

# A 4-dimensional deformation of tertiary Burniat surfaces

Jorge Neves  
CMUC – University of Coimbra

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— Joint work with Roberto Pignatelli —

# Known surfaces with $p_g = 0$ and $K^2 = 3$

$(\mathbb{Z}/2)^{\oplus 2} \times \mathbb{Z}/4$	irred 6-dim comp	[Keum88,MLP04]
$\mathbb{Q}_8 \times \mathbb{Z}/2$	1-dim family	[Bu66,Pet77,Inoue94]
$\mathbb{Z}_6 \times \mathbb{Z}_2$	family	[BauerPign11]
$\mathbb{Z}_4 \times \mathbb{Z}_2$	family	[BauerPign11]
$H_1 = \mathbb{Z}/8$	family	[BauerPign11]
$\mathbb{Z}/6$	family	[BauerPign11]
$\mathbb{Z}/2$	example	[CartSteg09]
$\mathbb{Z}/2$	family	[KeumLeePark10]
$H_1 = \mathbb{Z}/2$	family	[ParkParkShin10]
$\{1\}$	family	[ParkParkShin09]

## Theorem (–, Pignatelli)

Consider  $W = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^{15}$ .

- *There exists a family of simply connected smooth surfaces  $\tilde{S} = W \cap Z_1 \cap Z_2$  with a fixed point free action of  $\mathbb{Z}/2 \times Q_8$ .*
- *Their quotients form a 4-dimensional locus in  $\mathfrak{M}_{1,3}$  containing the locus of the Burniat surfaces with  $K^2 = 3$ .*

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- In parallel work, Bauer and Catanese have produced a 4-dim locus of  $\mathfrak{M}_{1,3}$  containing the Burniat surfaces whose closure is an irreducible component.
- We know that their locus and ours intersect in a subset of dim 4 but we don't know if the two loci coincide.

$$\tilde{S} \subset W$$

## The general idea

- $W = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^{15}$
- $\tilde{S} = W \cap Z_1 \cap Z_2,$
- $K_{\tilde{S}}^2 = 48$  and  $\chi(\mathcal{O}_{\tilde{S}}) = 16$  ( $q(\tilde{S}) = 0$ )
- $G = \mathbb{Z}/2 \times Q_8$

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$$\tilde{S} \subset Y \subset W$$

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$$T \subset V \dashrightarrow X \subset \mathbb{P}^6$$

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- Setting up the action of  $(\mathbb{Z}/2)^3$  on  $X$  is easier.
- To get  $V$ , use **parallel unprojection** on  $X$ .

## Unprojecting $\widehat{X} \subset \mathbb{P}^7$

$$\tilde{S} \subset Y \subset W$$



$$T \subset V \subset \widehat{V} \leftarrow \widehat{X} \subset \mathbb{P}^7$$



$$S = T/H = \tilde{S}/G$$

- Let  $\widehat{X}_{2,2,2} \subset \mathbb{P}^7$  be given by  
 $x_{00}x_{01} = x_{10}x_{11} = x_{20}x_{21} = x_{30}x_{31}$ .

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- Consider  $\{(i, j, k, l) \mid i+j+k+l \text{ even}\}$ . Get 8 rational functions, e.g:

$$\varphi_{0011} = \frac{x_{11}x_{20}x_{30}}{x_{00}} = \frac{x_{01}x_{20}x_{30}}{x_{10}} = \frac{x_{01}x_{11}x_{30}}{x_{21}} = \frac{x_{01}x_{11}x_{20}}{x_{31}}.$$

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$$y_{0011} \leftrightarrow \varphi_{0011} = \frac{x_{11}x_{20}x_{30}}{x_{00}} = \frac{x_{01}x_{20}x_{30}}{x_{10}} = \frac{x_{01}x_{11}x_{30}}{x_{21}} = \frac{x_{01}x_{11}x_{20}}{x_{31}}$$

- Identify these with the weight 2 variables of  $\mathbb{P}(1^8, 2^8)$ .
- Get equations for  $\widehat{V} \subset \mathbb{P}(1^8, 2^8)$ :

$$\begin{cases} x_{00}x_{01} = x_{10}x_{11} = x_{20}x_{21} = x_{30}x_{31} \\ y_{0011}x_{00} = x_{11}x_{20}x_{30}, y_{0011}x_{10} = x_{01}x_{20}x_{30}, \text{ etc.} \\ y_{0011}y_{0000} = \frac{x_{01}x_{11}x_{30}}{x_{21}} \frac{x_{01}x_{11}x_{21}}{x_{30}} = x_{01}^2 x_{11}^2, \text{ etc.} \end{cases}$$

## $(\mathbb{Z}/2)^3$ -action on $\widehat{X}$ and $\widehat{V}$

$$\tilde{S} \subset Y \subset W$$

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$\alpha_1$	$x_{00} \leftrightarrow x_{01}$	$x_{10} \leftrightarrow x_{11}$	$y_{abcd} \leftrightarrow y_{a'b'cd}$
$\alpha_2$	$x_{00} \leftrightarrow x_{01}$	$x_{20} \leftrightarrow x_{21}$	$y_{abcd} \leftrightarrow y_{a'bc'd}$
$\alpha_3$	$x_{00} \leftrightarrow x_{01}$	$x_{30} \leftrightarrow x_{31}$	$y_{abcd} \leftrightarrow y_{a'bcd'}$
$\beta_i$	$x_{i0} \rightarrow -x_{i0}$	$x_{i1} \rightarrow -x_{i1}$	$y_{abcd} \rightarrow -y_{abcd}$

$(\mathbb{Z}/2)^3$ -action on  $\widehat{X}$  and  $\widehat{V}$ 

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- $H = \langle \alpha_1\beta_2, \alpha_2\beta_3, \alpha_3\beta_1 \rangle \cong (\mathbb{Z}/2)^3 \implies H^0(\mathcal{O}_{\widehat{V}}(1)) \cong \text{Reg}(H)$ .

$(\mathbb{Z}/2)^3$ -action on  $\widehat{X}$  and  $\widehat{V}$ 

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- Then  $V = \widehat{V} \cap (x_{00} + x_{01} = 0)$ ,  $T = V \cap (q = 0)$  where

$$q = \nu_0 s_0 + \nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3 + \nu_4 \sum (-1)^a y_{abcd},$$

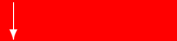
$$\text{with } s_i = \frac{x_{i0}^2 + x_{i1}^2}{2} \text{ and } \nu_0, \nu_1, \nu_2, \nu_3 \in \mathbb{C}.$$

## Lifting to $W$

$$\tilde{S} \subset Y \subset W$$



$$T \subset V \subset \widehat{V} = W/s$$



$$S = T/H = \tilde{S}/G$$

- $W = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $(u_1, v_1), \dots, (u_4, v_4)$  be coordinates.
- $\widehat{V}$  is the quotient of  $W$  by the involution  $s = (+, -), (+, -), (+, -), (+, -)$ .

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Lifting  $\alpha_1, \beta_2, \alpha_2, \beta_3, \alpha_3, \beta_1$  to  $W$ :

	$u_1$	$v_1$	$u_2$	$v_2$	$u_3$	$v_3$	$u_4$	$v_4$
$\tilde{\alpha}_1 \tilde{\beta}_2$	$-iu_2$	$v_2$	$u_1$	$iv_1$	$v_4$	$-iu_4$	$v_3$	$iu_3$
$\tilde{\alpha}_2 \tilde{\beta}_3$	$-iu_3$	$v_3$	$v_4$	$iu_4$	$u_1$	$iv_1$	$v_2$	$-iu_2$
$\tilde{\alpha}_3 \tilde{\beta}_1$	$-iu_4$	$v_4$	$v_3$	$-iu_3$	$v_2$	$iu_2$	$u_1$	$iv_1$

## Lifting to $W$

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$$T \subset V \subset \widehat{V} = W/s$$

$$\downarrow$$

$$S = T/H = \tilde{S}/G$$

- $W = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $(u_1, v_1), \dots, (u_4, v_4)$  be coordinates.
  - $\widehat{V}$  is the quotient of  $W$  by the involution  $s = (+, -), (+, -), (+, -), (+, -)$ .
- $\langle \tilde{\alpha}_1 \tilde{\beta}_2, \tilde{\alpha}_2 \tilde{\beta}_3, \tilde{\alpha}_3 \tilde{\beta}_1 \rangle \cong \mathbb{Z}/2 \times Q_8$ .  $(\alpha_1 \beta_2)^2 = (\alpha_2 \beta_3)^2 = (\alpha_3 \beta_1)^2 = s$ .

## Lifting to $W$

$$\tilde{S} \subset Y \subset W$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$T \subset V \subset \hat{V} = W/s$$

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- $\hat{V}$  is the quotient of  $W$  by the involution  $s = (+, -), (+, -), (+, -), (+, -)$ .

$$\bullet \langle \tilde{\alpha}_1 \tilde{\beta}_2, \tilde{\alpha}_2 \tilde{\beta}_3, \tilde{\alpha}_3 \tilde{\beta}_1 \rangle \cong \mathbb{Z}/2 \times Q_8. \quad (\alpha_1 \beta_2)^2 = (\alpha_2 \beta_3)^2 = (\alpha_3 \beta_1)^2 = s.$$

$$Z_1: v_1 u_2 u_3 u_4 + u_1 v_2 v_3 v_4 = 0$$

$$Z_2: \sum_{1 \leq i \leq 4} \nu_{i-1} (u_i^2 \prod_{j \neq i} v_j^2 + v_i^2 \prod_{j \neq i} u_j^2) - 2\nu_4 (u_1^2 u_2^2 u_3^2 u_4^2 - v_1^2 v_2^2 v_3^2 v_4^2 + \dots) = 0$$

- $\tilde{S} = W \cap Z_1 \cap Z_2$  is smooth and  $G = \mathbb{Z}/2 \times Q_8$  acts freely.
- Hence,  $S$  is of general type,  $K^2 = 3$ ,  $p_g = 0$ ,  $\pi_1 = \mathbb{Z}/2 \times Q_8$ .
- The construction depends on 5 parameters.

## The bicanonical map of $S$

$$\tilde{S} \subset Y \subset W$$

$$\downarrow \quad \downarrow$$
$$T \subset V \subset \mathbb{P}(1^7, 2^8)$$

$$\downarrow$$
$$S = T/H = \tilde{S}/G$$

$$\downarrow \varphi_{2K_S}$$
$$\mathbb{P}^3$$

- $H^0(2K_S) = \langle s_0, s_1, s_2, s_3 \rangle, s_i = \frac{x_{i0}^2 + x_{i1}^2}{2}$

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$$\mathbb{P}^3$$

- $H^0(2K_S) = \langle s_0, s_1, s_2, s_3 \rangle$ ,  $s_i = \frac{x_{i0}^2 + x_{i1}^2}{2}$
- $\hat{H} = \left\{ \prod_{i=1}^3 \alpha_i^{a_i} \beta_i^{b_i} \mid \sum (a_i + b_i) \text{ even} \right\}$
- $\Gamma \cong \hat{H}/H \cong (\mathbb{Z}/2)^2$ .

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$$S/\Gamma = \Sigma_3 \subset \mathbb{P}^3$$

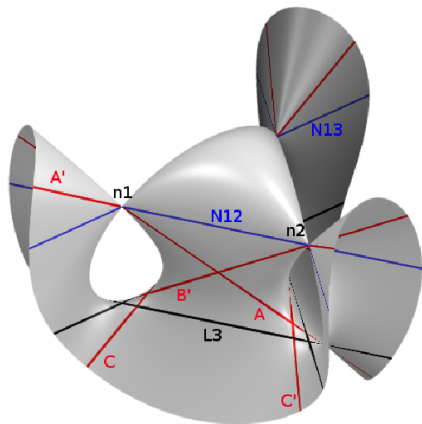
- $H^0(2K_S) = \langle s_0, s_1, s_2, s_3 \rangle$ ,  $s_i = \frac{x_{i0}^2 + x_{i1}^2}{2}$
- $\hat{H} = \left\{ \prod_{i=1}^3 \alpha_i^{a_i} \beta_i^{b_i} \mid \sum (a_i + b_i) \text{ even} \right\}$
- $\Gamma \cong \hat{H}/H \cong (\mathbb{Z}/2)^2$ .
- $T/\hat{H} = S/\Gamma = \Sigma_3 \subset \mathbb{P}^3$  given by

$$8\nu_4^2(s_1 - s_0)(s_2 - s_0)(s_3 - s_0) - s_0(\nu_0 s_0 + \nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3)^2 = 0.$$

- $S$  is a bidouble cover of  $\Sigma_3$ .

# $S$ as a bidouble cover of $\Sigma_3$

$$8\nu_4^2(s_1 - s_0)(s_2 - s_0)(s_3 - s_0) - s_0 \underbrace{(\nu_0 s_0 + \nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3)}_L^2 = 0$$



- $\Sigma_3$  has 3 nodes:

$$n_1 = (s_2 - s_0 = s_3 - s_0 = L = 0);$$

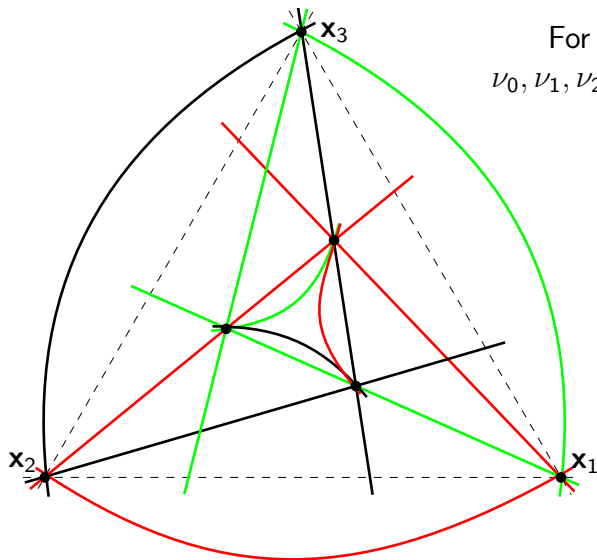
$$n_2 = (s_1 - s_0 = s_3 - s_0 = L = 0);$$

$$n_3 = (s_1 - s_0 = s_2 - s_0 = L = 0).$$



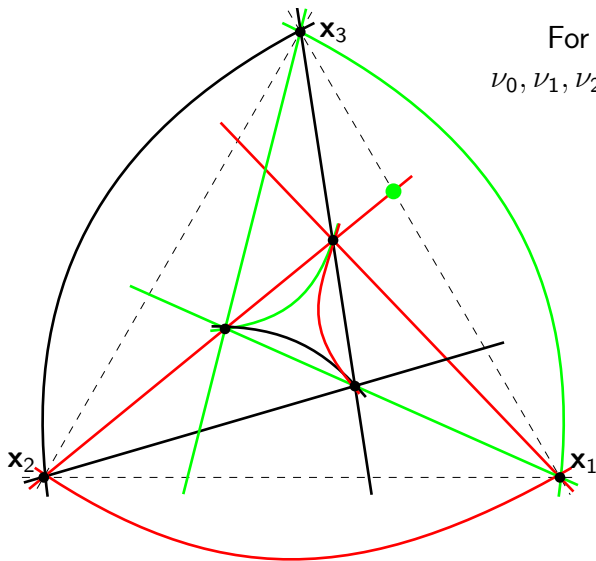


# $S$ as a bidouble cover of the plane



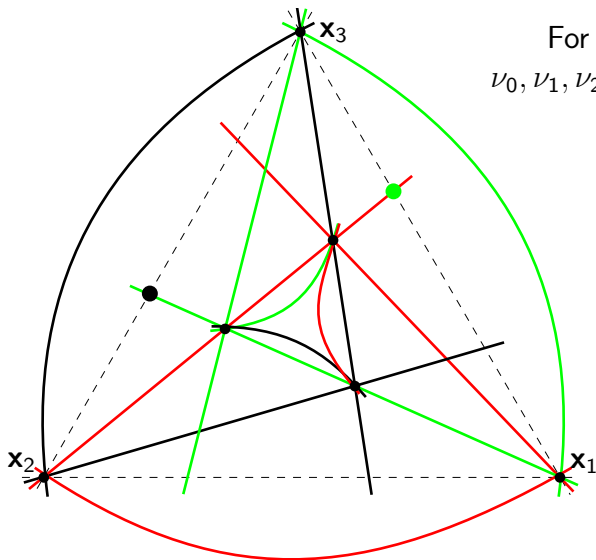
For general  
 $\nu_0, \nu_1, \nu_2, \nu_3, \nu_4 \in \mathbb{C}$

# $S$ as a bidouble cover of the plane



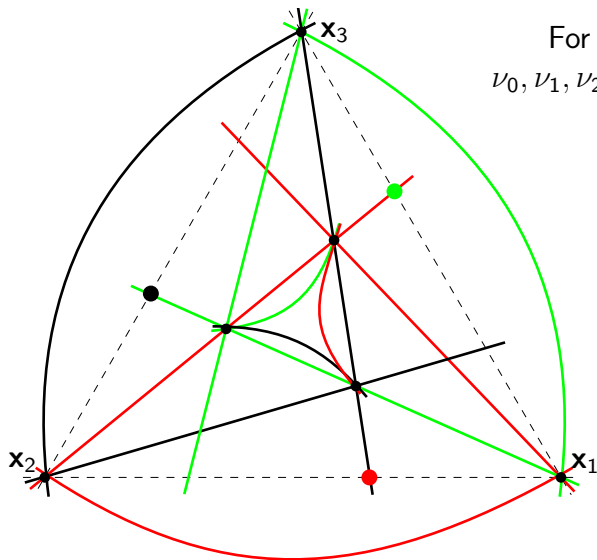
For general  
 $\nu_0, \nu_1, \nu_2, \nu_3, \nu_4 \in \mathbb{C}$

# $S$ as a bidouble cover of the plane



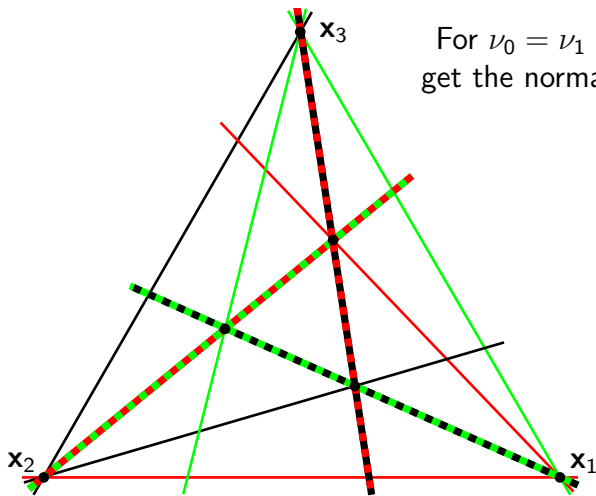
For general  
 $\nu_0, \nu_1, \nu_2, \nu_3, \nu_4 \in \mathbb{C}$

# $S$ as a bidouble cover of the plane



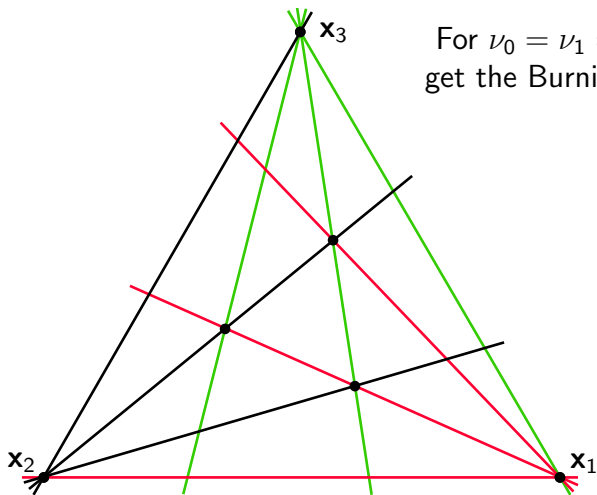
For general  
 $\nu_0, \nu_1, \nu_2, \nu_3, \nu_4 \in \mathbb{C}$

# $S$ as a bidouble cover of the plane



For  $\nu_0 = \nu_1 = \nu_2 = \nu_3$   
get the normalisation of:

# $S$ as a bidouble cover of the plane



For  $\nu_0 = \nu_1 = \nu_2 = \nu_3$   
get the Burniat surfaces