Topological semi-abelian algebras

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Abstract

Given an algebraic theory $\mathbb{T}$ whose category of models is semi-abelian, we study the category $\mathsf{Top}^{\mathbb{T}}$ of topological models of $\mathbb{T}$ and generalize to it various results on topological groups. In particular $\mathsf{Top}^{\mathbb{T}}$ is regular, Mal'cev and protomodular; every open subobject is closed and every quotient map is open. We devote special attention to the Hausdorff, compact, locally compact, connected, totally disconnected and profinite $\mathbb{T}$-algebras.

Introduction

Semi-abelian categories have been introduced in [15] as a formal context in which all diagram lemmas of universal algebra are valid, but also many properties characteristic of non-abelian situations: the theory of normal subobjects, of commutators, of semi-direct products, and so on. Of course all abelian categories are semi-abelian, but there are many more examples: the category of all groups, of rings without unit, of Ω-groups, of Heyting semi-lattices, of presheaves or sheaves of these, and so on. The algebraic theories $\mathbb{T}$ yielding semi-abelian categories $\mathsf{Mod}^{\mathbb{T}}$ of models have been characterized in [12]; in particular, they admit a unique constant which we write 0 and various operations which collectively recapture some of the properties of the addition and the subtraction in the case of groups. This paper investigates the properties of topological models of such theories, that is, models of the theory provided with a topology which makes all the operations of the theory continuous. We write $\mathsf{Top}^{\mathbb{T}}$ for the corresponding category. For example, when $\mathbb{T}$ is the theory of groups, we recapture the theory of topological groups.

In the case of topological groups, the multiplication by an element x is an homeomorphism, with inverse the multiplication by x^{-1}. When performing the

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quotient by a (normal) subgroup, this homeomorphism transforms the equivalence class of the unit in the equivalence class of x. The semi-abelian theories do not give rise to such homeomorphisms and our first task is to prove some substitutes for these results, which will turn out to be sufficient for generalizing most of the classical results known in the case of topological groups. This includes some purely algebraic lemmas, closely related to recent work in universal algebra (see [19]), and of which we present a direct (categorical) approach in an appendix section.

We start our study with that of subalgebras $B \subseteq A$ of a topological $\mathbb{T}$-algebra A, proving at once that every open topological subalgebra $B \subseteq A$ is also closed. Moreover the closure $\overline{B} \subseteq A$ of a subalgebra $B \subseteq A$ is another subalgebra and $\overline{B} \subseteq A$ is normal when $B \subseteq A$ is so.

Next we focus on the quotient of a topological $\mathbb{T}$-algebra A by a normal subalgebra $B \subseteq A$. The algebraic quotient A/B provided with the quotient topology is still a topological $\mathbb{T}$-algebra and the quotient map $q\colon A \twoheadrightarrow A/B$ is a continuous open mapping. When moreover the normal subalgebra B is compact, this mapping q is also a closed map. The openness of quotient maps implies the regularity of the category $\mathsf{Top}^{\mathbb{T}}$.

The category $\mathsf{Top}^{\mathbb{T}}$ is generally not exact, but it shares various other properties of the category $\mathsf{Mod}^{\mathbb{T}}$ of ordinary models, including some properties which one proves classically using the exactness of $\mathsf{Mod}^{\mathbb{T}}$. For example $\mathsf{Top}^{\mathbb{T}}$ is complete, cocomplete, Mal'cev and protomodular. The inverse image functors of the fibration of points in $\mathsf{Top}^{\mathbb{T}}$ are monadic, yielding so a good theory of topological semi-direct products.

We choose to call "proabelian" a finitely complete, pointed, regular and protomodular category with coequalizers. Every proabelian category is a Mal'cev category which satisfies all the basic diagram lemmas of homological algebra and admits good theories of normal subobjects and abelian objects.

The full subcategory of abelian topological $\mathbb{T}$-algebras inherits all the already mentioned properties of $\mathsf{Top}^{\mathbb{T}}$: it is thus proabelian, with semi-direct products. Moreover, if $B \subseteq A$ is an abelian subobject of a Hausdorff topological $\mathbb{T}$-algebra A, the closure $\overline{B} \subseteq A$ is still an abelian subalgebra.

The rest of the paper is devoted to the study of various classes of topological $\mathbb{T}$-algebras. A topological $\mathbb{T}$-algebra A is Hausdorff as soon as $0 \in A$ is a closed point. The quotient A/B by a normal subalgebra $B \subseteq A$ is Hausdorff precisely when the subalgebra B is closed. The Hausdorff reflection of a topological $\mathbb{T}$-algebra A is the quotient of A by the closure of $0 \in A$. The category of Hausdorff $\mathbb{T}$-algebras is complete, cocomplete and proabelian.

On the other hand, a topological $\mathbb{T}$-algebra A is discrete when $0 \in A$ is an open point. The category of discrete $\mathbb{T}$-algebras can be identified with $\mathsf{Mod}^{\mathbb{T}}$.

Turning our attention to the case of compact Hausdorff $\mathbb{T}$-algebras, we obtain this time a semi-abelian category, thus a corresponding abelian category of abelian compact $\mathbb{T}$-models. Locally compact $\mathbb{T}$-algebras present also interesting properties: in particular, they consititute a proabelian category. Even in the non Hausdorff case, a topological $\mathbb{T}$-algebra is locally compact as soon as 0 admits a compact neighborhood. Moreover in the Hausdorff case, every locally compact subalgebra is closed.

Next we devote some attention to the case of totally disconnected $\mathbb{T}$-algebras. The connected component $\Gamma(0)$ of 0 in a topological $\mathbb{T}$-algebra A is always a closed normal subgroup and the corresponding quotient $A/\Gamma(0)$ is the totally disconnected reflection of A. The category of totally disconnected $\mathbb{T}$-algebras is still another example of a proabelian category.

We particularize these results to the case of profinite (= compact totally disconnected) $\mathbb{T}$-algebras, yielding again this time a semi-abelian category of profinite $\mathbb{T}$-models, thus an abelian category of profinite abelian $\mathbb{T}$-algebras.

Let us mention also that in a short exact sequence of topological $\mathbb{T}$-algebras

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0$$

A is compact (respectively: Hausdorff, connected, totally disconnected) as soon as B and A/B are compact (respectively: Hausdorff, connected, totally disconnected).

All results on semi-abelian categories needed in this paper can be found in the survey paper [4]; various original papers are cited in the bibliography. Some few additional results, essentially inspired from universal algebra (in particular [19]), are shortly presented in the "Appendix" section.

1 Introducing topological semi-abelian algebras

Given an object X of a category $\mathcal{V}$, the corresponding category $\mathsf{Pt}_X(\mathcal{V})$ of points (see [8]) has for objects the triples (A, p, s) in $\mathcal{V}$

$$p\colon A \longrightarrow X, \quad s\colon X \longrightarrow A, \quad p \circ s = \mathsf{id}_X.$$

A morphism $f\colon (A, p, s) \longrightarrow (B, q, t)$ is a morphism of $\mathcal{V}$ such that

$$f\colon A \longrightarrow B, \quad q \circ f = p, \quad f \circ s = t.$$

When $\mathcal{V}$ has pullbacks, every morphism $v\colon X \longrightarrow Y$ in $\mathcal{V}$ induces an inverse image functor

$$v^*\colon \mathsf{Pt}_Y(\mathcal{V}) \longrightarrow \mathsf{Pt}_X(\mathcal{V}).$$

The category $\mathcal{V}$ is protomodular (see [9]) when it admits pullbacks and all the inverse image functors v^* between the categories of points reflect isomorphisms. The category $\mathcal{V}$ is semi-abelian (see [15]) when it is protomodular, Barr exact, and admits finite limits, finite coproducts and a zero object. In that case $\mathcal{V}$ has all finite colimits and the inverse image functors v^* between categories of points are monadic (see [11]), yielding a theory of semi-direct products.

An algebraic theory $\mathbb{T}$ has a semi-abelian category $\mathsf{Mod}^{\mathbb{T}}$ of models precisely when (see [12]) $\mathbb{T}$ has

1. a unique constant 0;

2. binary operations $\alpha_1(X, Y), \ldots, \alpha_n(X, Y)$ satisfying $\alpha_i(X, X) = 0$;

3. a $n+1$-ary operation $\theta(X_1, \dots, X_{n+1})$ satisfying

$$\theta\big(\alpha_1(X,Y), \dots, \alpha_n(X,Y), Y\big) = X.$$

This is in particular the case when $\mathbb{T}$ has a group operation $+$, in which case it suffices to choose

$$n = 1, \quad \alpha_1(X,Y) = X - Y, \quad \theta(X,Y) = X + Y.$$

Thus groups, Ω-groups, modules on a ring, rings without unit, all these theories with additional sup and/or inf semi-lattice structure, Heyting semi-lattices for their own, and so on, yield examples of semi-abelian theories. Let us emphasize the fact that in general, $\mathbb{T}$ admits indeed many more operations than simply α_i and θ; moreover, the choice in $\mathbb{T}$ of operations α_i and θ as indicated is by no means unique. We shall in general refer to such an algebraic theory $\mathbb{T}$ as a "semi-abelian" theory and to the $\mathbb{T}$-algebras as "semi-abelian algebras".

Convention *Through this paper, given a semi-abelian theory $\mathbb{T}$, the notation α_i or θ will always indicate operations as above, with $n \in \mathbb{N}$ the corresponding number of operations α_i.*

Let us now introduce the topic of the present paper:

Definition 1 *Let $\mathbb{T}$ be an algebraic theory. By a topological model of $\mathbb{T}$, or a topological $\mathbb{T}$-algebra, we mean a topological space A provided with the structure of a $\mathbb{T}$-algebra, in such a way that every operation $\tau\colon T^n \longrightarrow T$ of $\mathbb{T}$ induces a continuous mapping*

$$\tau_A\colon A^n \longrightarrow A, \quad (a_1, \dots, a_n) \mapsto \tau(a_1, \dots, a_n).$$

We write $\mathsf{Top}^{\mathbb{T}}$ for the category of topological $\mathbb{T}$-algebras and continuous $\mathbb{T}$-homomorphisms between them.

For example when $\mathbb{T}$ is the theory of groups, $\mathsf{Top}^{\mathbb{T}}$ is the category of topological groups. The theory of topological groups uses in an intensive way the fact that given an element $g \in G$ of a topological group G (written additively), the mapping

$$- + g\colon G \longrightarrow G, \quad x \mapsto x + g$$

is an homeomorphism mapping 0 on g. This "homogeneity property" of the topology can be partly recaptured in the case of a semi-abelian theory:

Proposition 2 *Let $\mathbb{T}$ be a semi-abelian theory. For every element $a \in A$ of a topological $\mathbb{T}$-algebra,*

$$A \rightarrowtail A^n, \quad x \mapsto \big(\alpha_1(x,a), \dots, \alpha_n(x,a)\big)$$

presents A as a topological retract of A^n, with thus the induced topology, and maps the element $a \in A$ on $(0, \dots, 0) \in A^n$.

Proof It suffices to observe that

$$A^n \longrightarrow A, \quad (a_1, \ldots, a_n) \mapsto \theta(a_1, \ldots, a_n, a)$$

is a retraction of the given map in the category of topological spaces. $\square$

Notice that the inclusion given in proposition 2 is by no means a $\mathbb{T}$-homomorphism: it does not preserve the constant 0.

Corollary 3 *Let $\mathbb{T}$ be a semi-abelian theory. Given an element $a \in A$ of a topological $\mathbb{T}$-algebra A, the subsets*

$$\bigcap_{i=1}^{n} \alpha_i(-, a)^{-1}(U), \quad U \textit{ open neighborhood of } 0$$

constitute a fundamental system of open neighborhoods of a.

Proof Every open neighborhood of $(0, \ldots, 0) \in A^n$ contains a neighborhood of the form U^n, with $U \subseteq A$ open neighborhood of 0. One concludes by proposition 2. $\square$

Metatheorem 4 *Let $\mathbb{T}$ be a semi-abelian theory and P, a topological property stable under finite limits. If the property P is valid at the neighborhood of 0 in a given semi-abelian algebra A, that property P is valid at the neighborhood of every point of A.*

Proof By proposition 2, since every retract of A^n is the equalizer of the identity and an idempotent morphism on A^n. $\square$

Another useful property of topological groups is that every neighborhood V of 0 contains a symmetric neighborhood W such that $W + W \subseteq V$. The generalization to the semi-abelian case is easy:

Lemma 5 *Let $\mathbb{T}$ be a semi-abelian theory and V, an open neighborhood of 0 in a topological $\mathbb{T}$-algebra A. Let τ be a $(k+l)$-ary operation of the theory and $a_1, \ldots, a_k$, elements of A such that $\tau(a_1, \ldots, a_k, 0, \ldots, 0) = 0$. Then there exists an open neighborhood U of 0 in A such that*

$$b_1, \ldots, b_l \in U \Rightarrow \tau(a_1, \ldots, a_k, b_1, \ldots, b_l) \in V.$$

Proof The function

$$f\colon A^l \longrightarrow A, \quad (X_1, \ldots, X_l) \mapsto \tau(a_1, \ldots, a_k, X_1, \ldots, X_l)$$

is continuous and maps $(0, \ldots, 0)$ on 0. Therefore $f^{-1}(V)$ is an open neighborhood of $(0, \ldots, 0)$ in A^l and this neighborhood contains one of the form U^l, with U neighborhood of 0 in A. $\square$

Using as usual $\overline{(\ )}$ to indicate the topological closure, let us immediately observe that

Proposition 6 *Let $\mathbb{T}$ be a semi-abelian theory. Every topological $\mathbb{T}$-algebra is a regular topological space.*

Proof By the metatheorem 4, it suffices to prove that every open neighborhood V of 0 in A contains the closure of an open neighborhood W of 0. We choose the neighborhood W given by lemma 5 applied to the function $\theta(X_1, \ldots, X_{n+1})$ and prove that $\overline{W} \subseteq V$. If $a \in \overline{W}$,

$$Z = \bigcap_{i=1}^{n} \alpha_i(a, -)^{-1}(W)$$

is open and contains a. Since $a \in \overline{W}$, this proves the existence of some $b \in Z \cap W$. For each index i, we have $\alpha_i(a, b) \in W$ because $b \in Z$; on the other hand $b \in W$. By lemma 5, this implies

$$a = \theta\big(\alpha_1(a, b), \ldots, \alpha_n(a, b), b\big) \in V. \qquad \square$$

2 On topological subalgebras

We focus first on the properties of subalgebras $B \subseteq A$ of a topological algebra A, still in the case of a semi-abelian theory $\mathbb{T}$. Obviously, every subalgebra B of the topological algebra A, provided with the induced topology, is a topological algebra on its own. As usual when we mention that the subalgebra B is open, or closed, or compact, or whatever, this is always for the topology induced by that of A.

First, let us generalize a celebrated result on topological groups.

Proposition 7 *Let $\mathbb{T}$ be a semi-abelian theory. Every open subalgebra $B \subseteq A$ of a topological algebra A is closed.*

Proof Given $a \in A \backslash B$, we must prove the existence of an open subset $U \subseteq A \backslash B$ containing a. It suffices to put

$$U = \bigcap_{i=1}^{n} \alpha_i(a, -)^{-1}(B).$$

This subset is open, as finite intersection of open subsets. It contains a because $\alpha_i(a, a) = 0 \in B$ for each index i. Moreover $U \cap B = \emptyset$, because $b \in U \cap B$ would imply

$$a = \theta\big(\alpha_1(a, b), \ldots, \alpha_n(a, b), b\big) \in B$$

since then each $\alpha_i(a, b)$ and b itself would be in the subalgebra B. $\square$

Corollary 8 *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a subalgebra. The following conditions are equivalent:*

1. *B is a neighborhood of 0;*
2. *B is an open neighborhood of 0;*
3. *B is a closed neighborhood of 0.*

Proof (2) $\Rightarrow$ (3) follows from proposition 7 and (3) $\Rightarrow$ (1) is trivial. If B is a neighborhood of 0 and $b \in B$,

$$U = \bigcap_{i=1}^{n} \alpha_i(-,b)^{-1}(B)$$

is a neighborhood of b; it is contained in B because

$$x \in U \Rightarrow x = \theta\big(\alpha_1(x,b), \ldots, \alpha_n(x,b), b\big) \in B$$

since B is a subalgebra. Thus B is open. □

Let us now investigate the behaviour of subalgebras with respect to topological closure.

Proposition 9 *Let $\mathbb{T}$ be a semi-abelian theory. The closure $\overline{B} \subseteq A$ of every subalgebra $B \subseteq A$ of a topological $\mathbb{T}$-algebra A is still a subalgebra.*

Proof Let $\tau(X_1, \ldots, X_m)$ be a m-ary operation of the theory $\mathbb{T}$. Define P to be the topological pullback

$$\begin{array}{ccc} P & \longrightarrow & \overline{B} \\ \downarrow & & \downarrow \\ A^m & \xrightarrow[\tau]{} & A \end{array}$$

The topological subspace $P \subseteq A^m$ is closed because $\overline{B} \subseteq A$ is closed. Moreover $B^m \subseteq P$ because

$$b_1, \ldots, b_m \in B \Rightarrow \tau(b_1, \ldots, b_m) \in B \subseteq \overline{B}.$$

This implies $\overline{B}^m = \overline{B^m} \subseteq P$ because P is closed. This means exactly

$$a_1, \ldots, a_m \in \overline{B} \Rightarrow \tau(a_1, \ldots, a_m) \in \overline{B}$$

and $\overline{B}$ is stable in A for all the operations of the theory $\mathbb{T}$. □

Analogously, we obtain:

Proposition 10 *Let $\mathbb{T}$ be a semi-abelian theory. The closure $\overline{B} \subseteq A$ of every normal subalgebra $B \subseteq A$ of a topological $\mathbb{T}$-algebra A is still a normal subalgebra.*

Proof Using theorem 64, let us consider an operation $\tau(X_1, \ldots, X_k, Y_1, \ldots, Y_l)$ of the theory satisfying the axiom $\tau(X_1, \ldots, X_k, 0, \ldots, 0) = 0$. As in proposition 9, we consider the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, tail] & \overline{B} \arrow[d, tail] \\ A^k \times A^l \arrow[r, "\tau"'] & A \end{tikzcd}$$

to get P closed in $A^k \times A^l$. This time $A^k \times B^l \subseteq P$ because B is normal in A and thus $A^k \times \overline{B}^l \subseteq P$ because P is closed. By theorem 64, this shows that $\overline{B}$ is normal in A. $\square$

3 On topological quotients and regularity

The following proposition generalizes a key property of topological groups:

Proposition 11 *When $\mathbb{T}$ is a semi-abelian theory, the coequalizer $q\colon B \twoheadrightarrow Q$ of two morphisms $f, g\colon A \rightrightarrows B$ of $\mathsf{Top}^{\mathbb{T}}$ is computed as in $\mathsf{Mod}^{\mathbb{T}}$ and provided with the quotient topology. Moreover, the continuous surjection q is also an open map.*

Proof Consider first two morphisms $f, g\colon A \rightrightarrows B$ in $\mathsf{Top}^{\mathbb{T}}$ and their coequalizer $q\colon B \twoheadrightarrow Q$ in $\mathsf{Mod}^{\mathbb{T}}$ provided with the quotient topology; this makes already q continuous. The regular epimorphism q in $\mathsf{Mod}^{\mathbb{T}}$ is the cokernel of its kernel $k\colon K \rightarrowtail B$. If U is open in B, we must prove first that $q(U)$ is open in $Q \cong B/K$, that is, $q^{-1}\big(q(U)\big)$ is open in B. By proposition 65,

$$q^{-1}\big(q(U)\big) = \bigcup_{k_1, \ldots, k_n \in K} \theta(k_1, \ldots, k_n, -)^{-1}(U)$$

is indeed open, as a union of open subsets.

Next we prove that Q, provided with the quotient topology, is a topological $\mathbb{T}$-algebra. If $\tau(X_1, \ldots, X_k)$ is a k-ary operation of the theory $\mathbb{T}$ and $U \subseteq Q$ is open for the quotient topology

$$\begin{aligned} \tau^{-1}(U) &= \big\{\big([b_1], \ldots, [b_k]\big) \big| \tau\big([b_1], \ldots, [b_k]\big) \in U\big\} \\ &= \big\{q^k(b_1, \ldots, b_k) \big| q\tau(b_1, \ldots, b_n) \in U\big\} \\ &= q^k\big(\tau^{-1}q^{-1}(U)\big) \end{aligned}$$

and this last subset is open because q and τ are continuous and q^k is open, since so is q. Therefore the operation τ is continuous on the quotient Q.

It is now trivial to conclude that $q = \mathsf{Coker}\,(f, g)$ in $\mathsf{Top}^{\mathbb{T}}$. □

The category Top of topological spaces is not Barr regular, nevertheless:

Theorem 12 *The category $\mathsf{Top}^{\mathbb{T}}$ of topological models of a semi-abelian theory $\mathbb{T}$ is Barr regular.*

Proof In the category of topological spaces, every open surjection yields necessarily the quotient topology and open surjections are stable under pullbacks. One concludes by proposition 11. □

4 Introducing proabelian categories

The category $\mathsf{Top}^{\mathbb{T}}$, for a semi-abelian theory $\mathbb{T}$, is generally not semi-abelian, because it is not Barr exact. Indeed, the kernel pair of a morphism $f\colon A \longrightarrow B$ is its set-theoretical kernel pair provided with the topology induced by that of $A \times A$. Providing this kernel pair with a finer $\mathbb{T}$-topology (for example, the discrete one), yields an equivalence relation in $\mathsf{Top}^{\mathbb{T}}$ which is not a kernel pair.

This paper intends also to give evidence that there is a notion of good interest, more general than semi-abelianity and which recaptures many of the properties of semi-abelian categories:

Definition 13 *A category $\mathcal{V}$ is proabelian when*

1. *$\mathcal{V}$ is finitely complete;*
2. *$\mathcal{V}$ admits all coequalizers;*
3. *$\mathcal{V}$ has a zero object;*
4. *$\mathcal{V}$ is Barr regular;*
5. *$\mathcal{V}$ is protomodular.*

The proabelian category $\mathcal{V}$ is said to have semi-direct products when the inverse image functors of the fibration of points are monadic.

Let us recall that when $v\colon 0 \longrightarrow Y$ is such that v^* is monadic, every algebra (A, ξ) for the corresponding monad yields a point $\big(p, s\colon B \leftrightarrows Y\big)$ and B is defined to be the semi-direct product $Y \rtimes (A, \xi)$.

Of course every semi-abelian category is proabelian; but we shall now prove that the topological models of a semi-abelian theory constitute a proabelian category with semi-direct products. Other examples of proabelian categories will be presented in the subsequent sections.

Theorem 14 *A proabelian category $\mathcal{V}$ is in particular a Mal'cev category satisfying the five lemma, the nine lemma, the snake lemma and the Jordan-Hölder theorem.*

Proof Observe that the corresponding proofs given in [4] and [5] use only the proabelian axioms. $\square$

Proposition 15 *A proabelian category $\mathcal{V}$ with semi-direct products is finitely cocomplete.*

Proof It suffices to prove the existence of binary coproducts. This follows from proposition 4 in [7]: in a category with finite limits, if the inverse image functors of the fibration of points have left adjoints, these adjoints are computed by pushouts. But in the presence of a zero object, pushing out along a morphism $v\colon 0 \longrightarrow Y$ is taking the coproduct with V. $\square$

Let us prove now a useful property for constructing proabelian categories. In the following proposition, and in the whole paper, "epireflective" means always "regular epireflective": the unit of the adjunction is a regular epimorphism.

Proposition 16 *If $\mathcal{V}$ is a proabelian category, every epireflective subcategory $\mathcal{W} \subseteq \mathcal{V}$ is proabelian as well.*

Proof The category $\mathcal{W}$ is finitely complete, has coequalizers and a zero object, by reflexivity in $\mathcal{V}$. It is regular by epireflexivity. It is protomodular since the inclusion $\mathcal{W} \subseteq \mathcal{V}$ is full and preserves pullbacks. $\square$

Proposition 17 *When $\mathbb{T}$ is an algebraic theory, the forgetful functor*

$$U\colon \mathsf{Top}^{\mathbb{T}} \longrightarrow \mathsf{Mod}^{\mathbb{T}}$$

is topological. As a consequence, the category $\mathsf{Top}^{\mathbb{T}}$ of topological $\mathbb{T}$-algebras is complete and cocomplete; limits and colimits are computed as in $\mathsf{Mod}^{\mathbb{T}}$.

Proof If $(f_i\colon A \longrightarrow A_i)_{i\in I}$ is a class of morphisms of $\mathbb{T}$-algebras, with each A_i a topological $\mathbb{T}$-algebra, the corresponding initial topology on A (which exists, whatever the size of I) provides A with the structure of a topological $\mathbb{T}$-algebra. Indeed if $\tau(X_1, \ldots, X_k)$ is an operation of the theory, the continuity of τ in A is equivalent to the continuity of each $f_i \circ \tau_A$, which is the case since $f_i \circ \tau_A = \tau_{A_i} \circ f_i^k$. This forces the conclusion, since $\mathsf{Mod}^{\mathbb{T}}$ is complete and cocomplete (see [3], II-7.3). $\square$

Proposition 18 *When $\mathbb{T}$ is an algebraic theory yielding a protomodular category $\mathsf{Mod}^{\mathbb{T}}$ of models, the models of $\mathbb{T}$ in every category $\mathcal{C}$ with finite limits constitute again a protomodular category.*

Proof This is a standard "Yoneda" argument. Indeed, the functors

$$\mathcal{C}(C, -)\colon \mathsf{Mod}^{\mathbb{T}}(\mathcal{C}) \longrightarrow \mathsf{Mod}^{\mathbb{T}}, \quad A \mapsto \mathcal{C}(C, A)$$

individually preserve pullbacks and collectively reflect isomorphisms. Therefore the protomodularity of $\mathsf{Mod}^{\mathbb{T}}$ implies that of $\mathsf{Mod}^{\mathbb{T}}(\mathcal{C})$. $\square$

Theorem 19 *When $\mathbb{T}$ is a semi-abelian theory, the category $\mathsf{Top}^{\mathbb{T}}$ of topological $\mathbb{T}$-algebras is complete, cocomplete, proabelian with semi-direct products.*

Proof Using theorem 12, proposition 17 and proposition 18 (with $\mathcal{C} = \mathsf{Top}$, the category of topological spaces), it remains to check the existence of semi-direct products.

Given $v\colon X \longrightarrow Y$ in $\mathsf{Top}^{\mathbb{T}}$, the functor

$$v^*\colon \mathsf{Pt}_Y\big(\mathsf{Top}^{\mathbb{T}}\big) \longrightarrow \mathsf{Pt}_X\big(\mathsf{Top}^{\mathbb{T}}\big)$$

has a left adjoint, namely, the pushout along v. The functor v^* reflects isomorphisms because $\mathsf{Top}^{\mathbb{T}}$ is protomodular, by proposition 18. By the Beck criterion, we still have to check a condition on some coequalizers. But coequalizers in the categories $\mathsf{Pt}_Y\big(\mathsf{Top}^{\mathbb{T}}\big)$ and $\mathsf{Pt}_X\big(\mathsf{Top}^{\mathbb{T}}\big)$ are computed as in $\mathsf{Pt}_Y\big(\mathsf{Mod}^{\mathbb{T}}\big)$ and $\mathsf{Pt}_X\big(\mathsf{Mod}^{\mathbb{T}}\big)$, that is as in $\mathsf{Mod}^{\mathbb{T}}$, and provided with the quotient topology. Now

$$v^*\colon \mathsf{Pt}_Y\big(\mathsf{Mod}^{\mathbb{T}}\big) \longrightarrow \mathsf{Pt}_X\big(\mathsf{Mod}^{\mathbb{T}}\big)$$

preserves the coequalizers involved in the Beck criterion, because the category $\mathsf{Mod}^{\mathbb{T}}$ is semi-abelian. Moreover v^* preserves open surjections, as every topological pullback. We conclude by proposition 11. $\square$

Theorem 20 *When $\mathbb{T}$ is a semi-abelian theory, the forgetful functor*

$$U\colon \mathsf{Top}^{\mathbb{T}} \longrightarrow \mathsf{Top}$$

to the category of topological spaces is monadic.

Proof Given a topological space X, consider the free $\mathbb{T}$-algebra $F(X)$ on the set X and, for every continuous mapping $f\colon X \longrightarrow A$ to a topological $\mathbb{T}$-algebra A, the corresponding factorization $f'\colon F(X) \longrightarrow A$. The initial topological $\mathbb{T}$-algebra structure on $F(X)$ for all these mappings f' (see proposition 17) yields the left adjoint functor of U.

Every homeomorphic $\mathbb{T}$-homomorphism of topological $\mathbb{T}$-algebras is an isomorphism, thus U reflects isomorphisms.

Consider next a reflexive pair $f, g\colon A \rightrightarrows B$ of morphisms in $\mathsf{Top}^{\mathbb{T}}$ admitting a split coequalizer $q\colon B \twoheadrightarrow Q$ in Top. Coequalizers in $\mathsf{Top}^{\mathbb{T}}$ and in Top are constructed, respectively, as in $\mathsf{Mod}^{\mathbb{T}}$ and in Set, and are provided in both cases with the quotient topology. Thus being a coequalizer in $\mathsf{Top}^{\mathbb{T}}$ is being a coequalizer both in $\mathsf{Mod}^{\mathbb{T}}$ and in Top (see proposition 11). The Beck criterion applied to the forgetful functor $\mathsf{Mod}^{\mathbb{T}} \longrightarrow \mathsf{Set}$ indicates that $q = \mathsf{Coker}\,(f, g)$ in $\mathsf{Mod}^{\mathbb{T}}$. Since moreover $q = \mathsf{Coker}\,(f, g)$ in Top by assumption, $q = \mathsf{Coker}\,(f, g)$ in $\mathsf{Top}^{\mathbb{T}}$ and one concludes by again the Beck criterion. $\square$

5 Abelian topological algebras

The notion of abelian object makes sense in a proabelian category. Given an object A, the commutator $[A,A]$ is the kernel of the composite $q \circ (\mathsf{id}_A, 0) = q \circ (0, \mathsf{id}_A)$

$$
\begin{tikzcd}
{[A,A]} \arrow[r, tail, "k"] & A \arrow[r, shift left, "(\mathsf{id}_A,0)"] \arrow[r, shift right, "(0,\mathsf{id}_A)"'] & A \times A \arrow[r, two heads, "q"] & \rho(A)
\end{tikzcd}
$$

where $q = \mathsf{Coker}\big((\mathsf{id}_A, 0), (0, \mathsf{id}_A)\big)$. The object A is abelian when $A = [A,A]$ (see [4]). This is equivalent to the existence of a (necessarily unique) structure of internal abelian group on A.

The abelian objects in a semi-abelian category constitute an epireflective subcategory which is abelian. The corresponding topological result is:

Proposition 21 *Let $\mathcal{V}$ be a proabelian category. The abelian objects of $\mathcal{V}$ constitute an additive epireflective proabelian subcategory $\mathsf{Ab}(\mathcal{V})$ of $\mathcal{V}$. When $\mathcal{V}$ has semi-direct products, $\mathsf{Ab}(\mathcal{V})$ has semi-direct products as well.*

Proof The proofs in the semi-abelian case given in [4] remain valid to show that the coequalizer $\rho(A)$ above yields the expected epireflection ρ. By proposition 16, $\mathsf{Ab}(\mathcal{V})$ is proabelian. And of course $\mathsf{Ab}(\mathcal{V})$ is additive, since every object is internally an abelian group.

Suppose now that $\mathcal{V}$ has semi-direct products. By proposition 15, it is finitely cocomplete. Thus $\mathsf{Ab}(\mathcal{V})$ is finitely cocomplete as well, by reflexivity. To prove that the inverse image functors of the fibration of points of $\mathsf{Ab}(\mathcal{V})$ are monadic, we use once more the Beck criterion. Pushing out a point along v yields the left adjoint of v^*. On the other hand v^* reflects isomorphisms by protomodularity. Moreover since $\mathsf{Ab}(\mathcal{V})$ is closed in $\mathcal{V}$ under regular quotients, coequalizers in $\mathsf{Ab}(\mathcal{V})$ are computed as in $\mathcal{V}$. Next in a category of points, coequalizers are computed as in the base category. This forces at once the condition on coequalizers in the Beck criterion, since it holds in $\mathcal{V}$. $\square$

Proposition 22 *Let $\mathbb{T}$ be a semi-abelian theory and A an abelian topological $\mathbb{T}$-algebra. The operations*

$$
\begin{aligned}
a + b &= \theta\big(\alpha_1(a,0), \ldots, \alpha_n(a,0), b\big) \\
-a &= \theta\big(\alpha_1(0,a), \ldots, \alpha_n(0,a), 0\big)
\end{aligned}
$$

describe the internal abelian group structure of A and thus provide in particular A with the structure of a topological abelian group.

Proof Let us write

$$\oplus \colon A^2 \longrightarrow A, \quad \ominus \colon A \longrightarrow A$$

for the internal abelian group operations of A and

$$p(X,Y,Z) = \theta\big(\alpha_1(X,Y), \ldots, \alpha_n(X,Y), Z\big)$$

for the Mal'cev operation of $\mathbb{T}$ inherited from the semi-abelian structure. Since $\oplus$ and $\ominus$ are $\mathbb{T}$-homomorphisms, we get

$$\begin{aligned} a \oplus b &= p(a,0,0) \oplus p(0,0,b) \\ &= p\big(a \oplus 0, 0 \oplus 0, 0 \oplus b\big) \\ &= p(a,0,b) \\ &= a + b \\ a \oplus (-a) &= p(a,0,0) \oplus p(0,a,0) \\ &= p(a \oplus 0, 0 \oplus a, 0 \oplus 0) \\ &= p(a,a,0) \\ &= 0 \end{aligned}$$

which proves the result. $\square$

Theorem 23 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Ab}\big(\mathsf{Top}^{\mathbb{T}}\big)$ of abelian topological $\mathbb{T}$-algebras is complete, cocomplete, additive and proabelian with semi-direct products.*

Proof By proposition 21 and theorem 19. $\square$

Given a semi-abelian theory $\mathcal{T}$, the category $\mathsf{Ab}\big(\mathsf{Top}^{\mathbb{T}}\big)$ is generally not exact, since $\mathsf{Top}^{\mathbb{T}}$ is not. This prevents $\mathsf{Ab}\big(\mathsf{Top}^{\mathbb{T}}\big)$ to be abelian.

Theorem 24 *Let $\mathbb{T}$ be a semi-abelian theory. The forgetful functor*

$$U\colon \mathsf{Ab}\big(\mathsf{Top}^{\mathbb{T}}\big) \longrightarrow \mathsf{Top}$$

is monadic.

Proof We use the Beck criterion. The left adjoint of U is the composition of the left adjoints in theorems 20 and 21 (with $\mathcal{V} = \mathsf{Top}^{\mathbb{T}}$). It is again trivial that U reflects isomorphisms. The condition on coequalizers holds by the Beck criterion applied to the situation of theorem 20, since $\mathsf{Ab}\big(\mathsf{Top}^{\mathbb{T}}\big)$ is closed in $\mathsf{Top}^{\mathbb{T}}$ under regular quotients. $\square$

6 On Hausdorff algebras

We investigate now the properties of those $\mathbb{T}$-algebras which are Hausdorff spaces.

Proposition 25 *Let $\mathbb{T}$ be a semi-abelian theory. For a topological $\mathbb{T}$-algebra A, the following conditions are equivalent:*

1. *$\{0\}$ is closed in A;*

2. *A is a* T_0*-topological space;*

3. *A is a* T_1*-topological space;*

4. *A is a Hausdorff space.*

Proof (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious. Let us prove (2) $\Rightarrow$ (1). If A is T_0 but $0 \in A$ is not closed, choose $0 \neq a \in \overline{\{0\}}$. Every neighborhood of a contains 0, thus by the T_0-axiom there exists a neighborhood V of 0 which does not contain a. Let U be the neighborhood of lemma 5 corresponding to the function $\theta(X_1, \ldots, X_n, 0)$. Consider

$$W = \bigcap_{i=1}^{n} \alpha_i(a, -)^{-1}(U).$$

This is an open neighborhood of $a \in \overline{\{0\}}$, thus it contains 0. This means $\alpha_i(a, 0) \in U$ for each index i, thus

$$a = \theta\big(\alpha_1(a, 0), \ldots, \alpha_n(a, 0), 0\big) \in V$$

by construction of U. This is a contradiction.

(1) $\Rightarrow$ (3) holds by our metatheorem 4 while (3) $\Rightarrow$ (4) holds because every regular T_1-space is Hausdorff (see proposition 6). □

Proposition 26 *Let* $\mathbb{T}$ *be a semi-abelian theory and* B*, an abelian subalgebra of a Hausdorff* $\mathbb{T}$*-algebra* A*. The closure* $\overline{B} \subseteq A$ *is still an abelian subalgebra.*

Proof We must prove that the operations $+$ and $-$ of proposition 22, restricted to $\overline{B}$, are homomorphisms of $\mathbb{T}$-algebras. This means, for every operation $\tau(X_1, \ldots, X_k)$ of the theory, the equality for all elements of $\overline{B}$ of the following functions, defined and continuous for all elements of A

$$\tau(X_1, \ldots, X_k) + \tau(Y_1, \ldots, Y_k) = \tau(X_1 + Y_1, \ldots, X_k + Y_k)$$
$$-\tau(X_1, \ldots, X_k), = \tau(-X_1, \ldots, -X_k).$$

The equalities hold in B, thus they hold in $\overline{B}$, by continuity of the various functions and Hausdorffness of A. □

Proposition 27 *Let* $\mathbb{T}$ *be a semi-abelian theory and* A *a topological* $\mathbb{T}$*-algebra. For a subalgebra* $B \subseteq A$*, the following conditions are equivalent:*

1. *B is closed in A;*

2. *the quotient topological* $\mathbb{T}$*-algebra* A/B *is Hausdorff.*

Proof By proposition 25, the quotient A/B is Hausdorff when $[0]$ is closed in it. When this is the case, B is closed in A as inverse image of $[0]$ by the quotient map $q\colon A \twoheadrightarrow A/B$. Conversely if B is closed in A, its image $[0] \in A/B$ is a closed point because B is saturated and the quotient map q is open (see proposition 11). □

Corollary 28 *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a normal subalgebra. If B and A/B are Hausdorff $\mathbb{T}$-algebras, A is a Hausdorff algebra as well.*

Proof By proposition 25, $0 \in B$ is closed and by proposition 27, $B \subseteq A$ is closed as well. Thus $0 \in A$ is closed. $\square$

Corollary 29 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Haus}^{\mathbb{T}}$ of Hausdorff topological $\mathbb{T}$-algebras is epireflective in the category $\mathsf{Top}^{\mathbb{T}}$ of topological $\mathbb{T}$-algebras. In particular, it is complete and cocomplete.*

Proof Given a topological $\mathbb{T}$-algebra A, it follows at once from proposition 10 that $\overline{\{0\}}$ is the smallest closed normal subobject of A. Therefore $A/\overline{\{0\}}$ is the Hausdorff reflection of A, by proposition 27. One concludes by proposition 17. $\square$

Theorem 30 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Haus}^{\mathbb{T}}$ of Hausdorff $\mathbb{T}$-algebras is complete, cocomplete, proabelian and the forgetful functor*

$$U\colon \mathsf{Haus}^{\mathbb{T}} \longrightarrow \mathsf{Haus}$$

to the category of Hausdorff spaces is monadic.

Proof The category $\mathsf{Haus}^{\mathbb{T}}$ is proabelian by corollary 29 and proposition 16.

Let X be a Hausdorff space. The $\mathbb{T}$-Hausdorff reflection (corollary 29) of the free topological $\mathbb{T}$-algebra on X (theorem 20) yields the adjoint functor of U. The functor U reflects trivially isomorphisms.

To conclude by the Beck criterion, consider a reflexive pair $f, g\colon A \rightrightarrows B$ in $\mathsf{Haus}^{\mathbb{T}}$ which admits a split coequalizer $q\colon B \twoheadrightarrow Q$ in Haus. The split coequalizer is thus also a coequalizer in Top and of course, (f, g) is a reflexive pair in $\mathsf{Top}^{\mathbb{T}}$. By theorem 20, $q = \mathsf{Coker}\,(f, g)$ in $\mathsf{Top}^{\mathbb{T}}$. Since Q is a Hausdorff space, this is also a coequalizer in $\mathsf{Haus}^{\mathbb{T}}$. $\square$

Theorem 31 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Ab}(\mathsf{Haus}^{\mathbb{T}})$ of Hausdorff abelian $\mathbb{T}$-algebras is complete, cocomplete, additive, proabelian and the forgetful functor*

$$U\colon \mathsf{Ab}(\mathsf{Haus}^{\mathbb{T}}) \longrightarrow \mathsf{Haus}$$

to the category of Hausdorff spaces is monadic.

Proof The category $\mathsf{Ab}(\mathsf{Haus}^{\mathbb{T}})$ is proabelian by theorem 30 and proposition 21. The monadicity of U is proved as in theorem 24. $\square$

7 On compact algebras

Let us make clear that we do not include Hausdorffness in compactness. First of all, an obvious observation:

Proposition 32 *Let $\mathbb{T}$ be a semi-abelian theory. Every quotient of a compact $\mathbb{T}$-algebra is again compact.*

Proof Every continuous image of a compact is compact. $\square$

Here is an striking property of quotients, to be compared with proposition 11.

Proposition 33 *Let $\mathbb{T}$ be a semi-abelian theory and A, a topological $\mathbb{T}$-algebra. When $B \subseteq A$ is a compact normal subalgebra, the quotient $q\colon A \twoheadrightarrow A/B$ is a closed map.*

Proof Consider a closed subset $C \subseteq A$; we must prove that its saturation $\widetilde{C} = q^{-1}\big(q(C)\big)$ is closed as well. By proposition 65, we know that

$$\widetilde{C} = \big\{a \in A \big| \exists b_1, \ldots, b_n \in B \ \theta(b_1, \ldots, b_n, a) \in C\big\}.$$

Considering the continous mappings

$$A \xleftarrow{p_A} B^n \times A \rightarrowtail^{\iota} A^{n+1} \xrightarrow{\theta} A$$

where ι is the canonical inclusion, we have thus

$$\widetilde{C} = p_A\Big(\iota^{-1}\big(\theta^{-1}(C)\big)\Big).$$

Since C is closed, $\iota^{-1}\big(\theta^{-1}(C)\big)$ is closed as well. Since B^n is compact, the projection p_A is a closed map (see [6]) and therefore $\widetilde{C}$ is closed. $\square$

Proposition 34 *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a normal subalgebra B. If B and A/B are compact (resp. compact Hausdorff), A is compact (resp. compact Hausdorff) as well.*

Proof By proposition 65, for every element $a \in A$, the corresponding equivalence class is given by

$$[a] = \theta(B^n, a) = \big\{\theta(b_1, \ldots, b_n, a)\big| b_1, \ldots, b_n \in B\big\}.$$

This equivalence class is compact, as continuous direct image of the compact B^n. Therefore q is a closed continuous map (see proposition 33) with compact fibres $[a]$; thus q is a proper map and therefore, reflects compact subspaces (see [6] or [14]). In particular, $A = q^{-1}(A/B)$ is compact.

The Hausdorff case follows now from proposition 28. $\square$

In order to investigate further the properties of the category $\mathsf{HComp}^{\mathbb{T}}$ of compact Hausdorff $\mathbb{T}$-algebras, let us first observe that:

Proposition 35 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{HComp}^{\mathbb{T}}$ of compact Hausdorff $\mathbb{T}$-algebras is reflective in the category $\mathsf{Top}^{\mathbb{T}}$ of topological $\mathbb{T}$-algebras. In particular, the category $\mathsf{HComp}^{\mathbb{T}}$ is complete and cocomplete.*

Proof The category HComp of compact Hausdorff spaces is closed for limits in the category Top of topological spaces (it is even reflective in it). Therefore $\mathsf{HComp}^{\mathbb{T}}$ is closed in $\mathsf{Top}^{\mathbb{T}}$ under limits (see proposition 17). To get the expected adjoint functor, it remains to check the solution set condition.

If A is a fixed topological $\mathbb{T}$-algebra, every morphism $f\colon A \longrightarrow C$ in $\mathsf{Top}^{\mathbb{T}}$, with $C \in \mathsf{HComp}^{\mathbb{T}}$, factors through $\overline{f(A)} \subseteq C$, which is still a compact $\mathbb{T}$-algebra. Every point of $\overline{f(A)}$ is the limit of an ultrafilter in $f(A)$ and the cardinal of $f(A)$ is less than the cardinal of A. Write λ for the cardinal of the set of ultrafilters in A. There is only a set of compact $\mathbb{T}$-algebras with cardinal at most λ and, as we have just seen, they constitute a solution set for A. $\square$

Theorem 36 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{HComp}^{\mathbb{T}}$ of compact Hausdorff $\mathbb{T}$-algebras is complete, cocomplete, semi-abelian and the forgetful functor*

$$U\colon \mathsf{HComp}^{\mathbb{T}} \longrightarrow \mathsf{HComp}$$

to the category of compact Hausdorff spaces is monadic.

Proof The category $\mathsf{HComp}^{\mathbb{T}}$ is complete and cocomplete by proposition 35; limits are computed as in $\mathsf{Top}^{\mathbb{T}}$, thus as in $\mathsf{Haus}^{\mathbb{T}}$. By proposition 32, $\mathsf{HComp}^{\mathbb{T}}$ is closed in $\mathsf{Haus}^{\mathbb{T}}$ for regular quotients; therefore $\mathsf{HComp}^{\mathbb{T}}$ is regular by theorem 30. It is also protomodular by proposition 18.

To prove the exactness of $\mathsf{HComp}^{\mathbb{T}}$, consider in $\mathsf{HComp}^{\mathbb{T}}$ an equivalence relation $r\colon R \rightarrowtail A \times A$. Since R is compact, it is homeomorphic to its image $r(R)$, thus provided with the induced topology. But being compact, R is closed in $A \times A$. In particular the equivalence class of 0 is closed, as inverse image of R along $(\mathsf{id}_A, 0)$. Thus the corresponding quotient in $\mathsf{Top}^{\mathbb{T}}$ is compact Hausdorff (see propositions 27 and 32) and R is the kernel pair of this quotient, since this is the case in $\mathsf{Mod}^{\mathbb{T}}$ and R has the induced topology.

Given a compact Hausdorff space X, the $\mathbb{T}$-compact reflection (proposition 35) of the free topological $\mathbb{T}$-algebra on X (theorem 19) yields the adjoint functor of U. The functor U reflects trivially isomorphisms. The condition on coequalizers in the Beck criterion is satisfied in the Hausdorff case (theorem 30), thus also in the compact Hausdorff case, where coequalizers are computed in the same way (proposition 32). $\square$

Theorem 37 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Ab}\big(\mathsf{HComp}^{\mathbb{T}}\big)$ of compact Hausdorff abelian $\mathbb{T}$-algebras is complete, cocomplete, abelian and the forgetful functor*

$$U\colon \mathsf{Ab}\big(\mathsf{HComp}^{\mathbb{T}}\big) \longrightarrow \mathsf{HComp}$$

to the category of compact Hausdorff spaces is monadic.

Proof The category $\mathsf{Ab}\big(\mathsf{HComp}^{\mathbb{T}}\big)$ is abelian by semi-abelianity of $\mathsf{HComp}^{\mathbb{T}}$ (theorem 36). The monadicity of U is proved as in theorem 24. $\square$

8 On locally compact algebras

Again we do not include Hausdorffness in local compactness.

Proposition 38 *Let $\mathbb{T}$ be a semi-abelian theory. For a $\mathbb{T}$-algebra A, the following conditions are equivalent:*

1. *0 has a compact neighborhood;*
2. *A is locally compact.*

Proof (2) $\Rightarrow$ (1) is obvious.

Assume now that V is a compact neighborhood of some point $a \in A$. If U is an arbitrary neighborhood of a, by regularity (see proposition 6), we consider closed neighborhoods $V' \subseteq V$ and $U' \subseteq U$ of a. Then $U' \cap V' \subseteq U$ is a closed neighborhood of a which is compact, as a closed subset of the compact V. Thus a admits a fundamental system of compact neighborhoods. This proves that assuming (1), it suffices to show that every point $a \in A$ admits a compact neighborhood.

Given $a \in A$ and K a compact neighborhood of 0,

$$\theta(K^n, a) = \big\{\theta(k_1, \ldots, k_n, a) \big| k_1, \ldots, k_n \in K\big\}$$

is compact, as continuous image of the compact K^n. To prove that $\theta(K^n, a)$ is a neighborhood of a, it suffices to show that

$$\bigcap_{i=1}^{n} \alpha_i(-, a)^{-1}(K) \subseteq \theta(K^n, a)$$

(see corollary 3). Indeed if $\alpha_i(x, a) \in K$ for each index i

$$x = \theta\big(\alpha_1(x, a), \ldots, \alpha_n(x, a), a\big) \in \theta(K^n, a). \qquad \square$$

Proposition 39 *Let $\mathbb{T}$ be a semi-abelian theory and A a Hausdorff $\mathbb{T}$-algebra. Every locally compact subalgebra B of A is closed.*

Proof Given $a \in \overline{B}$, we must prove that $a \in B$. For this we choose a compact neighborhood Z of 0 in B, which has thus the form $Z = U \cap B$ for some neighborhood U of 0 in A. The continuous image of the compact $U \cap B \subseteq B$ in A is compact, thus closed. In other words, $Z = U \cap B$ is closed in A. We choose further an open neighborhood $U' \subseteq U$ of 0 in A. We consider then the open subset

$$V = \bigcap_{i=1}^{n} \alpha_i(a, -)^{-1}(U')$$

which is a neighborhood of $a \in \overline{B}$, thus meets B:

$$\exists b \in B\ \forall i\ \alpha_i(a,b) \in U'.$$

Let us prove now that $\alpha_i(a,b) \in B$ for each index i. For this it suffices to prove that

$$\alpha_i(a,b) \in U' \cap \overline{B} \subseteq \overline{U' \cap B} \subseteq \overline{U \cap B} = U \cap B \subseteq B,$$

where the first inclusion holds because U' is open. By choice of b, $\alpha_i(a,b) \in U'$. Since $a, b \in \overline{B}$, $\alpha_i(a,b) \in \overline{B}$ by proposition 9.

One concludes now that

$$a = \theta\big(\alpha_1(a,b), \ldots, \alpha_n(a,b), b\big) \in B$$

since b and all the $\alpha_i(a,b)$ are in the subalgebra B. □

Proposition 40 *Let $\mathbb{T}$ be a semi-abelian theory and A a locally compact $\mathbb{T}$-algebra. Every topological quotient $\mathbb{T}$-algebra of A is still locally compact.*

Proof Because every open (see proposition 5) continuous image of a locally compact space is locally compact. □

Proposition 41 *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a normal subalgebra B. If B is compact and A/B is locally compact, A is locally compact.*

Proof The same argument as in proposition 34 shows that the quotient map $q\colon A \twoheadrightarrow A/B$ reflects compact subspaces, thus also compact neighbourhoods. One concludes by proposition 38. □

Theorem 42 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{HLComp}^{\mathbb{T}}$ of locally compact Hausdorff $\mathbb{T}$-algebras is proabelian.*

Proof A closed subspace of a locally compact space is locally compact. Since in the Hausdorff case equalizers are closed subspaces, the category $\mathsf{HLComp}^{\mathbb{T}}$ is closed in $\mathsf{Haus}^{\mathbb{T}}$ for equalizers; it is also obviously closed for finite products. By proposition 40, HLComp is also closed in $\mathsf{Haus}^{\mathbb{T}}$ for coequalizers, thus regular quotients. Therefore HLComp is regular, since so is $\mathsf{Haus}^{\mathbb{T}}$ (see theorem 30). Finally HLComp is protomodular by proposition 18. □

Theorem 43 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Ab}\big(\mathsf{HLComp}^{\mathbb{T}}\big)$ of locally compact Hausdorff abelian $\mathbb{T}$-algebras is additive proabelian.*

Proof By proposition 21 and theorem 42. □

9 On discrete algebras

The category $\mathsf{Mod}^{\mathbb{T}}$ of $\mathbb{T}$-algebras can be identified with the category of discrete $\mathbb{T}$-algebras. Observe at once that

Proposition 44 *Let $\mathbb{T}$ be a semi-abelian theory. The forgetful functor*

$$U\colon \mathsf{Top}^{\mathbb{T}} \longrightarrow \mathsf{Mod}^{\mathbb{T}}, \quad U(A) = A$$

has both a left and a right adjoint which map a $\mathbb{T}$-algebra A on A provided with, respectively, the discrete or the indiscrete topology.

Proof This is trivial. □

Let us observe further that:

Proposition 45 *Let $\mathbb{T}$ be a semi-abelian theory. For a topological $\mathbb{T}$-algebra A, the following conditions are equivalent:*

1. *$\{0\}$ is open in A;*
2. *A is a discrete topological space.*

Proof (1) $\Rightarrow$ (2) holds by our metatheorem 4 while the converse is obvious. □

Proposition 46 *Let $\mathbb{T}$ be a semi-abelian theory and A a topological $\mathbb{T}$-algebra. For a subalgebra $B \subseteq A$, the following conditions are equivalent:*

1. *B is open in A;*
2. *the quotient topological $\mathbb{T}$-algebra A/B is discrete.*

Proof By proposition 45, the quotient A/B is discrete when $[0]$ is open in it. When this is the case, B is open in A as inverse image of $[0]$ by the quotient map $q\colon A \twoheadrightarrow A/B$. Conversely if B is open in A, its image $[0] \in A/B$ is an open point because the quotient map q is open (see proposition 11). □

Of course the category of discrete $\mathbb{T}$-algebras is semi-abelian and monadic over the category Set of sets, since it is isomorphic to the category $\mathsf{Mod}^{\mathbb{T}}$ of $\mathbb{T}$-algebras.

10 Connected or totally disconnected algebras

We recall that a space is totally disconnected when the connected component of each point is reduced to that point.

Lemma 47 *Let $\mathbb{T}$ be a semi-abelian theory and A, a topological $\mathbb{T}$-algebra. Writing $\Gamma(a)$ for the connected component of a point $a \in A$,*

$$\Gamma(a) = \theta\big(\Gamma(0)^n, a\big) = \big\{\theta(b_1, \ldots, b_n, a) \big| b_1, \ldots, b_n \in \Gamma(0)\big\}.$$

Proof The subset $\theta\big(\Gamma(a)^n, a\big) \subseteq A$ is connected as direct image of the connected space $\Gamma(a)^n$ by a continuous function. It contains $a = \theta(0, \ldots, 0, a)$ by lemma 63. Thus it is contained in the connected component $\Gamma(a)$.

Conversely, let $b \in \Gamma(a)$. Each set $\alpha_i\big(\Gamma(a), a\big)$ contains $0 = \alpha_i(a, a)$ and is connected, as direct image of the connected space $\Gamma(a)$ by a continuous function. Thus $\alpha_i\big(\Gamma(a), a\big) \subseteq \Gamma(0)$. Therefore

$$b = \theta\big(\alpha_1(b, a), \ldots, \alpha_n(b, a), a\big) \in \theta\big(\Gamma(0)^n, a\big). \qquad \square$$

Proposition 48 *Let $\mathbb{T}$ be a semi-abelian theory and A a topological $\mathbb{T}$-algebra. The following conditions are equivalent:*

1. *the connected component of 0 is reduced to $\{0\}$;*
2. *A is totally disconnected.*

Proof By lemma 47. $\square$

Proposition 49 *Let $\mathbb{T}$ be a semi-abelian theory and A, a topological $\mathbb{T}$-algebra. The connected component of 0 in A is a closed normal subalgebra.*

Proof The connected component of a point is always a closed subset. Let us write B for the connected component of 0 in A. By theorem 64, it suffices to prove that for every operation

$$\tau(X_1, \ldots, X_k, Y_1, \ldots, Y_l) \text{ such that } \tau(X_1, \ldots, X_k, 0, \ldots, 0) = 0$$

one has

$$\forall a_1, \ldots, a_k \in A,\ \forall b_1, \ldots, b_l \in B\ \tau(a_1, \ldots, a_k, b_1, \ldots, b_l) \in B.$$

The case $k = 0$ proves in particular that B is a subalgebra. We prove this statement by induction on l.

When $l = 0$, the statement reduces to $0 \in B$. Assuming the result for $l - 1$ and considering the operation

$$\tau(X_1, \ldots, X_k, Y_1, \ldots, Y_{l-1}, 0),$$

we know by inductive assumption that

$$\forall a_1, \ldots, a_k \in A,\ \forall b_1, \ldots, b_l \in B\ \tau(a_1, \ldots, a_k, b_1, \ldots, b_{l-1}, 0) \in B.$$

Thus B is also the connected component of $\tau(a_1, \ldots, a_k, b_1, \ldots, b_{l-1}, 0)$. Therefore

$$\tau(a_1, \ldots, a_k, b_1, \ldots, b_{l-1}, b_l) \in \tau(a_1, \ldots, a_k, b_1, \ldots, b_{l-1}, -)(B) \subseteq B$$

since the continuous image of a connected subset is connected. $\square$

Proposition 50 *Let $\mathbb{T}$ be a semi-abelian theory. Every quotient of a connected topological $\mathbb{T}$-algebra is again connected.*

Proof The direct continuous image of a connected space is connected. $\square$

Lemma 51 *Let $\mathbb{T}$ be a semi-abelian theory and A, a topological $\mathbb{T}$-algebra. If $B \subseteq A$ is a connected normal subobject, every equivalence class $[a]$ of an element $a \in A$ is connected and every clopen $U \subseteq A$ is saturated for the equivalence relation corresponding to the quotient $q\colon A \twoheadrightarrow A/B$.*

Proof Given $a \in U$, we consider the continuous function

$$\varphi\colon A^n \longrightarrow A, \quad (X_1, \ldots, X_n) \mapsto \theta(X_1, \ldots, X_n, a).$$

By proposition 65, we know that $[a] = \varphi(B^n)$; thus $[a]$ is connected as direct image of the connected subspace $B^n \subseteq A^n$. In particular, if $[a]$ intersects a clopen U, by connectedness, $[a] \subseteq U$. This proves that U is saturated. $\square$

Proposition 52 *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$, a normal subalgebra. If both B and A/B are connected, then A is connected as well.*

Proof Write $q\colon A \twoheadrightarrow A/B$ for the quotient map. Let U be a clopen of A. By lemma 51, U is saturated, thus $q(U)$ is a clopen of A/B. This forces $q(U) = \emptyset$ or $q(U) = A/B$, that is, $U = \emptyset$ or $U = A$. $\square$

Proposition 53 *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$, a normal subalgebra. If both B and A/B are totally disconnected, then A is totally disconnected as well.*

Proof Write $q\colon A \twoheadrightarrow A/B$ for the quotient. Since $q\big(\Gamma(0)\big)$ is connected and contains $[0]$, it is reduced to that element, because A/B is totally disconnected. This implies $\Gamma(0) \subseteq B$ and since B is totally disconnected, this forces $\Gamma(0) = \{0\}$. One concludes by proposition 48. $\square$

Proposition 54 *Let $\mathbb{T}$ be a semi-abelian theory and A, a topological $\mathbb{T}$-algebra. The quotient of A by the connected component of 0 is a totally disconnected $\mathbb{T}$-algebra.*

Proof By proposition 49, the connected component $\Gamma(0)$ of 0 is a closed normal subobject of A. Consider the following diagram, where the right hand square is a pullback and $k = \mathsf{Ker}\,(p)$.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, dotted, "r"'] & C \arrow[r, two heads, "p"] \arrow[d, tail, "s"'] & \Gamma\big([0]\big) \arrow[r] \arrow[d, tail, "t"] & 0 \\
0 \arrow[r] & \Gamma(0) \arrow[r, tail, "i"'] & A \arrow[r, two heads, "q"'] & A/\Gamma(0) \arrow[r] & 0
\end{tikzcd}$$

By theorem 12, p is a regular epimorphism in $\mathsf{Top}^{\mathbb{T}}$, thus the cokernel of its kernel k. Since pullbacks commute with kernels, the left hand square is a pullback as well, thus an intersection.

Now $q \circ i = 0 = t \circ 0$, thus i factors through the right hand pullback, yielding $\Gamma(0) \subseteq C$. This implies $K = \Gamma(0) \cap C = \Gamma(0)$. Next $K = \Gamma(0)$ and $\Gamma\big([0]\big)$ are connected components, thus by proposition 52, the algebra C is connected. But since C is connected and contains 0, $C \subseteq \Gamma(0)$ and finally, $C = \Gamma(0)$. Therefore

$$\Gamma\big([0]\big) = q(C) = q\big(\Gamma(0)\big) = [0].$$

One concludes by proposition 48. $\square$

Corollary 55 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{TotDisc}^{\mathbb{T}}$ of totally disconnected $\mathbb{T}$-algebras is epireflective in the category $\mathsf{Top}^{\mathbb{T}}$ of all topological $\mathbb{T}$-algebras and also in the category $\mathsf{Haus}^{\mathbb{T}}$ of Hausdorff $\mathbb{T}$-algebras. In particular, the category $\mathsf{TotDisc}^{\mathbb{T}}$ is complete and cocomplete.*

Proof Let $f\colon A \longrightarrow C$ be a morphism of topological $\mathbb{T}$-algebras, with C totally disconnected. Since the direct image of a connected subspace is a connected subspace, the connected component of $0 \in A$ is mapped in the connected component of $0 \in C$, that is, on the singleton 0. Therefore f factors through the quotient of proposition 54, which is thus the expected totally disconnected reflection of A.

Every totally disconnected $\mathbb{T}$-algebra is a Hausdorff $\mathbb{T}$-algebra. The totally disconnected reflection of a Hausdorff $\mathbb{T}$-algebra A is its reflection as topological $\mathbb{T}$-algebra. $\square$

Theorem 56 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{TotDisc}^{\mathbb{T}}$ of totally disconnected $\mathbb{T}$-algebras is complete, cocomplete, proabelian and the forgetful functor*

$$U\colon \mathsf{TotDisc}^{\mathbb{T}} \longrightarrow \mathsf{TotDisc}$$

to the category of totally disconnected spaces is monadic.

Proof The category $\mathsf{TotDisc}^{\mathbb{T}}$ is proabelian by corollary 55 and propositions 19 (or 30) and 16. The monadicity of U is proved as in theorem 30. $\square$

Theorem 57 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Ab}\big(\mathsf{TotDisc}^{\mathbb{T}}\big)$ of totally disconnected abelian $\mathbb{T}$-algebras is complete, cocomplete, proabelian and the forgetful functor*

$$U\colon \mathsf{Ab}\big(\mathsf{TotDisc}^{\mathbb{T}}\big) \longrightarrow \mathsf{TotDisc}$$

to the category of totally disconnected spaces is monadic.

Proof The category $\mathsf{Ab}\big(\mathsf{TotDisc}^{\mathbb{T}}\big)$ is proabelian by theorem 56 and proposition 21. The monadicity of U is proved as in theorem 24. $\square$

11 On profinite algebras

A compact, totally disconnected space is also called a profinite space, or a Stone space.

Proposition 58 *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$, a normal subalgebra. If both B and A/B are profinite, then A is profinite as well.*

Proof By propositions 34 and 53. $\square$

Proposition 59 *Let $\mathbb{T}$ be a semi-abelian theory and A a profinite $\mathbb{T}$-algebra. If $B \subseteq A$ is a closed normal subalgebra, the quotient topological $\mathbb{T}$-algebra A/B is still profinite.*

Proof By proposition 27, the quotient A/B is Hausdorff; it is also compact, as continuous image of the compact A. Each equivalence class $[a]$ is closed – thus compact – in A as inverse image of the closed point $[a]$ of the Hausdorff space A/B. Notice also that B is compact, as a closed subspace of a compact Hausdorff one. By propositions 11 and 33, the quotient map $q\colon A \twoheadrightarrow A/B$ is both open and closed.

Given elements $[a] \neq [b] \in A/B$, the compact subsets $[a]$ and $[b]$ can be included in disjoint clopens U, V of A, by profiniteness of the space:

$$[a] \subseteq U, \quad [b] \subseteq V, \quad U \cap V = \emptyset.$$

Since the projection $q\colon A \twoheadrightarrow A/B$ is open and closed, $q(U)$ is a clopen in A/B and thus its saturation $q^{-1}\big(q(U)\big)$ is a clopen in A.

Since $q^{-1}\big(q(U)\big)$ is a saturated clopen, so is its complement. Of course these saturated clopens are disjoint and it remains to prove that

$$[a] \subseteq q^{-1}\big(q(U)\big), \quad [b] \subseteq \complement q^{-1}\big(q(U)\big).$$

The first assertion is clear. To prove the second, it suffices to show that $b \notin q^{-1}\big(q(U)\big)$, that is, $U \cap [b] = \emptyset$. This is the case because $U \cap [b] \subseteq U \cap V = \emptyset$. $\square$

Corollary 60 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Prof}^{\mathbb{T}}$ of profinite $\mathbb{T}$-algebras is epireflective in the category $\mathsf{HComp}^{\mathbb{T}}$ of compact Hausdorff $\mathbb{T}$-algebras, thus also reflective in the category $\mathsf{Top}^{\mathbb{T}}$ of topological $\mathbb{T}$-algebras. In particular, the category $\mathsf{Prof}^{\mathbb{T}}$ is complete and cocomplete.*

Proof Consider a compact Hausdorff algebra A and the connected component $B \subseteq A$ of 0. The quotient A/B is totally disconnected by proposition 54 and compact by proposition 32. Thus A/B is profinite. One concludes as for corollary 55 that $\mathsf{Prof}^{\mathbb{T}}$ is epireflective in $\mathsf{HComp}^{\mathbb{T}}$. It remains to compose with the reflection of proposition 35. $\square$

Theorem 61 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Prof}^{\mathbb{T}}$ of profinite $\mathbb{T}$-algebras is complete, cocomplete, semi-abelian and the forgetful functor*

$$U\colon \mathsf{Prof}^{\mathbb{T}} \longrightarrow \mathsf{Prof}$$

to the category of profinite spaces is monadic.

Proof The category $\mathsf{Prof}^{\mathbb{T}}$ is complete and cocomplete by corollary 60 and proposition 17. It is proabelian by corollary 60, proposition 16 and theorem 36. The exactness of $\mathsf{Prof}^{\mathbb{T}}$ follows from that of $\mathsf{HComp}^{\mathbb{T}}$ (see theorem 36) since $\mathsf{Prof}^{\mathbb{T}}$ is closed in $\mathsf{HComp}^{\mathbb{T}}$ for finite limits, but also for quotients by proposition 59.

The monadicity of U is proved as in theorem 36, replacing Hausdorff spaces by completely disconnected ones. $\square$

Theorem 62 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Ab}\big(\mathsf{Prof}^{\mathbb{T}}\big)$ of profinite abelian $\mathbb{T}$-algebras is complete, cocomplete, abelian and the forgetful functor*

$$U\colon \mathsf{Ab}\big(\mathsf{Prof}^{\mathbb{T}}\big) \longrightarrow \mathsf{Prof}$$

to the category of profinite spaces is monadic.

Proof The category $\mathsf{Ab}\big(\mathsf{Prof}^{\mathbb{T}}\big)$ is abelian by semi-abelianity of $\mathsf{Prof}^{\mathbb{T}}$ (theorem 61). The monadicity of U is proved as in theorem 24. $\square$

12 Appendix

This section contains some purely algebraic results on semi-abelian theories: most of them can be found, in possibly rather different form, in a series of papers on universal algebra due to Ursini (see in particular [19]). We give here direct (categorical) proofs.

Lemma 63 *Let $\mathbb{T}$ be a semi-abelian theory. Given elements a, b, c of a $\mathbb{T}$-algebra A:*

$$\begin{aligned} \big(\forall i \quad \alpha_i(a,c) = \alpha_i(b,c)\big) &\Rightarrow \big(a = b\big), \\ \big(\forall i \quad \alpha_i(a,b) = 0\big) &\Rightarrow \big(a = b\big), \\ \theta(0,\dots,0,a) &= a. \end{aligned}$$

Proof The first case is the injectivity condition in proposition 2; the second case is obtained from the first one by putting $c = b$. The third assertion is obtained by writing $0 = \alpha_i(a,a)$. $\square$

Notice that the implication

$$\big(\forall i \quad \alpha_i(c,a) = \alpha_i(c,b)\big) \Rightarrow \big(a = b\big)$$

has no reason to hold in general.

Let us now recall that a Mal'cev operation is a ternary operation $p(X,Y,Z)$ such that

$$p(X,X,Y) = Y, \quad p(X,Y,Y) = X.$$

In a semi-abelian theory $\mathbb{T}$, the formula

$$p(X,Y,Z) = \theta\big(\alpha_1(X,Y),\dots,\alpha_n(X,Y),Z\big)$$

defines a Mal'cev operation (see lemma 63). The following result – valid in particular for semi-abelian theories – is borrowed from [19]; we propose here a direct proof.

Theorem 64 *Let $\mathbb{T}$ be an algebraic theory containing a unique constant 0 and a Mal'cev operation $p(X,Y,Z)$. For a subalgebra $B \subseteq A$, the following conditions are equivalent:*

1. *B is the kernel of some morphism $q\colon A \longrightarrow Q$ of $\mathbb{T}$-algebras;*
2. *for every operation $\tau(X_1,\dots,X_k,Y_1,\dots,Y_l)$ of the theory satisfying the axiom $\tau(X_1,\dots,X_k,0,\dots,0) = 0$ and for all elements $a_1,\dots,a_k \in A$, $b_1,\dots,b_l \in B$, one has $\tau(a_1,\dots,a_k,b_1,\dots,b_l) \in B$.*

Proof The necessity of the condition is obvious. Conversely, consider the subalgebra $R \subseteq A \times A$ generated by all the pairs

$$(a,a) \text{ for } a \in A, \quad (b,0) \text{ for } b \in B.$$

By construction, R is a reflexive relation in $\mathsf{Mod}^{\mathbb{T}}$, thus a congruence by the Mal'cev property (see [13]). Define $q\colon A \twoheadrightarrow Q$ to be the quotient of A by R. The kernel of q contains B since each pair $(b,0)$, for $b \in B$, is in R. Conversely, if $a \in A$ is such that $q(a) = 0$, the pair $(a,0)$ is in R and therefore is an algebraic combination of the generators of R: there exists an operation γ and elements $a_i \in A$, $b_j \in B$ such that

$$\begin{aligned}(a,0) &= \gamma\big((a_1,a_1),\dots,(a_k,a_k),(b_1,0),\dots,(b_l,0)\big)\\ &= \big(\gamma(a_1,\dots,a_k,b_1,\dots,b_l),\gamma(a_1,\dots,a_k,0,\dots,0)\big).\end{aligned}$$

The operation

$$\begin{aligned}\tau(X_1,\dots,X_k,&Y_1,\dots,Y_l)\\ &= p\big(\gamma(X_1,\dots,X_k,Y_1,\dots,Y_l),\gamma(X_1,\dots,X_k,0,\dots,0),0\big)\end{aligned}$$

satisfies the conditions of assumption 2 and

$$\begin{aligned}
a &= \gamma(a_1,\ldots,a_k,b_1,\ldots,b_l)\\
&= p\big(\gamma(a_1,\ldots,a_k,b_1,\ldots,b_l),0,0\big)\\
&= p\big(\gamma(a_1,\ldots,a_k,b_1,\ldots,b_l),\gamma(a_1,\ldots,a_k,0,\ldots,0),0\big)\\
&= \tau(a_1,\ldots,a_k,b_1,\ldots,b_l)
\end{aligned}$$

and this last term is in B by assumption 2. □

For example, when $\mathbb{T}$ is the theory of groups, the operation

$$\tau(X,Y) = X + Y - X$$

satisfies $\tau(X,0) = 0$ and we know that a subgroup $B \subseteq A$ is a kernel (i.e. is normal) precisely when

$$\forall a \in A\ \forall b \in B\ \tau(a,b) \in B.$$

When $\mathbb{T}$ is the theory of rings with unique constant 0, the operations

$$\tau_1(X,Y) = XY, \quad \tau_2(X,Y) = YX$$

satisfy $\tau_i(X,0) = 0$ and a subring $B \subseteq A$ is a kernel (= a two-sided ideal) precisely when

$$\forall a \in A\ \forall b \in B\ \tau_1(a,b) \in B,\ \tau_2(a,b) \in B.$$

Finally let us describe more precisely the quotient by a normal subobject:

Proposition 65 *Let $\mathbb{T}$ be a semi-abelian theory and $B \subseteq A$ a normal subalgebra. Given an arbitrary subset $X \subseteq A$, the saturation $\widetilde{X}$ of X for the corresponding quotient $q\colon A \longrightarrow A/B$ is given by*

$$\begin{aligned}
\widetilde{X} &= q^{-1}\big(q(X)\big)\\
&= \big\{a \in A \big| \exists x \in X\ \forall i\ \alpha_i(a,x) \in B\big\}\\
&= \big\{a \in A \big| \exists x \in X\ \forall i\ \alpha_i(x,a) \in B\big\}\\
&= \big\{a \in A \big| \exists b_1,\ldots,b_n \in B\ \theta(b_1,\ldots,b_n,a) \in X\big\}\\
&= \big\{\theta(b_1,\ldots,b_n,x) \big| b_1,\ldots,b_n \in B,\ x \in X\big\}.
\end{aligned}$$

In particular, for every $x \in A$,

$$[x] = \theta(B^n,x) = \big\{\theta(b_1,\ldots,b_n,x)\big| b_1,\ldots,b_n \in B\big\}.$$

Proof By semi-abelianity of $\mathsf{Mod}^{\mathbb{T}}$, given elements $a, c \in A$

$$[a] = [c] \in A/B \quad \Leftrightarrow \quad \forall i\ \big[\alpha_i(a,c)\big] = \alpha_i\big([a],[c]\big) = 0 \quad \Leftrightarrow \quad \forall i\ \alpha_i(a,c) \in B$$

where the first equivalence holds by lemma 63. This condition is of course left-right symmetric since so is the equality $[a] = [c]$.

Next if $\theta(b_1, \ldots, b_n, a) \in X$, we have in A/B (see lemma 63)

$$\begin{aligned} [a] &= \big[\theta(0, \ldots, 0, a)\big] \\ &= \theta\big([0], \ldots, [0], [a]\big) \\ &= \theta\big([b_1], \ldots, [b_n], [a]\big) \\ &= \big[\theta(b_1, \ldots, b_n, a)\big] \\ &\in q(X) \end{aligned}$$

thus $a \in q^{-1}\big(q(X)\big)$. Conversely if $a \in q^{-1}\big(q(X)\big)$, there exists $x \in X$ such that $[x] = [a]$, that is by the first part of the proof, $\alpha_i(x, a) \in B$ for each index i. This implies

$$\theta\big(\alpha_1(x, a), \ldots, \alpha_n(x, a), a\big) = x \in X$$

and it suffices to choose $b_i = \alpha_i(x, a)$.

Finally when $a \in \widetilde{X}$, we have already observed that

$$a = \theta\big(\alpha_1(a, x), \ldots, \alpha_n(a, x), x\big)$$

with $x \in X$ and $\alpha_i(a, x) \in B$ for each index i. Conversely if $x \in X$ and $b_i \in B$ for each index i, using lemma 63 we obtain

$$\big[\theta(b_1, \ldots, b_n, x)\big] = \theta\big([b_1], \ldots, [b_n], [x]\big) = \theta\big([0], \ldots, [0], [x]\big) = [x]$$

thus $\theta(b_1, \ldots, b_n, x) \in \widetilde{X}$. $\square$

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