Towards the socle of Eulerian ideals

Jorge Neves

Centre and Department of Mathematics
of the University of Coimbra

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- Let G be a simple graph.
- Def. $C \subseteq E_G$ is called Eulerian if $\deg_C(v)$ is even, $\forall\, v \in V_G$.
- Def. $J \subseteq E_G$ is a join iff

  $$|J \cap C| \leqslant \frac{|C|}{2}, \ \forall\, C \subseteq E_G \text{ Eulerian.}$$

  [Frank (1993). Combinatorica]

- J is a join iff J is a minimum cardinality T-join, where T is (some) even set of vertices. [by Guan's Lemma, (1960)]

- Consider $K[E_G] = K[t_e : e \in E_G]$. If $J \subseteq E_G$ let $\mathbf{t}_J = \prod_{e \in J} t_e$.

- $\mathbf{t}_J - \mathbf{t}_L \in K[E_G]$ is said an Eulerian binomial iff

  $J \cap L = \emptyset, \quad |J| = |L|, \quad J \cup L$ is Eulerian.

$$\begin{aligned} J(G) &= I(G) + (t_\ell^2) \\ &= (\{\text{Eulerian binomials}\} \cup \left\{ t_e^2 : e \in E_G \right\}). \end{aligned}$$

- Then, $\left(\dim_K \frac{K[E_G]_d}{J(G)_d} \right)_{d \geqslant 0} = \ (h_0, h_1, h_2, \ldots, h_r, 0, \ldots).$

- Question: can r be related directly to G?

- Thm. If G is bipartite then

$$r = \max\{|J| : J \text{ is a join of } G\}$$

[Neves, Vaz Pinto, Villarreal (2020). Journal of Algebra]

- Idea of the proof $\leqslant$. Let $\deg(\mathbf{t}^\alpha) = \max + 1$.

1. Can assume $\mathbf{t}^\alpha = \mathbf{t}_L$ for some $L \subseteq E_G$.
2. Let $C \subseteq E_G$ be Eulerian s.t. $|C \cap L| > {}^{|C|}/_2$.
3. Choose $L' \subsetneq C \cap L$, with $|L'| = {}^{|C|}/_2$, and take $\mathbf{t}_{L'} - \mathbf{t}_{C \setminus L'}$.
4. Then $\mathbf{t}_L = \mathbf{t}_{L \setminus L'}(\mathbf{t}_{L'} - \mathbf{t}_{C \setminus L'}) + \mathbf{t}_{L \setminus L'}\mathbf{t}_{C \setminus L'} \in J(G)$. $\square$

- Def. $J \subseteq E_G$ is a parity join of G iff $|J \cap C| \leqslant \frac{|C|}{2}$, for every, C, Eulerian subset of even cardinality.

  [Neves (in press). Commun. Alg.]

  Note :

  - If G is bipartite, then: join $\iff$ parity join
  - If G is non-bipartite, then: join $\implies$ parity join

- Thm. If G is any graph

  $$r = \max\{|J| : J \text{ is a parity join of } G\}$$

  [Neves (in press). Commun. Alg.]

- Question: What about the non-zero values of

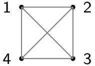
$$\left(\dim_K \frac{K[E_G]_d}{J(G)_d}\right)_{d\geqslant 0} = \;(h_0, h_1, h_2, \ldots, h_r, 0, \ldots)?$$

- Def. Fix an ordering of E_G. $J \subseteq E_G$ is a reduced parity join iff J is a parity join and whenever $|J \cap C| = {}^{|C|}/_{2}$, for some even cardinality Eulerian C, the last edge of C belongs to J.

- Thm. $\{J \text{ reduced parity join}, |J| = d\} \rightsquigarrow$ basis of $\frac{K[E_G]_d}{J(G)_d}$.

[Neves, Vaz Pinto (in press). São Paulo J. Math. Sci.]

Example



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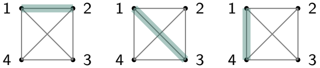
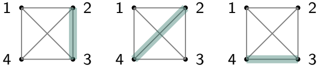
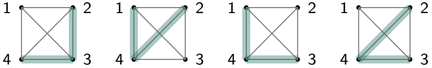


$t_{12},\ t_{13},\ t_{14}$

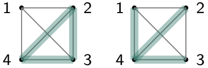


$t_{23},\ t_{24},\ t_{34}$

$t_{12}t_{34},\ t_{13}t_{34},\ t_{23}t_{34}$



$t_{23}t_{34},\ t_{14}t_{24},\ t_{14}t_{34},\ t_{24}t_{34}$



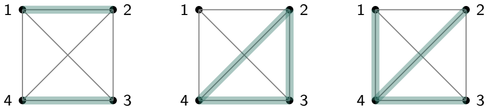
$t_{23}t_{24}t_{34},\ t_{14}t_{24}t_{34}$

- Def. The socle of $\frac{K[E_G]}{J(G)}$ is

$$\left\{ u \in \frac{K[E_G]}{J(G)} : t_e u = 0, \forall\, e \in E_G \right\}.$$

- Questions:

  1. Characterize reduced parity joins yielding socle.
  2. Compute the dimension of the socle.
  3. Classify graphs with level socle.



Thank you.