Split extensions of bialgebras

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An elementary result in the theory of modules says that in any short exact sequence

$$0 \longrightarrow K \xrightarrow{k} X \underset{s}{\overset{f}{\rightleftarrows}} Y \longrightarrow 0 \qquad\qquad f \circ s = 1_Y$$

where the cokernel f admits a section s, the middle object X decomposes as a direct sum $X \cong K \oplus Y$. If, however, the given sequence is a short exact sequence of, say, groups or Lie algebras, then this is no longer true, and we can at most deduce that X is a semidirect product $K \rtimes Y$ of K and Y. In a fundamental way, this interpretation depends on, or even amounts to, the fact that X is generated by its subobjects $k(K)$ and $s(Y)$. This idea gave rise to a characterization of groups amongst monoids: *a monoid M is a group if and only if all split extensions over it are (stably) strong*, [1, 4].

The objective of this talk is to extend this characterization. Since monoids (resp. groups) in the category of cocommutative coalgebras are cocommutative bialgebras (resp. Hopf algebras) one could expect to have a similar result in this context. Indeed, we will prove that, over an algebraically closed field of characteristic zero, *a cocommutative bialgebra B is a Hopf algebra if and only if all split extensions over it are (stably) strong.* Finally, we will explain that even though the category of cocommutative Hopf algebras is protomodular (it is actually semi-abelian [3]), when we remove cocommutativity we lose protomodularity.

References

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*Joint work with Tim Van der Linden

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