EXTENDING STONE DUALITY TO METRIC SPACES

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The classical duality results

$$\mathsf{Stone} \simeq \mathsf{Bool}^{\mathrm{op}} \qquad\qquad \mathsf{Spec} \simeq \mathsf{DLat}^{\mathrm{op}}$$

of Stone [4, 5] assert a dual equivalence between a category of certain topological spaces on one side (that is, the basic structure can be taken as a convergence relation $UX \times X \to 2$) and categories of ordered sets with some (co)completeness properties on the other (here the basic structure can be taken as an order relation $X \times X \to 2$). Moreover, the involved equivalence functors are liftings of the hom-functor into 2. Due to this fact, we can only expect duality results for categories somehow cogenerated by 2 (with appropriate structure). If we want to have a duality theorem for all (ordered) compact Hausdorff spaces instead of Stone spaces resp. spectral spaces, we need to use a cogenerator of (ordered) compact Hausdorff spaces instead of the 2-element discrete space (with $0 \leq 1$). Consequently, as a general framework, we might want to consider structures enriched in $[0, \infty]$ or $[0, 1]$; that is, approach spaces and metric compact Hausdorff spaces on the left-hand side and metric spaces (with some (co)completeness properties) on the right-hand side.

To get there, it seems to be beneficial to consider a more general situation. Recall that the dualities above can be extended to categories of spaces and continuous *relations* (see, for instance, [1]); where "continuous relation" can be defined as morphism in the Kleisli category $\mathsf{Stone}_{\mathbb{V}}$ for the Vietoris monad $\mathbb{V}$. If we want to base our constructions on $[0, \infty]$ or $[0, 1]$, it might be more natural(?) to consider also an $[0, \infty]$-enriched (or $[0, 1]$-enriched) version of the Vietoris monad; and such monads are described in [2]. Denoting this enriched Vietoris monad by $\mathbb{V}$ as well, the morphisms in $\mathsf{CompHaus}_{\mathbb{V}}$ are "continuous" distance functions and, in analogy to the Stone/Halmos case, we hope for a full embedding

$$\mathsf{CompHaus}_{\mathbb{V}} \to (\text{"finitely cocomplete" metric spaces})^{\mathrm{op}}.$$

A first valuable hint towards such generalisations we found in [3] where the author gives a "functional representation" of the classical Vietoris monad on $\mathsf{CompHaus}$. Building on [3] we will be able to proof the desired embedding results.

References

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[5] ———, *Topological representations of distributive lattices and Brouwerian logics*, Časopis pro pěstování matematiky a fysiky, 67 (1938), pp. 1–25.