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The regularity of Eulerian ideals

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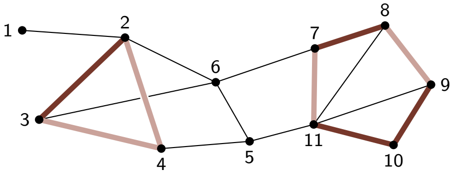
- G a simple graph;
- $V_G = \{1, \ldots, n\}$, $E_G \subset \{\{i,j\} \mid_{i \neq j}\}$;
- K a field, $K[V_G] = K[x_1, \ldots, x_n]$, $K[E_G] = K[t_e \mid_{e \in E_G}]$;
- $\varphi \colon K[E_G] \to K[V_G]$ defined by $t_{\{i,j\}} \mapsto x_i x_j$.
- Def. The *Eulerian ideal* of G is $I_G \subset K[E_G]$ given by:

$$I_G = \varphi^{-1}(x_i^2 - x_j^2 : {}_{i,j \in V_G}).$$

- $t_e^2 - t_f^2$, for all $e, f \in E_G$.
- *Eulerian binomials*, i.e., $\mathbf{t}^\alpha - \mathbf{t}^\beta$, s.t.
  1. $\alpha, \beta \in \{0,1\}^{E_G}$ $\leftrightarrow$ $\mathbf{t}^\alpha, \mathbf{t}^\beta$ are square-free.
  2. $\langle \alpha, \beta \rangle = 0$ $\leftrightarrow$ $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$.
  3. $\sum_{f \in E_G} \alpha(f) = \sum_{f \in E_G} \beta(f)$ $\leftrightarrow$ $\mathbf{t}^\alpha - \mathbf{t}^\beta$ homogeneous.
  4. $\mathrm{supp}(\alpha) \sqcup \mathrm{supp}(\beta) \subset E_G$ yields an Eulerian subgraph.



E.g. $\mathbf{t}^\alpha = t_{24} t_{34} t_{7,11} t_{89}$
and $\mathbf{t}^\beta = t_{23} t_{78} t_{9,10} t_{10,11}$.

Gröbner bases: the case of I_G

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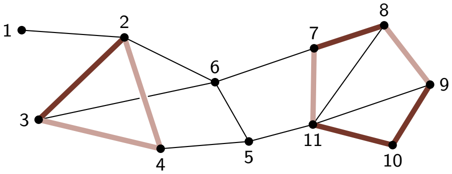
- Endow E_G with a total order: $t_{e_1} \succ \cdots \succ t_{e_s}$.
- Fix the graded reverse lexicographic order in $K[E_G]$ with respect to this.

  In particular, if $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are coprime of the same degree, $\mathbf{t}^\alpha \succ \mathbf{t}^\beta$ iff $\mathrm{supp}(\beta)$ contains the last edge of $\mathrm{supp}(\alpha) \sqcup \mathrm{supp}(\beta)$.



$$\min\left\{t_{23}, t_{24}, \ldots, t_{\{9,10\}}, t_{\{10,11\}}\right\} = t_{\{10,11\}}$$

$\mathbf{t}^\alpha = t_{24} t_{34} t_{7,11} t_{89}$ [light brown],

$\mathbf{t}^\beta = t_{23} t_{78} t_{9,10} t_{10,11}$ [dark brown],

$\mathrm{lt}(\mathbf{t}^\alpha - \mathbf{t}^\beta) = \mathbf{t}^\alpha$.

- Thm. $\{t_e^2 - t_f^2 : e, f \in E_G\} \cup \left\{\mathbf{t}^\alpha - \mathbf{t}^\beta, \text{Eulerians}\right\}$ is a Gröbner basis.
  [N., *Eulerian ideals*, submitted]

  Denote
  $\{g_1, \ldots, g_k\} = \{t_e^2 - t_f^2 : e, f \in E_G\} \cup \left\{\mathbf{t}^\alpha - \mathbf{t}^\beta, \text{Eulerians}\right\}$,
  $\mathcal{B}_d = \{\mathbf{t}^\gamma : \deg(\mathbf{t}^\gamma) = d \text{ and } \operatorname{lt}(g_i) \nmid \mathbf{t}^\gamma, \text{ for all } i\}$.

- Prop. Let $\mathbf{t}^\gamma \in \mathcal{B}_d$. Then $J = \mathrm{supp}(\gamma) \subset E_G$ is s.t. $|J \cap E_C| \leqslant {}^{|E_C|}/_2$ for every Eulerian subgraph $C \subset G$ with *even* $|E_C|$.

  Proof. Let $C \subset G$ be Eulerian with $|E_C| = 2m$.

  1. Assume that $|J \cap E_C| > m$.
  2. Let $\mathbf{t}^\alpha =$ product of the first m edges in $J \cap E_C$.
  3. Let $\mathbf{t}^\beta$ be the product of the remaining edges of C.
  4. $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is an Eulerian binomial with leading term $\mathbf{t}^\alpha$.
  5. $\mathbf{t}^\alpha \mid \mathbf{t}^\gamma \rightsquigarrow$ Contradiction. ■

- Def. $J \subset E_G$ is called a *parity join* if $|J \cap E_C| \leqslant {}^{|E_C|}/_2$ for every Eulerian subgraph, $C \subset G$, with even $|E_C|$.

- Thm. G any graph. $\text{reg}(G) = \max\{|J| : J \text{ is a parity join}\} - 1$.
  [N., *Eulerian ideals*, submitted]

Proof. Denote $\ell \in E_G$ last edge; $\mu = \max\{|J| : J \text{ is a parity join}\}$.

1. $\mathbf{t}^\gamma \in \mathcal{B}_d$ is square-free on t_f, $f \neq \ell$ and $\text{supp}(\gamma) \subset E_G$ is a parity join.
2. Hence $d \geqslant \mu \implies t_\ell \mid \mathbf{t}^\gamma$, for all $\mathbf{t}^\gamma \in \mathcal{B}_d$.
3. Therefore, $d \geqslant \mu \implies \mathcal{B}_d \subset t_\ell \mathcal{B}_{d-1} \subset \mathcal{B}_d \implies |\mathcal{B}_d| = |\mathcal{B}_{d-1}|$.

- Prop. Let $J \subset E_G$ be a parity join. Denote $d = |J|$ and let $\mathbf{t}^\alpha$ be the product of its edges. Then, $\mathbf{t}^\gamma \in \mathcal{B}_d$, the remainder of the division of $\mathbf{t}^\alpha$ by the Gröbner basis is square-free. Hence $\operatorname{supp}(\gamma)$ is a parity join of cardinality d.

Proof of Thm. (continued). $\mu = \max\{|J| : J \text{ is a parity join}\}$.

3. $|\mathcal{B}_{\mu-1}| = |\mathcal{B}_\mu| = |\mathcal{B}_{\mu+1}| = \cdots$
4. $\exists\, \mathbf{t}^\gamma \in \mathcal{B}_\mu$ square-free, s.t. $t_\ell \mid \mathbf{t}^\gamma$ and hence $t_\ell \nmid \mathbf{t}^\gamma t_\ell^{-1} \in \mathcal{B}_{\mu-1}$.
5. $t_\ell \mathcal{B}_{\mu-2} \subset \mathcal{B}_{\mu-1}$ is proper and hence $|\mathcal{B}_{\mu-2}| < |\mathcal{B}_{\mu-1}|$.
6. $\operatorname{reg}(G) = \mu - 1$. ■