\mathbb{H}$ and $\mathbb{G}$, and two equivariant group homomorphisms

$$\phi\colon \pi_0\mathbb{H} \to \pi_0\mathbb{G}, \qquad \psi\colon \pi_1\mathbb{H} \to \pi_1\mathbb{G}$$

- determine if the set of monoidal functors $F\colon \mathbb{H} \to \mathbb{G}$ such that
$$\Pi(F) = (\phi, \psi)$$
is nonempty;
- in this case, determine such a set.

In my talk, I will adopt a formal approach to the problem, in order to develop a purely formal obstruction theory for a fiber-wise cofibration between two fibrations (this is a notion that generalizes that of a two-sided discrete fibration). Applications and examples will be provided.

[1] A. S. Cigoli and G. Metere, *Extension theory and the calculus of butterflies*, Journal of Algebra, 2016.
[2] S. MacLane, *Homology*, Springer Verlag (1975).
[3] E. Vitale, *Categorical groups: a bit of theory and some applications to homological and homotopical algebra*, notes for the summer school Category Theory and Algebraic Topology, 11-13 September 2014, EPFL – Lausanne.

**Andrea Montoli** - *The Nine Lemma and the push forward construction for special Schreier extensions of monoids*

We show that the Nine Lemma holds for special Schreier extensions of monoids. This fact is used to obtain a push forward construction for special Schreier extensions with abelian kernel. This construction permits to give a functorial description of the Baer sum of such extensions.

Joint work with Nelson Martins-Ferreira and Manuela Sobral.

**Pedro Nora** - *Stone-type dualities beyond lattice structures I*

One of the common features of the duality theorems

$$\mathsf{Stone} \simeq \mathsf{Boole}^{\mathsf{op}}, \qquad \mathsf{Spec} \simeq \mathsf{DLat}^{\mathsf{op}}, \qquad \mathsf{Stone}_{\mathbb{V}} \simeq \mathsf{Boole}^{\mathsf{op}}_{\perp,\vee}, \qquad \ldots$$

is that all are obtained using the two-element space and the two-element lattice as a dualising object. Due to this fact, we can only expect dualities for categories somehow cogenerated by 2. In order to capture all (ordered) compact Hausdorff spaces, we will combine duality theory and enriched category theory. Motivated by the fact that ordered sets are categories enriched in the two-element quantale and metric spaces are categories enriched in the $[0,\infty]$ quantale, our thesis is: *the passage from the two-element space to the compact Hausdorff space* $[0,\infty]$ *should be matched by a move from ordered structures to metric structures on the other side.* Accordingly, in this talk we present duality theory for ordered compact Hausdorff spaces and (suitably defined) finitely cocomplete metric spaces.

**Diana Rodelo** - *An object-wise approach in categorical algebra*

The categorical-algebraic characterisation of groups amongst monoids discussed by Tim leads to the development of an object-wise approach to certain important properties occurring in categorical algebra. The aim of this talk is to give an overview of a basic theory involving what we call *unital* and *strongly unital* objects, *subtractive* objects, *Mal'tsev* objects and *protomodular* objects. We explore some of the connections between these new notions and give examples and counterexamples.

**Tim Van der Linden** - *Towards a characterisation of groups amongst monoids*

While it is quite easy to express, in categorical-algebraic terms, when a monoid is an abelian group, characterising non-abelian groups amongst monoids turns out to be a difficult problem. After discussing several approaches we present a solution based on an object-wise viewpoint of categorical-algebraic notions such as protomodularity and the Maltsev property. I shall focus on the characterisation of groups inside the category of monoids, and Diana will tell us more about the categorical-algebraic properties themselves.

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