Regularity of the vanishing ideal of a bipartite nested ear decomposition

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Definition of I_G

Vanishing ideals over graphs

- G is a simple graph without isolated vertices.
- $V_G = \{1, 2, \ldots, n\}$.
- K a finite field of cardinality q.
- $K[t_{ij} \mid \{i, j\} \in E_G] = K[E_G]$.
- Let I_G be the ideal generated by the polynomials

$$f \text{ hom. and } f(t_{ij} \mapsto u_i u_j) = 0, \ \forall_{u_1,\ldots,u_n \in K^*}$$

[Rentería, Simis, Villarreal, 2011]

Example [Using Macaulay2]

Generators of I_G

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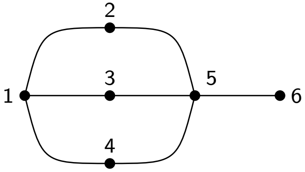
$$f(t_{ij} \mapsto u_i u_j) = 0, \ \forall_{u_1,\dots,u_n \in K^*}$$



I_G has the following minimal generating set:

- $t_{13}^{q-1} - t_{12}^{q-1}, \dots, t_{56}^{q-1} - t_{12}^{q-1}$
- $t_{13}t_{25} - t_{12}t_{35}, \ \ t_{14}t_{35} - t_{13}t_{45}, \ \ t_{14}t_{25} - t_{12}t_{45}$
- $t_{12}^a t_{25}^b - t_{13}^a t_{35}^b, \ t_{12}^a t_{25}^b - t_{14}^a t_{45}^b$ with $a + b \equiv 0 \pmod{q-1}$
- $t_{12}^a t_{13}^b t_{14}^c - t_{25}^a t_{35}^b t_{45}^c$, with $a + b + c \equiv 0 \pmod{q-1}$.

(Can assume $0 \leq a, b, c \leq q-2$.)

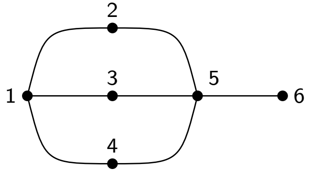
Binomials in I_G

Caracterization of binomials

Notation:

Given $\alpha = (\alpha\{i,j\} \mid \{i,j\} \in E_G)$, $\alpha\{i,j\} \in \mathbb{N}$, denote:

$$\mathbf{t}^\alpha = \prod t_{ij}^{\alpha\{i,j\}}.$$



Lemma

Given α, β multi-indices, such that $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is homogeneous,

$$\mathbf{t}^\alpha - \mathbf{t}^\beta \in I_G \iff \sum_{i \in N_G(v)} \alpha\{v,i\} \equiv \sum_{i \in N_G(v)} \beta\{v,i\} \mod q-1, \quad \forall_{v \in V_G}$$

[N., Vaz Pinto, 2014]

Properties of I_G

in [Rentería, Simis, Villarreal, 2011]

- I_G is a radical, binomial, graded ideal of $K[E_G]$.
- Denoting $s = |E_G| = \dim K[E_G]$,

$$\left\{ t_{ij}^{q-1} - t_{12}^{q-1} \;|_{\{i,j\}\in E_G\setminus\{1,2\}} \right\} \subset I_G$$

$$\Downarrow$$

$$\dim K[E_G]/I_G = \dim K[E_G] - (s-1) = 1.$$

- Recall:

Hilbert function: $\varphi(n) = \dim\left(K[E_G]/I_G\right)_n, \quad n \geq 0.$

$\dim R = d \iff$ the Hilbert function of R is polynomial of degree $d-1$.

- Since $\dim K[E_G]/I_G = 1$ the Hilbert function of $K[E_G]/I_G$, becomes constant for $n \geq r$, for some r (index regularity).
- For the earlier example taking, say, $q = 5$:

$$\varphi(n) = (1, 7, 25, 65, 116, 170, 216, 240, 252, 256, 256, \dots)$$

Regularity of a graph

Main goal

- Define the regularity of a graph as:

$$\operatorname{reg} G = \text{ index of regularity of } K[E_G]/I_G$$

- Question:

Can we relate $\operatorname{reg} G$ with a invariant of G?

Regularity of a graph

Some initial results

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- reg $\mathcal{K}_{a,b} = (\max\{a, b\} - 1)(q - 2)$
  [González, Rentería, 2008]
- reg $\mathcal{C}_{2k} = (k - 1)(q - 2)$
  [N., Vaz Pinto, Villarreal, 2015]
- $G =$ tree or $\mathcal{C}_{2k+1}$, reg $G = (s - 1)(q - 2)$;
  [Sarmiento, Vaz Pinto, Villarreal, 2011]

Computing the regularity

Artinian Reduction

- Let $t_{k\ell} \in E_G$. Then $\dim K[E_G]/(I_G, t_{k\ell}) = 0$ and the Hilbert Function of this quotient is zero iff $n \geq r + 1$.

  In the earlier example,

  $n=10$

  $$\varphi(n) = (1, 6, 18, 40, 51, 54, 46, 24, 12, 4, 0, \dots)$$

- Hence

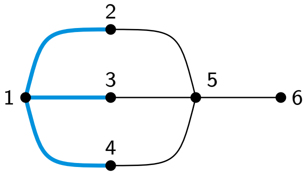
$$\operatorname{reg} G \geq d \iff (I_G, t_{k\ell})_d \neq K[E_G]_d$$

$$\operatorname{reg} G \leq d \iff (I_G, t_{k\ell})_{d+1} = K[E_G]_{d+1}$$

Worked Example

reg $G \geq 3(q-2)$

- Proof:
  - Consider $\mathbf{t}^\alpha = t_{12}^{q-2} t_{13}^{q-2} t_{14}^{q-2}$.
  - Suppose that $\mathbf{t}^\alpha \in (I_G, t_{56})$.
    Then: $\exists\, \mathbf{t}^\beta$, s. t:
    - $t_{56} \mid \mathbf{t}^\beta \iff \beta\{5,6\} > 0$,
    - $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I_G$,
    - $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is homogeneous.
  - Then $\left.\begin{aligned} \beta\{1,2\} + \beta\{2,5\} &\geq q-2, \\ \beta\{1,3\} + \beta\{3,5\} &\geq q-2, \\ \beta\{1,4\} + \beta\{4,5\} &\geq q-2, \\ \beta\{5,6\} &> 0 \end{aligned}\right\} \implies \deg \mathbf{t}^\beta > 3(q-2)$. ■



- Lemma: If H_1, H_2 are subgraphs of G s.t. $E_G = E_{H_1} \cup E_{H_2}$ and $E_{H_1} \cap E_{H_2} \neq \emptyset$ then $\operatorname{reg} G \leq \operatorname{reg} H_1 + \operatorname{reg} H_2$.

  [Macchia, N., Vaz Pinto, Villarreal]

  Proof:

  - Fix some $t_{k\ell} \in E_{H_1} \cap E_{H_2}$.
  - Let $\mathbf{t}^\alpha \in K[E_G]$ be of degree $\operatorname{reg} H_1 + \operatorname{reg} H_2 + 1$.
  - Write $\mathbf{t}^\alpha = \mathbf{t}^\beta \mathbf{t}^\gamma$ with $\mathbf{t}^\beta \in K[E_{H_1}]$ and $\mathbf{t}^\gamma \in K[E_{H_2}]$.
  - W.l.o.g. $\deg(\mathbf{t}^\beta) \geq \operatorname{reg} H_1 + 1$.
  - Then $\mathbf{t}^\beta \in (I_{H_1}, t_{k\ell}) \subset (I_G, t_{k\ell})$.
  - Hence $\mathbf{t}^\alpha = \mathbf{t}^\beta \mathbf{t}^\gamma \in (I_G, t_{k\ell})$. ∎

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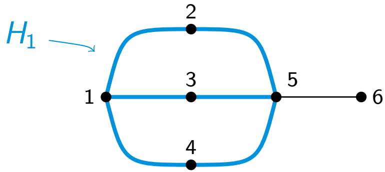

- reg $G \leq$ reg H_1 + reg $H_2 \leq$ reg C_1 + reg $C_2 + (q-2)$.
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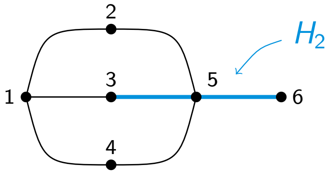

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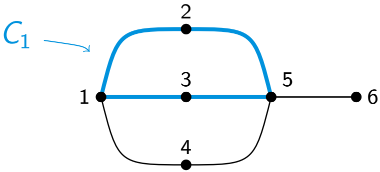

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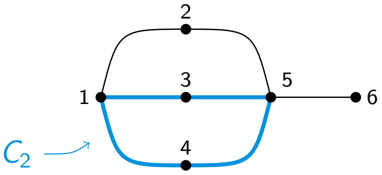

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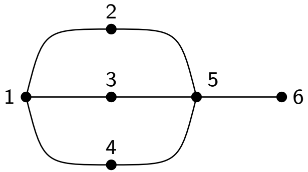

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2-connected graphs

Block decomposition

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- Recall: A graph is 2-(vertex)-connected if $|V_G| \geq 3$ and, for every $v \in V_G$, the graph $V_G - v$ is connected.
- Any graph decomposes into a set of subgraphs (blocks) consisting of either isolated vertices; single edges or maximal 2-connected subgraphs.
- Theorem: If G is bipartite and $H_1, \ldots, H_m$ are its blocks:

  $\operatorname{reg} G = \sum \operatorname{reg} H_i + (m-1)(q-2)$.

  [N., Vaz Pinto, Villarreal, 2014]

  Proof: Key idea $I_G \longleftrightarrow I_{H_1} + \cdots + I_{H_m}$. ■

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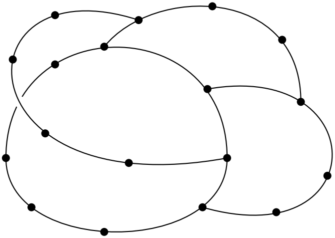
- [Whitney's theorem] G is 2-connected iff it is endowed with an (open) ear decomposition starting from any cycle.



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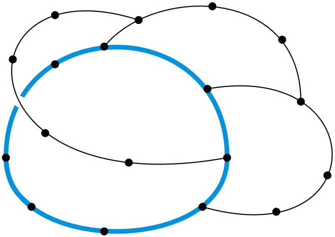

Ear decompositions

Whitney's theorem

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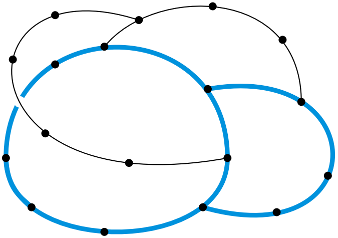

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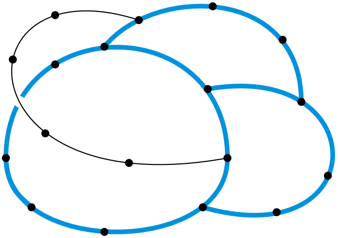

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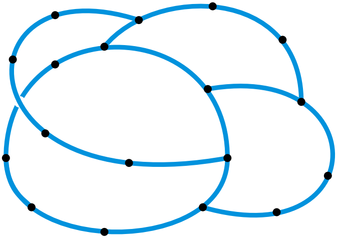


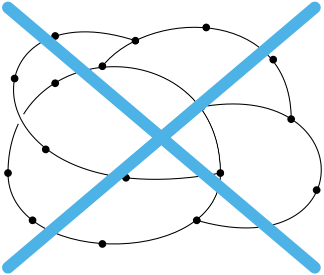
- Definition: A *nested* ear decomposition of a graph $G = P_0 \cup P_1 \cup \cdots \cup P_r$ is an ear decomposition of G such that:

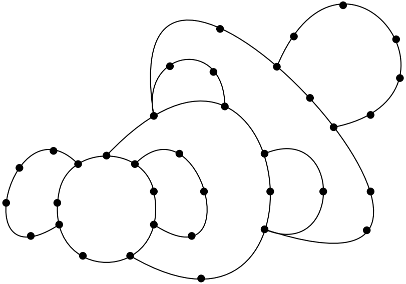
  1. $\forall\, i{>}0$, the endpoints of P_i belong to P_j, for some $j < i$.
  2. If two paths are nested in P_j, their endpoints determine open intervals in P_j which are either nested or disjoint.

  [Eppstein, 1992]

Ear decompositions

Nesting ears





Theorem (N.)

Let G be a bipartite graph endowed with a nested ear decomposition $P_0, \ldots, P_r$ starting from a vertex P_0. Let ϵ denote the number of even length paths in $P_1, \ldots, P_r$. Then,

$$\operatorname{reg} G = \tfrac{n+\epsilon-3}{2}(q-2).$$

Corollary

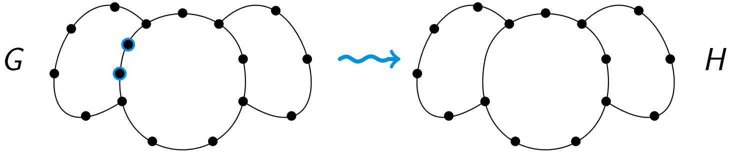
In a nested ear decomposition of a bipartite graph, starting from a vertex, the number of even length ears is constant.

Within the proof

Bipartite ear modifications

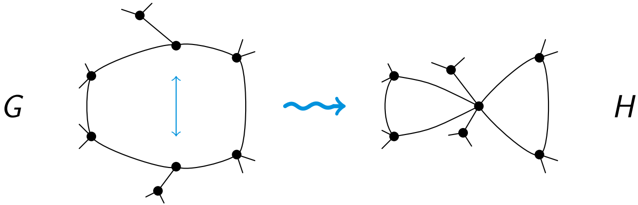
- If $v_1, v_2 \in V_G$ are two degree 2 *adjacent* vertices then the *smoothing* of v_1 and v_2 produces a graph H such that

$$\operatorname{reg} H = \operatorname{reg} G - (q-2).$$



- Proposition: If H is obtained from G by identifying two nonadjacent vertices, then $\operatorname{reg} G \geq \operatorname{reg} H$.

  [Macchia, N., Vaz Pinto, Villarreal]



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