Halfcanonical rings on algebraic curves and applications to surfaces of general type

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Contents

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Declaration

Except for Chapter II which is largely expository, I declare that, to the best of my knowledge, the material in this thesis is my original work and has not been submitted for a degree at another university.

Summary

In this thesis we prove that the canonical model of a surface of general type in an open family of surfaces in each of the birational classes

$$\text{(I)} \quad q = 0, \quad p_g = 4 \quad \text{and} \quad K^2 = 7$$
$$\text{(II)} \quad q = 0, \quad p_g = 6 \quad \text{and} \quad K^2 = 13$$

is a complete intersection in a generalised weighted Grassmannian. For each of the families the canonical ring has Pfaffian 5×5 format. In other words, it is a quotient of a polynomial ring by an ideal generated by the five submaximal Pfaffians of a 5×5 skew matrix. Our methods involve a generalisation of Mukai's vector bundle method. We construct a vector bundle on the surface which, as a result of our analysis, is the restriction of the universal orbi-bundle on a generalised weighted Grassmannian. As in Mukai's work, this bundle yields an embedding of the canonical model of S as a quasilinear section of that generalised weighted Grassmannian. Chapter I is an introduction to the context of the problem. Chapter III focuses on curves with a halfcanonical divisor. We prove preliminary results concerning the numerical structure of the halfcanonical ring, such as the degrees of the generators of $R(\mathcal{C}, A)$ and of the ideal I_A. We also give an intrinsic description of all the the complete intersection halfcanonical ideals of codimension ≤ 3 on an algebraic curve. Chapter IV sees the first application of the vector bundle method. There we show how to embed a curve of genus 8 with a halfcanonical net and a curve of genus 14 with a halfcanonical divisor with $h^0(A) = 5$ into generalised weighted Grassmannians. Each of these corresponds to a member of the canonical linear system of the surfaces mentioned above. In the same chapter we define generalised weighted Grassmannians and give a detailed introduction to the vector bundle technique. Finally, Chapter V is dedicated to the construction of the canonical model of surfaces of general type in the birational classes (I) and (II).

CHAPTER I

Introduction

Of all those working in the subject of algebraic surfaces the Italian school of G. Castelnuovo and F. Enriques is undoubtedly responsible for pioneering a systematic treatment of surfaces of general type. In his book, [**En**] Enriques sets up most of the general theory of algebraic surfaces and linear systems on an algebraic surface that is still in use to this day. Chapter VII of [**En**] is dedicated to the study of regular surfaces of general type* with geometric genus $p_g \geq 4$. The condition on the genus is a necessary condition to study surfaces with birational canonical map φ_{K_S}. The interplay between the geometry of hyperplane curve sections of the image of the surface under the canonical map and the geometry of the surface itself justifies the requirement that the irregularity $q = h^1(\mathcal{O}_S) = h^1(K_S)$ be zero.

In Chapter V of this thesis we construct the canonical model of a surface in an open family in each of the birational classes

$$\begin{array}{ll}(\mathrm{I}) & q=0, \quad p_g=4 \quad \text{and} \quad K^2=7\\ (\mathrm{II}) & q=0, \quad p_g=6 \quad \text{and} \quad K^2=13\end{array}$$

as a complete intersection inside a generalised weighted Grassmannian.

The surfaces in (I) were well known to Enriques† *et al.* We can find a classical treatment in chapter VII of [**En**]. The author constructs examples of surfaces in (I) with a free canonical linear system and with a canonical linear system with a simple base point. He also goes quite far in the description of the component of the moduli space of (I) corresponding to surfaces with a free canonical linear system. In this case, the image of the canonical morphism is a surface Σ in $\mathbb{P}^3$ having a curve of singularities γ which at a general point of γ are double points. An *adjoint* surface to Σ is by definition any surface passing through γ (in analogy with the notion of adjoint curve to a given plane curve). The surface Σ has a unique minimal degree adjoint

*See page 122 for our conventions.

†Enriques cites the work of his student Franchetta, *Su alcuni esempi di superficie canoniche.*

surface. It is a quadric cone Q with its vertex at the (unique) triple point of γ (and of Σ). The data (Σ, γ, Q) is enough to describe the component of the moduli space. The adjuvant element is, of course, the rationality of Q. A modern treatment of the moduli space of (I) can be found in the work of Bauer [**Bau**]. The method of the adjoint surface in conjunction with the theory of symmetric determinantal hypersurfaces is employed by Ciliberto in [**C81**] to give a description of the component of the moduli space of surfaces of general type with $p_g = 4$, birational canonical morphism and K^2 in the set $\{5, 6, 7, 8, 9, 10\}$. His method relies on the fact that both Σ and its minimal degree adjoint Σ' are *generic symmetric* with the degree of Σ' equal to $\deg(\Sigma) - 5$; and it fails for $K^2 > 10$. Notice that whilst for $K^2 \leq 8$ the minimal degree adjoint surface is rational, for $K^2 > 10$ the adjoint may well be of general type. The question of existence of regular canonical surfaces with $p_g = 4$ and $K^2 \geq 11$ is still an open problem. (See [**C**]).

Surfaces in the birational class (II) have been constructed by Tonoli in [**Ton**] as one of many canonical surfaces in $\mathbb{P}^5$. They also appear in the work of Catanese [**C**]. Their existence follows by an application of Buchsbaum–Eisenbud's theorem for Gorenstein ideals of codimension 3 (see below) and as such has been know for quite some time. To our knowledge the moduli space of the surfaces in (II) remains to be investigated.

What will bring together (I) and (II) in Chapter V is the fact the fact that under generality assumptions, the canonical ring $R(S, K_S)$ is a Pfaffian 5×5 ring and can be expressed as the quotient by a regular sequence of the homogeneous ring of a generalised weighted Grassmannian.

The canonical ring. For a surface, S, of general type, the canonical ring

$$R(S, K_S) = \bigoplus_{n \geq 0} H^0(S, nK_S),$$

encodes information about the surface, its canonical image, its bicanonical image, questions of projective normality of these images and so forth. It was defined by D. Mumford in [**Mum**] and subsequently used by many authors as a tool in the description of the moduli space of surfaces of general type. In the case of surfaces of general type, the canonical ring is a finitely generated Noetherian ring and the scheme $\operatorname{Proj} R(S, K_S)$, called the canonical model of S, is birational to the surface S. It is a normal surface with at most du Val singularities. (See [**Mum**]). Hence, describing the birational class of a surface of general type can be done by studying $R(S, K_S)$. This principle appears for the first time in an effective way in the work of Miles Reid. In the article [**R78**] Reid gives a complete treatment of regular surfaces of general type with $p_g(S) = 0$, $K_S^2 = 1$ and $\operatorname{Tors} S = \mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_5$. These surfaces are also known as Godeaux surfaces. Those with Severi group, $\operatorname{Tors} S$, equal to $\mathbb{Z}_5$ were constructed by Godeaux, and those with Severi group equal to $\mathbb{Z}_4$ by Miyaoka. In this birational class the higher the dimension of $\operatorname{Tors} S$ the simpler is the geometry of S and the algebra of $R(S, K_S)$. To construct his surface Godeaux construct the $\mathbb{Z}_5$-covering of S that is a quintic hypersurface in $\mathbb{P}^3$. Miyaoka's construction is similar. One constructs the $\mathbb{Z}_4$-covering of S, which is a complete intersection in weighted projective space. Mirroring this approach Reid describes the *covering ring* of the surface S associated with the Severi group $\operatorname{Tors} S$:

$$R(S, K_S, \operatorname{Tors} S) = \bigoplus_{\substack{n \geq 0 \\ \mathfrak{d} \in \operatorname{Tors} S}} H^0(S, nK_S + \mathfrak{d})$$

The ring $R(S, K_S, \operatorname{Tors} S)$ has a $\operatorname{Tors} S$-action and the canonical ring of S is the invariant ring under this action:

$$R(S, K_S) = R(S, K_S, \operatorname{Tors} S)^{\operatorname{Tors} S}.$$

In this light, Reid gives a complete treatment of the case $\operatorname{Tors} S = \mathbb{Z}_3$. The article [**R78**] points out a systematic method for the study of surfaces of general type.

There have been many follow-ups to this work. Notably R. Barlow in [**Ba85**] has constructed a simply connected Godeaux surface, and in [**Ba84**] a 4-parameter family of Godeaux surfaces with $\operatorname{Tors} S = \mathbb{Z}_2$. Her treatment involves a slight modification of the classical covering method, allowing for ramification at isolated points and thus introducing a singular cover. The action is not that of $\operatorname{Tors} S$ since this group is either trivial or too small. Instead we take a Galois covering of a non Abelian group. This variation of Reid's construction is very subtle and it does not extend easily to other examples. However, the guiding principle is still that of Reid's work [**R78**]. To study S we study $R(S, K_S)$.

Recently Bauer, Catanese and Pignatelli in [**BCP**] have described the canonical ring for surfaces of general type with $p_g = 4$ and $K^2 = 7$ for which the canonical linear system has a base point.

The subject of graded rings on algebraic varieties has seen many developments and is increasingly attracting more interest. Besides the canonical ring of a surface of general type, the anticanonical ring of a $\mathbb{Q}$-Fano 3-fold ([**Fl88, Fl00, CPR**]) and the ring of a polarising divisor on a K3 surface ([**ABR**]) are other examples of graded rings in active areas of research.

The algebra of the canonical ring. The study of moduli of algebraic varieties via graded rings is tainted by a fundamental difficulty. Be it for the canonical ring of a surface of general type or for any other of the examples of graded rings mentioned before, the construction of a graded ring from numerical data is hindered by the absence of structure theories for codimension ≥ 4 ideals. The desire to have a structure theory for codimension 4 is justified by the existing results in codimension 2 (the theorem of Hilbert–Burch) and in codimension 3.

Theorem I.1 (Buchsbaum–Eisenbud). *Let $A = \mathbb{C}[x_1, \ldots, x_n]$ be a graded polynomial ring, I Gorenstein homogeneous ideal of codimension 3. Then there exist an odd integer $k = 2n + 1$ and a $k \times k$ skew matrix M such that*

$$A/I \leftarrow A \xleftarrow{\operatorname{Pf} M} \bigoplus^{k} A(-a_i) \xleftarrow{M} \bigoplus^{k} A(-b_i) \xleftarrow{\operatorname{Pf} M^t} A(-t) \leftarrow 0 \tag{0.1}$$

is the minimal free resolution of A/I as an A-module. Conversely any codimension 3 ideal which allows such a minimal resolution is Gorenstein.

Proof. See [**BE**] or [**BH**, p. 119–123]. □

Let us spare a few words on the assumptions of this theorem. The Gorenstein assumption is a regularity assumption. It implies the Cohen–Macaulay property, renowned for bridging the gap between commutative algebra and algebraic geometry. As an example one should think of an ideal defining a hypersurface in projective space with an embedded component. Despite the codimension of such ideal be 1, the ideal is definitely *not* generated by a single form. Off course that in this situation the quotient ring is not Cohen–Macaulay. This counter-example to an "assumption-free" structure theory of codimension 1 ideals, which is drawn from a geometrical point of view, reveals only part of the problem. The Gorenstein assumption expresses extra homological regularity. (See Section II.2). The free resolution of a Gorenstein ideal is self-dual.

Most of the homogeneous ideals of codimension ≥ 4 relevant to the problem of moduli of varieties have been constructed without the safety net of a theorem like Theorem I.1. The main techniques for doing this are unprojection and the use of key varieties. These techniques base themselves on two archetypal operations of geometry; unprojection on that of projection and the use of key varieties on that of taking a linear section.

Key varieties *versus* unprojection. In terms of graded rings a key variety corresponds to a *key* graded ring, i.e. a graded ring to be used as raw material in the construction of more graded rings. An example of this is the homogeneous ring of $\mathrm{G}(2,5) \subset \mathbb{P}^9$ in its Plücker embedding. Constructing graded rings using a key ring in the simplest cases amounts to taking the quotient by a regular sequence. Letting $\mathrm{ev}\colon \mathbb{C}[x_i] \twoheadrightarrow R$ be a minimal surjection corresponding to a choice of generators of a key graded ring R and considering R has a quotient of $\mathbb{C}[x_i]$, there is a way in which taking a regular sequence in R can preserve codimension of the quotienting ideal, by a process of elimination of (some) generators of R. We should be terming it as a quasilinear regular sequence in analogy to the terminology of quasilinear section of Corti and Reid's paper [**CR**].

Definition I.2. Let R be a graded[‡] ring. Fix $\mathrm{ev}\colon \mathbb{C}[x_1,\dots,x_n] \to R$ a minimal surjection. A regular sequence $(f_1,\dots,f_n)$ in R is said to be quasilinear if each f_i is homogeneous and a preimage (and indeed all) of f_i under ev is a polynomial F_i containing, for some integer j, the term x_j. It can be checked that this notion does not depend on the surjection ev.

In view of Theorem I.1 above, roughly speaking, Gorenstein ideals of codimension 3 (including complete intersections of length 3!) are the ideals generated by the submaximal Pfaffians of a certain skew matrix of odd dimension. Unlike the ideals in a polynomial ring generated by regular sequences that have a geometrical realisation

[‡]See page 14 for our conventions on graded rings.

in complete intersections, the Pfaffian ideals, until very recently, had no geometrical counterpart. They correspond to the weighted Grassmannians of Corti and Reid [**CR**]. Despite applying only to the case of Pfaffian 5×5 ideals, these varieties are the corner stone of a geometrical theory of Pfaffian 5×5 ideals. They are *key varieties* and their homogeneous rings are *key rings*. Let us give an example of a quotient of a key ring by quasilinear regular sequence.

Let $\mathbb{C}[m_{ij}, n_i]$ be a polynomial ring of 10 variables of weights[§]

$$\operatorname{wt}(M) = \operatorname{wt}\begin{pmatrix} m_{12} & m_{13} & m_{14} & n_1 \\ & m_{23} & m_{24} & n_2 \\ & & m_{34} & n_3 \\ & & & n_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{pmatrix}$$

and consider the key ring R given as the quotient of $\mathbb{C}[m_{ij}, n_i]$ by the ideal generated by the 5 submaximal Pfaffians of M (see Section II.3 for a definition). Denote

$$J = (\mathrm{Pf}_1, \ldots, \mathrm{Pf}_5).$$

Then, from Theorem I.1 we know that $R = \mathbb{C}[m_{ij}, n_i]/J$ is Gorenstein. The weighted Grassmannian $\mathbb{G}$ is by definition $\operatorname{Proj} R$. We take the quotient of R by the ideal

$$I = (n_1 - q_1 + J, n_2 - q_2 + J, n_3 - q_3 + J, n_4 - q_4 + J) \subset R$$

with each q_i a general quadratic form in m_{ij}. The ideal I is generated by a quasilinear regular sequence. Define a surjective homomorphism $\lambda\colon \mathbb{C}[m_{ij}, n_i] \to \mathbb{C}[m_{ij}]$ by setting $n_i \mapsto q_i$ and consider the diagram

$$\begin{tikzcd} & \mathbb{C}[m_{ij}, n_i] \arrow[dl, "\lambda"'] \arrow[dr, "\sigma"] & \\ \mathbb{C}[m_{ij}] & & R \end{tikzcd}$$

The codimension of I is 4. However, since I is generated by a quasilinear regular sequence, the ideal $G = \lambda(\sigma^{-1}(I))$ has codimension 3 in $\mathbb{C}[m_{ij}]$. The variety $S = \operatorname{Proj} \mathbb{C}[m_{ij}]/G$ is a codimension 3 subscheme of $\mathbb{P}^5$. It was obtain as a quasilinear section of the weighted Grassmannian $\mathbb{G} = \operatorname{Proj} R$. As such, from the geometry of $\mathbb{G}$ (see Proposition IV.5 on page 79) we deduce that S is a regular surface of general type

[§]We use the convention to write only the upper triangle of a skew matrix, diagonal non-inclusive.

with $p_g = 6$ and $K^2 = 13$. In other words an example of a surface in the birational class (II).

Gorenstein unprojection was introduced by Kustin Miller in [**KM**] and later given a geometrical rendition in the work of Papadakis and Reid [**PR**]. As the title of [**KM**] says, it is a procedure of constructing Gorenstein ideals from smaller codimensional ones. The set up is that of an inversion of projection. One starts with the end product of the projection of a projectively Gorenstein variety in (weighted) projective space: (X, D); the projected variety X and a divisor D image of the exceptional divisor of the resolution of indeterminacy at the centre of projection. (Assume, for simplicity, one projects from a point). Then there exists a section $s \in \operatorname{Hom}(\mathcal{I}_D, \omega_X)$ which determines the unprojection of D in X. The unprojected variety is given by

$$Y = \operatorname{Spec} \mathcal{O}_X[s].$$

(See Papadakis and Reid's article [**PR**] for details). The starting and most fruitful examples occur with $\mathbb{Q}$-Fano 3-folds. Let us illustrate how one can obtain a surface of general type in the birational class (II) using the technique of unprojection.

Let $V_{2,3}$ be the Fano 3-fold given as the complete intersection of two hypersurfaces F_2, F_3 of degrees 2 and 3 in $\mathbb{P}^5$. Suppose that $V_{2,3}$ contains a plane $\pi \simeq \mathbb{P}^2$, whose equations are $x_1 = x_2 = x_3 = 0$. Then[¶] we can write:

$$\begin{pmatrix} F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} L_1 & L_2 & L_3 \\ Q_1 & Q_2 & Q_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{0.2}$$

where $L_i, Q_i \in \mathbb{C}[x_1, \ldots, x_6]$ are linear and quadratic respectively. Unprojecting π means adjoining a new variable s of degree 2 to the polynomial ring $\mathbb{C}[x_1, \ldots x_6]$ and writing down the birational map

$$V_{2,3} \dashrightarrow V \subset \mathbb{P}(1^6, 2), \text{ with } (x_1, \ldots, x_6) \mapsto (x_1, \ldots, x_6, s);$$

where s is the degree 2 rational form:

$$s = \frac{L_2 Q_3 - L_3 Q_2}{x_1} = \frac{L_3 Q_1 - L_1 Q_3}{x_2} = \frac{L_1 Q_2 - L_2 Q_1}{x_3}. \tag{0.3}$$

[¶]Notice that this will produce singularities of $V_{2,3}$ along π.

The rational form s is everywhere defined except at π and we can derive it from (0.2) using Cramer's rule. The unprojection of $V_{2,3}$, i.e. the variety Y, is the proper transform of $V_{2,3}$ under this map. Its equations follow immediately from (0.3):

$$sx_i = \bigwedge\nolimits_i^2 \begin{pmatrix} L_1 & L_2 & L_3 \\ Q_1 & Q_2 & Q_3 \end{pmatrix}, \text{ plus } F_2, F_3. \tag{0.4}$$

We can put these equations in the following Pfaffian format:

$$\mathrm{Pf}\begin{pmatrix} x_1 & -x_3 & L_3 & Q_3 \\ & x_2 & L_2 & Q_2 \\ & & L_1 & Q_1 \\ & & & -s \end{pmatrix}. \tag{0.5}$$

From either (0.5) or (0.4) we deduce that Y meets the singularity of $\mathbb{P}(1^6, 2)$ transversely and thus has a singularity of type $\frac{1}{2}(1,1,1)$ at the point $(0,\ldots,0,1)$. This variety is an anticanonically embedded Fano 3-fold of Gorenstein index 2 and genus 4. (These Fano varieties have been classified by Takagi in [**T00**] and many more Fano 3-folds can be obtained in the same way as here, by unprojection). The canonical ring of a general member $S \in |-2K_Y|$ obtained from the anticanonical ring of the Y, by adding the equation $s = q(x_1 \ldots, x_6)$, for a general quadratic form $q(x_1, \ldots, x_6)$, is Gorenstein of codimension 3. The surface S is nonsingular, projectively normal and canonically embedded in $\mathbb{P}^5$. There is a small computation to get the degree, namely, $-K_Y^3 = 2 \cdot 3 + \frac{1}{2} = \frac{13}{2}$ and $K_S^2 = 2 \cdot (-K_Y^3) = 13$.

The use of unprojection in the construction of Gorenstein graded rings follows a preliminary process of induction. One starts with the numerics of the graded ring we expect to obtain, sees how many times and what type of projection leads to a smaller ring and only then attempts to construct the targeted ring via unprojection.

For some cases, the method of key varieties can be used "deductively" applying the vector bundle method as we shall see in this thesis. (See Chapter IV and V). But for most cases it remains as yet essentially not inductive. (The reader should compare this with the idea of finding a variety with given numerical invariants among the set of complete intersections of projective space). On the other hand the method of key varieties is remarkably simple. Taking a quasilinear regular sequence in a key ring, or should we say, a quasilinear section in a key variety, makes the geometry transparent.

The vector bundle method. The idea to use key varieties is inspired by Mukai's linear section theorem. This result, which the reader can find on page 82, is the unification of several research papers [**M95a, M95b, M95c, M93, M88, M89**]. It asserts that every indecomposable Fano 3-fold with at most Gorenstein canonical singularities and genus $6 \leq \mathrm{g} \leq 10$ or $\mathrm{g} = 12$ is a linear section of an appropriate homogeneous space. In his list (see page 82) Mukai considers Grassmannians, orthogonal Grassmannians and Spinor varieties.

The technique Mukai uses to proof the linear section theorem is known as the vector bundle method. Given a Fano 3-fold, V, Mukai constructs a bundle on it, tautological with respect to the corresponding homogeneous space, and derives from this bundle an embedding of V into that homogeneous space. There is an important reduction step. Mukai considers a ladder

$$C \subset T \subset V \tag{0.6}$$

where T is a nonsingular K3 surface and C is a nonsingular *canonical* curve. The ladder consists in taking a linear section at each step with respect to the anticanonical embedding of V by $|-K_V|$, and therefore makes it possible to work with either C or T. The ladder of varieties of (0.6) corresponds to a chain of surjections of graded rings:

$$R(V, -K_V) \twoheadrightarrow R(T, -K_{V|T}) \twoheadrightarrow R(C, K_C)$$

each given by quotienting by a nonzero divisor of degree 1. In Mukai's linear section theorem (except for for genus 6 case where we need to take a quadric section) all sections are linear and therefore the final regular sequence is made up of linear homogeneous forms. Replacing linear by quasilinear and homogeneous by quasihomogeneous we arrive to the concepts described in the previous paragraph.

The tautological orbi-bundle. To formulate a vector bundle technique that would extend Mukai's results to weighted Grassmannians one has to describe the weighted homogeneous version of the Grassmannian variety $\mathrm{G}(2,5)$ and especially its tautological vector bundle. The tautological (orbi)-bundle on a weighted Grassmannian is defined by Corti and Reid in [**CR**].

Consider $\mathrm{G}(2,5)$ the ordinary Grassmannian of 2-dimensional subspaces of a fixed vector space V of dimension 5. Let $\mathscr{F}$ be the universal subbundle, i.e. the subbundle of $V \otimes \mathcal{O}_{\mathrm{G}(2,5)}$ whose fibre over a point $[L] \in \mathrm{G}(2,5)$ is L. We denote by $\mathscr{E}$ the quotient bundle of $V \otimes \mathcal{O}_{\mathrm{G}(2,5)}$ by $\mathscr{F}$. We have

$$0 \to \mathscr{F} \to V \otimes \mathcal{O}_{\mathrm{G}(2,5)} \to \mathscr{E} \to 0. \tag{0.7}$$

Let us write $\mathrm{aG}(2,5)\backslash 0$ for the punctured affine cone of $\mathrm{G}(2,5)$ in its Plücker embedding. The sequence (0.7) descends to the punctured affine cone $\mathrm{aG}(2,5)\backslash 0$ and yields a sheaf of rank 2 (that we keep denoting by $\mathscr{E}$). By $\mathbb{C}[\mathrm{aG}(2,5)]$ we mean the coordinate ring of $\mathrm{aG}(2,5)$ in affine space $\mathbb{A}^{10}$. If we denote the variables of affine space by x_{ij}, the ring $\mathbb{C}[\mathrm{aG}(2,5)]$ is the quotient of $\mathbb{C}[x_{ij}]$ by the ideal of submaximal Pfaffians of the skew matrix

$$M = \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ & x_{23} & x_{24} & x_{25} \\ & & x_{34} & x_{35} \\ & & & x_{45} \end{pmatrix}.$$

The Serre module of $\mathscr{E}$,

$$E = \bigoplus_{k \geq 0} H^0(\mathrm{aG}(2,5), \mathscr{E}(k)),$$

over the ring $\mathbb{C}[\mathrm{aG}(2,5)]$ is generated by 5 elements identified with either 5 column vectors $s_i = \binom{a_i}{b_i}$ or with the five columns of M. These generators are yoked by the following 10 relations:

$$x_{ij}s_k - x_{ik}s_j + x_{jk}s_i \quad \text{for} \quad 1 \leq i < j < k \leq 5. \tag{0.8}$$

These come from the map $\bigwedge^2 H^0(\mathrm{aG}(2,5), \mathscr{E}) \twoheadrightarrow H^0(\mathcal{O}_{\mathrm{aG}(2,5)}(1))$. In fact, we have $s_i \wedge s_j = x_{ij} \in H^0(\mathrm{aG}(2,5), \mathcal{O}_{\mathrm{aG}(2,5)}(1))$ and the relations of (0.8) are expressing the tautology

$$(s_i \wedge s_j)s_k - (s_i \wedge s_k)s_j + (s_j \wedge s_k)s_i. \tag{0.9}$$

It is easy to see that at a point of $\mathrm{aG}(2,5)$ either $s_i = s_j = s_k$ and (0.9) is trivially true, or, say s_i and s_j span the fibre of $\mathscr{E}$, in which case $s_k = \alpha s_j + \beta s_j$ at p and by direct computation we derive (0.9).

The point of this description of E is to enable the setting up of a grading compatible with the grading on $\mathbb{C}[\mathrm{aG}(2,5)]$ yielding a weighted Grassmannian. (See Section IV.1 for a definition of weighted Grassmannian). Then, via the Serre functor, the graded module E will correspond to the tautological orbi-bundle on the weighted Grassmannian.

Let $c_i \in \frac{1}{2}\mathbb{Z}$ be half-integers such that $\mathrm{wt}(x_{ij}) = c_i + c_j$ is the grading of $\mathbb{C}[\mathrm{aG}(2,5)]$ giving the weighted Grassmannian. Denote $\mathbb{C}[\mathrm{aG}(2,5)]$ with this grading by R. Then the assignment $\widetilde{\mathrm{wt}}(s_i) = c_i$ defines a $\mathbb{Z}$-grading on E by setting

$$E_n = \left\{ v \in E \mid v \text{ is homogeneous and } \widetilde{\mathrm{wt}}(v) = n + 1/2 \right\}$$

where by homogeneous we mean a sum of elements $\sum f_i s_i$ where $f_i \in R$ is homogeneous and $\deg(f_i) + \widetilde{\mathrm{wt}}(s_i)$ is constant when varying i. Notice that we have

$$R_m E_n \subset E_{m+n}$$

and that the relations of (0.9) are homogeneous with respect to this grading.

Let $\widetilde{E}$ be the sheaf on the weighted Grassmannian $\mathbb{G} = \operatorname{Proj} R$ corresponding to the graded module E under Serre's functor. For a subvariety, $X \subset \mathbb{G}$ the restriction of $\widetilde{E}$ to X can be a vector bundle of rank 2. We will see how to embed curves with a halfcanonical divisor A into generalised weighted Grassmannians using a bundle of rank 2 whose Serre module is isomorphic to E described above. This will be done in more detail in Chapter IV. For example, for a curve $\mathcal{C}$ of genus 14 with a halfcanonical divisor A with $\dim H^0(A) = 5$ we can find a vector bundle $\mathscr{E}$ on $\mathcal{C}$ of rank 2 and determinant A such that its Serre module E over the ring $R(\mathcal{C}, A)$ has four generators $\langle s_1, s_2, s_3, s_4 \rangle \subset H^0(\mathscr{E})$ in degree 0 and one generator $t \in H^0(\mathscr{E}(A))$, in degree 1. If this is to be induced by $\widetilde{E}$, the tautological orbi-bundle of certain weighted Grassmannian wG, we infer from numerical inspection that the half-integers giving the grading of $\mathbb{C}[\mathrm{aG}(2,5)]$ must be $(c_1, c_2, c_3, c_4, c_5) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$. If we trace back our definition of the graded module E such weighting yields

$$\widetilde{\mathrm{wt}}(s_i) = \frac{1}{2} \quad \text{for } i = 1..4, \quad \text{and} \quad \widetilde{\mathrm{wt}}(s_5) = \frac{3}{2}$$

In other words, E has four generators in degree 0 and one generator in degree 1.

Reduction to halfcanonical curves. Once we have made up our mind to use the vector bundle method and the notion of quasilinear section, the idea to studying halfcanonical curves is a natural reduction step. Just as in Mukai's canonical curves, halfcanonical curves are a few steps (in fact, exactly one) down the ladder of quasilinear sections containing a surface of general type.

In general the halfcanonical ring of a curve in the canonical linear system of a surface of general type is simpler to compute than the canonical ring of the surface. Supporting this assertion is the fact that the geometry of nonsingular curves is far simpler than of surfaces of general type. The same applies to the theory of coherent sheaves on a nonsingular curve. However, working with the halfcanonical ring on a curve is harder than working with its canonical ring. In the halfcanonical case the sheer numerical structure given by the theorems of Noether and Petri in the canonical curve case is unavailable. (These theorems have a subtle presence in Mukai's work). In Chapter III we prove some incipient results in this direction. In most important cases they fell short on giving the whole information that, say, a Petri type theorem for the halfcanonical ring $R(\mathcal{C}, A)$ would give. This has prompted in Chapter III the study of all the "easy" cases of halfcanonical rings of codimension ≤ 3. When taken a step up the ladder to the canonical ring of a surface of general type, these "easy" cases correspond to the canonical models of surfaces of general type that have been know since the time of Enriques.

CHAPTER II

Preliminaries

In this work, all varieties are defined over $\mathbb{C}$.

Definition II.1. Let V be a finite dimensional vector space over $\mathbb{C}$.

(i) A polynomial ring in n indeterminates will be denoted by $\mathbb{C}[x_1, \dots, x_n]$.

(ii) When a choice of basis for V is clear, the polynomial ring associated to V is denoted by $\mathbb{C}[V]$ and by definition is the $\mathbb{C}$-module

$$\mathbb{C}[V] = \bigoplus_{n \geq 0} \mathrm{S}^n V.$$

(iii) Affine space of dimension n is denoted by $\mathbb{A}^n$; when is necessary to exhibit a system of coordinates, by $\mathbb{A}[x_1, \dots, x_n]$ or to include the underlying vector space, by $\mathbb{A}[V]$.

(iv) Projective space of dimension n is denoted by $\mathbb{P}^n$, $\mathbb{P}[x_1, \dots, x_n]$, $\mathbb{P}[V]$, following similar conventions as in the previous item.

Let $\mathbb{C}[x_1, \dots, x_n]$ be a polynomial ring generated in arbitrary degrees.

(v) Weighted projective space, by definition $\operatorname{Proj} \mathbb{C}[x_1, \dots, x_n]$, is denoted by $\mathbb{P}(1^a, 2^b, \dots)$ where a is the number of variables of weight 1, b the number of those of weight 2 and so forth.

Let $\mathcal{C}$ be a nonsingular curve.

(vi) The notation g^r_d is for a linear system of dimension r and degree d, i.e. for a $(r+1)$-dimensional subspace of global sections of a line bundle on $\mathcal{C}$ of degree d.

(vii) The algebraic variety of linear systems of dimension r and degree d is denoted by $W^r_d(\mathcal{C})$.

(viii) The gonality of $\mathcal{C}$, denoted $\operatorname{gon}(\mathcal{C})$, is the least integer d for which there exists a g^1_d on $\mathcal{C}$.

We introduce some additional notation in this and subsequent chapters. An index for all the notation used in this work can be found on page 145.

II.1. Graded rings, Hilbert series and Hilbert numerator

Definition II.2 (Compare with [**GW**]). In this work a *graded ring* R is understood to be a finitely generated $\mathbb{C}$-algebra which decomposes as a direct sum of $\mathbb{C}$-vector spaces $R = \bigoplus_{n\geq 0} R_n$ such that

(i) $R_0 = \mathbb{C}$;
(ii) $R_m R_n \subset R_{m+n}$, for $m, n \geq 0$.

The summands R_i are called the *homogeneous components* of R and their elements are referred to as being *homogeneous.* (If R is a polynomial ring we will also use the terminology *quasihomogeneous forms*). An ideal of R is said to be *homogeneous* if it can be generated by homogeneous elements. We denote by $\mathfrak{m}$ the homogeneous maximal ideal $\bigoplus_{n>0} R_n$. A *graded module* over R is an R-module with a grading $\bigoplus_{n\in\mathbb{Z}} M_n$ of $\mathbb{C}$-modules such that $R_i M_j \subset M_{i+j}$ for all $j \in \mathbb{Z}$ and for all $i \geq 0$. For a graded R-module M and an integer d we define the *shift* by d to be the module M endowed with the grading $\bigoplus_{n\in\mathbb{Z}} M_{n+d}$. We denote this new graded module by $M(d)$.

Remark. As a convention we treat R as a graded R-module by setting $R_i = 0$ for all negative i. A similar convention applies to the quotient of R by a homogeneous ideal R/I.

II.1.1. The graded ring of a divisor. Suppose that we are given X a variety, D a line bundle on X and a set $s_1, \dots, s_k$ of global sections of D. A classical problem is to study the image of X under the map

$$X \supset X^0 \ni p \mapsto (s_1(p), \dots, s_k(p)) \in \mathbb{P}^{k-1}$$

where X^0 is an open set of X where the map can be defined. From the initial question of which equations define the image of X in $\mathbb{P}^{k-1}$ to the more sophisticated issue of projective normality, all can be rephrased in terms of the algebraic properties of $s_1, \dots, s_k$. E.g., the image of X under the map above is contained in a hypersurface of degree d if and only if there exists a relation of the same degree among $s_1, \dots, s_k$.

We take all the algebraic information on *any* set of sections of *any* multiple nD into to the graded ring associated to D.

DEFINITION II.3. Let X be a variety and D a divisor on X. The *graded ring of D on X* is the ring

$$R(X,D) = \bigoplus_{n\geq 0} H^0(X, nD).$$

The ring $R(X,D)$ is a very useful tool even when D is not ample. For example, the pair (S, K_S) consisting of a (nonsingular) surface of general type and its canonical divisor produces the canonical model $X = \operatorname{Proj} R(S, K_S)$. The surface X is embedded in weighted projective space and has at most du Val singularities resulting from the contraction of -2-cycles on S by the natural map $S \to \operatorname{Proj} R(S, K_S)$. (See [**Mum**]).

II.1.2. Example. Consider a nonsingular curve $\mathcal{C}$ of genus 6 with a divisor A such that $2A = K_{\mathcal{C}}$ and that $h^0(A) = 3$. A nonsingular quintic curve in the projective plane is a curve of genus 6, and indeed the hyperplane section is the a divisor A with $\dim H^0(A) = 3$. In this example we will see how to recover the plane model of $(\mathcal{C}, A)$ from the original data.

A curve $\mathcal{C}$ in this conditions is said to have a halfcanonical divisor. In general, a curve $\mathcal{C}$ is said to have a subcanonical divisor if there exists a divisor A such that for some integer k, we have $kA = K_{\mathcal{C}}$. The degree of a halfcanonical divisor is uniquely determined by the genus: $\deg(A) = \mathrm{g}(\mathcal{C}) - 1$, and for $n \geq 3$ the dimension of $H^0(nA)$ is given by the Riemann–Roch theorem. For $n = 1$ the theorem tells us nothing about the dimension of the space of global sections of $\mathcal{O}_{\mathcal{C}}(A)$:

$$h^0(A) - h^1(A) = 1 - g + g - 1 = 0;$$

except that $h^0(D) = h^1(D)$ which we already knew by Serre duality. This is typical in the study of subcanonical divisors on varieties: there is a range of "initial values" of $h^0(nA)$ which we have to specify.

We reconstruct the plane quintic and its hyperplane section divisor by constructing the graded ring $R(\mathcal{C}, A)$. We start by writing the dimension of the spaces $H^0(nA)$

$$\dim H^0(nA) = \begin{cases} 3 \text{ if } n = 1 \\ 6 \text{ if } n = 2 \\ 5(n-1) \text{ if } n \geq 3 \end{cases}$$

Consider the following generators for $H^0(nA)$ when $n \leq 3$:

$$\begin{aligned} H^0(A) &= \langle x_1, x_2, x_3 \rangle, \\ H^0(2A) &\supset \left\langle \mathrm{S}^2(x_1, x_2, x_3) \right\rangle, \\ H^0(3A) &\supset \left\langle \mathrm{S}^3(x_1, x_2, x_3) \right\rangle. \end{aligned}$$

Let us assume that the divisor A is free. Denote by $\langle s_1, s_2 \rangle \subset H^0(A)$ a pair of sections spanning a free pencil. Then Castelnuovo's free-pencil trick shows that the kernel of

$$\langle s_1, s_2 \rangle \otimes H^0(nA) \to H^0((n+1)A) \tag{1.1}$$

is isomorphic to $H^0((n-1)A)$ and in particular, we deduce that for $n \geq 4$ the map (1.1) is surjective. The upshot is that if $\left\langle \mathrm{S}^2(x_1, x_2, x_3) \right\rangle$ and $\left\langle \mathrm{S}^3(x_1, x_2, x_3) \right\rangle$ span the whole of $H^0(2A)$ and $H^0(3A)$, respectively, then we have no need for extra generators for $R(\mathcal{C}, A)$. This happens if and only if the image of $\mathcal{C}$ by the map φ_A does not map to a plane conic or to a plane cubic. If $\varphi_A(\mathcal{C})$ is a conic or a cubic then A cannot be free. We deduce that

$$A \text{ free } \implies R(\mathcal{C}, A) \text{ is generated by } H^0(A).$$

All we need now to describe $R(\mathcal{C}, A)$ is the ideal I_A, of relations holding among x_1, x_2, x_3. A relation will only occur when the number elements of $\mathrm{S}^n \langle x_1, x_2, x_3 \rangle$ exceeds the dimension of $h^0(nD)$, in other words, when

$$\binom{n+2}{n} \geq 5(n-1).$$

This happens first for $n = 5$. Therefore the ideal I_A is generated by a single quintic relation, as expected.

II.1.3. Speeding up the calculation.

Definition II.4. Let M be a finite graded module over a graded ring R. The *Hilbert function of M* is defined by setting $\phi_M(n) = \dim_{\mathbb{C}} M_n$ and the *Hilbert series* by

$$H_M(t) = \sum_{n \in \mathbb{Z}} \phi_M(n) t^n.$$

For example, take $M = R = \mathbb{C}[x_1, \ldots, x_d]$. Then

$$H_M(t) = \frac{1}{(1-t)^d}. \tag{1.2}$$

To see this, expand each of $\frac{1}{1-t}$ on the right-hand side as $1 + t + t^2 + \cdots$, and then expand the product $(1 + t + t^2 + \cdots)^d$. For each k the coefficient of t^k is exactly the dimension of R_k. Assume now that $I \subset R$ is a homogeneous ideal and take for M the R-module R/I. Since $R \twoheadrightarrow R/I$ the same line of reasoning applies here, except that we must correct numerator of (1.2) to account for the relations, 1st syzygies, 2nd syzygies, and so forth. Thus we can write

$$H_M(t) = \frac{1 + a_1 t + a_2 t^2 + \cdots}{(1-t)^d}. \tag{1.3}$$

Theorem II.5 (Hilbert). *Let M be a finite graded R-module of dimension δ. Assume that R is generated over $\mathbb{C}$ by elements of degree 1. Then ϕ_M is of polynomial type of degree $\delta - 1$.*

Proof. See [**BH**, p. 147]. □

This theorem shows that the numerator of (1.3) is indeed a 'finite' polynomial. The reason is that multiplying the Hilbert series of M by $(1-t)^\delta$ is the same as computing

$$\sum_{k \geq 0} \Delta^\delta \phi_M(k) t^k$$

and as ϕ_M is polynomial of degree $\delta - 1$, for large enough k, $\Delta^\delta \phi_M(k) = 0$. (We go through a similar computation in more detail a few lines below, where we define the operator Δ).

Besides the clear advantage of being compact notation, writing the Hilbert series of M as in (1.3) is a natural byproduct of a free resolution of M. Consider one such exact complex of free graded R-modules:

$$0 \leftarrow R/I \leftarrow R \leftarrow \bigoplus R(-a_i) \leftarrow \bigoplus R(-b_i) \leftarrow \cdots \tag{1.4}$$

Taking dimensions over graded components we see that

$$H_M(t) = \frac{1}{(1-t)^d} - \sum H_{R(-a_i)}(t) + \sum H_{R(-b_i)}(t) - \cdots$$

and as $H_{R(-a)} = \frac{t^a}{(1-t)^d}$ we deduce that

$$H_M(t) = \frac{1 + \sum t^{a_i} - \sum t^{b_i} + \cdots}{(1-t)^d}.$$

Conversely having the Hilbert series of R/I in a fractional form with denominator $(1-t)^d$ (corresponding to the generators of R) takes us one step closer to unfolding the structure of R/I.

Proposition II.6. *Let $\mathcal{C}$ be a nonsingular curve of genus* g *with a halfcanonical divisor A with $h = h^0(A)$. Then the Hilbert series of $R(\mathcal{C}, A)$ is given by*

$$\frac{1 + (h-2)t + (\mathrm{g} - 2h + 1)t^2 + (h-2)t^3 + t^4}{(1-t)^2} \tag{1.5}$$

Proof. By Riemann–Roch,

$$H(t) = 1 + ht + \mathrm{g}\, t^2 + \sum_{k\geq 3}(k-1)(\mathrm{g} - 1)t^k,$$

thus

$$(1-t)H(t) = 1 + (h-1)t + (\mathrm{g} - h)t^2 + (\mathrm{g} - 2)t^3 + \sum_{k\geq 4}(\mathrm{g} - 1)t^k$$

and hence

$$(1-t^2)H(t) = 1 + (h-2)t + (\mathrm{g} - 2h + 1)t^2 + (h-2)t^3 + t^4. \quad \square$$

If we apply this to our example, $(\mathcal{C}, A)$ of a curve of genus 6 and halfcanonical net, we obtain

$$H_{R(\mathcal{C},A)} = \frac{1 + t + t^2 + t^3 + t^4}{(1-t)^2}.$$

As $h = 3$ there is at least one more generator in degree 1 in any minimal surjection $\mathbb{C}[x_i] \twoheadrightarrow R(\mathcal{C}, A)$ we might set up. Multiplication by $(1-t)$ leads to

$$\frac{1-t^5}{(1-t)^3}.$$

If this were the starting point of the computations of our example we would now haste the guess that there is a surjection $\mathrm{ev}\colon S \twoheadrightarrow R$ with $S = \mathbb{C}[x_1, x_2, x_3]$; in other words, that $R(\mathcal{C}, A)$ is generated by $H^0(A)$, and furthermore, that there is a resolution

$$0 \leftarrow R(\mathcal{C}, A) \leftarrow S \leftarrow S(-5) \leftarrow 0, \tag{1.6}$$

i.e. $R(\mathcal{C}, A) = \mathbb{C}[x_1, x_2, x_3]/(F_5)$. In a certain sense, what the Hilbert series does is to go through the table-like computation of Paragraph II.1.2 and compute the general case given the numerical data of $(\mathcal{C}, A)$. However it will not tell us what assumptions on the geometry of $(\mathcal{C}, A)$ in fact lead to this general situation. As we saw before the assumption that A be a free divisor played an important part.

Consider the case when A has a simple base point. Since the genus of $\mathcal{C}$ is ≥ 2, the canonical divisor $2A = K_{\mathcal{C}}$ is free of base points, hence it is not true that $\mathrm{S}^2\langle x_1, x_2, x_3\rangle$ generate the space $H^0(2A)$. Therefore there has to be a new generator $y \in H^0(2A)$ and a quadratic relation $F_2 \in \mathrm{S}^2\langle x_1, x_2, x_3\rangle$. The statements we made about the generators and resolution of $R(\mathcal{C}, D)$ are no longer right. The polynomial ring S must be substituted for $\mathbb{C}[x_1, x_2, x_3, y]$ and the resolution of (1.6) for*

$$0 \leftarrow R(\mathcal{C}, A) \leftarrow S \leftarrow S(-2) \oplus S(-5) \leftarrow S(-7) \leftarrow 0.$$

Yet the Hilbert series are identical:

$$\frac{1-t^5}{(1-t)^3} = \frac{1-t^2-t^5+t^7}{(1-t)^3(1-t^2)}. \tag{1.7}$$

This is a first, rather artificial, example of **masking**. We mention this in later chapters and we give examples where the general case is 'masked'. Roughly, masking occurs when generators of two terms of a resolution, either adjacent or an odd number of terms apart happen to have the same degree and thus cancel out in the Hilbert series. Just like in (1.7):

$$\frac{1-t^5}{(1-t)^2} = \frac{1-t^2+t^2-t^5}{(1-t)^3} = \frac{1-t^2-t^5+t^7}{(1-t)^3(1-t^2)}.$$

II.1.4. Polynomial rings generated in arbitrary degree.

Definition II.7. Compare with [**BH**, p. 167]. A function $\phi\colon \mathbb{Z} \to \mathbb{Q}$ is said to have *quasipolynomial type (of period g)* if there exist a positive integer g and polynomials $P_i \in \mathbb{Q}[t]$, $i = 0, \ldots, g-1$, of equal degree and equal leading coefficient, such that for $m \gg 0$, $\phi(mg+i) = P_i(mg+i)$, for $0 \leq i \leq g-1$.

It is an easy exercise to extend the results of [**BH**] to this setting. Namely, if we define the operator Δ_g by $\Delta_g\phi(n) = \phi(n+g) - \phi(n)$ for any function $\phi\colon \mathbb{Z} \to \mathbb{Q}$, then:

Proposition II.8. *The following are equivalent:*

(i) $\Delta_g^d\phi(n) = c$, $c \neq 0$, *for all* $n \gg 0$*;*

---

*The new term, $A(-7)$, accounts for the syzygy: $F_5 \cdot F_2 - F_2 \cdot F_5$.

(ii) *ϕ is a quasipolynomial function of period g and degree d.*

Proof. Adapt proof of [**BH**, 4.1.2]. □

This is an analogue of Theorem II.5:

THEOREM II.9. *Let S be a graded ring generated in arbitrary positive degrees and M be a finite graded S-module of dimension δ. Then ϕ_M is of quasipolynomial type of degree $\delta - 1$.*

Proof. See [**BH**, 4.3.5]. □

Let us illustrate this theorem with a few examples. Take

$$M = S = \mathbb{C}[x_1, \dots, x_d]$$

where the weight of the variable x_i is the integer $b_i > 0$. Expanding the product

$$(1 + t^{b_1} + t^{2b_1} + \cdots)(1 + t^{b_2} + t^{2b_2} + \cdots) \cdots (1 + t^{b_d} + t^{2b_d} + \cdots)$$

what we get as the coefficient of t^k is exactly the dimension of the space of quasihomogeneous forms of S of degree k. Thus,

$$H_S(t) = \frac{1}{\prod_{i=1}^{d}(1 - t^{b_i})}.$$

Notice that, like in the case of a polynomial ring generated in degree 1, multiplying the Hilbert series by $(1 - t^b)$ is equivalent to computing

$$\sum_{k \geq 0} \Delta_b \phi_M(k) t^k.$$

Fixing b the least common multiple of $\{b_i \mid 1 \leq i \leq d\}$, so that for all i, $(1-t^{b_i})$ divides $(1 - t^b)$ we deduce that

$$(1 - t^b)^d H_S(t) = \frac{(1 - t^b)^d}{\prod_{i=1}^{d}(1 - t^{b_i})}$$

is a polynomial. In virtue of Theorem II.9, for any homogeneous ideal $I \subset S$, writing, just as in Paragraph II.1.3, the Hilbert series of S/I as

$$\frac{1 + a_1 t + a_2 t + \cdots}{\prod_{i=1}^{d}(1 - t^{b_i})}$$

we deduce that the numerator of this rational function is a polynomial. Which brings us to the definition:

Definition II.10. Let $S = \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial ring generated in arbitrary positive degrees. Denote the degree of the variable x_i by b_i. Let M be a graded S-module of finite type. Then the polynomial

$$H_M(t)\prod_{i=1}^{d}(1-t^{b_i})$$

is called the *Hilbert numerator* of $H_M(t)$, or simply the Hilbert numerator of M. We denote this polynomial by $Q_M(t)$.

We end this section with a standard result computing the degree of graded S-module M directly from the Hilbert numerator.

Definition II.11. Let S be as above, a polynomial ring generated in arbitrary positive degrees, and M a graded S-module of dimension $d-1$. The degree of M is by definition $c_d/d!$ where c_d is the leading coefficient of the quasi-polynomial $\phi_M(n)$. It follows that $\deg M$ equals the constant value of $\Delta_g^d \phi_M(k)$ for large k.

The aim is to show a way to compute the degree of M from the Hilbert polynomial of M. We start with a preliminary result.

Proposition II.12. *Let $A = \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial ring generated in degree 1 and M a graded A-module. Denote the codimension of M by c and its Hilbert numerator by*

$$Q_M(t) = a_0 + a_1 t + \cdots + a_r t^r.$$

Then

$$\deg M = \sum_{i=0}^{r} a_i \binom{r-i}{c}.$$

Proof. For each i, let us define a numerical function:

$$\phi_i(k) = a_i \binom{k-i+d-1}{d-1}.$$

The binomial coefficient $\binom{k-i+d-1}{d-1}$ counts the number of homogeneous forms of degree $k-i$ in d variables, thus its associated series $\sum_{k\in\mathbb{Z}} \phi_i(k)t^k$ equals $\frac{a_i t^i}{(1-t)^d}$. Therefore, it follows that

$$H_M(t) = \sum_{k\in\mathbb{Z}} \left(\sum_{i=1}^{r} \phi_i(k)\right) t^k,$$

and therefore

$$\phi_M(k) = \sum_{i=1}^{r} \phi_i(k). \tag{1.8}$$

Thus to compute degree we have only to apply Δ^{d-c-1} to both sides and decide for which "large enough" k does $\Delta^{d-c-1}\phi_M(k)$ becomes a *nonzero* constant. This can be done using (1.8). Notice that we can use the extension to negative integers by zero, since for $k < 0$ the binomial coefficient $\binom{k-i+d-1}{d-1}$ is zero for any $0 \leq i \leq r$. Now, by an elementary binomial identity it follows that

$$\Delta^{d-c-1}\phi_i(k) = \binom{k-i+c}{c}.$$

The function $\sum_{i=1}^{r} \Delta^{d-c-1}\phi_i(k)$ will settle down in some nonzero constant value for k big enough. With what we have done so far it is easy to give a precise value k_0 for which this first happens. Just notice that

$$(1-t)^{c+1} \sum_{k\in\mathbb{Z}} \left(\sum_{i=1}^{r} \phi_i(k) \right) t^k = Q_M(t) = a_0 + a_1 t + \cdots + a_r t^r$$

and therefore if

$$\sum_{k\in\mathbb{Z}} \left(\sum_{i=1}^{r} \Delta^{d-c-1}\phi_i(k) \right) t^k = b_0 + b_1 t + \cdots b_{k_0} t^{k_0} + \sum_{k\geq k_0+1} b_{k_0} t^k$$

then $r = k_0 + c$ and we finally conclude that

$$\deg M = \sum_{i=1}^{r} \phi_i(r-c) = \sum_{i=0}^{r} a_i \binom{r-i}{c}. \quad \square$$

Corollary II.13. *Let $S = \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial ring and M a graded S-module. Denote the degree of the variable x_i by b_i, the codimension of M by c and the Hilbert numerator of M by*

$$Q_M(t) = a_0 + a_1 t + \cdots + a_r t^r.$$

Then

$$\deg M = \frac{\sum_{i=0}^{r} a_i \binom{r-i}{c}}{\prod_i^d b_i}.$$

Proof. This is a corollary of the proof of the previous proposition. By definition we have

$$\prod_{i=1}^{d} (1 - t^{b_i}) H_M(t) = Q_M(t) \tag{1.9}$$

By the identity

$$(1 + t + t^2 + \cdots t^{b-1})(1 - t) = (1 - t^b)$$

equation (1.9) is equivalent to

$$(1 - t)^d \prod_{i=1}^{d} (1 + t + \cdots + t^{b_i - 1}) H_M(t) = Q_M(t).$$

Since

$$\prod_{i=1}^{d} (1 + t + \cdots + t^{b_i - 1}) H_M(t) = \sum_{k \in \mathbb{Z}} \left\{ \prod_{i=1}^{d} (1 + t + \cdots + t^{b_i - 1}) \phi_M(k) \right\} t^k$$

we deduce that the numerical function

$$\psi(k) = \sum_{0 \leq j_i \leq b_i - 1} \phi_M(k - j_1 - j_2 - \cdots - j_d) \tag{1.10}$$

is of polynomial type (of some degree $\leq d$) and from (the proof of) Proposition II.12 that the leading coefficient of ψ times the factorial of its the degree equals

$$\sum_{i=0}^{r} a_i \binom{r - i}{c}.$$

The degree of ψ is the same as that of the quasipolynomial function $\phi_M(k)$ and in terms of the leading coefficient of coefficient of $\phi_M(k)$, that of ψ comes multiplied by $\prod_{i=1}^{d} b_i$ as we deduce from (1.10). $\square$

II.1.5. Remark. When we introduce Gorenstein ideals, this formula will become slightly easier to remember:

$$\deg S/I = \frac{\sum_{i=0}^{r} a_i \binom{i}{c}}{\prod_i^d b_i}.$$

with notations as in Corollary II.13.

II.2. The attributes of Cohen–Macaulay and Gorenstein

The notions of a Cohen–Macaulay and Gorenstein play an important role in the geometry of graded rings. In this section we introduce the notions and state the results that we use in subsequent chapters. The books of Eisenbud [**Ei**] and Bruns and Herzog [**BH**] are standard reference texts that do the general theory of Cohen-Macaulay modules. As far as duality on graded rings is concerned we follow the work of Goto and Watanabe [**GW**].

Definition II.14 (See [**Ei**]). A ring R is Cohen–Macaulay if and only if

$$\operatorname{depth} P = \operatorname{codim} P$$

for every maximal ideal P of R. Where the codimension of P is the supremum of lengths of chains of primes descending from P and the depth of P is the length of a *maximal* R-sequence contained in P.

Besides being a measure of how close to *good* a ring is the notion of Cohen–Macaulay yields useful results when dealing with intersections. Suppose that R is a graded ring and $f_1, \dots, f_n$ are homogeneous forms in R. Then the ideal $I = (f_1, \dots, f_n)$ defines a subscheme of $\operatorname{Proj} R$ given by $\operatorname{Proj} R/I$. From the onset we would like to know the dimension of R/I and more specifically what are the primary components of the scheme $\operatorname{Proj} R/I$. Roughly speaking $\operatorname{Proj} R/I$ is made of irreducible components with some multiplicity plus some embedded components. This decomposition corresponds to a primary decomposition of $I \subset R$ (recall that primary decompositions are not unique). The Unmixedness result says that if I is generated by a regular sequence the scheme $\operatorname{Proj} R/I$ has no embedded components. We start by addressing the issue of dimension.

Theorem II.15. *Let R be a local Cohen-Macaulay ring. Then $\mathbf{x} = x_1, \dots, x_r$ is an R-sequence if and only if* $\dim R/\mathbf{x} = \dim R - r$

Proof. See [**BH**, 2.1.2]. □

We can recover this result for graded rings by localising at the graded maximal ideal $\mathfrak{m}$. Noticing that if $(\underline{x})$ is a sequence of homogeneous elements of R then

$$(\underline{x}) \text{ is an } R\text{-sequence} \Longleftrightarrow (\underline{x}_{\mathfrak{m}}) \text{ is an } R_{\mathfrak{m}}\text{-sequence.}$$

Theorem II.16 (Unmixedness Theorem). *Let R be a ring. If $I = (x_1, \dots, x_n)$ is an ideal generated by n elements such that* $\operatorname{codim} I = n$, *then all minimal primes of I have codimension n. If R is Cohen-Macaulay, then every associated prime of I is minimal over I.*

Proof. See [**Ei**, 18.14]. □

II.2.1. Example. These two results can be used to compute the homogeneous ring of a projective variety. Here is an example that *almost* works. Recall the if X is a projective reduced variety in (weighted) projective space $\mathbb{P}$ then the homogeneous ideal of X, denoted by $I(X)$ is the ideal of quasihomogeneous forms $F \in \mathbb{C}[x_1, \ldots, x_n]$ that vanish on X. The ideal $I(X)$ is a radical ideal. By the Nullstellensatz, the zero set of two ideals A, B coincide if and only if $\operatorname{Rad} A = \operatorname{Rad} B$. If we have the variety $X \subset \mathbb{P}$ and an ideal of quasihomogeneous forms $(f_1, \ldots, f_n)$ that cuts out X set-theoretically X then $\operatorname{Rad}(f_1, \ldots, f_n) = I(X)$. Take for example the twisted cubic. The curve $C_3 \subset \mathbb{P}^3$ is the image of $\mathbb{P}^1$ by the third Veronese embedding:

$$(u, v) \mapsto (u^3, u^2v, uv^2, v^3).$$

The homogeneous ideal of $C_3 \subset \mathbb{P}^3 = \mathbb{P}[x, y, z, w]$ is generated by 3 quadrics given by

$$\operatorname{rank} \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix} \leq 1.$$

Now let us take the quadric $q = xz - y^2$ and the cubic $f = \det \begin{pmatrix} x & y & z \\ y & z & w \\ z & w & 0 \end{pmatrix}$. Since q is irreducible and f is not a zero divisor in $\mathbb{C}[x, y, z, w]/(q)$, by definition, (q, f) are a regular sequence. Since $\mathbb{C}[x, y, z, w]$ is a Cohen–Macaulay graded ring, by Theorem II.15 the dimension of $\mathbb{C}[x, y, z, w]/(q, f)$ is $3 - 2 = 1$. By the Unmixedness Theorem we know that all associated primes of (q, f) are minimal over (q, f). Let us now show that there is only one minimal prime over (q, f) corresponding to the irreducible component C_3. On the one hand it is clear that q and f go through C_3. (The condition on the rank implies that the determinant of the matrix defining f is zero). On the other hand if

$$q = 0 \quad \text{and} \quad f = z \det \begin{pmatrix} y & z \\ z & w \end{pmatrix} - w \det \begin{pmatrix} x & z \\ y & w \end{pmatrix} = 0$$

then from $q = 0$ we deduce that there exists $k \in \mathbb{C}$ such that $(x, y) = (ky, kz)$ or $(y, z) = (kx, ky)$. With no loss in generality, we can assume that $(y, z) = (kx, ky)$. Then $y = kx$, $z = k^2x$ and from $f = 0$ we get $kz = w$, i.e. $w = k^3x$. This means that the rank of the matrix $\begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}$ is ≤ 1. We have shown that q and f cut out set-theoretically C_3. Scheme-theoretically, (q, f) cuts out a subscheme of degree 6. (The degree is easily computed from the fact that f, g are a regular sequence. The scheme-theoretic intersection is $2C_3$, in other words $q = 0$ and $f = 0$ meet with multiplicity 2 at C_3. The twisted cubic is a classical example of a self-linked curve). However, if the degree of $R/(q,f)$ were 3 then we would deduce that $I(X) = (q,f)$. This sketches a type of reasoning that will be applied in a later chapter.

PROPOSITION II.17. *Let S be a polynomial ring (generated in arbitrary positive degrees). The degree of an homogeneous ideal $I \subset S$ is by definition the degree of the graded S-module S/I. Assume that I is primary. Let P be the prime ideal* $\operatorname{Rad} I$. *Suppose that* $\deg(I) = \deg(P)$. *Then $I = P$.*

Proof. Since I is primary the set of associated primes of A/I consists of $\{P\}$. Consider a *dévissage* of A/I:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = A/I$$

where $M_i/M_{i-1} \simeq A/P$. (See Reid's textbook [**R95**, pag 103]). In particular we have $\deg(P) = \deg(M_i) - \deg(M_{i-1})$ and therefore

$$\deg(A/I) = n \deg(P).$$

We deduce that $n = 1$ and therefore $A/I = A/P$. $\square$

Remark. When S is a polynomial ring generated in degree 1, the integer n is by definition the multiplicity of P in I and corresponds to the geometric multiplicity of the irreducible component given by P. For a general ideal I with a set of minimal primes $\{P_1, \dots, P_r\}$ the multiplicity of P_r in I can be defined as the length of a *dévissage* of the A_{P_r}-module $(A/I)_{P_r}$. It can be checked this notion is well defined (see [**H**, I.7.4]).

DEFINITION II.18. [**BH**, §3.4] Let R be a Noetherian regular local ring. An ideal $I \subset R$ is a Gorenstein ideal of codimension c if R/I is Cohen–Macaulay and $\operatorname{Ext}^c_R(R/I, R) \cong R/I$. A Noetherian ring R is Gorenstein if each localisation at maximal ideals is a Gorenstein local ring.

Since,

$$c = \operatorname{codim} I = \min \left\{ i \mid \operatorname{Ext}^i_R(R/I, R) \neq 0 \right\}$$

(see [**BH**, 1.2.5]) if

$$R/I \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_c \leftarrow 0 \tag{2.1}$$

is a minimal free resolution of a Gorenstein ideal, the dual complex of (2.1) will have zero cohomology at every degree except at degree c and the cohomology at this degree is $\mathrm{Ext}^c_R(R/I, R)$, which by assumption is again R/I. Therefore by uniqueness of minimal free resolutions, $F_{c-i}^\vee \cong F_i$ and this is expressed by saying that (2.1) is self-dual.

II.2.2. The canonical module of a graded ring. In [**GW**] the canonical module of a graded ring is defined. This is a global version of the canonical module of a local Noetherian ring, as in [**Ei**] or [**BH**]. Its explicit definition would require some more homological algebra. In this account we only sketch some of its properties. Let us denote the canonical module of a graded ring by ω_R for any graded ring R as in Definition II.2. If R is a graded ring and M is a graded R-module we denote the associated sheaf to M on $\mathrm{Proj}(R)$, image of M by Serre's functor, by $\widetilde{M}$. (See [**H**, p 116]).

Theorem II.19. [**GW**, 5.1.8] *Let $X = \mathrm{Proj}(R)$. Then $\widetilde{\omega_R} = \omega_X$.* $\square$

Notice that $\mathrm{Proj}\, R$ is a projective scheme and has a dualising sheaf $\omega_X = \mathscr{E}xt^c_{\mathbb{P}}(\mathcal{O}_X, \omega_{\mathbb{P}})$ given that we can embed X as a subscheme of projective space $\mathbb{P}$ of some codimension c. (See [**H**, 7.5]). The next two results will be used in the proof of Theorem II.23.

Theorem II.20. [**GW**, 2.1.3] *If R is a Cohen-Macaulay graded ring, then R is Gorenstein if and if $\omega_R = R(d)$ for some $d \in \mathbb{Z}$.*

Theorem II.21. [**GW**, 2.1.6] *Let R be a Cohen-Macaulay graded ring. Then for any homogeneous ideal I of codimension c, such that R/I is Cohen-Macaulay*

$$\mathrm{Ext}^j_R(R/I, \omega_R) = \begin{cases} 0 & \text{if } i < c \\ \omega_{R/I} & \text{if } i = c \end{cases}$$

Definition II.22. Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be graded polynomial ring. A subscheme $X \subset \mathrm{Proj}(S) = \mathbb{P}$ is called projectively Gorenstein if the homogeneous ideal of X is a Gorenstein ideal. In particular a projectively Gorenstein subscheme of $\mathbb{P}$ is arithmetically Cohen–Macaulay, i.e. its homogeneous ideal is Cohen–Macaulay.

Theorem II.23. *Let $X \subset \mathbb{P}$ be a projectively Gorenstein scheme of codimension c. Then the minimal free resolution of $\mathcal{O}_X$ as an $\mathcal{O}_{\mathbb{P}}$-module has length c:*

$$\mathcal{O}_X \leftarrow \mathcal{L}_0 \leftarrow \cdots \leftarrow \mathcal{L}_c \leftarrow 0$$

and moreover there is an integer—the adjunction number—such that

(i) $\mathcal{L}_{c-i}^\vee \otimes \mathcal{O}_{\mathbb{P}}(k) = \mathcal{L}_i$ *(Gorenstein symmetry)*
(ii) $\omega_X = \omega_{\mathbb{P}} \otimes \mathcal{L}_c^\vee$ *(Adjunction)*

Proof. Let S denote the homogeneous ring of $\mathbb{P}$ and I the homogeneous ideal of X. Let

$$S/I \leftarrow L_0 \leftarrow \cdots \leftarrow L_c \leftarrow 0$$

be a minimal free resolution of S/I. Apply the functor $\mathrm{Hom}_S(\_,\omega_S)$:

$$0 \to L_0^\vee \otimes \omega_S \to \cdots \to L_c^\vee \otimes \omega_S \to 0.$$

Since S/I is Cohen–Macaulay, this complex is exact at every degree except in degree c, where the cokernel is $\omega_{S/I}$. By the Gorenstein assumption $\omega_{S/I} = (S/I)(d)$; and by uniqueness of minimal resolutions, $L_{c-i}^\vee \otimes \omega_S(-d) = L_i$. In particular,

$$\omega_{S/I} = (S/I)(d) = L_0(d) = L_c^\vee \otimes \omega_S \quad \square$$

II.3. Pfaffians

Let $M = (m_{ij})$ be a 5×5 skew matrix. We define the submaximal Pfaffians of M, using the notation Pf_i for the ith submaximal Pfaffian by

$$\mathrm{Pf}_i = (-1)^{i+1} (m_{hj}m_{kl} - m_{hk}m_{jl} + m_{hl}m_{jk})$$

where the indices i, h, j, k, l are such that $\{i, h, j, k, l\} = \{1, 2, 3, 4, 5\}$. Modulo a plus or minus sign the submaximal Pfaffian Pf_i is the Pfaffian of the skew matrix obtained from M by removing the row and column i. Given that a skew matrix has even rank, the determinant of M and therefore the Pfaffian of M are zero. Expanding along each row we obtain the following Pfaffian[†] identity:

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \end{pmatrix} \begin{pmatrix} \mathrm{Pf}_1 \\ \mathrm{Pf}_2 \\ \mathrm{Pf}_3 \\ \mathrm{Pf}_4 \\ \mathrm{Pf}_5 \end{pmatrix} = 0$$

If we regard Pf_i as defining polynomial equations in a ring $\mathbb{C}[m_{ij}]$ then this identity gives a set of 5 syzygies. In connection with this recall the theorem of Buchsbaum–Eisenbud on page 4. As shorthand notation we denote the 5-tuple consisting of the submaximal Pfaffians of M by $\mathrm{Pf}\, M$.

[†]It is our convention to write only the upper triangle of a given skew matrix.

CHAPTER III

Halfcanonical Curves

Throughout this work $\mathcal{C}$ is a nonsingular abstract algebraic curve. *A priori*, $\mathcal{C}$ is not embedded in a projective space. The initial datum is the genus of $\mathcal{C}$. One step further we consider a polarising divisor of a particular kind.

Definition III.1. Let $\mathcal{C}$ be a nonsingular algebraic curve.

(i) A divisor A is called an *halfcanonical divisor** if $2A = K_C$.

In this work we restrict our attention to effective halfcanonical divisors, i.e. those for which $h^0(A) > 0$. On an elliptic curve, there is only one effective halfcanonical divisor. We restrict to curves of genus ≥ 2. Let A be a halfcanonical divisor on a curve $\mathcal{C}$ of genus ≥ 2. Since $\mathrm{g}(\mathcal{C}) \geq 2$ and $2A = K_{\mathcal{C}}$, the divisor A is ample. To the polarising divisor A we associate

(ii) the halfcanonical ring $R(\mathcal{C}, A) = \bigoplus_{n\geq 0} H^0(nA)$.

Let S be a nonsingular regular surface of general type. A general member $\mathcal{C}$ of the canonical linear system $|K_S|$ is a nonsingular curve and by adjunction it has a halfcanonical divisor $A = K_{S|\mathcal{C}}$. If S is a regular, it is left with two invariants p_g and K^2. (According to our conventions on page 122, S is minimal). Additionally, the genus of $\mathcal{C}$ is given by $K_S^2 + 1$ and the invariant p_g of S is transferred onto $\mathcal{C}$ as $\dim H^0(A) = p_g - 1$. There is one more way in which we should think of $(\mathcal{C}, A)$ as being given with two invariants, and this is related to the numerical structure of $R(\mathcal{C}, A)$. The dimensions of the graded components of this ring are by definition $\dim H^0(nA)$. Hence on a first attempt at computing $R(\mathcal{C}, A)$ we must own this data. The solution is given by the Riemann–Roch theorem (which for most of this work we abbreviate to RR). For large n, the divisor nA is non special and so

$$\dim H^0(nA) = 1 - \mathrm{g}(\mathcal{C}) + n\deg(A).$$

*In the literature theta characteristic is also used to designate a halfcanonical divisor.

The genus of $\mathcal{C}$, $h^0(2A)$, has to be given and additionally so does $h^0(A)$.

The focus of this and later chapters will be, on the one hand, the construction of curves with a halfcanonical divisor and on the other hand, almost equivalently, to study the structure of the halfcanonical ring. We finish this introduction with a proposition that we use many times in this work.

PROPOSITION III.2. *Let $\mathcal{C}$ be a nonsingular curve with a halfcanonical divisor, A. Let D be a divisor and $\langle s_1, s_2\rangle \subset H^0(D)$ a base-point free pencil. Then*

(i) $h^0(A-D) \geq h^0(A) - \frac{\deg(D)}{2}$;

(ii) *if* $\deg(D)$ *is odd,* $h^0(A-2D) > h^0(A) - \deg(D)$;

Proof. Castelnuovo's free-pencil trick amounts to saying that, since the system $\langle s_1, s_2\rangle$ is free of base points, the map $2\mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(D)$ given by multiplication with s_1 and s_2, is surjective. The kernel of this map is a torsion free sheaf of rank 1 and since $\mathcal{C}$ is a nonsingular curve this is equivalent to invertibility. Therefore,

$$0 \to \mathcal{O}_{\mathcal{C}}(-D) \to 2\mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(D) \to 0.$$

Finally, tensoring this with $\mathcal{O}_{\mathcal{C}}(A)$, we get $h^0(A-D) + h^0(A+D) \geq 2h^0(A)$; so that by Riemann–Roch and Serre duality, $h^0(A-D) \geq h^0(A) - \frac{\deg(D)}{2}$. If $\deg(D)$ is odd then indeed $h^0(A-D) > h^0(A) - \frac{\deg(D)}{2}$. But another application of Castelnuovo's free-pencil trick yields:

$$0 \to H^0(A-2D) \to H^0(D) \otimes H^0(A-D) \to H^0(A).$$

Hence, $2h^0(A-D) \leq h^0(A-2D) + h^0(A)$. Which implies that

$$h^0(A-2D) > h^0(A) - \deg(D). \quad \square$$

III.1. Noether's theorem for halfcanonical divisors

In this section we give a few elementary results on the degree of the generators of $R(\mathcal{C}, A)$. First recall Noether's theorem.

THEOREM III.3 (Noether). *Let $\mathcal{C}$ be a nonsingular curve. If $\mathcal{C}$ is not hyperelliptic then the maps $\mathrm{S}^n H^0(K_C) \to H^0(nK_C)$ are surjective.* $\square$

Without any assumptions on $(\mathcal{C}, A)$ the following is as good a result as we can get.

Proposition III.4. *Let $\mathcal{C}$ be a nonsingular curve of genus ≥ 2 and A a halfcanonical divisor on $\mathcal{C}$. Then the ring $R(\mathcal{C}, A)$ is generated by elements of degree up to 5.*

Proof. The linear system $2A = K_{\mathcal{C}}$ is free and at least 2-dimensional. Take a pencil of sections $V \subset H^0(2K_{\mathcal{C}})$ and apply Castelnuovo's free pencil trick. The kernel of the map

$$H^0(nA) \otimes V \to H^0((n+2)A)$$

is isomorphic to $H^0((n-2)A)$. For $n \geq 5$, by RR and Serre duality, its dimension is

$$(n-3)\deg(A) = 2h^0(nA) - h^0((n+2)A). \quad \square$$

Notice that $C_{10} \subset \mathbb{P}(1,2,5)$ is a nonsingular curve with a halfcanonical divisor $A = \mathcal{O}(1)$ halfcanonically embedded, i.e. the homogeneous ring of C_{10} equals $R(\mathcal{C}, A)$, and therefore this ring has a generator in degree 5.

Proposition III.5. *Let $\mathcal{C}$ be a nonsingular curve with a halfcanonical divisor such that $h^0(A) \geq 2$. Assume that the divisor A is free. Denote by $\mathbf{d}$ the degree in which the ring $R(\mathcal{C}, A)$ is generated. Then $\mathbf{d} \leq 4$. Furthermore,*

(i) *If $\dim H^0(A) = 2$ then $\mathbf{d} \leq 2$ if and only if $\mathcal{C}$ is not hyperelliptic.*
(ii) *If $\dim H^0(A) \geq 3$ then $\mathbf{d} \leq 3$.*
(iii) *If $\dim H^0(A) \geq 3$ and $2\,\mathrm{gon}(\mathcal{C}) > \deg(A)$ then $\mathbf{d} \leq 2$.*

Proof. To show that the maps

$$H^0(A) \otimes H^0(nA) \to H^0((n+1)A)$$

are surjective for every $n \geq 4$ we use the next lemma, which we use again later in this chapter.

Lemma III.6. *Suppose $|A|$ is free. Let $\langle s_1, s_2 \rangle \subset H^0(A)$ be a pair of sections spanning a free pencil. Then, for $n \geq 4$, the map*

$$\langle s_1, s_2 \rangle \otimes H^0(nA) \to H^0((n+1)A) \tag{1.1}$$

is surjective.

Proof of the lemma. This is a straightforward application of Castelnuovo's free-pencil trick. The kernel of such a map is isomorphic to $H^0((n-1)A)$, and thus, for $n \geq 4$ its dimension (by RR and Serre duality) equals to $(n-2)\deg(A)$. Hence the image of (1.1) has dimension $2(n-1)\deg(A) - (n-2)\deg(A) = n\deg(A)$, which is the dimension of $H^0((n+1)A)$. $\square$

This shows that $R(\mathcal{C}, A)$ is generated in degree 4, as long as A is free and thus in all of the cases (i)–(iv).

Proof of (i). Since A is free, if $\mathcal{C}$ is a hyperelliptic curve, then by Proposition III.2, $h^0(A - g^1_2) \geq 1$ so that A is the hyperelliptic divisor. Then, neither of

$$H^0(2A) \otimes H^0(2A) \to H^0(4A)$$
$$H^0(A) \otimes H^0(3A) \to H^0(4A)$$

is surjective. Conversely If $\mathcal{C}$ is not hyperelliptic then $H^0(2A) \otimes H^0(2A) \to H^0(4A)$ is surjective and by Castelnuovo's free–pencil trick $H^0(A) \otimes H^0(2A) \to H^0(3A)$ is also surjective.

Proof of (ii). We show that the map

$$H^0(A) \otimes H^0(3A) \to H^0(4A) \tag{1.2}$$

is surjective. We start by proving a lemma which we use, often without mention, throughout this work.

LEMMA III.7. *Let A and B be two divisors on an algebraic curve $\mathcal{C}$. The extension bundles of $\mathcal{O}_{\mathcal{C}}(B)$ by $\mathcal{O}_{\mathcal{C}}(A)$ with maximum number of global sections are parametrised by the cokernel of the multiplication map*

$$H^0(K_{\mathcal{C}} - A) \otimes H^0(B) \to H^0(K_{\mathcal{C}} + B - A).$$

Proof of the lemma. The group classifying extensions of $\mathcal{O}_{\mathcal{C}}(B)$ by $\mathcal{O}_{\mathcal{C}}(A)$ is $\mathrm{Ext}^1(B, A)$. By Serre duality we have:

$$\mathrm{Ext}^1(B, A) = \mathrm{Ext}^1(K_{\mathcal{C}} + B - A, K_{\mathcal{C}}) \simeq H^0(K_{\mathcal{C}} + B - A)^{\vee}.$$

On the other hand, extensions of $\mathcal{O}_{\mathcal{C}}(B)$ by $\mathcal{O}_{\mathcal{C}}(A)$ with maximum number of global sections,

$$0 \to \mathcal{O}_{\mathcal{C}}(A) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(B) \to 0$$

have zero connecting homomorphism $H^0(B) \to H^1(A)$. In other words $\mathscr{F}$ has maximum number of global sections if and only if its class $[\mathscr{F}] \in \operatorname{Ext}^1(B, A)$, under the canonical morphism $\operatorname{Ext}^1(B, A) \to \operatorname{Hom}(H^0(B), H^1(A))$ maps to zero. Which is to say, $\mathscr{F}$ has maximum number of global sections if and only if

$$[\mathscr{F}] \in \operatorname{Ker}\left\{\operatorname{Ext}^1(B, A) \to H^0(B)^\vee \otimes H^1(A)\right\}. \tag{1.3}$$

Again by Serre duality we have

$$H^1(A) \simeq \operatorname{Ext}^0(A, K_{\mathcal{C}})^\vee = \operatorname{Ext}^0(0, K_{\mathcal{C}} - A)^\vee = H^0(K_{\mathcal{C}} - A)^\vee.$$

Finally, by dualising the statement of (1.3) we conclude that the classes of $\operatorname{Ext}^1(B, A)$ corresponding to extensions with maximum number of global sections are in bijection with the cokernel of the map $H^0(K_{\mathcal{C}} - A) \otimes H^0(B) \to H^0(K_{\mathcal{C}} + B - A)$. □

Let us apply this lemma to our case. Let $\mathscr{F}$ be an extension of $\mathcal{O}_{\mathcal{C}}(A)$ by $\mathcal{O}_{\mathcal{C}}(-A)$ corresponding to an element of the cokernel of (1.2). We have

$$0 \to \mathcal{O}_{\mathcal{C}}(-A) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A) \to 0 \tag{1.4}$$

with $h^0(\mathscr{F}) = h^0(A) \geq 3$. Since $h^0(\mathscr{F}) > 2$ there exists a section of $\mathscr{F}$ with a nontrivial divisor of zeros, δ. Such section gives rise to an embedding $\mathcal{O}_{\mathcal{C}}(\delta) \hookrightarrow \mathscr{F}$, which by saturation yields

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(-\xi) \to 0, \tag{1.5}$$

for some effective divisor $\delta \subset \xi \neq 0$. From this sequence we immediately conclude that $h^0(\xi) = h^0(A)$. As $\mathcal{O}_{\mathcal{C}}(\xi)$ cannot be a subsheaf of $\mathcal{O}_{\mathcal{C}}(-A)$, the composition of $\mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{F}$ with the map $\mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A)$ of (1.4) is injective. Since A is free $\xi \simeq A$. This means that $\mathscr{F}$ is the split extension. We deduce that the cokernel of that map (1.2) is trivial.

Proof of (iii). Suppose that $\dim H^0(A) \geq 3$ and $2\operatorname{gon}(\mathcal{C}) > \deg(A)$. We prove that the map

$$H^0(A) \otimes H^0(2A) \to H^0(3A) \tag{1.6}$$

is surjective. Consider $\mathscr{F}$, an extension of $\mathcal{O}_{\mathcal{C}}(A)$ by $\mathcal{O}_{\mathcal{C}}$ corresponding to an element of the cokernel of the map (1.6). Since $\dim H^0(\mathscr{F}) > 2$, as before, we can take a global section of $\mathscr{F}$ with a nontrivial divisor of zeros. The same construction as above now yields:

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A-\xi) \to 0. \tag{1.7}$$

Since A is free (1.7) implies that $h^0(A-\xi) \leq h^0(A) - 1$ and as $h^0(\mathscr{F}) = 1 + h^0(A)$, we deduce that $h^0(\xi) \geq 2$. By $2\operatorname{gon}(\mathcal{C}) > \deg(A)$, we get $2\deg(\xi) > \deg(A)$ and then that $2\deg(A-\xi) < \deg(A) < 2\operatorname{gon}(\mathcal{C})$. Hence $h^0(A-\xi) \leq 1$ and so $h^0(\xi) = h^0(A)$. As before we have an embedding $\mathcal{O}_{\mathcal{C}}(\xi) \hookrightarrow \mathcal{O}_{\mathcal{C}}(A)$ and since A is free we deduce that $\xi \simeq A$. In other words $\mathscr{F}$ is the split extension. $\square$

The statement and proof of this proposition are inspired in the article [**GL**] of Green and Lazarsfeld. Their result characterises the normal generation of a line bundle L (equivalently gives a condition for the graded ring of a complete linear series to be generated in degree 1) in terms of the Clifford index, $\operatorname{Cliff}(\mathcal{C})$.

Theorem III.8 (Green–Lazarsfeld). *Let L be a very ample line bundle on $\mathcal{C}$, with*

$$\deg(L) \geq 2\,\mathrm{g}+1 - 2 \cdot h^1(L) - \operatorname{Cliff}(\mathcal{C})$$

(and hence $h^1(L) \leq 1$). Then L is normally generated.

Notice that for the purposes of halfcanonical rings $R(\mathcal{C}, A)$ we will only expect to have a normally generated halfcanonical divisor for relatively high values of $h^0(A)$. On the other hand, by Serre duality $h^1(A) = h^0(A)$ and therefore it will virtually impossible to apply their theorem directly. However the proof, involving vector bundles, has been easy to adapt to our case. We will need to refine the result in one or two occasions when $R(\mathcal{C}, A)$ is still generated in degree ≤ 2 despite the fact that $2\operatorname{gon}(\mathcal{C}) \leq \deg(A)$.

Notation. Throughout this work we use the notation

$$\operatorname{sym}^2\colon \mathrm{S}^2 H^0(A) \to H^0(2A)$$

for the second symmetric product of $H^0(A)$. We use it mainly when A is a halfcanonical divisor on a curve $\mathcal{C}$. In the last chapter the same notation is used for $\mathrm{S}^2 H^0(K_S) \to H^0(2K_S)$ in the context of surfaces of general type.

Let V be a finite vector space. The subvariety of $\mathbb{P}[\mathrm{S}^2 V]$ parametrising symmetric tensors of rank $\leq k$ will be denoted by $\mathcal{Q}_k$.

The following Proposition goes some way in the direction of characterising the kernel of sym^2, and therefore of deciding when $R(\mathcal{C}, A)$ is generated in degree 1.

Proposition III.9. *Let $\mathcal{C}$ be a nonsingular curve and A a divisor on $\mathcal{C}$ such that $h^0(A) \geq 3$. Consider the linear subspace of $\mathbb{P}[\mathrm{S}^2 H^0(A)]$ given as the projectivised kernel of the map $\mathrm{sym}^2\colon \mathrm{S}^2 H^0(A) \to H^0(2A)$ and denote it by $\mathbb{P}[\mathrm{Ker}\,\mathrm{sym}^2]$. Let $d = \lfloor \frac{\deg(A)}{2} \rfloor$. Then,*

(i) $\mathbb{P}[\mathrm{Ker}\,\mathrm{sym}^2] \cap \mathcal{Q}_4 \neq \emptyset \implies W^1_d(\mathcal{C}) \neq \emptyset$;

Suppose that A is a halfcanonical divisor and let $d' \leq 2h^0(A) - 3$. Then,

(ii) $W^1_{d'}(\mathcal{C}) \neq \emptyset \implies \mathbb{P}[\mathrm{Ker}\,\mathrm{sym}^2] \cap \mathcal{Q}_4 \neq \emptyset$.

Additionally, suppose that $d < d'$ and that $\xi \in W^1_{d'}(\mathcal{C})$ is a free pencil. Then there exists a quadric of rank 4 in the intersection $\mathbb{P}[\mathrm{Ker}\,\mathrm{sym}^2] \cap \mathcal{Q}_4$.

Proof. The free part of $|A|$ yields a morphism, $\varphi\colon \mathcal{C} \to C_t \subset \mathbb{P}^{h-1}$ onto a curve of degree t, where we have denoted $h^0(A)$ by h. We have the inequality

$$\deg(\varphi) \cdot t \leq \deg(A).$$

Let us suppose that $\mathbb{P}[\mathrm{Ker}\,\mathrm{sym}^2] \cap \mathcal{Q}_3 \neq \emptyset$. In other words assume that C_t is contained in a quadric Q, of rank 3. Projecting from the vertex of Q and composing with φ,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\varphi"] \arrow[dr] & C_t \subset Q \subset \mathbb{P}^{h-1} \arrow[d, "\pi"] \\ & C_2 \subset \mathbb{P}^2 \end{tikzcd}$$

yields a pencil of degree $< \frac{\deg(A)+1}{2}$, since

$$\deg(\pi) \cdot 2 \leq t \Longrightarrow \deg(\varphi) \cdot \deg(\pi) \leq \frac{\deg(A)}{t} \cdot \frac{t}{2}.$$

Next, consider the case when the rank of Q is 4. Again, project from the vertex of Q to obtain:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\varphi"] \arrow[dr] & C_t \subset Q \subset \mathbb{P}^{h-1} \arrow[d, "\pi"] \\ & C_s \subset Q_4 \subset \mathbb{P}^3 \end{tikzcd}$$

where C_s is a curve of degree $s \leq t$. As $C_s \subset Q_4$, denoting by L_1 and L_2 the two rulings of Q_4, we can write $C_s = aL_1 + bL_2$ with $a, b \geq 0$ and $a + b = s$. But then either

$a < \frac{s+1}{2}$ or $b < \frac{s+1}{2}$ and thus the corresponding ruling gives a pencil of degree strictly less than $\frac{s+1}{2} \leq \frac{t+1}{2} \leq \frac{\deg(A)+1}{2}$. In both instances we conclude that $W^1_d(\mathcal{C}) \neq \emptyset$.

Proof of (ii). Let us now assume that A is a halfcanonical divisor. Let ξ be a base-point free pencil of degree $d' \leq 2h^0(A) - 3$. Proposition III.2 yields that

$$\dim H^0(A-\xi) \geq \dim H^0(A) - \frac{\deg(\xi)}{2} \geq \dim H^0(A) - \dim H^0(A) + \frac{3}{2} > 1.$$

We deduce that $h^0(A-\xi) \geq 2$. We choose two pencils of sections in each of $H^0(\xi)$ and $H^0(A-\xi)$:

$$\langle s_1, s_2\rangle \subset H^0(\xi) \quad \text{and} \quad \langle t_1, t_2\rangle \subset H^0(A-\xi).$$

LEMMA III.10. *The kernel of the map*

$$\langle s_1, s_2\rangle \otimes \langle t_1, t_2\rangle \to H^0(A) \tag{1.8}$$

is at most 1*-dimensional.*

Proof of the lemma. We apply Castelnuovo's linear-bilinear principle. Arguing by contradiction, suppose that the kernel of (1.8) is is 2-dimensional. Then, its projectivised in $\mathbb{P}[\langle s_1, s_2\rangle \otimes \langle t_1, t_2\rangle]$ is a line. Therefore it must intersect the variety of rank 1 tensors: $\mathbb{P}[\langle s_1, s_2\rangle] \times \mathbb{P}[\langle t_1, t_2\rangle]$, as this variety is 2-dimensional. But the image of a tensor $w \otimes v$ is simply $w \cdot v$, hence, never zero unless one of w or v is zero. We conclude that the kernel of the map above is at most 1-dimensional. $\square$

Next, notice that the elements $s_1t_1, s_1t_2, s_2t_1, s_2t_2$ of $H^0(A)$ readily produce a symmetric tensor,

$$\sigma = (s_1t_1)\cdot(s_2t_2) - (s_1t_2)\cdot(s_2t_1) \in \mathrm{S}^2 H^0(A)$$

which is in the kernel of the map sym^2. By the lemma, σ has rank 3 or 4.

Finally suppose additionally, that $d < d'$ and $\xi \in W^1_{d'}(\mathcal{C})$ is free. Applying Castelnuovo's free-pencil trick, we see that the kernel of the map

$$\langle s_1, s_2\rangle \otimes H^0(A-\xi) \to H^0(A)$$

is isomorphic to $H^0(A-2\xi)$. We have

$$\deg(A-2\xi) = \deg(A) - 2d' = 2\left(\frac{\deg(A)}{2} - 1 - d'\right) + 2 \leq 2(d-d') + 2 \leq 0.$$

If $\deg(A-2\xi)=0$ then $d'=\frac{\deg(A)}{2}=d$. Hence $\deg(A-2\xi)<0$ and therefore $\dim H^0(A-2\xi)=0$. Thus the image of ξ under the map of item (ii), Q, is a quadric of rank 4. $\square$

III.2. Petri's theorem for halfcanonical divisors

Notation. Let $S=\mathbb{C}[x_i]$ be a polynomial ring. We denote a surjection of $\mathbb{C}[x_i]$ onto $R(\mathcal{C},A)$ by $\mathrm{ev}\colon \mathbb{C}[x_i]\to R(\mathcal{C},A)$. The notation ev is mostly used for minimal surjections, i.e., surjections corresponding to a minimal choice of generators of $R(\mathcal{C},A)$. In Chapter V the same notation is used for a surjection onto the canonical ring $R(S,K_S)$.

Let $\mathrm{ev}: S=\mathbb{C}[x_i]\to R(\mathcal{C},A)$ be a minimal surjection. By definition I_A is the kernel of the map ev. By an abuse of language, we will sometimes say the ring $R(\mathcal{C},A)$ has codimension c when c is the codimension of I_A. We denote by $I'_{A,d}$ the subspace of $I_{A,d}$ given as

$$I'_{A,d}=\sum_{k=1}^{d-1} S_k\cdot I_{A,d-k}.$$

Recall Petri's theorem on the homogeneous ideal of the canonical model $C_{2\,\mathrm{g}-2}$ of a nonsingular curve.

Theorem III.11 (Petri). *Let $\mathcal{C}$ be a nonsingular curve of genus $g\geq 3$. Assume that $\mathcal{C}$ is not hyperelliptic. Then the homogeneous ideal of $\mathcal{C}_{2\,\mathrm{g}-2}$ is generated by quadrics and cubics, and by quadrics only in case $\mathcal{C}$ is neither trigonal nor a nonsingular plane quintic.* $\square$

Petri's analysis is composed of two steps. In the first step one shows that the canonical ideal $I_{K_\mathcal{C}}$ is generated by forms of degree ≤ 3. This step is purely algebraic and somewhat easier. The second step is the core of Petri's analysis. The argument relates the geometry of $C_{2\,\mathrm{g}-2}$ with the gonality of $\mathcal{C}$. It relies on two auxiliary results. One is Petri's identity expressing a relation between the generators of $I_{K_\mathcal{C}}$. The other establishes a relation between gonality of $\mathcal{C}$ and the set cut by the generators of degree 2 of $I_{K_\mathcal{C}}$. Petri's identity has an important role. Without it, the fact that the quadrics relations cut out the canonical model would have to be stated set-theoretically. When the quadric relations cut out the canonical model, Petri's identity turns the statement into a scheme-theoretical statement which is the conclusion of the theorem. (See [**S**] for a proof of Petri's theorem).

Proposition III.12. *Let $\mathcal{C}$ be a nonsingular curve equipped with a free halfcanonical divisor A. Let $\mathbf{d}$ be the maximum degree of a minimal set of generators of the halfcanonical ideal I_A. Let m be the maximum degree of the generators in a (minimal) set of generators of $R(\mathcal{C}, A)$ Then $\mathbf{d} \le 4 + m$. This is, in full display:*

(i) $\mathbf{d} \le 8$ *when $R(\mathcal{C}, A)$ is generated in degree 4;*
(ii) $\mathbf{d} \le 7$ *when $R(\mathcal{C}, A)$ is generated in degree 3;*
(iii) $\mathbf{d} \le 6$ *when $R(\mathcal{C}, A)$ is generated in degree 2;*
(iv) $\mathbf{d} \le 5$ *when $R(\mathcal{C}, A)$ is generated in degree 1.*

Proof. The proof relies entirely on the elementary observation of Lemma III.6. Let $\mathrm{ev}\colon \mathbb{C}[X_i] \twoheadrightarrow R(\mathcal{C}, A)$ denote a minimal surjection. Since we assume that A is free, by Proposition III.5, the ring $R(\mathcal{C}, A)$ is generated in degree 4 and therefore

a) our list (i)–(iv) exhausts all possibilities,
b) $\mathrm{wt}(X_i) \le 4$ and
c) $m = \max\{\mathrm{wt}(X_i)\}$.

Fix X_1 and X_2 two variables of weight 1 such that $\mathrm{ev}(X_1) = s_1$ and $\mathrm{ev}(X_2) = s_2$ for $\langle s_1, s_2\rangle \subset H^0(A)$ spanning a free pencil. The next thing to do is to reduce any polynomial in the variables X_i to a convenient form.

Lemma III.13. *Let $F \in \mathbb{C}[X_i]$ be a form of degree $d \ge 5 + m$. Then, there exist $G, H \in \mathbb{C}[X_i]$, quasihomogeneous forms of degree $d - 1$ such that*

$$F \equiv GX_1 + HX_2 \mod I'_{A,d}$$

Proof of the lemma. We can write F as

$$F = GX_1 + HX_2 + \sum_{k\ge 3} X_k F_k$$

where $\deg(F_k) = d - \mathrm{wt}(X_k) \ge 5$. By Lemma III.6 there exist $G_k, H_k \in \mathbb{C}[X_i]$ forms of degree $d - \mathrm{wt}(X_k) - 1$ such that

$$F_k = G_k X_1 + H_k X_2 \mod I_{A,j_k},$$

where $j_k = d - \mathrm{wt}(X_k)$. $\square$

If F is a quasihomogeneous form of degree $d \geq 5+m$, such that $\mathrm{ev}(F)=0$, then $s_1\,\mathrm{ev}(G)+s_2\,\mathrm{ev}(H)=0$ and thus by Castelnuovo's free-pencil trick, there exists $l \in H^0((d-1)A-A) \simeq H^0((d-2)A)$ such that

$$\mathrm{ev}(G)=-s_2 l \quad \text{and} \quad \mathrm{ev}(h)=s_1 l.$$

Let $L \in \mathbb{C}[X_i]$ be a quasihomogeneous form of degree $d-2$ such that $\mathrm{ev}(L)=l$. Then,

$$G=-X_2 L \mod I_{A,d-1} \quad \text{and} \quad H=X_1 L \mod I_{A,d-1}$$

and thus $F \equiv 0 \mod I'_{A,d}$. Hence if $m=4$, we get that $\mathbf{d}$ (as in the statement of this proposition) is ≤ 8; since any quasihomogeneous form of degree ≥ 9 in the kernel of ev can be reduced to a combination of forms of smaller degree in the kernel of ev. Similarly, $\mathbf{d} \leq 4+m$. $\square$

Despite the fact that in most interesting cases our estimate of $\mathbf{d}$ is far from good, we point out that in the context of curves with a free halfcanonical divisor the estimate of $\mathbf{d}$ is sharp. We see this by considering a plane quintic, which has $m=1$ (the notation is taken from Proposition III.12) and $\mathbf{d}=5$. Also, as we see in Section III.4, a sextic nonsingular curve $C_6 \subset \mathbb{P}(1^2,3)$ is an example for which $m=2$ and $\mathbf{d}=6$. This is the only curve of genus 4 with a free halfcanonical pencil. Finally an octic curve of in $\mathbb{P}(1^2,4)$ has $m=4$ and $\mathbf{d}=8$. All of these are "easy cases" in the sense of Section III.4. Table III.1 on page 42, contains a list of candidate pairs $(\mathcal{C},A)$, consisting of a nonsingular curve $\mathcal{C}$ and a halfcanonical divisor A such that the ring $R(\mathcal{C},A)$ is simple.

Theorem III.14 (Petri's Theorem for a halfcanonical divisor). *Let $\mathcal{C}$ be a nonsingular curve with an effective halfcanonical divisor A. Assume that $R(\mathcal{C},A)$ is generated in degree ≤ 2 and that $\mathcal{C}$ is not hyperelliptic. Consider the morphism given by a choice of generators of $R(\mathcal{C},A)$, $\varphi_{A^+}\colon \mathcal{C} \to \mathbb{P}$. Then,*

(i) *$\varphi_{A^+}(\mathcal{C})$ is cut out set-theoretically by quasihomogeneous forms of degree 6. Additionally if $\mathcal{C}$ is not trigonal neither a plane quintic,*

(ii) *$\varphi_{A^+}(\mathcal{C})$ is cut out set-theoretically by quasihomogeneous forms of degree 4.*

Proof. Denote by W a complementary subspace to the image of the map sym^2. By our assumptions the pluri-halfcanonical map, φ_{A^+} maps to $\mathbb{P}[H^0(A) \oplus W]$. It is an embedding fitting into the commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\varphi_{A+}"] \arrow[d, "\varphi_{K_\mathcal{C}}"'] & \mathbb{P}[H^0(A) \oplus W] \arrow[d, "\mathrm{v}_2"] \\ \mathbb{P}[H^0(K_\mathcal{C})] \arrow[r, "\mathbb{P}(\beta)"] & \mathbb{P}[\mathrm{S}^2 H^0(A) \oplus W] \end{tikzcd}$$

The map v_2 is the second Veronese embedding and the map $\mathbb{P}(\beta)$ is the linear embedding induced by the surjection

$$\beta\colon\ \mathrm{S}^2 H^0(A) \oplus W \to H^0(K_\mathcal{C}).$$

Since $\mathcal{C}$ is not hyperelliptic, by Petri's theorem, the image of $\mathcal{C}$ under the canonical embedding is cut out by cubics. Therefore the image of $\mathcal{C}$ under $\mathbb{P}(\beta) \circ \varphi_{K_\mathcal{C}}$ is cut out by forms of degree ≤ 3. Using the commutativity of the diagram we deduce that the image of $\mathcal{C}$ under $\mathrm{v}_2 \circ \varphi_{A+}$ is cut out cubic forms in the variables $\mathrm{S}^2 H^0(A) \oplus W$ which implies that $\mathcal{C}$ under φ_{A+} is cut out by forms of degree ≤ 6. This shows (i). Item (ii) follows by a similar argument. $\square$

To get a theorem of the same calibre for the halfcanonical ring, i.e. to prove that the ideal I_A is cut out by forms of degree ≤ 6 (and by forms of degree ≤ 4 in the nontrigonal and "non-plane-quintic" case) we have to run a Petri analysis on the generators of $R(\mathcal{C}, A)$. The easy part of Petri's analysis is done in Proposition III.12. From there we recover the result that when $R(\mathcal{C}, A)$ is generated in degree 2, I_A is generated in degree 6 (allowing that we assume A to be free). The two results collected in this section are slightly diverse in nature. However, even the weaker form of Petri's theorem as stated here will serve our purposes. In general, and especially in Chapter IV, we use it for complete intersections in weighted projective space or in weighted Grassmannians, when the set theoretic result of Theorem III.14 combined with Bézout's theorem from intersection theory (see [**Fu**, p 10]) allow us to describe the generators of I_A.

III.3. Hilbert numerators

If A is an halfcanonical divisor on a nonsingular curve then by Clifford's Theorem we know that $h^0(A) \leq \frac{\mathrm{g}(\mathcal{C})-1}{2} + 1$ with equality taking place if and only if the curve is hyperelliptic. This upper bound allows us to limit our search of curves whose

$g(\mathcal{C})-1$	$h^0(A)$	Hilbert Numerator	Denominator
1	1	$1-t^{10}$	$(1-t)(1-t^2)(1-t^5)$
2	1	$1-t^4-t^6+t^{10}$	$(1-t)(1-t^2)^2(1-t^3)$
2	2	$1-t^8$	$(1-t)^2(1-t^4)$
3	2	$1-t^6$	$(1-t)^2(1-t^2)$
4	2	$1-2t^4+t^8$	$(1-t)^2(1-t^2)^2$
4	3	$1-t^2-t^6+t^8$	$(1-t)^3(1-t^3)$
5	3	$1-t^5$	$(1-t)^3$
6	3	$1-t^3-t^4+t^5$	$(1-t)^3(1-t^2)$
8	4	$1-t^2-t^4+t^6$	$(1-t)^4$
9	4	$1-2t^3+t^6$	$(1-t)^4$
11	5	$1-3t^2-2t^4+2t^3+3t^5-t^7$	$(1-t)^5$
12	5	$1-2t^2-t^3+t^4+2t^5-t^7$	$(1-t)^5$

TABLE III.1. Codimension ≤ 3 easy cases.

$g(\mathcal{C})-1$	$h^0(A)$	Hilbert Numerator	Denominator
5	2	$1-5t^4+5t^6-t^{10}$	$(1-t)^2(1-t^2)^3$
7	3	$1-2t^3-3t^4+3t^5+2t^6-t^9$	$(1-t)^3(1-t^2)^2$
10	4	$1-4t^3+4t^5-t^8$	$(1-t)^4(1-t^2)$
13	5	$1-t^2-4t^3+4t^4+t^5-t^7$	$(1-t)^5$

TABLE III.2. Codimension 3 Pfaffian 5×5.

halfcanonical ring $R(\mathcal{C}, A)$ has low codimension. In the two tables and in Figure 1 of this section we collect all pairs $(\mathcal{C}, A)$ whose halfcanonical ring $R(\mathcal{C}, A)$ has expected codimension ≤ 3.

By inspection of Table III.1, we see that all the rings of codimension 2 are expected to be complete intersections. There are 4 clear† 5×5 Pfaffian plus a single 7×7 Pfaffian case. The result of the computation of the Hilbert series for a pairs $(\mathcal{C}, A)$ consisting

†The case $g = 12$ and $h^0(A) = 5$ is an "easy" 5×5 Pfaffian ring.

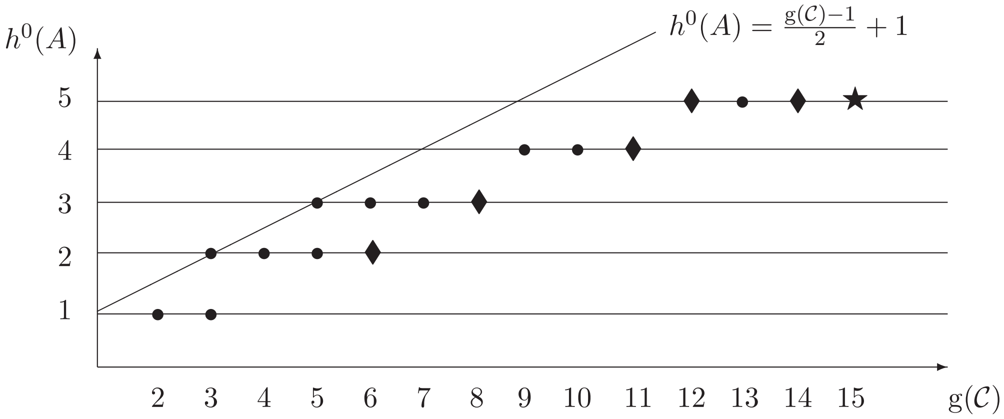

FIGURE 1. All halfcanonical rings of codimension ≤ 3.

of a smooth curve and a halfcanonical divisor can be displayed as a correspondence between $g(\mathcal{C})$ and $h^0(A)$ an in Figure 1. In this graph, points $(g(\mathcal{C}), h^0(A))$ marked with a bullet correspond to rings $R(\mathcal{C}, A)$ of codimension ≤ 3. The diamonds indicate the cases whose corresponding rings are expected to have Pfaffian 5×5 format. The star signals a Pfaffian 7×7 halfcanonical ring.

THEOREM III.15. *The pairs $(\mathcal{C}, A)$ as depicted in Figure 1 exhaust all possible cases of halfcanonical rings of codimension ≤ 3 on nonsingular curves.*

Proof. It is clear that if $R(\mathcal{C}, A)$ has codimension ≤ 3 necessarily $h^0(A) \leq 5$. The proof now breaks into an analysis of each value of $\dim H^0(A)$.

Case $\dim H^0(A) = 5$**.** The dimension of the space $S^2 H^0(A)$ is 15. Therefore if $g(\mathcal{C}) > 15$, even if sym^2 is injective we still need a new generator in degree 2. Therefore $g(\mathcal{C}) \leq 15$. It remains to show that $g(\mathcal{C}) \geq 12$.

LEMMA III.16. *Let $\mathcal{C}$ be a nonsingular curve of genus $9 \leq g(\mathcal{C}) \leq 11$. Let A be a halfcanonical divisor on $\mathcal{C}$ with $h^0(A) = 5$. Then the halfcanonical ring $R(\mathcal{C}, A)$ has codimension ≥ 4.*

Proof of the lemma. Let $\mathcal{C}$ be a nonsingular curve of genus $9 \leq g(\mathcal{C}) \leq 11$ with a halfcanonical divisor A such that $h^0(A) = 5$. We argue by contradiction. Suppose that $R(\mathcal{C}, A)$ has codimension 3. In particular A is very ample and the image of $\mathcal{C}$

under the embedding given by $|A|$ is projectively normal. We deduce that $\mathcal{C}$ can be realised as nonsingular curve in projective space of dimension 4.

Theorem (Castelnuovo's bound). *Let $\mathcal{C}$ be a nonsingular curve that admits a birational map onto a nondegenerate curve of degree d in $\mathbb{P}^r$. Then*

$$\mathrm{g}(\mathcal{C}) \leq \pi(d, r)$$

where Castelnuovo's number $\pi(d, r)$ is defined by

$$\pi(d, r) = \frac{m(m-1)}{2}(r-1) + m\varepsilon,$$

with $m = \left\lfloor \frac{d-1}{r-1} \right\rfloor$ and $\varepsilon = d - 1 - m(r-1)$.

Proof. See [**ACGH**, pp 113–116]. □

We apply Castelnuovo's bound to each of the cases $9 \leq \mathrm{g}(\mathcal{C}) \leq 11$. Castelnuovo's number, $\pi(d, 4)$, equals 9, 7 and 5 for $\mathrm{g}(\mathcal{C}) = 11$, 10 and 9 respectively. Applying the Theorem we get a contradiction. □

Remark. If $\mathrm{g}(\mathcal{C}) = 12$, then $m = 3$, $\varepsilon = 1$ and thus $\pi(11, 4) = 12$. The curve $\mathcal{C}$ is therefore an extremal‡ curve. For any $r \geq 3$ and $d \geq 2r + 1$, extremal curves exists and are completely classified. See [**ACGH**, p. 122]. In our case, an extremal curve of degree 11 in $\mathbb{P}^4$ is a nonsingular member of $|4H - L|$ of the cubic scroll $\mathbb{F}(1, 2) \subset \mathbb{P}^4$. See page 63 for details.

Case $\dim H^0(A) = 4$**.** The dimension of the space $\mathrm{S}^2 H^0(A)$ is 10. Hence if $\mathrm{g}(\mathcal{C}) > 11$, we still need two generators in degree 2. It follows that $\mathrm{g}(\mathcal{C}) \leq 11$.

Lemma III.17. *Let $\mathcal{C}$ be a nonsingular curve of genus 7 with a halfcanonical divisor A such that $h^0(A) = 4$. Then $R(\mathcal{C}, A)$ has codimension ≥ 4.*

Proof of the lemma. The kernel of sym^2 is at least 3-dimensional. If the kernel of sym^2 is 4-dimensional then its projectivised must meet $\mathcal{Q}_2 \subset \mathbb{P}^9[\mathrm{S}^2 H^0(A)]$ since this variety has dimension 6. The elements of $\mathcal{Q}_2$ are reducible quadrics, hence this cannot happen. We deduce that sym^2 must be surjective. In degree 3 the dimension of $\mathrm{S}^3 H^0(A)$ is 20. The space of multiples of the kernel of sym^2 has dimension $3 \cdot 4 = 12$.

‡Castelnuovo extremal.

Suppose that there are at least 3 first syzygies between the 3 quadrics generating $\operatorname{Ker}\operatorname{sym}^2$. Each syzygy can be written as

$$AQ_1 + BQ_2 + CQ_3 \tag{3.1}$$

where $Q_i \in \operatorname{Ker}\operatorname{sym}^2$ and $A, B, C \in \langle x_1, \ldots, x_5\rangle$. In view of (3.1) we can identify the space of first syzygies with a subspace of the variety $\operatorname{Hom}(H^0(A), \langle Q_1, Q_1, Q_3\rangle^\vee)$. Saying that the linear space of syzygies has dimension ≥ 3 is equivalent to saying that its (projective) dimension is ≥ 2. Therefore the space of syzygies meets the subvariety of $\operatorname{Hom}(H^0(A), \langle Q_1, Q_1, Q_3\rangle^\vee)$ parametrising homomorphisms of rank ≤ 2. (Since the codimension of this variety is $(4-2)(3-2) = 2$). This means that there exists a nontrivial syzygy of the form $AQ + BQ' = 0$ with $Q, Q' \in \operatorname{Ker}\operatorname{sym}^2$. This is a contradiction since Q and Q' are irreducible. Therefore the space of first syzygies between $\langle Q_1, Q_2, Q_3\rangle$ is at most 2 dimensional and accordingly, the span of $\mathrm{S}^3 H^0(A)$ in $H^0(3A)$ has dimension $20 - 12 + 2 = 10$. We conclude that we will need two new generators in degree 2, which implies that $R(\mathcal{C}, A)$ has codimension ≥ 4. $\square$

Lemma III.18. *Let $\mathcal{C}$ be a nonsingular curve of genus 8 with a halfcanonical divisor A such that $h^0(A) = 4$. Then $R(\mathcal{C}, A)$ has codimension ≥ 4.*

Proof. We claim that $\operatorname{gon}(\mathcal{C}) \leq 3$. This follows from an application of Proposition III.9. The kernel of sym^2 has dimension ≥ 2 thus $W^1_d(\mathcal{C}) \neq \emptyset$, where $d = \left\lfloor \frac{\deg(A)}{2} \right\rfloor = 3$, in other words $\operatorname{gon}(\mathcal{C}) \leq 3$.

Now suppose that $\mathcal{C}$ has a free g^1_3. Then by Proposition III.2 (on page 31) we deduce that the space $H^0(A - 2g^1_3)$ has dimension > 1, which means that there exists an effective divisor, $A - 2g^1_3$, of degree 1 with at least 2 global sections. In other words $\mathcal{C}$ is rational. This is a contradiction. Therefore $\mathcal{C}$ is hyperelliptic. (Notice that the point (8,4) is not on the Clifford line).

If $\mathcal{C}$ is a hyperelliptic curve with a halfcanonical divisor A such that $h^0(A) = 4$ then Proposition III.2 gives that $h^0(A - 3g^1_2) \geq 1$. Hence A is not free and there exists a point p such that $A = 3g^1_2 + p$. This implies that sym^2 is not surjective, since $K_{\mathcal{C}}$ is free. By the same token, the map

$$H^0(A) \otimes H^0(2A) \to H^0(3A)$$

cannot be surjective. In conclusion, besides $\langle x_1, x_2, x_3, x_4\rangle$ we will have at least one new generator in degree 2 and at least one new generator in degree 3. The ring $R(\mathcal{C}, A)$ has codimension ≥ 4. $\square$

There are three more cases before we finish the proof of Theorem III.15.

Case $\dim H^0(A) = 3$**.** The dimension of the space $\mathrm{S}^2 H^0(A)$ is 6. When $\mathrm{g}(\mathcal{C}) > 8$, we still lack at least three generators in degree 2. Therefore $\mathrm{g}(\mathcal{C}) \leq 8$.

Case $\dim H^0(A) = 2$**.** The dimension of the space $\mathrm{S}^2 H^0(A)$ is 3. If $\mathrm{g}(\mathcal{C}) > 6$, we have to add at least four new generators in degree 2. Therefore $\mathrm{g}(\mathcal{C}) \leq 6$.

Case $\dim H^0(A) = 1$**.** We need x to generate $H^0(A)$ and a further $y_1 \dots y_{g-1}$ to generate $H^0(2A)$. In degree 3 we have

$$x^3, xy_i \quad \text{for} \quad i = 1, \dots, g-1.$$

Altogether, g elements. The dimension of $H^0(3A)$ is $2g-2$ hence we need an extra set of $g-2$ generators in degree 3. Up to now, the number of generators is $1+g-1+g-2 = 2g-2$. If $g \geq 4$ then the number of elements needed to generate $R(\mathcal{C}, A)$ is ≥ 6 which means that $R(\mathcal{C}, A)$ has codimension ≥ 4.

We have finished the proof of Theorem III.15. $\square$

III.4. Halfcanonical rings: easy cases

We refer the reader to Table III.1 on page 42 for a list of the easy cases. In this section we construct the halfcanonical ring $R(\mathcal{C}, A)$ of a pair $(\mathcal{C}, A)$, in each the classes of Table III.1. We will need to make some generality assumptions on the pair $(\mathcal{C}, A)$. In most cases this will exclude the occurrence of masking of the Hilbert numerator.

Genus 2**.**

Theorem III.19. *Let $\mathcal{C}$ be a nonsingular curve of genus* 2*. For every ramification point $p \in \mathcal{C}$ of the hyperelliptic system, the divisor $A = p$ is a halfcanonical divisor. The ring $R(\mathcal{C}, A)$ is generated by x, y, z of degrees* 1*,* 2 *and* 3*, respectively, The ideal of relations I_A is generated by a single quasihomogeneous form of degree* 10*. In particular, $\mathcal{C}$ is isomorphic to a curve of degree* 10 *in $\mathbb{P}(1, 2, 3)$.*

Proof. A nonsingular curve of genus 2 is hyperelliptic. Let us start the analysis of the ring $R(\mathcal{C}, A)$ by writing

$$\begin{aligned} H^0(A) &= \langle x \rangle \\ H^0(2A) &= \langle x^2, y \rangle \\ H^0(3A) &= \langle x^3, xy \rangle \\ H^0(4A) &= \langle x^4, x^2y, y^2 \rangle. \end{aligned}$$

The element x is a generator of $H^0(A)$ and $\langle y \rangle$ is a complementary basis to $\langle x^2 \rangle \subset H^0(2A)$. Then x^3, xy have to be linearly independent. Moreover, x^4, x^2y, y^2 are also linearly independent, as any linear dependence relation would imply that y is zero on the divisor of zeros of x which contradicts the fact that x^2, y are linearly independent. In the next step, $h^0(5A) = 4$ and we already own a 3-dimensional subspace of $H^0(5A)$ given by $\langle x^5, x^3y, x^2y \rangle$. Therefore there exists a generator $u \in H^0(5A)$ complementary to this space. By Proposition III.4 these are all the generators of the ring $R(\mathcal{C}, A)$.

As far as the ideal I_A is concerned, firstly, there is no relation in degree 6: the space $H^0(6A)$ contains the elements x^6, x^4y, x^3y, xu which are clearly linearly independent. Furthermore, $y^3 \in H^0(6A)$ and cannot be involved in any linear relation with x^6, x^4y, x^3y, xu, otherwise y would vanish at $\mathrm{div}_0(x)$. Therefore

$$H^0(6A) = \langle x^6, x^4y, x^3y, xu, y^3 \rangle .$$

The divisor $2A = K_{\mathcal{C}}$ is free. Thus, by Castelnuovo's free-pencil trick the map

$$\langle x^2, y \rangle \otimes H^0(nA) \to H^0((n+2)A) \tag{4.1}$$

has a kernel of dimension $h^0((n-2)A)$. But then, for $n \geq 5$ we have

$$2h^0(nA) - h^0((n-2)A) = (n+1)\deg(A) = h^0((n+2)A),$$

i.e., the map (4.1) is surjective. Since $u^n \in H^0(5nA)$ and is not a multiple of $\langle x^2, y \rangle$, we deduce that there is a relation for each degree divisible by 5 bigger than 10. These have to be multiples of the original relation in degree 10. $\square$

Genus 3 and $h^0(A) = 1$.

Let A be a halfcanonical divisor with $h^0(A) = 1$. If $\mathcal{C}$ is nonhyperelliptic, the canonical morphism embeds $\mathcal{C}$ as a plane quartic $C_4 \subset \mathbb{P}^2$ and then A is supported on the intersection of C_4 and a bitangent line (see Figure 2 below). Scheme-theoretically the intersection is $2A = K_{\mathcal{C}}$ since C_4 is a canonical curve. In particular we deduce that the number of halfcanonical divisors on a nonhyperelliptic curve of genus 3 with 0-dimensional associated linear system equals the number of bitangents of a plane quartic, i.e. 28. If $\mathcal{C}$ is hyperelliptic and the canonical morphism $\varphi_{K_{\mathcal{C}}}$ is 2-to-1 onto a plane conic $C_2 \subset \mathbb{P}^2$. Let L be a secant to C_2 at two branching points and denote $L \cap C_2$ by $P_1 + P_2$, with $P_1 \neq P_2$. The pulled-back divisors $\varphi^*_{K_{\mathcal{C}}}(P_1)$ and $\varphi^*_{K_{\mathcal{C}}}(P_2)$ are denoted by $2p_1$ and $2p_2$ respectively. Then $A = p_1 + p_2$. Conversely, a halfcanonical divisor A, determines a unique secant line L at two branching points. We deduce that in the hyperelliptic case there are $\binom{r}{2}$ halfcanonical divisors on $\mathcal{C}$, where r is the degree of the ramification divisor on $\mathcal{C}$. By Hurwitz' formula,

$$r = 2\,\mathrm{g} + 2 = 8 \implies \binom{r}{2} = 28.$$

Incidentally, in the hyperelliptic case, $\mathcal{C}$ has a unique halfcanonical pencil (the g^1_2) and 35 noneffective halfcanonical divisors

$$-g^1_2 + p_{i_1} + p_{i_2} + p_{i_3} + p_{i_4}$$

where $\{i_1, i_2, i_3, i_4\} \subset \{p_1, \dots, p_8\} = R$, where R is the support of the ramification divisor. As written there are $\binom{8}{4} = 70$ such expressions, but this number should be halved on account of the relation:

$$-g^1_2 + p_{i_1} + p_{i_2} + p_{i_3} + p_{i_4} \sim -g^1_2 + p_{j_1} + p_{j_2} + p_{j_3} + p_{j_4}$$

where $\{j_1, j_2, j_3, j_4\}$ is a complementary set to $\{i_1, i_2, i_3, i_4\}$. Altogether the number of halfcanonical divisors on a hyperelliptic curve of genus 3 is $28 + 1 + 35 = 2^{2\cdot 3}$ as expected.

Theorem III.20. *Let $\mathcal{C}$ be a nonsingular curve of genus 3 and A a halfcanonical divisor with $h^0(A) = 1$. Then the ring $R(\mathcal{C}, A)$ is generated by x, y_1, y_2, z of degrees $1, 2, 2$ and 3, respectively. The ideal of relations I_A is generated by two quasihomogeneous forms F_4, F_6 of degrees 4 and 6, respectively.*

Proof of Theorem III.20. Let us write

$$\begin{aligned} H^0(A) &= \langle x \rangle \\ H^0(2A) &= \left\langle x^2, y_1, y_2 \right\rangle \\ H^0(3A) &= \left\langle x^3, xy_1, xy_2, z \right\rangle = (\dagger) \\ H^0(4A) &= \left\langle x(\dagger), y_1^2, y_1y_2, y_2^2 \right\rangle = (*) \\ H^0(5A) &= \langle x(*), y_1z, y_2z \rangle \end{aligned}$$

In the above display, x is a generator of $H^0(A)$, y_1, y_2 are complementary to $\left\langle x^2 \right\rangle \subset H^0(2A)$ and z is complementary to $\left\langle x^3, xy_1, xy_2 \right\rangle \subset H^0(3A)$. However in what follows it will be important to choose y_1, y_2, z in general conditions. Firstly we want the pencil $\langle y_1, y_2 \rangle$ to be free. Since $2A$ is free there is no impediment for this to happen. As we can see in Figure 2, geometrically this corresponds to choosing y_1, y_2 to define two lines meeting outside the image of $\mathcal{C}$ by φ_A. Additionally, in the hyperelliptic case, if we denote $\varphi^*_{K_\mathcal{C}}(P_1) = 2p_1$ and $\varphi^*_{K_\mathcal{C}}(P_2) = 2p_2$, with $A = p_1 + p_2$, we can choose z such that $z(p_i) \neq 0$. This is because the divisor $3A$ is free. (In fact, very ample as we will see). The set $(*)$ has 7 elements and the space itself is only 6-dimensional. Hence we find a relation in degree 4.

Lemma III.21. *There is only one relation in degree* 4.

Proof of the lemma. We argue by contradiction. Suppose there are two relations in degree 4. Then, we can eliminate from one of them the term xz. In other words there exists a relation involving only $x^4, x^2y_1, x^2y_2, y_1^2, y_1y_2$ and y_2^2, which means that the image of $\mathcal{C}$ under the canonical map $\varphi_{K_\mathcal{C}}$ is contained in a quadric, i.e. $\mathcal{C}$ is hyperelliptic. Let us write a relation in degree 4 in the form:

$$\alpha zx = q(x^2, y_1, y_2).$$

where q is a quadratic polynomial. For at least one of the relations the scalar α is nonzero, thus $q(x^2, y_1, y_2)$ is not a relation. However, in this situation, q goes through P_1 and P_2 so that $p_1 + q_1 + p_2 + q_2 \subset \mathrm{div}_0(q(x^2, y_1, y_2))$, which implies that $q_1 + q_2 \subset \mathrm{div}_0(z)$, which is a contradiction. $\square$

Let us denote this new relation by F_4. Equally, there is only one relation in $\langle x(*), y_1z, y_2z \rangle$ and this is the multiple F_4. A relation that is not a multiple F_4 is

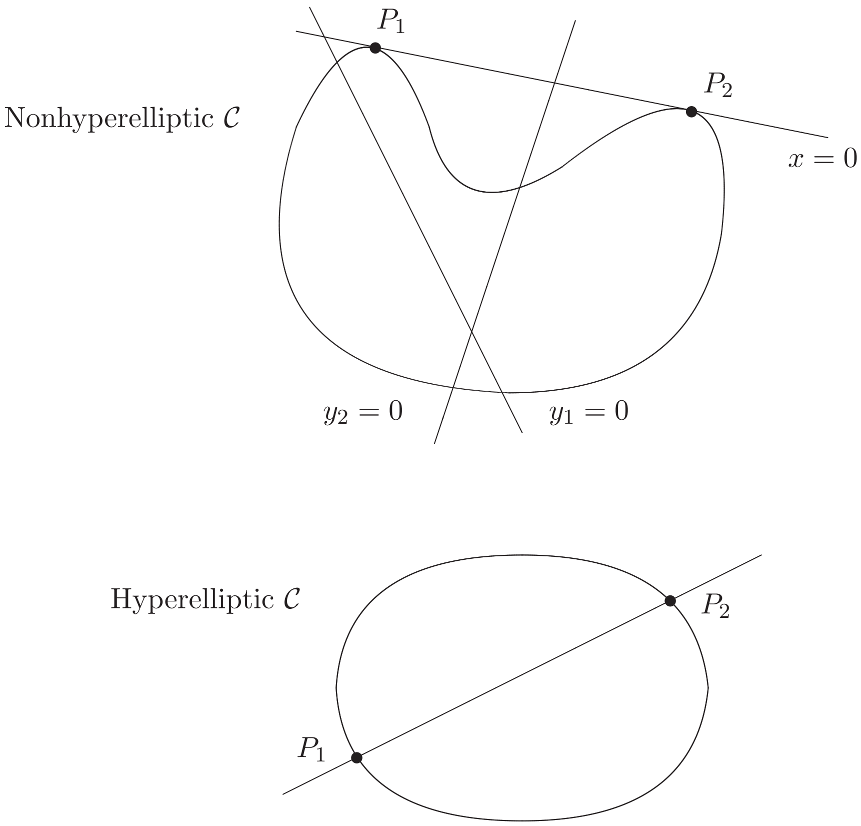

FIGURE 2. The image of $\mathcal{C}$ under $\varphi_{K_\mathcal{C}}$

necessarily of the form

$$z(\alpha y_1 + \beta y_2) = xA$$

where A is some appropriate polynomial expression of x, y_1, y_2, z. Since $\mathrm{div}_0(z)$ misses out on $\mathrm{div}_0(x)$ such a relation implies that $\alpha y_1 + \beta y_2 \in H^0(2A - \mathrm{div}_0(x)) = \langle x^2 \rangle$ and this is a contradiction. Accordingly, there are no new generators in degree 4 and 5 and so by Proposition III.4 the ring $R(\mathcal{C}, A)$ is generated by x, y_1, y_2, z. We now deal with the generators of the ideal I_A. We have already a generator of degree 4. The full symmetric power of degree 6 of the generators x, y_1, y_2, z of the ring $R(\mathcal{C}, A)$ is based by

$$\left\{x^6, x^3 z, z^2\right\} \cup \{y_1, y_2\} \cdot \left\{zx, x^4\right\} \cup \mathrm{S}^2\{y_1, y_2\} \cdot x^2 \cup \mathrm{S}^3\{y_1, y_2\}. \tag{4.2}$$

Subtracting the 3 multiples of F_4 we obtain an 11-dimensional space. As the dimension of $H^0(6A)$ is 10, we get a new relation in degree 6. Since the ring $R(\mathcal{C}, A)$ has no new generators in degree 6 we conclude that there are no more relations in degree 6. Let us denote this relation by F_6.

Lemma III.22. *There is no syzygy between F_4 and F_6 in degree 7.*

Proof of the lemma. If $xF_6 + l(x^3, y_1x, y_2x, z)F_4 = 0$ then

$$x \mid l(x^3, y_1x, y_2x, z)F_4 \implies x \mid l(x^3, y_1x, y_2x, z) \quad \text{or} \quad x \mid F_4.$$

In the first case, we obtain $F_6 + a(x^2, y_1, y_2)F_4 = 0$ and this is clearly not true. In the second case, we obtain a relation between the elements of (†) which again is not true. $\square$

By the usual numerical argument this lemma enables us to show that there are no new generators of $I_{A,d}$ in degree 7.

Lemma III.23. *The map*

$$\langle y_1, y_2\rangle \otimes H^0(nA) \to H^0((n+2)A)$$

is surjective for all $n \geq 5$ and for $n = 4, 3$ has a 1-dimensional cokernel.

Proof of the lemma. By Castelnuovo's free-pencil trick, the kernel of this map is isomorphic to the space $H^0((n-2)A)$. For $n \geq 5$ all divisors involved in computation of the dimensions of the spaces are nonspecial. The cokernel has dimension $(n+1)\deg(A) - 2(n-1)\deg(A) + (n-3)\deg(A) = 0$. If $n = 4$ then the kernel has dimension 1 more than expected. And if $n = 3$ the kernel is 1-dimensional. $\square$

For clarity, we make extra notation. Let $\mathrm{ev}\colon \mathbb{C}[x, y_1, y_2, z] \to R(\mathcal{C}, A)$ be the minimal surjection corresponding to the generators[§] x, y_1, y_2, z.

Corollary III.24. *Let $F \in \mathbb{C}[x, y_1, y_2, z]$ be a polynomial of degree $d \geq 10$. Then*[¶]*,*

$$F \in I_{A,d} \implies F \in I'_{A,d}$$

[§]We keep the same notation for the sections as elements of $R(\mathcal{C}, A)$ and the variables of the polynomial ring $\mathbb{C}[x, y_1, y_2, z]$, as no confusion is likely to arise. In particular $\mathrm{ev}(x) = x$ and so on.

[¶]Recall the notation introduced on page 38 for $I'_{A,d}$.

Proof. Let us write F as

$$F = Ay_1 + By_2 + Cx + Dz$$

where $A, B, C, D \in \mathbb{C}[x, y_1, y_2, z]$ are quasihomogeneous forms of degrees $d-2$, $d-2$, $d-1$ and $d-3$ respectively. In particular, $\deg(D)$ and $\deg(C)$ are ≥ 7. By Lemma III.23 we can write $D \equiv A_1y_1 + B_1y_2 \mod I_{A,d-3}$ and therefore

$$F \equiv A_2y_1 + B_2y_2 + Cx \mod I'_{A,d}.$$

Writing $C \equiv A_3y_1 + B_3y_2 \mod I_{A,d-1}$ we deduce that $F \equiv A_3y_1 + B_3y_2 \mod I'_{A,d}$. Consequently, if $F \in I_{A,d}$ we conclude that $\mathrm{ev}(A_3)y_1 + \mathrm{ev}(B_3)y_2 = 0$. By Castelnuovo's free-pencil trick, there exists a section $k \in H^0((d-2)A - 2A)$ such that

$$\mathrm{ev}(A_3) = -ky_2 \quad \text{and} \quad \mathrm{ev}(B_3) = ky_1.$$

Let $K \in \mathbb{C}[x, y_1, y_2, z]$ be such that $\mathrm{ev}(K) = k$. Then,

$$A_3 \equiv -Ky_2 \mod I_{A,d-2} \quad \text{and} \quad B_3 \equiv Ky_2 \mod I_{A,d-2}$$

and this implies that $F \equiv 0 \mod I'_{A,d}$. In other words, $F \in I'_{A,d}$. $\square$

This corollary shows that any relation in degree ≥ 10 is a multiple of a relation of smaller degree.

Lemma III.25. *Let $F \in \mathbb{C}[x, y_1, y_2, z]$ be a quasihomogeneous form of degree 8. Then,*

$$F \in I_{A,8} \implies F \in I'_{A,8}$$

Proof of the lemma. If F has degree 8, then we can write it as

$$F = Ay_1 + By_2 + \alpha x^8 + \beta x^5 z + \delta x^2 z^2.$$

All three last summands have a factorisation of the form $x \cdot G$ with $\deg(G) = 7$, therefore by Lemma III.23, $F \equiv A_1y_1 + A_2y_2 \mod I'_{A,8}$. To finish we only have to repeat the argument of Corollary III.24. $\square$

Finally we have only to analyse the the component of degree 9 of I_A.

LEMMA III.26. *Let* $F \in \mathbb{C}[x, y_1, y_2, z]$ *be a quasihomogeneous form of degree* 9. *Then,*

$$F \in I_{A,9} \implies F \in I'_{A,9}$$

Proof of the lemma. We write

$$F = Ay_1 + By_2 + x^9 + x^6 z + x^3 z^2 + z^3$$

and as in the previous lemma we readily reduce this expression to

$$F \equiv A_1 y_1 + A_2 y_2 + z^3 \mod I'_{A,9}.$$

The problem is that we cannot apply Lemma III.23 to z^2 since its degree is only 6. We go around this matter by observing that

$$\langle x \rangle \otimes H^0(5A) \hookrightarrow H^0(6A)$$

has a cokernel of dimension 2 which must be spanned by $\left\{y_1^3, y_1 y_1^2\right\}$. The argument is a repetition of a previous argument: if $\alpha y_1^3 + \beta y_1^2 y_2$ lies in the image of $\langle x \rangle \otimes H^0(5A)$ then $\alpha y_1 + \beta y_2 \in H^0(2A - \operatorname{div}_0(x))$ which is a contradiction. By this we can write

$$z^2 \equiv xC + \alpha_1 y_1^3 + \beta_1 y_1^2 y_2 \mod I_{A,6};$$

and the lemma follows. $\square$

We deduce that we have determined all the generators of I_A. This finishes the proof of Theorem III.20. $\square$

Remark. There is an alternative way of proving Theorem III.20 that relies on some results of Commutative Algebra introduced in Chapter II. In later theorems, when the analysis of I_A does not rend itself so easily we will have to use these arguments. An alternative proof can be described in the following way. Since $R(\mathcal{C}, A)$ is generated in degree 3, the map $\varphi_{A^+} : \mathcal{C} \to \mathbb{P}(1, 2^2, 3)$ is an embedding. The homogeneous ideal $I(\varphi_{A^+}(\mathcal{C}))$ is I_A. Suppose we have determined two relations F_4 and F_6 and we know they form a regular sequence. Denote by $I = (F_4, F_6)$. Clearly $I \subset I_A$. Since F_4, F_6 are a regular sequence their zero locus is a 1-dimensional. Write the scheme defined by I as a sum of irreducible components $Z = \sum \alpha_i Z_i$. By Bézout's theorem we know that $\sum \alpha_i \deg(Z_i) \leq \frac{4 \cdot 6}{2^2 \cdot 3} = 2$. But $\deg(\varphi_{A^+}(\mathcal{C})) = \deg(A) = 2$, hence Z has only one irreducible component of multiplicity 1. Since the homogeneous ring of $\mathbb{P}(1, 2^2, 3)$

is Cohen–Macaulay and I is generated by a regular sequence, Z has no embedded components (Unmixedness). Hence $Z = \varphi_{A^+}(\mathcal{C})$, and consequently $I = I_A$. (See Proposition IV.2 on page 75 for a more general and precise version of this type of reasoning. There we have replaced Bézout's theorem with an elementary argument).

This example is special in more than one way. To begin with, notice that the general curve of genus 3 has a halfcanonical divisor whose linear system is 0-dimensional (simply consider a bitangent). By Gieseker's result (see [**Gie, ACGH**]) if a curve $\mathcal{C}$ is general in moduli, the map

$$H^0(A) \otimes H^0(A) \to H^0(2A)$$

is injective. Therefore we can be sure that whenever in the remainder of this work we consider curves with halfcanonical divisors whose linear systems have dimension ≥ 1, the curve is not general in moduli. So in this sense, curves of genus 3 special in the context of this work. Another feature of this case is that it disproves the assertion that going from $R(\mathcal{C}, K_{\mathcal{C}})$ to $R(\mathcal{C}, A)$ decreases the codimension of the ring. For large genus, this is verified *empirically* to be true in this work. Here, $R(\mathcal{C}, K_{\mathcal{C}})$ is a ring of codimension 1 whereas $R(\mathcal{C}, A)$ has codimension one more. Of course that $R(\mathcal{C}, A)$ should be thought of a ring of a polarising divisor on $\mathcal{C}$ plus some information about a bitangent to C_4 (assume $\mathcal{C}$ is nonhyperelliptic). Which brings us to our third remark. The problem of reconstructing a plane quartic from (a subset of) its 28 bitangents is classical. The most recent studies are those by Caporaso and Sernesi [**CS**] and by Lehavi [**Le**]. To our knowledge none of these accounts considers the ring $R(\mathcal{C}, A)$. In the context of halfcanonical rings the bitangents to a plane quartic give 28 embeddings of $\mathcal{C}$ into $\mathbb{P}(1, 2^2, 3)$ and the properties of these embeddings seem worthwhile investigating.

Genus 3 and $h^0(A) = 2$.

Since $\mathcal{C}$ is not rational, a divisor of degree 2 on $\mathcal{C}$ is necessarily free. Indeed A is the unique g^1_2 on $\mathcal{C}$.

Theorem III.27. *A nonsingular hyperelliptic curve $\mathcal{C}$ of genus 3 is isomorphic to the halfcanonical curve $C_8 \subset \mathbb{P}(1^2, 4)$.*

Proof. By Proposition III.5, the ring $R(\mathcal{C}, A)$ is generated in degree ≤ 4. Moreover we have

$$\begin{aligned} H^0(A) &= \langle x_1, x_2 \rangle \\ H^0(2A) &= \langle x_1^2, x_1x_2, x_2^2 \rangle \\ H^0(3A) &= \langle x_1^3, x_1^2x_2, \ldots .x_2^3 \rangle \\ H^0(4A) &= \langle x_1^4, \ldots, x_2^4, w \rangle \\ H^0(5A) &= \langle x_1^5, \ldots, x_2^5, x_1w, x_2w \rangle \end{aligned}$$

Clearly for any n the n-symmetric power of x_1, x_2 is linearly independent, and this takes care of the first three degrees. When we come to degree 4 we must add a generator w complementary to $\langle x_1^4, \ldots, x_2^4 \rangle$. These are all the generators we need. Moreover there is no excess of generators up to (not including) degree 10. In degree 10 we find the first relation. By Proposition III.12 this is the only relation. Therefore $R(\mathcal{C}, A) = \mathbb{C}[x_1, x_2, w]/(F_{10})$ and the theorem follows. $\square$

Remark. These two cases illustrate the different structures of $R(\mathcal{C}, A)$ obtained in the same curve considering diverse halfcanonical divisors A. For hyperelliptic curves of genus 3 we can consider $R(\mathcal{C}, A_1)$ where A_1 has 0-dimensional associated linear system and thus determines a secant line at branching points to the image of $\mathcal{C}$ under the canonical morphism, and this leads to $\mathbb{C}[x, y_1, y_2, z]/(F_4, F_6)$; or we can consider $R(\mathcal{C}, A_2)$ where A_2 is a pencil of lines going through a point of $\varphi_{K_\mathcal{C}}(\mathcal{C}) = C_2 \subset \mathbb{P}^2$, which leads to $\mathbb{C}[x_1, x_2, w]/(F_8)$.

Genus 4 and $h^0(A) = 2$.

If $\mathcal{C}$ is a nonsingular curve of genus 4 with a halfcanonical pencil A, then $\mathcal{C}$ is nonhyperelliptic if and only if A is free. From the halfcanonical model point of view, $C_6 \subset \mathbb{P}(1^2, 2)$, it is convenient to phrase the following theorem with the assumption that A is free, though the reader might want to substitute this by the 'nonhyperelliptic' assumption. Notice however that a free halfcanonical pencil in this case is a trigonal system.

Theorem III.28. *Let $\mathcal{C}$ be a nonsingular curve of genus 4. Then $\mathcal{C}$ has a free halfcanonical pencil if and only if $\mathcal{C}$ is isomorphic to $C_6 \subset \mathbb{P}(1^2, 2)$.*

Proof. A general sextic curve $C_6 \subset \mathbb{P}(1^2, 2)$ is nonsingular as we can always make sure that $(0, 0, 1) \notin C_6$. By adjunction C_6 has a halfcanonical divisor given as the restriction of $\mathcal{O}(1)$ to C_6. Since as we mentioned, $(0, 0, 1) \notin C_6$ this divisor is free.

Conversely assume that $\mathcal{C}$ is a nonsingular curve of genus 4 with a free halfcanonical pencil. Since A is a free pencil we deduce that $\mathcal{C}$ is not hyperelliptic (Proposition III.2). Applying Proposition III.5 we conclude that $R(\mathcal{C}, A)$ is generated in degree ≤ 2. In addition, we have:

$$\begin{aligned}
H^0(A) &= \langle x_1, x_2\rangle \\
H^0(2A) &= \mathrm{S}^2\langle x_1, x_2\rangle + \langle y\rangle = (*) \\
H^0(3A) &= \mathrm{S}^3\langle x_1, x_2\rangle + \langle yx_1, yx_2\rangle = (\dagger) \\
H^0(4A) &= \mathrm{S}^4\langle x_1, x_2\rangle + y\cdot(*) = (\S) \\
H^0(5A) &= \mathrm{S}^5\langle x_1, x_2\rangle + y\cdot(\dagger) \\
H^0(6A) &= \mathrm{S}^6\langle x_1, x_2\rangle + y\cdot(\S)
\end{aligned}$$

By Proposition III.12, the ideal I_A has a minimal set of generators of degree ≤ 6. On the left hand side, for $4 \le n \le 6$ the dimension of $H^0(nA)$ is 9, 12 and 15, respectively. Except for degree 6 they equal the dimension of the spaces on the right hand side. We conclude that I_A is generated by a single element of degree 6. Therefore $\mathcal{C} \simeq \operatorname{Proj} R(\mathcal{C}, A)$ is isomorphic to $C_6 \subset \mathbb{P}(1^2, 2)$. $\square$

Genus 5 and $h^0(A) = 2$.

We consider the case when $\mathcal{C}$ is not trigonal. This implies that A is free and also by Proposition III.5 that the ring $R(\mathcal{C}, A)$ is generated in degree ≤ 2.

Theorem III.29. *A nonsingular curve $\mathcal{C}$ of genus 5 and gonality strictly bigger than 3 has a halfcanonical pencil if and only if $\mathcal{C}$ is isomorphic to a complete intersection $C_{4,4} \subset \mathbb{P}(1^2, 2^2)$.*

Proof. A general complete intersection $C_{4,4} \subset \mathbb{P}(1^2, 2^2)$ is nonsingular since we can always make sure that $C_{4,4} \cap \mathbb{P}[y_1, y_2] = \emptyset$. The sheaf $\mathcal{O}(1)$ restricts to $C_{4,4}$ as a halfcanonical divisor. This divisor is free by the fact we have just mentioned. In particular, by Proposition III.2 this implies that $C_{4,4}$ is not trigonal.

Conversely, suppose that $\mathcal{C}$ is a nonsingular curve of genus 5 of gonality strictly bigger that 3 with a halfcanonical pencil. Let us start by writing

$$\begin{aligned}
H^0(A) &= \langle x_1, x_2\rangle \\
H^0(2A) &= \mathrm{S}^2\langle x_1, x_2\rangle + \langle y_1, y_2\rangle \\
H^0(3A) &= \mathrm{S}^3\langle x_1, x_2\rangle + \langle y_1, y_2\rangle \cdot \langle x_1, x_2\rangle \\
H^0(4A) &= \mathrm{S}^4\langle x_1, x_2\rangle + \mathrm{S}^2\langle x_1, x_2\rangle \cdot \langle y_1, y_2\rangle + \mathrm{S}^2\langle y_1, y_2\rangle\,.
\end{aligned}$$

We have written down 8 generators for $H^0(3A)$ from the generators of $H^0(A)$ and $H^0(2A)$. As $R(\mathcal{C}, A)$ is generated in degree ≤ 2 (Proposition III.5) there are no relations holding among these. Finally, when it comes to $H^0(4A)$ we own two more generators than the dimension thus there are exactly 2 quartic relations. Let us denote them by F_4 and G_4. Since these two quartics are linearly independent they form a regular sequence. The ring $R(\mathcal{C}, A)$ is generated by x_1, x_2, y_1, y_2 and thus the map $\varphi_{A^+}\colon \mathcal{C} \to \mathbb{P}(1^2, 2^2)$ given by x_1, x_2, y_1, y_2 is an embedding of $\mathcal{C}$ and its image is contained in the scheme-theoretic intersection of two quartics. Since $\deg(\varphi_{A^+}(\mathcal{C})) = 4 = \frac{4\cdot 4}{2^2}$, by Bézout's theorem the scheme-theoretic intersection of the hypersurfaces $F_4 = 0$ and $G_4 = 0$ has only one irreducible component of multiplicity one. Since $\mathbb{C}[x_1, x_2, y_1, y_2]$ is Cohen–Macaulay $\mathrm{Z}(F_4, G_4)$ has no embedded components. Hence, $(F_4, G_4) = I(\varphi_{A^+}(\mathcal{C})) = I_A$. $\square$

Genus 5 and $h^0(A) = 3$.

By Clifford's theorem, a curve of genus 5 with a halfcanonical net is necessarily hyperelliptic and moreover $A = 2g^1_2$. So that A is free and in particular, there exists a unique halfcanonical net on $\mathcal{C}$.

Theorem III.30. *Let $\mathcal{C}$ be a nonsingular curve of genus 5. Then $\mathcal{C}$ has a halfcanonical net A, if and only if, $\mathcal{C}$ is hyperelliptic and $A = 2g^1_2$, if and only if $\mathcal{C}$ is isomorphic to a complete intersection $C_{2,6} \subset \mathbb{P}(1^3, 3)$.*

Proof. The first equivalence is clear. Likewise, that a complete intersection $C_{2,6}$ has a halfcanonical net is straightforward. Thus let us concentrate on showing that a nonsingular curve $\mathcal{C}$ with a halfcanonical net is isomorphic to a complete intersection $C_{2,6} \subset \mathbb{P}(1^3, 3)$ by explicitly computing its halfcanonical ring. By our preamble A must be free. Therefore $R(\mathcal{C}, A)$ is generated in degree ≤ 3. Denote by

$$\langle x_1, x_2, x_3\rangle = H^0(A)$$

a basis of the component in degree 1. The second symmetric power

$$\mathrm{S}^2\langle x_1, x_2, x_3\rangle \subset H^0(2A)$$

has an excess of one generator, thus we get a quadric relation. Denote it by F_2. Thus the halfcanonical net maps $\mathcal{C}$ onto a plane conic. This is clearly the only relation in degree 2. In degree 3 the third symmetric power $\mathrm{S}^3\langle x_1, x_2, x_3\rangle$ has dimension 10 and any cubic relation between these elements must be a multiple of F_2. Hence modulo these multiples, $\mathrm{S}^3\langle x_1, x_2, x_3\rangle$ generates a subspace of $H^0(3A)$ of dimension 7. Thus there is a single new generator in degree 3 which we denote by z. By Proposition III.5, $R(\mathcal{C}, A)$ is generated by x_1, x_2, x_3 and z. The map

$$\varphi_{A^+}\colon \mathcal{C} \to \mathbb{P}(1^3, 3) \tag{4.3}$$

given by x_i and z is an embedding into a curve of degree 4. The relation F_2 is a generator of the halfcanonical ideal. If there is a new relation in degree 4 then under the map $\varphi_{|A|^+}$ of (4.3) the curve $\mathcal{C}$ maps to the intersection of a cubic and a quartic hypersurface. Two hypersurfaces in 3-dimensional projective space forming a regular sequence cut out a one dimensional scheme. We deduce that if there exists a new quartic relation then $\varphi_{A^+}(\mathcal{C})$ is a component of a one dimensional subscheme of $\mathbb{P}(1^3, 3)$ whose degree, by Bézout's theorem is $\leq \frac{2\cdot 4}{3}$. However $\deg(\varphi_{A^+}(\mathcal{C})) = 4$. The same argument shows that there is no quintic relation. By Lemma III.6 the map

$$\langle x_1, x_2, x_3\rangle \otimes \left\{\mathrm{S}^5\langle x_1, x_2, x_3\rangle + \mathrm{S}^2\langle x_1, x_2, x_3\rangle \cdot z\right\} \twoheadrightarrow H^0(6A)$$

is surjective and $z^2 \notin \langle x_1, x_2, x_3\rangle \otimes \left\{\mathrm{S}^5\langle x_1, x_2, x_3\rangle + \mathrm{S}^2\langle x_1, x_2, x_3\rangle \cdot z\right\}$ we deduce that there is a relation in degree 6. Let us denote it by F_6. Together with F_2 these cut out a subscheme of $\mathbb{P}[x_1, x_2, x_3, z]$ of dimension 1 of degree $4 = \deg(\varphi_{A^+}(\mathcal{C}))$. We conclude that the ideal I_A is generated by F_2 and F_6. $\square$

Genus 6 **and** $h^0(A) = 3$.

THEOREM III.31. *Let $\mathcal{C}$ be a nonsingular curve of genus 6. Then $\mathcal{C}$ has a* free *halfcanonical net if an only if $\mathcal{C}$ is isomorphic to a plane quintic.*

Proof. This is done in Example II.1.2 of Chapter II, on page 15. $\square$

Genus 7 and $h^0(A) = 3$.

THEOREM III.32. *Let $\mathcal{C}$ be a nonsingular curve of genus 7 such that* $\mathrm{gon}(\mathcal{C}) > 3$. *Then $\mathcal{C}$ has a halfcanonical net if and only if $\mathcal{C}$ is isomorphic to to a complete intersection $C_{3,4} \subset \mathbb{P}(1^3, 2)$.*

Proof. Suppose that $\mathcal{C}$ is not trigonal and has a halfcanonical net A. By Proposition III.5 the ring $R(\mathcal{C}, A)$ is generated in degree ≤ 2. Since $\mathcal{C}$ is nontrigonal the map sym^2 must have null kernel. This implies that we only need one new generators in degree 2. Thus $R(\mathcal{C}, A)$ is generated by x_1, x_2, x_3 in degree 1 and y in degree 2.

$$H^0(A) = \langle x_1, x_2, x_3 \rangle$$
$$H^0(2A) = \left\langle \mathrm{S}^2 H^0(A), y \right\rangle$$

Accordingly, there can only be one relation in the set $\mathrm{S}^3\{x_1, x_2, x_3\} \cup \{yx_1, yx_2, yx_3\}$. And likewise, only one new relation in $\mathrm{S}^4\{x_1, x_2, x_3\} \cup \mathrm{S}^2\{x_1, x_2, x_3\} \cdot y \cup \left\{y^2\right\}$. By our assumption $\mathcal{C}$ is not trigonal, nor it is a plane quintic. Consequently, by Theorem III.14, denoting by φ_{A^+} the embedding yielded by our choice of generators of $R(\mathcal{C}, A)$, $\varphi_{A^+}(\mathcal{C})$ is cut out set-theoretically by a cubic and a quartic in $\mathbb{P}(1^3, 2)$ forming a regular sequence. Let I be the ideal generated by the cubic and quartic forms. By Unmixedness, I has no embedded primes and using Bézout's theorem we conclude that I has a unique minimal prime of multiplicity 1. Hence $I = I_A$ (see Proposition II.17).

Conversely a complete intersection $C_{3,4} \subset \mathbb{P}(1^3, 2)$ is a nonsingular curve of genus 7 with a halfcanonical divisor, $\mathcal{O}(1)_{|C_{3,4}}$, such that sym^2 is injective, i.e., such that $\mathrm{gon}(\mathcal{C}) > 3$. Note that $C_{3,4}$ does not contain $(0, 0, 0, 1) \in \mathbb{P}(1^3, 2)$. In this situation $C_{3,4}$ would be singular. Moreover, if A had base locus, then $\mathcal{C}$ would be hyperelliptic. $\square$

Genus 9 and $h^0(A) = 4$.

THEOREM III.33. *Let $\mathcal{C}$ be a nonsingular curve of genus 9. Let A be a halfcanonical divisor with* $\dim H^0(A) = 4$. *If* $\dim W^1_4(\mathcal{C}) > 0$ *then $R(\mathcal{C}, A)$ has codimension ≥ 3. If* $\dim W^1_4(\mathcal{C}) = 0$ *then $R(\mathcal{C}, A)$ is a codimension 2 ring generated in degree 1 by x_1, x_2, x_3, x_4 and the ideal of relations I_A is generated by F_2 of degree 2 and F_4 of degree 4.*

Proof. Suppose that $\dim W^1_4(\mathcal{C}) > 0$. We want to show that $R(\mathcal{C}, A)$ is not generated in degree 1. Suppose that $\mathrm{gon}(\mathcal{C}) \leq 3$. Let ξ be a free pencil of degree $d \leq 3$. If $d = 3$ then by (ii) of Proposition III.2, the dimension of $H^0(A - 2\xi)$ is $> 4 - 3$, i.e. $|A - 2\xi|$ is a linear system of dimension ≥ 1 and degree 2. We conclude that $\mathcal{C}$ is hyperelliptic. For a hyperelliptic curve $R(\mathcal{C}, A)$ cannot be generated in degree 1. Hence we can assume that $\mathrm{gon}(\mathcal{C}) = 4$. Let ξ_1 and ξ_2 be two free base-point free pencils. By Proposition III.9 each of them yields a symmetric tensor in the kernel of sym^2. Since $W^1_4(\mathcal{C}) > 0$ we may assume they yield distinct symmetric tensors. Hence $\dim \mathrm{Ker}\,\mathrm{sym}^2 \geq 2$ and thus $R(\mathcal{C}, A)$ is not generated in degree 1.

Suppose that $\dim W^1_4(\mathcal{C}) = 0$. Then, by the reasoning above, the dimension of $\mathrm{Ker}\,\mathrm{sym}^2$ is 1. From Proposition III.5 we deduce that $R(\mathcal{C}, A)$ is generated in degree ≤ 3. Hence we must show that the map

$$H^0(A) \otimes H^0(2A) \to H^0(3A) \tag{4.4}$$

is surjective. We will use a refinement of the argument of the proof of Proposition III.5. Consider a element of the cokernel of (4.4). According to Lemma III.7 it corresponds to an extension

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A) \to 0$$

with 5 global sections. Consider the divisor of zeros δ of a section through two general points p, q of $\mathcal{C}$. Saturating the embedding $\mathcal{O}_{\mathcal{C}}(p + q) \subset \mathcal{O}_{\mathcal{C}}(\delta) \hookrightarrow \mathscr{F}$ we obtain

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A - \xi) \to 0$$

where $\xi \supset p+q$ is an effective divisor. Since $p+q$ are general we obtain $h^0(A-\xi) \leq 2$. Consequently $h^0(\xi) \geq 3$. Thus $\deg(\xi) \geq 6$ and accordingly $\deg(A - \xi) \leq 2$, so that $\dim H^0(A - \xi) \leq 1$. We deduce that $h^0(\xi) \geq h^0(A)$.

To finish the proof of our claim, by showing that $\mathscr{F}$ must be the split extension, we need to show that $|A|$ is base point free. Suppose that $|A|$ has a base point. Then φ maps $\mathcal{C}$ onto a space curve of degree ≤ 7. The degree of φ_A is ≤ 2. Suppose $\deg \varphi_A = 2$. Then the image of $\mathcal{C}$ by φ_A is the rational normal curve of degree 3. Which means that $\mathcal{C}$ is hyperelliptic. Thus $\deg \varphi_A = 1$ and the image of $\mathcal{C}$ by φ_A has degree ≤ 7. Additionally we know that $\varphi_A(\mathcal{C})$ is contained in a quadric. This implies that there exists a g^1_3 on $\varphi_A\mathcal{C}$ and therefore on $\mathcal{C}$. This is a contradiction. We conclude that A is free.

Therefore if $\dim W^1_4(\mathcal{C}) = 0$ the map sym^2 has a 1-dimensional kernel and the ring $R(\mathcal{C}, A)$ is generated in degree 1. In particular φ_A is very ample. Moreover the curve $\varphi_A(\mathcal{C})$ has a 1-dimensional family of 4-secants and as such is not contained in any irreducible cubic. In degree 4 we have

$$H^0(4A) \supset \mathrm{S}^4 H^0(A).$$

On the right hand side, the space has dimension 35. Subtracting 10 to account for multiples of the quadric relation we obtain 25. The space $H^0(4A)$ has dimension $3 \cdot 8 = 24$. Therefore there is a new quartic relation in I_A. Denote the two relations we have found so far by F_2 and F_4. Let S denote the intersection $(F_2 = 0) \cap (F_4 = 0)$. By Unmixedness, S has no embedded components. By the Bézout's theorem we have

$$8 = \deg \varphi_{|A|}(\mathcal{C}) \leq \sum_{C_i \in S} \alpha_i \deg(C_i) \leq 8.$$

In other words S is irreducible and coincides with $\varphi_A(\mathcal{C})$, which implies that I_A is generated by F_2, F_4. This finishes the proof of Theorem III.33. □

Genus 10 and $h^0(A) = 4$.

Theorem III.34. *A nonsingular curve $\mathcal{C}$ of genus 10 and $\mathrm{gon}(\mathcal{C}) \geq 6$ has a halfcanonical divisor A with $h^0(A) = 4$ if and only if $\mathcal{C}$ is isomorphic to a complete intersection $C_{3,3} \subset \mathbb{P}^3$.*

Proof. By Proposition III.9 for a complete intersection $C_{3,3}$ of two cubics in $\mathbb{P}^3$ we have $\mathrm{gon}(C_{3,3}) \geq 6$. Conversely suppose that $\mathcal{C}$ is a nonsingular curve of genus 10 with a halfcanonical divisor A such that $\dim H^0(A) = 4$. Assume that $\mathrm{gon}(\mathcal{C}) \geq 6$. Since in this case the inclusion $\mathcal{Q}_4 \subset \mathbb{P}[\mathrm{S}^2 H^0(A)]$ is an equality (all quadrics of 3-dimensional projective space have rank ≤ 4) Proposition III.9 yields the equivalence:

$$\mathbb{P}[\mathrm{Ker}\,\mathrm{sym}^2] \neq \emptyset \Longleftrightarrow \mathrm{gon}(\mathcal{C}) \leq 4.$$

In fact this proposition also shows that for a nonsingular curve with a halfcanonical divisor A with $\dim H^0(A) = 5$ the case $\mathrm{gon}(\mathcal{C}) = 4$ does not happen. Hence our assumption that $\mathrm{gon}(\mathcal{C}) \geq 6$. Additionally, by the next lemma, $\mathrm{gon}(\mathcal{C}) \geq 6$ implies that A is very ample.

LEMMA III.35. *Let $\mathcal{C}$ be a nonsingular curve of genus* 10 *with a halfcanonical divisor A with* $\dim H^0(A) = 4$. *Then*

$$\mathrm{gon}(\mathcal{C}) \geq 6 \Longleftrightarrow |A| \text{ is very ample.}$$

Proof of the lemma. Suppose that $|A|$ has a base locus. Then it contains a free g^3_d where $d \leq 8$. By Castelnuovo's bound (see page 44) a nondegenerate curve of degree ≤ 8 in $\mathbb{P}^3$ has geometric genus ≤ 9, which means that it cannot be birational to $\mathcal{C}$. Hence a g^3_d with $d \leq 8$ must give a morphism of degree ≥ 2. In this situation it maps $\mathcal{C}$ onto a nondegenerate curve in $\mathbb{P}^3$ of degree ≤ 4. There is a finite list of curves to go through. They are all rational or elliptic. In case the image is a rational curve we get straightforwardly that $\mathrm{gon}(\mathcal{C}) \leq 4$. If the image is a quartic elliptic curve, then it is contained by the intersection of two linearly independent quadrics. But we know that if $\mathrm{gon}(\mathcal{C}) \geq 6$ then sym^2 has no kernel, hence this cannot happen. We have shown

$$\mathrm{gon}(\mathcal{C}) \geq 6 \implies |A| \text{ is free.}$$

By our previous argument,the image of $\mathcal{C}$ under the morphism $|A|$ cannot have degree 3 so that φ_A is a birational morphism onto a space curve C_9 of degree 9. If it is not an embedding then C_9 has at least a singular point. Projecting off such a point we obtain a free g^2_d where $d \leq 7$. Since $\mathrm{gon}(\mathcal{C}) \geq 6$ we must have $d = 7$ and such system yields a birational morphism onto a plane curve of degree 7. By the genus formula this plane septic must be singular and therefore projecting off a singular point yields a g^1_5.

Conversely suppose that $|A|$ is very ample and the image of $\mathcal{C}$ under φ_A is contained in a quadric. If the rank of the quadric is 3 then the genus of C_9 is 12. (Blow up the quadric at the vertex and use the genus formula on the blown up surface). Likewise if C_9 is contained in a quadric of rank 4 then there exist two positive integers a, b such that $a + b = 9$ (degree) plus $ab - a - b + 1 = 10$ (genus formula). These have the solution $\{a, b\} = \{3, 6\}$ which implies that $\mathcal{C}$ is trigonal. But then by Proposition III.2 the dimension of the space $H^0(A - 2g^1_3)$ is > 1. Since the genus of $\mathcal{C}$ is big enough (≥ 5) there exists a single trigonal system. Hence $A = 3g^1_3$. But then

$\deg(\varphi_A)$ is 3 and not 1. Hence

$$|A| \text{ is very ample} \implies \mathrm{sym}^2 \text{ is injective} \implies \mathrm{gon}(\mathcal{C}) \geq 6. \quad \square$$

If $\mathrm{gon}(\mathcal{C}) \geq 6$ then by the lemma $|A|$ is very ample and by Proposition III.5 the ring $R(\mathcal{C}, A)$ is generated in degree 2. Moreover since sym^2 is injective a dimension count shows that $R(\mathcal{C}, A)$ is indeed generated in degree 1. Denote a basis of $H^0(A)$ by $x_1, \dots x_4$. The space

$$\mathrm{S}^3 H^0(A) \subset H^0(3A)$$

is generated by $\binom{6}{3} = 20$ elements, two more than the dimension of $H^0(3A)$. As there are no new generators in degree 3 we get exactly two relations in degree 3. We deduce that $\varphi_A(\mathcal{C})$ is contained in a complete intersection of two cubics in $\mathbb{P}^3$. By Unmixedness and Bézout's theorem we deduce that I_A is generated by two cubic forms. This finishes the proof of Theorem III.34. $\square$

Genus 12 and $h^0(A) = 5$.

Suppose that φ_A is a birational morphism. Then as we have mentioned before, Castelnuovo's bound (page 44) classifies $\mathcal{C}$ as an extremal curve and therefore according to a classification of extremal curves, the image of $\mathcal{C}$ by φ_A is a divisor in the cubic scroll $\mathbb{F}(1, 2)$. Let us give an elementary proof of this.

Proposition III.36. *Let $\mathcal{C}$ be a nonsingular curve of genus 12. Assume that $\mathcal{C}$ has halfcanonical divisor A with $\dim H^0(A) = 5$. Let $C \subset \mathbb{P}^4$ be the image of $\mathcal{C}$ under the map φ_A. Then one of the following possibilities occurs:*

(i) $C \simeq C_4 \simeq \mathbb{P}^1 \iff \mathcal{C}$ *is hyperelliptic.*

(ii) $C \simeq C_5 \subset \mathbb{F}(1, 2)$ *and* $C_5 \in |H + 2L| \iff \mathcal{C}$ *is bielliptic.*

(iii) $C \simeq C_{11} \in\subset \mathbb{F}(1, 2)$ *and* $C_{11} \in |4H - L| \iff \dim W^1_4(\mathcal{C}) = 0$.

where H is the hyperplane section of $\mathbb{F}(1, 2)$ and L the class of its ruling. The equivalences hold under the assumption of existence of A. Moreover a nonsingular curve $C_{11} \in |4H - L|$ of $\mathbb{F}(1, 2)$ is a curve of genus 12 with a unique g^1_4 and a halfcanonical divisor A such that $\dim H^0(A) = 5$.

Proof. Since C is nondegenerate it is clear that $\deg(C) \geq 4$. By Castelnuovo's bound (page 44) if φ_A is birational then $\deg(C) = 11$. (This is the point of calling $\mathcal{C}$

extremal). Therefore the possibilities for the degree of C are 4, 5 or 11. Hence in the statement of the theorem we list all possible cases.

By a straightforward numerical argument, C is contained in a net of quadrics. Denote this net by $\pi \subset \mathbb{P}[\mathrm{S}^2 H^0(A)]$. Equivalently there are at least 3 quadric relations in $R(\mathcal{C}, A)$. In the first place this implies that there exists a quadric of rank ≤ 4 through C and therefore $\mathrm{gon}(\mathcal{C}) \leq 5$ (Proposition III.9). We will see here that indeed $\mathrm{gon}(\mathcal{C}) \leq 4$. Let $\mathcal{L} \subset \pi \subset \mathbb{P}[\mathrm{S}^2 H^0(A)]$ be the locus of singular quadrics through C. The scheme $\mathcal{L}$ is a nondegenerate subscheme of the plane π, of dimension ≥ 1. The curve C is contained in a component of the intersection the quadrics of π. Let us denote this component by S. Suppose that S has dimension 1. We rule out the possibility that $\deg(C) = 5$. If $\deg(C) = 5$ then for every $Q \in \mathcal{L}$ there exists $p \in C \cap (Q = 0)$. On the other hand if $p \in \bigcap_{Q \in \mathcal{L}} \mathrm{Sing}\, Q$ then any secant line to C through p must be contained in C. In other words C is the union of two or more lines, which is not true. We deduce that for every point of C there exists a singular quadric Q_p of π such that $p \in \mathrm{Sing}\, Q_p$. This implies that S is singular along C and therefore C has multiplicity ≥ 2 in S. By Bézout's theorem we have $5 \cdot 2 \leq 2^3$ which gives a contradiction. Therefore if $\dim S = 1$ then $\deg(C) = 4$. Then, $C \simeq C_4$ the rational normal curve of degree 4. Since $\mathcal{C}$ is not rational the map φ_A must have degree $d = 2$. This means that $\mathcal{C}$ is hyperelliptic. Conversely if $\mathcal{C}$ is hyperelliptic then $\dim H^0(A - 4g^1_2) \geq 1$ by Proposition III.2 on page 31, and by Clifford's theorem $\dim H^0(4g^1_2) = 5$, so that the set-theoretic support of $|A - 4g^1_2|$ is the base locus of A and φ_A maps $\mathcal{C}$ onto the rational normal curve of degree 4.

The next possibility happens for $\dim S = 2$. Since S is contained in two quadrics and is a nondegenerate variety of $\mathbb{P}^4$ (it contains C) it can only be one of two varieties: $\mathbb{F}(0,3)$, the cone over the rational normal curve of degree 3 or $\mathbb{F}(1,2)$ the cubic scroll. In the first case, projecting from the vertex we deduce that $\deg(C)$ must be divisible by 3. This is a contradiction. Hence the case $S = \mathbb{F}(3,0)$ does not happen for this genus. The final case is $S = \mathbb{F}(2,1)$. Given that $K_{\mathbb{F}} = -2H + L$, where H is the hyperplane section and L a class of the ruling of $\mathbb{F}(1,2)$, if we are to have $C = aH + bL$ we deduce from the genus formula that

$$2p_a(C) - 2 = K_{\mathbb{F}} C_{11} + C_{11}^2 = -5a - 2b + 3a^2 + 2ab. \tag{4.5}$$

Since $\deg(C) = 5$ or 11, there are two possibilities:

(a) $3a + b = 5$ and $p_a(C) \geq 0$

(b) $3a + b = 11$ and $p_a(C) \geq 12$

In case (a) the quadratic equation of (4.5) reduces to

$$3a^2 - 11a + 8 \leq 0$$

which has integer solutions for $a = 1$ or $a = 2$. If $a = 1$ then $p_a(C) = 0$ and therefore C is a smooth rational curve. Then, the morphism φ_A is 2-to-1 onto C. But this means that $\mathcal{C}$ is hyperelliptic and we have dealt with this case. (If $\mathcal{C}$ is hyperelliptic the image of $\mathcal{C}$ under φ_A is the rational normal curve of degree 4). If $a = 2$ then $C \in |2H - L|$ gets a hyperelliptic system from the ruling of $\mathbb{F}(1,2)$. Note that, as above, we can rule out the possibility of C being rational. In fact, in this case C must be a quintic elliptic curve. In this case $\mathcal{C}$ is bielliptic. Conversely, suppose that $\mathcal{C}$ is not trigonal with $\dim W^1_4(\mathcal{C}) > 0$ and let ξ_1, ξ_2 denote two complete free pencils of degree 4 on $\mathcal{C}$. By Proposition III.2 we know that $\dim H^0(A - \xi_1 - \xi_2) \geq 5 - 4 = 1$. On the other hand, by the fact that $\xi_1 \not\sim \xi_2$ we have that $\dim H^0(\xi_1 + \xi_2) \geq 4$ (see [**ACGH**, p. 137]). This implies that A is not free and therefore φ_A maps $\mathcal{C}$ to a quintic. This proves item (ii).

Finally suppose that we have (b). The quadratic equation of (4.5) reduces to

$$3a^2 - 23a + 44 \leq 0$$

whose only integer solution is $a = 4$. Therefore $p_a(C) = 12$ and $C_{11} \in |4H - L|$. We deduce that in this case the morphism φ_A is an embedding. In particular by Proposition III.2, $\mathcal{C}$ is nontrigonal. The curve C_{11} has a tetragonal linear system given by the ruling of $\mathbb{F}(1,2)$, and by reasoning as above this system is unique. Hence $\dim W^1_4(\mathcal{C}) = 0$. Conversely suppose that $\dim W^1_4(\mathcal{C}) = 0$. Then, by what we have shown so far φ_A is an embedding and we recover $C_{11} \in |4A - L|$.

Finally a nonsingular curve in $\mathcal{C} \in |4H - L|$ is halfcanonical by adjunction. The halfcanonical divisor A is given as the restriction of H to $\mathcal{C}$. We deduce that $h^0(A) = h^0(\mathbb{F}, H) = 5$. The divisor A is very ample and therefore applying the argument of a few lines above we deduce that $\dim W^1_4(\mathcal{C}) = 0$. This finishes the proof of the proposition. $\square$

Corollary III.37. *Let $\mathcal{C}$ be a nonsingular curve of genus* 12*. If* $\operatorname{gon}(\mathcal{C}) \geq 5$ *then there are no halfcanonical divisors with* $\dim H^0(A) = 5$. $\square$

Remark. Observe that if A is not free then the ring $R(\mathcal{C}, A)$ is not generated in degree 1 and hence not a codimension 3 ring. This justifies the assumptions of the next theorem.

Theorem III.38. *Let $\mathcal{C}$ be a nonsingular curve of genus* 12*. Let A be a halfcanonical divisor on $\mathcal{C}$ with* $\dim H^0(A) = 5$*. Assume that* $\dim W^1_4(\mathcal{C}) = 0$*. Then the ring $R(\mathcal{C}, A)$ is generated in degree* 1 *by x_1, x_2, x_3, x_4, x_5 and the ideal I_A is generated by three quadric relations F_2, G_2, H_2 and two quartic relations F_4, G_4.*

Proof. By Proposition III.36 the map φ_A is an embedding onto $C_{11} \subset \mathbb{F}(1,2)$ lying on the cubic scroll and belonging to the linear system $|4H - L|$. In particular there are no more quadrics through C_{11} (other than those defining $\mathbb{F}(1,2)$) and this means that there are no new generators in degree 2. A cubic form not contained in the ideal $\mathscr{I}_{\mathbb{F}}(3)$ cuts out in $\mathbb{F}(1,2)$ a subscheme of degree 9 hence there are no generators of I_A in degree 3. The dimension of $\mathrm{S}^3 H^0(A) = \mathrm{S}^3 \langle x_1, \ldots, x_5 \rangle$ is 35. The space of multiples of the quadric relations is 15-dimensional and there are exactly 2 syzygies holding between these. Given that $\dim H^0(3A) = 22$ we deduce that there are no new generators in degree 3. By Proposition III.5 we deduce that $R(\mathcal{C}, A)$ is generated in degree 1 and therefore it is a codimension 3 ring. Since $C_{11} \in |4H - L|$ and $\dim H^0(\mathbb{F}, 4H - (4H - L)) = 2$ we deduce that there is a pencil of quartic hypersurfaces through C_{11}. Two different quartics in this pencil cut out subschemes on $\mathbb{F}(1,2)$ which contain C_{11} and differ by two distinct lines of the ruling. Hence these quartics cut out set-theoretically C_{11} on $\mathbb{F}(1,2)$. Let F_4 and G_4 be two distinct quartics through C_{11}. Denote the quadratic generators of I_A by Q_1, Q_2, Q_3. We want to show that $I_A = (Q_1, Q_2, Q_3, F_4, G_4)$. Consider

$$J_1 = (Q_1, Q_2, Q_3, F_4) \quad \text{and} \quad J_2 = (Q_1, Q_2, Q_3, G_4).$$

Since $\mathbb{F}(1,2)$ is arithmetically Cohen–Macaulay and the projection of each of the ideals J_i in $\mathbb{C}[\mathbb{F}(1,2)] = \mathbb{C}[x_1, \ldots, x_4]/(Q_1, Q_2, Q_3)$ is principal we deduce that both $J_i \subset \mathbb{C}[x_1, \ldots, x_5]$ have no embedded primes. Since they cut out a subscheme of $\mathbb{P}^4$

containing $\varphi_A(\mathcal{C})$, each of J_i must be contained in I_A, the minimal prime corresponding to the irreducible component $\varphi_A(\mathcal{C})$. The degree of J_i equals $4 \cdot \deg \mathbb{F}(1,2) = 12$. Hence in a primary decomposition of each J_i, I_A appears with multiplicity 1. Moreover since J_i cut out a line outside $\varphi_A(\mathcal{C})$ we conclude that

$$J_1 = I_A + P_1 \quad \text{and} \quad J_2 = I_A + P_2,$$

(where P_i are distinct prime ideals defining disjoint lines in $\mathbb{P}^4$) are primary decompositions of J_1 and J_2. Clearly $J_1 + J_2 \subset I_A$. On the other hand,

$$I_A = I_A \mathbb{C}[x_1, \ldots, x_4] = I_A(P_1 + P_2) \subset (I_A + P_1) + (I_A + P_2) = J_1 + J_2$$

hence $I_A = J_1 + J_2$. $\square$

Remark. To begin with, notice that this is an example where the interpretation of the Hilbert numerator is not completely straightforward. The Hilbert series multiplied by $(1-t)^5$ is $1 - 3t^2 + 2t^3 - 2t^4 + 3t^5 - t^7$ which would suggest that we need 2 more generators in degree 3. In the end it turns out that this Hilbert numerator should be written as $1 - 3t^2 - 2t^4 + 2t^3 + 3t^5 - t^7$.

This is the first case of a Pfaffian 5×5 ring of which we give a full description in this Chapter. The reason is that in this case, writing the equations in a Pfaffian format is trivial. In the cases of Chapter IV to be able to write the 5 Pfaffians we need to employ the vector bundle method.

Let us first write the equations of $\mathbb{F}(1,2)$. In coordinates of $\mathbb{P}^4[x_1, \ldots, x_5]$ these are given by

$$\operatorname{rank} \begin{bmatrix} x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{bmatrix} \leq 2. \tag{4.6}$$

Suppose that one of the quartics is given as

$$F_4 = x_1 A + x_3 B + x_4 C.$$

Consider

$$G_4 = x_2 A + x_4 B + x_5 C.$$

Since the vectors (x_1, x_3, x_4) and (x_2, x_4, x_5) are proportional on $\mathbb{F}(1,2)$, the intersection G_4 with $\mathbb{F}(1,2)$ contains C_{11}. Moreover it contains a different line from the ruling of $\mathbb{F}(1,2)$. We deduce that F_4, G_4 span the pencil of quartics through C_{11}. Now we check that the three quadrics of (4.6) and these two quartics can be written as the 5 submaximal Pfaffians of the following skew matrix

$$\begin{pmatrix} 0 & x_1 & x_3 & x_4 \\ & x_2 & x_4 & x_5 \\ & & C & -B \\ & & & A \end{pmatrix}.$$

Let us take this opportunity to give a brief preview of the type of arguments used in Chapter IV.

Proposition III.39. *Let $\mathcal{C}$ be a nonsingular curve of genus 12. Assume that $\dim W^1_4(\mathcal{C}) = 0$. Let A be a halfcanonical divisor on $\mathcal{C}$ such that $\dim H^0(A) = 5$. Then there exists a vector bundle $\mathscr{E}$ of rank 2 and determinant $-A$ with the following properties:*

(i) $\dim H^0(\mathscr{E}) = 2$.

(ii) $H^0(A) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(A))$ *is surjective.*

(iii) $H^0(2A) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(2A))$ *has a 3-dimensional cokernel.*

Proof. Let ξ be the unique g^1_4 on $\mathcal{C}$. As we have shown in previous proofs we have $h^0(A - 2\xi) = 1$. Therefore the map

$$H^0(K_{\mathcal{C}} - \xi) \otimes H^0(-\xi - A) \to H^0(A - 2\xi)$$

has a 1-dimensional cokernel. Accordingly, by Lemma III.7 on page 33, we deduce that there exists a unique nonsplit extension

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{E} \to \mathcal{O}_{\mathcal{C}}(-\xi - A) \to 0$$

with 2 global sections. This shows (i). The dimension of the cokernel of the map $H^0(A) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(A))$ can be assessed by considering the adjoint maps. In fact, it is enough to show that

$$H^0(A) \otimes H^0(\xi) \to H^0(A + \xi) \tag{4.7}$$

is surjective. By Castelnuovo's free-pencil trick, the kernel of this map is isomorphic to $H^0(A - \xi)$, whose dimension, by RR and Serre duality equals $h^0(A + \xi) - 4$. Hence the cokernel of (4.7) has dimension $2h^0(A + \xi) - 14$. All we need to show is that $h^0(A + \xi) = 7$ or equivalently, that $h^0(A - \xi) = 3$. In fact, by Proposition III.2

we have $h^0(A-\xi) \geq 3$, and if $h^0(A-\xi) \geq 4$ then A is not very ample. This is a contradiction. Thus we have showed (ii).

To show (iii) we proceed analogously. This time, again, the map

$$H^0(2A) \otimes H^0(\xi) \to H^0(2A+\xi)$$

is surjective but

$$H^0(2A) \otimes H^0(-\xi - A) \to H^0(A-\xi)$$

has a 3-dimensional cokernel. $\square$

Choose $\langle s_1, s_2 \rangle = H^0(\mathscr{E})$ and $\langle t_1, t_2, t_3 \rangle \subset H^0(\mathscr{E}(2A))$ spanning a complementary set to $H^0(2A) \otimes H^0(\mathscr{E})$ and let us write down the skew matrix:

$$\begin{pmatrix} 0 & s_1 \wedge t_1 & s_1 \wedge t_2 & s_1 \wedge t_2 \\ & s_2 \wedge t_1 & s_2 \wedge t_2 & s_2 \wedge t_2 \\ & & t_1 \wedge t_2 & t_1 \wedge t_3 \\ & & & t_2 \wedge t_3 \end{pmatrix} \tag{4.8}$$

The zero entry comes from the fact that $s_1 \wedge s_2 = 0$ since these sections span a subbundle of $\mathscr{E}$. At every point of $\mathcal{C}$ this matrix has rank 2 since $\mathscr{E}$ is a vector bundle of rank 2. Moreover, the entries of (4.8) are elements of $R(\mathcal{C}, A)$. Hence its 5 submaximal Pfaffians represent relations between these elements. There are 3 quadric relations and 2 quartic relations. For complete details we refer the reader to Chapter IV. Let us only mention that there are still some details to be checked. For example that the 3 quadrics obtained in this way cut out $\mathbb{F}(1,2) \subset \mathbb{P}^4[H^0(A)]$. This is equivalent to proving that each of these Pfaffians is not a trivial relation, which is a consequence of the fact that the intersection $\langle t_1, t_2, t_3 \rangle \cap H^0(2A) \otimes H^0(\mathscr{E})$ is empty.

Genus 13 and $h^0(A) = 5$.

Proposition III.40. *Let $\mathcal{C}$ be a nonsingular curve of genus 13 with a halfcanonical divisor A such that $\dim H^0(A) = 5$. Then $\mathcal{C}$ has no g^2_8 if and only if sym^2 is surjective and if either is true then φ_A is a birational morphism.*

Proof. Assume that sym^2 is surjective. Then $|A|$ is necessarily free and using Proposition III.2 we see that $\mathcal{C}$ cannot be hyperelliptic. In fact we will show next that $\mathrm{gon}(\mathcal{C}) \geq 5$. First, let us prove a lemma which will be used also in the next chapter.

Lemma III.41. *Let $\mathcal{C}$ be a nonsingular curve and A a halfcanonical divisor for which* sym^2 *is surjective. Let D be an effective divisor on $\mathcal{C}$ and denote by d the dimension of its linear span in $\mathbb{P}[H^0(A)]$. Then we have the following inequality:*

$$\deg(D) - h^0(D) \leq \frac{1}{2}(d+2)(d+1) - 1$$

Proof. The number $\deg(D) - h^0(D)$ is the dimension of the linear span of D in canonical space, $\mathbb{P}(H^0(K_C))$. The number $\frac{1}{2}(d+1)d - 1$ is the dimension of the image of linear span of D in $\mathbb{P}[H^0(A)]$ by the second Veronese map. $\square$

We use this lemma to show that $\operatorname{gon}(\mathcal{C})$ cannot equal 3. For otherwise, assume that D is a divisor of degree 3 and with $h^0(D) = 2$. Then $h^0(A-D) \geq 4$ by Proposition III.2. Hence

$$\dim H^0(A) - \dim H^0(A-D) - 1 \leq 0.$$

From Lemma III.41 we deduce that $\deg(D) - h^0(D) \leq 0$ which implies that $\deg(D) \leq 2$. This is a contradiction. We conclude that $\operatorname{gon}(\mathcal{C}) \geq 4$.

Let us now show that $\operatorname{gon}(\mathcal{C}) \geq 5$. Consider a free g^1_4 on $\mathcal{C}$ and denote it by ξ. By Proposition III.2 we have $h^0(A-2\xi) \geq 1$. Suppose that $h^0(A-2\xi) = 1$. Denote $H^0(\xi)$ by $\langle s_1, s_2\rangle$ and $H^0(D)$ by $\langle t_1, t_2, t_3\rangle$. Then

$$H^0(\xi) \otimes H^0(D) \to H^0(A)$$

produces the following quadrics:

$$\operatorname{rank}\begin{bmatrix} s_1t_1 & s_1t_2 & s_1t_3 \\ s_2t_1 & s_2t_2 & s_2t_3 \end{bmatrix} \leq 1. \tag{4.9}$$

Since $h^0(A-2\xi) = 1$ we deduce that the kernel of sym^2 is at least 3-dimensional and hence sym^2 is not surjective. In fact we can see from (4.9) that $\varphi_A(\mathcal{C})$ is contained in the cubic scroll $\mathbb{F}(1,2)$. If $h^0(A-2\xi) = 2$ then $|A-2\xi|$ is a free g^1_4 on $\mathcal{C}$. Let us write $A = 2\xi + \xi'$ with ξ' a complete free g^1_4 (recall $\mathcal{C}$ is not trigonal). But now reverse the roles of ξ and ξ' to obtain $2\xi + \xi' = 2\xi' + \xi \implies \xi = \xi'$. Therefore $A = 3\xi$. In this situation we conclude that the projection from a point of $\mathbb{P}[H^0(A)]$ of the image of $\mathcal{C}$ by φ_A is a rational normal curve of degree 3. In other words, $\varphi_A(\mathcal{C})$ is contained in $\mathbb{F}(0,3)$, the cone over a rational normal curve of degree 3. But then sym^2 cannot be surjective. We conclude that $\operatorname{gon}(\mathcal{C}) \geq 5$.

Let us now show that there are no g^2_8 on $\mathcal{C}$. Let D be a divisor of degree ≤ 8 and $h^0(D) \geq 3$. By Proposition III.2, the linear system $|A - D|$ is effective. Hence we can apply Lemma III.41. Since

$$\dim H^0(A) - \dim H^0(A - (A - D)) - 1 \leq 1$$

we have $\deg(A - D) - h^0(A - D) \leq 2$. This implies that $h^0(A - D) \geq 2$. Therefore $\deg(D) = 8$ and the linear system $|A - D|$ is a free g^1_4. This is not possible.

We have shown

$$\mathrm{sym}^2 \text{ surjective} \implies \mathcal{C} \text{ has no } g^2_8$$

Assume now that $\mathcal{C}$ has no g^2_8. Clearly if there exists a g^1_4, then a suitable subsystem of $2g^1_4$ is a g^2_8. Hence we conclude that $\mathrm{gon}(\mathcal{C}) \geq 4$.

Let us describe the map φ_A. Let $B \subset |A|$ be the free part of $|A|$. The linear system $|B|$ yields of morphism of $\mathcal{C}$ onto a curve C in $\mathbb{P}^4$ of degree ≤ 12. The degree of φ_B is ≥ 3. If $\deg(\varphi_B) = 3$ then C is a rational normal curve of degree 4 and consequently $\mathcal{C}$ is trigonal, which is not true. If $\deg(\varphi_B) = 2$ then $\deg(C) \leq 6$ and a projection off a secant line to $C \subset \mathbb{P}^4$ composed with φ_B is a linear system of degree ≤ 8 and dimension 2, which we are assuming not to exist. Hence $\deg(\varphi_B) = 1$. In this situation the genus of C is 13. If $\deg(C) < 12$ then from Castelnuovo's bound (see page 44), the genus of $\varphi_B(\mathcal{C})$ is ≤ 12. We deduce that $B = A$ and φ_A is a birational morphism.

Let us now show that sym^2 is surjective. We argue by contradiction. Assume that sym^2 is not surjective. Choose 3 quadrics in the system of quadrics through $C \subset \mathbb{P}^4$, and denote their intersection by S. If $\dim S \leq 1$, then S is a scheme-theoretic complete intersection and by Bézout's theorem $\deg(C) \leq 8$. But by our previous discussion $\deg(C) = 12$. Therefore we can assume that $\dim S = 2$ and hence the component S_0 of S to which C belongs is 2-dimensional. The surface S_0 is contained in two linearly independent quadrics and accordingly has degree ≤ 4. If the degree is 4 then using Unmixedness and Bézout's theorem we deduce that S_0 is a complete intersection of two quadrics. But then it cannot be contained in an extra quadric as we are assuming. Thus $\deg(S_0) \leq 3$. Having minimal degree S_0 can only be either $\mathbb{F}(0,3)$ or $\mathbb{F}(1,2)$. In the first instance projecting from the vertex would yield a linear system of degree ≤ 4 on $\mathcal{C}$. This is a contradiction.

Finally suppose that $C \subset \mathbb{F}(1,2) \subset \mathbb{P}^4$. Let us write H for the hyperplane section of $\mathbb{F}(1,2)$ and L for the class of the ruling of $\mathbb{F}(1,2)$. Then there exist integers a, b such that $C = aH + bL$. With $3a + b = \deg(C) = 12$. Additionally, since $K_{\mathbb{F}} = -2H + L$ we have

$$24 \leq 2p_a(C) - 2 = K_{\mathbb{F}}C + C^2 = -5a - 2b + 3a^2 + 2ab.$$

substituting $b = 12 - 3a$ we get the following quadratic inequality:

$$3a^2 - 25a + 48 \leq 0$$

which has integers solutions $a = 3, 4$ and 5. If $a = 3$ then $b = 3$ and therefore there exists a g^1_3 which is not true. If $a = 4$ or $a = 5$ then $p_a(\mathcal{C}) = 15$ or 22 respectively. We deduce that $\mathcal{C}$ is singular at least at two distinct points. Projecting from a secant line at these points we obtain a g^2_8. We conclude that sym^2 must be surjective. $\square$

Theorem III.42. *Let $\mathcal{C}$ be a nonsingular curve of genus* 13*. Assume that $\mathcal{C}$ has no g^2_8. Then $\mathcal{C}$ has a halfcanonical divisor A with* $\dim H^0(A) = 5$ *if and only if $\mathcal{C}$ is isomorphic to a complete intersection of $C_{2,2,3} \subset \mathbb{P}^4$.*

Proof. Let $C_{2,2,3} \subset \mathbb{P}^4$ be a nonsingular complete intersection. By adjunction, $C = C_{2,2,3}$ has a halfcanonical divisor $A = \mathcal{O}(1)_{|C}$ with $\dim H^0(A) = 5$. Additionally sym^2 is surjective.

Conversely assume that $\mathcal{C}$ is a nonsingular curve of genus 13 with no g^2_8 and that $\mathcal{C}$ has a halfcanonical divisor A with $\dim H^0(A) = 5$. By Proposition III.40 we know that sym^2 is surjective and $|A|$ is a free linear system. Hence from Proposition III.5 we deduce that $R(\mathcal{C}, A)$ is generated in degree ≤ 3. We need to show that

$$H^0(A) \otimes H^0(2A) \to H^0(3A) \tag{4.10}$$

is surjective. The argument we use here is a refinement of the proof of Proposition III.5. By Lemma III.7 an element of the cokernel of (4.10) corresponds to an extension

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A) \to 0$$

with 6 global sections. Let p, q be two general points of $\mathcal{C}$ and denote by $\delta \supset p + q$ the divisor of zeros of a section of $\mathscr{F}$ vanishing at p and q. Let $\mathcal{O}_{\mathcal{C}}(\xi)$ be the saturation of $\mathcal{O}_{\mathcal{C}}(\delta) \hookrightarrow \mathscr{F}$. We have

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A-\xi) \to 0.$$

Since $p+q \subset \delta \subset \xi$ and p, q are general we deduce that $h^0(A-\xi) \leq 3$. Since $h^0(\mathscr{F}) \leq h^0(\xi) + h^0(A-\xi)$ we deduce that $h^0(\xi) \geq 3$. Since $\mathcal{C}$ has no g^2_8 this implies that $\deg(\xi) \geq 9$ and accordingly $\deg(A-\xi) \leq 3$. But then we must have $h^0(A-\xi) \leq 1$ and consequently $h^0(\xi) \geq 5$. Since $|A|$ is free, we deduce that $\xi = A$. In other words $\mathscr{F}$ is split. This shows that (4.10) is surjective.

Consequently $R(\mathcal{C}, A)$ is generated in degree 1. Since sym^2 is surjective the ideal I_A has two generators in degree 2. Let us denote them by Q_1 and Q_2. Since there are no linear syzygies holding between these quadratic generators a numerical argument shows that there exists a new cubic generator that we denote by F_3.

By the argument of the proof of Proposition III.40 the two quadrics Q_1, Q_2 cut out in $\mathbb{P}^4$ a reduced surface of degree 4. By Unmixedness and Bézout's theorem this intersection is indeed complete. Since F_3 is not a multiple of $\langle Q_1, Q_2 \rangle$ it cuts out on this surface a subscheme of dimension 1. We have $I = (Q_1, Q_2, F) \subset I_A$. By Unmixedness, I has no embedded primes. By Bézout's theorem I has a single minimal prime with multiplicity one. In other words $I = I_A$. $\square$

CHAPTER IV

Pfaffian 5×5 halfcanonical rings

In the context of Buchsbaum–Eisenbud's theorem a Pfaffian 5×5 ring is a the quotient of a polynomial ring by a homogeneous Gorenstein ideal of codimension 3 generated by the five submaximal Pfaffians of a skew matrix with entries in the polynomial ring. Recall from Table III.2 on page 42 that there are four (plus one) pairs $(\mathcal{C}, A)$ of a curve and a halfcanonical divisor for which the ring $R(\mathcal{C}, A)$, for a general pair, is expected to be a Pfaffian 5×5 ring. The "plus one" accounts for the case $\mathrm{g}(\mathcal{C}) = 12$ and $h^0(A) = 5$ which was already studied in the previous chapter (page 63). In this chapter we study the cases

$$\begin{aligned} \mathrm{g}(\mathcal{C}) &= 8 \quad \dim H^0(A) = 3 \\ \mathrm{g}(\mathcal{C}) &= 14 \quad \dim H^0(A) = 5. \end{aligned}$$

We prove that under some suitable generality assumptions on the pair $(\mathcal{C}, A)$, each of these is a complete intersection in a generalised weighted Grassmannian. We summarise the results of this chapter in the next table. The notation for the key varieties is taken from Section IV.1. Additionally, F_d denotes a general quasihomogeneous form of degree d in the variables of the ambient space.

$\mathrm{g}(\mathcal{C}) - 1$	$h^0(A)$	X, the key variety	Defining forms
7	3	$\mathbb{G}(\frac{1}{2}^3, \frac{1}{2}^2)$	$\bigcap_{i=1}^4 F_2^i \cap F_3$
13	5	$\mathbb{P}(1) \ltimes \mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$	$\bigcap_{i=1}^4 F_2^i \cap \bigcap_{j=1}^2 F_1^j$

Table IV.1. Sections of key varieties

Implicit in Table IV.1 is the notion of a complete intersection in a arithmetically Cohen–Macaulay scheme. We shall see below (Section IV.1) that generalised weighted Grassmannians are by definition automatically projectively Gorenstein schemes. In particular their homogeneous rings are Cohen–Macaulay and hence they are arithmetically Cohen–Macaulay.

Definition IV.1 (Complete intersection). Consider $S \subset X \subset \mathbb{P}$ with S a subscheme of an arithmetically Cohen–Macaulay scheme X in (weighted) projective space $\mathbb{P} = \operatorname{Proj} \mathbb{C}[x_i]$. Let $I(X)$ and $I(S)$ be the homogeneous ideals of X and S in $\mathbb{P}$. We say that S is a complete intersection in X if there exist $f_1, \dots, f_n \in \mathbb{C}[x_i]$ quasihomogeneous forms such that

$$n = \dim X - \dim S \quad \text{and} \quad I(S) = (f_1, \dots, f_n) + I(X)$$

In particular, since X is arithmetically Cohen–Macaulay $(f_i + I(X))$ is a regular sequence in $\mathbb{C}[x_i]/I(X)$ and therefore (f_i) is a regular sequence in $\mathbb{C}[x_i]$.

In this chapter we will see how we can use generalised Grassmannians and the notion of complete intersection to derive the structure of the ideal I_A associated to a halfcanonical divisor and to a choice of generators of the ring $R(\mathcal{C}, A)$.

The following result plays has a very important role. We have used it with $X = \mathbb{P}$ several times in the previous chapter. There are two reasons for only proving this result here. In the first place, taking a complete intersection in a weighted projective space is a more common operation than taking a complete intersection in an arithmetically Cohen–Macaulay scheme. Secondly we will only need the level of generality of its statement in this chapter.

Proposition IV.2. *Consider $S \subset X \subset \mathbb{P}$ with S an irreducible reduced variety, X an arithmetically Cohen–Macaulay subscheme and $\mathbb{P}$ weighted projective space. Let $\mathbb{C}[x_i]$ denote the homogeneous ring of $\mathbb{P}$ and $I(X)$, $I(S)$ the homogeneous ideals of X and S, respectively. Let $J \subset \mathbb{C}[x_i]$ be an ideal generated by a sequence $f_1, \dots, f_t$ of t quasihomogeneous forms in $\mathbb{C}[x_i]$. Suppose that*

(i) $\dim S = \dim(\mathrm{Z}(J) \cap X) = \dim X - t$

(ii) $S \subset \mathrm{Z}(J) \cap X$

(iii) $\deg(S) = \deg(X) \prod_{i=1}^{t} \deg(f_i)$

Then $I(S) = J + I(X)$.

Proof. The proof consists of three steps. In step 1 we show that $J + I(X)$ has no embedded primes. In step 2 we show that $\deg(J + I(X)) = \deg(X) \prod_{i=1}^{t} \deg(f_i)$ and finally in step 3 we show that $J + I(X)$ is primary and $\operatorname{Rad}(J + I(X)) = I(S)$. Then the proof follows from Proposition II.17 on page 26.

Step 1. Let $\widehat{J} \subset \mathbb{C}[x_i]/I(X)$ denote the ideal $(J + I(X))/I(X)$. Since we are assuming that $\dim(\mathrm{Z}(J) \cap X) = \dim X - t$, the ideal $\widehat{J}$ as codimension t. Moreover it is generated by the t elements $f_i + I(X)$. Since $\mathbb{C}[x_i]/I(X)$ is Cohen–Macaulay, by Unmixedness (Theorem II.16), $\widehat{J}$ has no embedded primes. Let $\pi\colon \mathbb{C}[x_i] \to \mathbb{C}[x_i]/I(X)$ be the quotient morphism. Consider a homogeneous primary decomposition of $\widehat{J}$

$$\widehat{J} = \widehat{P_1} \cap \cdots \cap \widehat{P_k}$$

Since $\widehat{J}$ has no embedded primes each $\operatorname{Rad} \widehat{P_i}$ is minimal over $\widehat{J}$. Denote the primary ideal $\pi^{-1}(\widehat{P_i})$ by P_i. Then

$$J + I(X) = P_1 \cap \cdots \cap P_k$$

is a homogeneous primary decomposition and for each i the prime ideal $\operatorname{Rad} P_i$ is minimal over $J + I(X)$. Therefore $J + I(X)$ has no embedded primes.

Step 2. By definition of degree of a homogeneous ideal

$$\deg(J + I(X)) = \deg \mathbb{C}[x_i]/(J + I(X)).$$

Denote by M_i the graded $\mathbb{C}[x_i]$ module $\mathbb{C}[x_i]/((f_1, \ldots, f_i) + I(X))$. Since $(f_i + I(X))$ is a regular sequence in $\mathbb{C}[x_i]/I(X)$ (Theorem II.15) we have for each integer i the exact sequence

$$0 \to M_i(-\deg(f_{i+1})) \to M_i \to M_{i+1} \to 0.$$

We deduce that $\deg(M_{i+1}) = \deg(f_i) \deg(M_i)$. By induction we get

$$\deg(I + I(X)) = \deg(X) \prod_{1=1}^{t} \deg(f_i).$$

Step 3. Let $J + I(X) = P_1 \cap \cdots \cap P_k$ be a primary decomposition. We have shown that each $\operatorname{Rad} P_i$ is a minimal prime. We are assuming that $S \subset \mathrm{Z}(J) \cap X$ and that S is irreducible. Therefore for some integer i, which we can assume to be k, $\operatorname{Rad} P_i = \operatorname{Rad} I(S) = I(S)$, as S is reduced. Since $J + I(X)$ has no embedded primes and for distinct primary components P_i we have distinct minimal primes $\operatorname{Rad}(P_i)$ one shows using induction on the exact sequence

$$0 \to \mathbb{C}[x_i]/(P_1 \cap P_2) \to \mathbb{C}[x_i]/P_1 \oplus \mathbb{C}[x_i]/P_2 \to \mathbb{C}[x_i]/(P_1 + P_2) \to 0$$

that

$$\deg(J + I(X)) = \deg(P_1) + \cdots + \deg(P_k).$$

From this we obtain

$$\deg(X)\prod_{i=1}^{y}\deg(f_1) \leq \deg(P_k) \leq \deg(S) = \deg(X)\prod_{i=1}^{y}\deg(f_i).$$

Therefore $P_1 = \cdots = P_{k-1} = 0$ and we conclude that $J + I(X)$ is primary. Additionally $\mathrm{Rad}(J + I(X)) = I(S)$. The result now follows from Proposition II.17 on page 26. $\square$

Throughout this chapter we deliberately do not apply Buchsbaum–Eisenbud's Theorem to the ring $R(\mathcal{C}, A)$ for each the cases considered. If we want to apply Buchsbaum–Eisenbud's theorem, besides having to assume Gorensteiness of $R(\mathcal{C}, A)$ from start, we would still have to analyse to which extent masking can occur in each case. Given a Hilbert numerator indicating a 5×5 Pfaffian format, *a priori*, there is no reason for no occurrence of masking of a 7×7 format. It should be interesting to find an example where this actually happens.

IV.1. Weighted Grassmannians

The variety of subspaces of dimension 2 of a fixed vector space of dimension 5, $\mathrm{G}(2,5)$ in its Plücker embedding, provides a key example of the structure theorem for a Gorenstein ideal of codimension 3 of Buchsbaum–Eisenbud [**BE**] — Theorem I.1. From this algebraic point of view $\mathrm{G}(2,5) \subset \mathbb{P}^9$ is the zero locus of

$$\mathrm{Pf}\begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ & x_{23} & x_{24} & x_{25} \\ & & x_{34} & x_{35} \\ & & & x_{45} \end{pmatrix} = \begin{pmatrix} \mathrm{Pf}_1 \\ \vdots \\ \mathrm{Pf}_5 \end{pmatrix}. \tag{1.1}$$

The matrix M above and its Pfaffians, Pf_i, is all we need to write down the free resolution of ideal generated by the five submaximal Pfaffians of M over $A = \mathbb{C}[x_{ij}]$:

$$A \xleftarrow{(\mathrm{Pf}_i)} 5A(-2) \xleftarrow{M} 5A(-3) \xleftarrow{(\mathrm{Pf}_i)^t} A(-5) \leftarrow 0. \tag{1.2}$$

The ordinary $G(2,5)$ corresponds to the choice of weights, $\mathrm{wt}(x_{ij}) = 1$. Making the weighting of the variables of A arbitrary, in a way that the Pfaffians remain homogeneous, we can generalise this construction to a weighted Grassmannian.

Proposition IV.3. *Let w_{ij} denote the weight of the variable x_{ij}. Then the submaximal Pfaffians of M are homogeneous with respect to this grading if and only if there exist $c_1, \ldots, c_5 \in \frac{1}{2}\mathbb{Z}$ such that $c_i + c_j$ is an integer and $w_{ij} = c_i + c_j$.*

Proof. Clearly a grading originating from such a choice leaves the Pfaffians homogeneous. Conversely, let $\mathrm{wt}(x_{ij}) = w_{ij}$ be a weighting that makes the Pfaffians homogeneous. Let us make the following extra notation: let b_i denote the degree of the i-th Pfaffian, Pf_i and w_{ji} for $i < j$ the 'weight' of the variable $-x_{ij}$, which is w_{ij}. The Pfaffian tautology,

$$\begin{pmatrix} x_{12} & \cdots & x_{15} \\ & \ddots & \vdots \\ & & x_{45} \end{pmatrix} \begin{pmatrix} \mathrm{Pf}_1 \\ \vdots \\ \mathrm{Pf}_5 \end{pmatrix} = 0$$

returns the following relations:

$$w_{il} + b_l = w_{ik} + b_k \text{ for any } k \neq j.$$

This means that $w_{il} - w_{ij}$ is independent of the choice of i. Suppose that (i, k, l) and (i, r, t) are two distinct ordered triples of integers in $\{1, \ldots, 5\}$. Then

$$w_{ik} - w_{kl} + w_{li} = w_{ir} - w_{rl} + w_{li} = w_{ir} - w_{rt} + w_{ti}.$$

Therefore the half-integers given by

$$c_i = \frac{w_{ik} - w_{kl} + w_{li}}{2}$$

for any ordered triple (i, k, l) of distinct integers of $\{1, \ldots, 5\}$, are well-defined. Then,

$$c_i + c_j = \frac{w_{ij} - w_{jl} + w_{li} + w_{ji} - w_{il} + w_{lj}}{2} = w_{ij}. \quad \square$$

We will denote $\sum_i c_i$ by k. A resolution of the Pfaffian ideal is readily at hand:

$$A \xleftarrow{(\mathrm{Pf}_i)} \bigoplus_{i=1}^{5} A(c_i - k) \xleftarrow{M} \bigoplus_{i=1}^{5} A(-c_i - k) \xleftarrow{(\mathrm{Pf}_i)^t} A(-2k) \leftarrow 0 \tag{1.3}$$

with Hilbert series:

$$\frac{1 - \sum_i t^{k-c_i} + \sum_i t^{k+c_i} - t^{2k}}{\prod_{i<j}(1 - t^{c_i+c_j})} \tag{1.4}$$

Definition IV.4. Let $(c_1, \ldots, c_5)$ be a collection of elements of $\frac{1}{2}\mathbb{Z}$. A *weighted Grassmannian* $\mathbb{G}(c_1, \ldots, c_5)$ is the projectively Gorenstein codimension 3 subscheme of $\mathbb{P}$ defined by the Pfaffian ideal of M for a choice of weights as in Proposition IV.3. (See [**CR**] for an equivalent definition).

Proposition IV.5. *Let* $\mathbb{G}$ *be a weighted Grassmannian for some choice of weights* $(c_1, \ldots, c_5)$. *Let* k *be the integer* $\sum_i c_i$. *Then*

(i) $\omega_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}}(-2k)$
(ii) $\deg \mathbb{G} = \frac{\sum \binom{k-c_i}{3} - \sum \binom{k+c_i}{3} + \binom{2k}{3}}{\prod_{i<j}(c_i+c_j)}$
(iii) $H^i(\mathcal{O}_{\mathbb{G}}(j)) = 0$ *for all* $0 < i < 6$.

Proof.

(i) We have $\omega_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-4k)$ and $2k$ is the adjunction number.
(ii) This follows from Corollary II.13.
(iii) We prove this using the fact that $\mathbb{G}$ is Cohen-Macaulay (hence the length of (1.3) equals the codimension of $\mathbb{G}$ in $\mathbb{P}$) and vanishing of $H^i(\mathcal{O}_{\mathbb{P}}(j))$ for $0 < i < 9$. □

Example IV.6. Let $\mathbb{G}$ be the weighted Grassmannian corresponding to the choice of weights:

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{pmatrix} \tag{1.5}$$

Proposition IV.7. *Denote by* x_{ij} *for* $1 \leq i < j \leq 4$ *the variables of weight* 1 *and by* y_i *the variables of weight* 2. *Then* $\mathbb{G}$ *is nonsingular away from* $\mathbb{P}(y_1, \ldots, y_4)$, *where it has a singularities of type* $\frac{1}{2}(1,1,1,0,0,0)$.

Proof. Away from $\mathbb{P}[y_1, \ldots, y_4]$ where, say for simplicity $x_{12} \neq 0$, and among the equations of $\mathbb{G}$ in $\mathbb{P}$ we can find $x_{12}x_{34} = \ldots$, $x_{12}y_3 = \ldots$, $x_{12}y_4 = \ldots$, for which we can apply the implicit function theorem. This means that the affine cone of $\mathbb{G}$ is smooth on the neighbourhood of $x_{12} = 1$ and hence $\mathbb{G}$ is also smooth. If we are at a point of $\mathbb{P}[y_1, \ldots, y_4]$ then in the same way, using the implicit function theorem we can eliminate three of the weight 1 coordinates and the affine cone will also be smooth. In this case, since the $\mathbb{C}^*$ action has a stabiliser we find at the corresponding point of $\mathbb{G}$ a singularity of type $\frac{1}{2}(1,1,1,0,0,0)$ where the 1 account for the remaining coordinates of weight 1. Alternatively one can simply say that since $\mathbb{G}$ and $G(2,5)$ have the same affine cone, the variety $\mathbb{G}$ is quasismooth and therefore it is nonsingular where the $\mathbb{C}^*$ action has trivial stabiliser. $\square$

IV.1.1. Generalised weighted Grassmannians. The theorem of Buchsbaum–Eisenbud, characterising Gorenstein homogeneous ideals of codimension 3 does not say that we should take all entries of the given skew matrix algebraically independent (as in the case of a weighted Grassmannian). In fact we take care of this detail by considering quasilinear sections of weighted Grassmannians. Unfortunately this still does not give all the possibilities for the entries of the skew matrix. Some generators of the ring may only appear as implicit functions in the entries of the matrix. Geometrically this corresponds to a hyperplane section in a cone variety. The constructions below of generalised weighted Grassmannians still give Gorenstein ideals of codimension 3.

Algebraically the definition of these varieties could not be simpler. Consider an injection of degree 0 of $\mathbb{C}[x_{ij}]$ into a graded ring $A = \mathbb{C}[\mathbf{x}]$. We only need notation for the weights of the variables x_{ij}: consider a collection of half-integers $(c_1, \dots, c_5)$; then

$$\mathrm{wt}(x_{ij}) = c_i + c_j.$$

This is a necessary and sufficient condition for the homogeneity of the submaximal Pfaffians of the following skew matrix (see Proposition IV.3):

$$M = \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ & x_{23} & x_{24} & x_{25} \\ & & x_{34} & x_{35} \\ & & & x_{45} \end{pmatrix} \tag{1.6}$$

The subscheme of $\mathbb{P}[\mathbf{x}] = \operatorname{Proj} A$ defined in this way is a projectively Gorenstein scheme of codimension 3.

There is a more geometrical definition of these varieties. Recall that $\mathrm{aG}(2,5)$, the affine cone over the Grassmannian $\mathrm{G}(2,5)$, is the affine subscheme of 10-dimensional affine space, $\mathbb{A}^{10}(x_{ij})$ with $1 \le i < j \le 5$, defined by the submaximal Pfaffians of the skew matrix (1.6). Let $\mathbb{A}[x_1, \dots x_n]$ be ordinary affine k-space. We use the symbol $\ltimes$ in the expression

$$\mathbb{A}^n \ltimes \mathrm{aG}(2,5)$$

to mean: make the cone over $\mathrm{aG}(2,5)$ with vertex at $\mathbb{A}[x_1, \dots, x_n]$. More precisely take the join of $\mathbb{A}[x_1, \dots, x_n]$ and $\mathrm{aG}(2,5)$ in $\mathbb{A}[x_1, \dots, x_n, x_{ij}]$.

DEFINITION IV.8. Let $(b_1, \dots, b_n)$ be a n-tuple of integers and $(c_1, \dots, c_5)$ a collection of five half-integers. With the notation introduced above, define the *generalised weighted Grassmannian* of weights $(b_1, \dots, b_n; c_1, \dots, c_5)$ to be the quotient

$$(\mathbb{A}^n \ltimes \mathrm{aG}(2,5) \backslash 0)/\mathbb{C}^*$$

where $\mathbb{C}^*$ acts by

$$x_l \mapsto \lambda^{b_l} x_l \text{ and } x_{ij} \mapsto \lambda^{c_i + c_j} x_{ij}$$

for $1 \le l \le n$ and $1 \le i < j \le 5$. We denote it by

$$\mathbb{P}(b_1, \dots, b_n) \ltimes \mathbb{G}(c_1, \dots, c_5).$$

The following proposition is very similar to Proposition IV.5. For this reason we keep the proof to a minimum.

PROPOSITION IV.9. *Let $\mathbb{G}$ be the generalised weighted Grassmannian $\mathbb{P}(b_1, \dots, b_n) \ltimes \mathbb{G}(c_1, \dots, c_5)$. Denote $\sum_{1=1}^5 c_i$ by k. Then*

(i) *$\mathbb{G}$ is a $(n+6)$-dimensional variety;*

(ii) *$\omega_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}}(-2k - \sum_i b_i)$;*

(iii) *$\deg \mathbb{G} = \frac{\sum \binom{k-c_i}{3} - \sum \binom{k+c_i}{3} + \binom{2k}{3}}{\prod_l b_l \cdot \prod_{i<j} (c_i + c_j)}$;*

(iv) *$H^i(\mathcal{O}_{\mathbb{G}}(j)) = 0$ for all $0 < i < n+6$.*

Proof. Item (i) is clear from the geometric definition or the algebraic definition of generalised weighted Grassmannian. Item (ii) comes straight out of the projective resolution of the ideal of Pfaffians:

$$A \xleftarrow{(\mathrm{Pf}_i)} \bigoplus_{i=1}^{5} A(c_i - k) \xleftarrow{M} \bigoplus_{i=1}^{5} A(-c_i - k) \xleftarrow{(\mathrm{Pf}_i)^t} A(-2k) \leftarrow 0 \tag{1.7}$$

where $A = \mathbb{C}[x_1, \dots x_n, x_{ij}]$ with $\mathrm{wt}(x_i) = b_i$ and $\mathrm{wt}(x_{ij}) = c_i + c_j$. The dualising module of $\mathbb{P}$ is $\mathcal{O}_{\mathbb{P}}(-4k - \sum_i b_i)$ and by Gorenstein adjunction $\omega_{\mathbb{G}} = \mathcal{O}(-2k - \sum_i b_i)$. Items (iii) and (iv) can be proved as in Proposition IV.5. $\square$

IV.2. The vector bundle method

Vector bundles played an important part in the preliminary results of Chapter III relating the gonality of a curve with a halfcanonical divisor and the halfcanonical ring $R(\mathcal{C}, A)$. We are referring in particular to Proposition III.5. Vector bundles appear in the study of special linear systems on algebraic curves in several places. For an overview see [**La87**].

However the vector bundle method we wish to describe here is of a complete different nature. The vector bundle method was used consistently in work of Mukai of classification of Gorenstein Fano 3-folds. In a long series of articles [**M89, M88, M93, M95c, M95b, M95a, M03**] Mukai describes a method of recovering the classical constructions of Fano 3-folds of coindex 3 by the use of a suitable vector bundle. If V is a nonsingular algebraic variety of dimension 3 whose anticanonical linear system $|-K_V|$ is ample, V is said to be a Fano 3-fold. For a Fano 3-fold of coindex 3 we define its genus to be the integer $1 - \frac{1}{2}(K_V^3)$, which is the genus of a generic linear curve section. We say that V is indecomposable if the anticanonical linear system is indecomposable. The following theorem is taken from [**M03**].

Theorem (Mukai). *Let V be an indecomposable Fano 3-fold with at most Gorenstein canonical singularities*[*]. *Then:*

(i) $\mathrm{g} \leq 10$ *or* $\mathrm{g} = 12$. *(Iskovskikh's genus bound in the smooth case).*

(ii)
$$
\begin{array}{ll}
\mathrm{g} = 2 & V = (6) \subset \mathbb{P}(1^3, 2, 3) \\
\mathrm{g} = 3 & V_4 \subset \mathbb{P}^4 \quad \textit{or} \\
 & V \xrightarrow{2:1} Q^3_{\mathrm{rank}\,5} \subset \mathbb{P}^4 \\
\mathrm{g} = 4 & V = (2) \cap (3) \subset \mathbb{P}^5 \\
\mathrm{g} = 5 & V = (2) \cap (2) \cap (2) \subset \mathbb{P}^6 \\
\mathrm{g} = 6 & V = \mathrm{G}(2,5) \cap H_1 \cap H_2 \cap Q \quad \textit{or} \\
 & V \xrightarrow{2:1} V_5 \subset \mathbb{P}^6,\ \textit{where}\ V_5 = \mathrm{G}(2,5) \cap H_1 \cap H_2 \cap H_3
\end{array}
$$

---

[*]See [**R87**] for a definition of Gorenstein canonical singularities.

$$
\begin{array}{ll}
\mathrm{g}=7 & V=\Sigma^{10}_{12}\cap H_1\cap\cdots\cap H_7 \textit{ where } \Sigma^{10}_{12}=\mathrm{OG}(5,10)\\
\mathrm{g}=8 & V=\mathrm{G}(2,6)\cap H_1\cap\cdots\cap H_5\\
\mathrm{g}=9 & V=\Sigma^{6}_{10}\cap H_1\cap H_2\cap H_3 \textit{ where } \Sigma^{6}_{10}=\mathrm{SpG}(3,6)\\
\mathrm{g}=10 & V=\Sigma^{5}_{18}\cap H_1\cap H_2 \textit{ where } \Sigma^{5}_{18}\subset\mathrm{G}(5,7)\\
\mathrm{g}=12 & V \textit{ is smooth and } V\simeq\mathrm{G}(3,7,N)\subset\mathrm{G}(3,7) \textit{ where}\\
 & N=\langle b_1,b_2,b_3\rangle\subset\bigwedge^2\mathbb{C}^7 \textit{ is a net of bivectors.}
\end{array}
$$

Mukai refers to (ii) of this theorem as the "linear section theorem". It states that a Fano 3-fold is a *linear* section of a Fano manifold of higher coindex. For genus $6,7,8,9,10$ and 12 the *key* Fano n-folds Σ appear as subvarieties of a Grassmannian or broadly speaking of a *homogeneous space*. To recover V as an embedded variety Mukai construct a vector bundle of the corresponding rank, determinant, space of global sections, etc.; and uses it to set up a map on V yielding an embedding into the correspondent homogeneous space.

In his own words [**M03**] Mukai describes the vector bundle method in a sequence of five steps:

(1) Construct the vector bundle on $S\in|-K_V|$.
(2) Extend the vector bundle on S to a vector bundle on X.
(3) Embed V into the corresponding homogeneous space.
(4) Regard the image as a subvariety of one of Σ in the Theorem.
(5) Prove that the image of V under this embedding $=\Sigma\cap H_1\cap\cdots\cap H_k$.

Let us illustrate Mukai's vector bundle method with one of Mukai's results. (See [**M93**]).

Theorem. *A curve $\mathcal{C}$ of genus 8 is a transversal linear section of the* 8*-dimensional Grassmannian* $\mathrm{G}(2,6)\subset\mathbb{P}^{14}$ *if and only if C has no g^2_7.*

The *only if* of the statement is checked directly by geometrical inspection of $\mathrm{G}(2,6)$. To prove that a curve $\mathcal{C}$ of genus 8 is a transversal linear section of $\mathrm{G}(2,6)$ Mukai constructs a bundle $\mathscr{E}$ on $\mathcal{C}$ of rank 2 and determinant $K_{\mathcal{C}}$ with 6-dimensional space of global sections. In this case one can show that a bundle satisfying these requirements is unique.

Lemma 1. *Suppose that* $\mathcal{C}$ *is nonsingular curve of genus* 8*. Assume that* $W^2_7(\mathcal{C}) = \emptyset$*. Then when* $\mathscr{E}$ *ranges over all stable bundles of rank* 2 *and canonical determinant the maximum value of* $\dim H^0(\mathscr{E})$ *is* 6 *and moreover there is unique bundle bundle for which* $\dim H^0(\mathscr{E}) = 6$

Proof of the lemma. (sketch) By the theory of Brill–Noether there exists a g^1_5 on $\mathcal{C}$. Let us denote it by ξ. By RR and Serre duality the adjoint linear system $K_{\mathcal{C}} - \xi$ has dimension 3. One shows that

$$H^0(K_{\mathcal{C}} - \xi) \otimes H^0(K_{\mathcal{C}} - \xi) \to H^0(2K_{\mathcal{C}} - 2\xi)$$

has a 1-dimensional cokernel. Then, from the Lemma III.7 there exists a unique nonsplit extension $\mathscr{E}$ of $\mathcal{O}_{\mathcal{C}}(K_{\mathcal{C}} - \xi)$ by $\mathcal{O}_{\mathcal{C}}(\xi)$ with $\dim H^0(\mathscr{E}) = 6$:

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{E} \to \mathcal{O}_{\mathcal{C}}(K_{\mathcal{C}} - \xi) \to 0. \tag{2.1}$$

Next one shows that $\mathscr{E}$ is stable and that any stable bundle of rank 2 and canonical determinant with a space of global sections at least 6-dimensional fits in the sequence (2.1) and therefore must be isomorphic to $\mathscr{E}$. $\square$

The vector bundle $\mathscr{E}$ is generated by its global sections, so that the map it defines is a morphism

$$\varphi_{|\mathscr{E}|} \colon C \to G(2, 6).$$

To get (5) of the vector bundle method, Mukai uses a skew version of Castelnuovo's linear-bilinear principle, equating

$$\mathbb{P}_* \operatorname{Ker} \left\{ \bigwedge^2 H^0(C) \to H^0(K_C) \right\} \cap G(2, H^0(\mathscr{E}))$$

with the set $W^1_5(C)$, and proving that the latter is finite. This is a consequence of $W^1_7(C) = \emptyset$. The proof that a curve of genus 8 having no g^2_7 has finite number of g^1_5 is somewhat intricate and we refer the reader to [**M93**] for details. This argument also shows that $\varphi_{|\mathscr{E}|}$ is an embedding as it is easily seen that $\varphi_{|\mathscr{E}|}$ composed with the Plücker map factors through the canonical map.

IV.2.1. Vector bundles and weighted Grassmannians. Let X be a nonsingular algebraic variety and $\mathscr{E}$ be a vector bundle of rank r on X, such that $\bigwedge^r \mathscr{E} = H$ is an ample line bundle. Consider E the Serre module of $\mathscr{E}$:

$$E = \bigoplus_{n \geq 0} H^0(X, \mathscr{E}(nH)).$$

Let $\langle s_i \rangle \in E$ be any subspace of sections of $\mathscr{E}$ of dimension $n \geq r$. We choose trivialisations for $\mathscr{E}$ and consider the induced trivialisations for $\mathscr{E}(nH)$. Notice that there is an induced trivialisation on $\bigwedge^r \mathscr{E} = H$. We can write down a rational map:

$$\varphi_{|\mathscr{E}|^+} \colon X \ni p \mapsto [s_{i_1} \wedge \cdots \wedge s_{i_r}(p)]_{i_1 \ldots i_r} \in \mathbb{P}$$

where the $(i_1, \ldots, i_r)$ ranges over all increasing r-tuples of $\{1, \ldots n\}$ and the projective space on the the right hand side is of dimension $\binom{n}{r}$. Then, the image of X under this map is contained in a weighted Grassmannian. The images of s_i after trivialisation are r-tuples of complex numbers, so that the target space is nothing but $\mathbb{C}_r^{r \times d}$ — the space of maximal rank $r \times d$ matrices — modulo a $\mathrm{GL}(r, \mathbb{C})$ action corresponding to a change of trivialisation. This map is not defined when s_i fail to give a matrix of maximal rank. The action of $N \in \mathrm{GL}(r, \mathbb{C})$ is

$$N \colon \begin{bmatrix} m_{11} & \ldots & m_{1d} \\ \vdots & & \vdots \\ m_{r1} & \ldots & m_{rd} \end{bmatrix} \mapsto$$

$$\mapsto \left[\det(N)^{\deg(s_i)} N \begin{bmatrix} m_{11} \\ \vdots \\ m_{r1} \end{bmatrix} \quad (\cdots) \quad \det(N)^{\deg(s_d)} \cdot N \begin{bmatrix} m_{1d} \\ \vdots \\ m_{rd} \end{bmatrix} \right]$$

that is, left-multiplication column-wise by $\det(N)^{\deg(s_i)} N$ where $\deg(s_i) = n$ if and only if $s_i \in \mathscr{E}(nH)$.

To see that this notion of weighted Grassmannian agrees with our previous definition of $\mathbb{G}$ simply embed this variety in weighted projective space using the *Plücker embedding*, i.e. the map

$$\mathbb{C}_r^{r \times d} \ni L \mapsto \wedge^r L \in \mathbb{P}(\bigwedge^r \mathbb{C}_r^{r \times d}).$$

The weights of the weighted projective space $\mathbb{P}$ on the right are determined by the distribution of $\{\deg(s_i)\}$, indeed, the $\mathbb{C}^*$ action giving $\mathbb{P}$ has eigenvalues λ^k for each k occurring as a power of $\det(N)$ in some maximal minor. For example, the weighted Grassmannian of Example IV.6 is the quotient $\mathbb{C}_2^{2\times 5}/\operatorname{GL}(2,\mathbb{C})$ with the action of $\operatorname{GL}(2,\mathbb{C})$:

$$\begin{bmatrix} a_1 & \dots & a_5 \\ b_1 & \dots & b_5 \end{bmatrix} \mapsto \left[N \begin{bmatrix} a_1 & \dots & a_4 \\ b_1 & \dots & b_4 \end{bmatrix} \middle| \det(N)\cdot N \begin{bmatrix} a_5 \\ b_5 \end{bmatrix} \right]$$

There are 6 minors of weight 1 (those in the 2×4 leftmost submatrix) and 4 of weight 2 corresponding to the maximal minors involving the last column. Under this embedding $\mathbb{C}_2^{2\times 5}/\operatorname{GL}(2,\mathbb{C}) = \mathbb{G}(\frac{1}{2}^4, \frac{3}{2}) \subset \mathbb{P}(1^6, 2^4)$, the codimension 3 projectively Gorenstein of Definition IV.4.

IV.2.2. A method that gives relations. In the previous paragraphs the emphasis was on the (rational) map given by the linear system $|\mathscr{E}|$ in the classical sense (as in Mukai's work) or by the extended linear system $|\mathscr{E}|^+$ composed of sections of the several twists $\mathscr{E}(nK_S)$ in the sense just described. However as was already visible in the illustration of the vector bundle method on page 83 the vector bundle method relies on a good knowledge of the graded ring $R(X, \det(\mathscr{E}))$. In Mukai's work all considerations could be reduced by an appropriate *ladder* of varieties to the canonical ring of a general algebraic curve of genus $\mathrm{g}(V)$ given as a linear section of V. The canonical ring of a nonsingular algebraic curve is an object that has been thoroughly analysed. Namely we have the theorems of Noether and Petri that give a precise numerical description of the generators of $R(\mathcal{C}, K_{\mathcal{C}})$ and $I_{K_{\mathcal{C}}}$. In the context of halfcanonical rings the results are less precise.

In the next example we illustrate the use of two bundles (as opposed to a single one) in finding the set of generators of the halfcanonical ring of a nonsingular curve of genus 10. In what follows the emphasis will be on exhibiting a set of relations for $R(\mathcal{C}, A)$ rather that setting up an embedding into a weighted Grassmannian. We refer the reader to the paragraph were we study these curves on page 61 for the necessary background.

Proposition IV.10. *Let $\mathcal{C}$ be a nonsingular curve of genus* 10*. Suppose that* $\operatorname{gon}(\mathcal{C}) = 6$ *and let ξ be a g^1_6 on $\mathcal{C}$. Let A be a halfcanonical divisor on $\mathcal{C}$ such that*

$\dim H^0(A) = 4$. *The map*

$$H^0(A+\xi) \otimes H^0(\xi) \to H^0(A+2\xi) \tag{2.2}$$

has a two dimensional cokernel. By Lemma III.7 there exists two nonsplit extensions, $\mathscr{E}_1$ *and* $\mathscr{E}_2$*, of* $\mathcal{O}_{\mathcal{C}}(\xi)$ *by* $\mathcal{O}_{\mathcal{C}}(A+\xi)$ *with* 3 *global sections. These vector bundles are stable.*

Proof. By Castelnuovo's free-pencil trick the kernel of the map (2.2) is isomorphic to $H^0(A)$ whose dimension is 4. The dimension of the tensor is 14 and the dimension of $H^0(A+2\xi)$ by RR is $2\deg(\xi) = 12$. Hence there is a two dimensional cokernel. By Lemma III.7 this cokernel classifies extensions

$$0 \to \mathcal{O}_{\mathcal{C}}(A-\xi) \to \mathscr{E} \to \mathcal{O}_{\mathcal{C}}(\xi) \to 0$$

with maximum number of global sections, this is with $\dim H^0(\mathscr{E}) = 3$. Let us now show that such extensions are stable. Let $\eta \subset \mathscr{E}$ be a subbundle. Suppose that $\dim H^0(\eta) \geq 2$. Then $\eta \subset \xi$. Since ξ is free this would imply that $\mathscr{E}$ splits which is not true. Hence $\dim H^0(\eta) \leq 1$. Since

$$0 \to \mathcal{O}_{\mathcal{C}}(\eta) \to \mathscr{E} \to \mathcal{O}_{\mathcal{C}}(A-\eta) \to 0$$

we have $\dim H^0(A-\eta) \geq 2$ and so

$$\deg(A-\eta) \geq 6 \Longleftrightarrow \deg(\eta) \leq 4 < \frac{\deg(\mathscr{E})}{2}. \quad \square$$

Proposition IV.11. *Let* $\mathcal{C}$ *be a nonsingular curve with a halfcanonical divisor* A *such that* $\dim H^0(A) = 4$. *Assume that* $\operatorname{gon}(\mathcal{C}) = 6$. *Let* $\mathscr{E}$ *be a stable bundle of rank* 2 *with* $\det(\mathscr{E}) = A$ *and* $\dim H^0(\mathscr{E}) = 3$. *Then*

(i) *the map* $\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$ *is injective.*
(ii) *The map* $H^0(A) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(A))$ *has a* 1*-dimensional cokernel.*

Proof. Suppose that there exist $s_1, s_2 \in H^0(\mathscr{E})$ such that $s_1 \wedge s_2 = 0$. Then the saturation η of the bundle $\mathcal{O}_{\mathcal{C}} \cdot s_1 + \mathcal{O}_{\mathcal{C}} \cdot s_2 \subset \mathscr{E}$ yields a subbundle $\eta \subset \mathscr{E}$ with $\dim H^0(\eta) \geq 2$. But by stability $\deg(\eta) \leq 4$ which is not possible, since $\operatorname{gon}(\mathcal{C}) = 6$. This shows (i).

To describe the cokernel of the map

$$H^0(A) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(A)) \tag{2.3}$$

we look into the adjacent morphisms

$$\begin{aligned}\sigma_1 \colon H^0(A) \otimes H^0(A-\xi) &\to H^0(2A-\xi)\\ \sigma_2 \colon H^0(A) \otimes H^0(\xi) &\to H^0(A+\xi)\end{aligned}$$

It is easily seen using RR that σ_1 has a 1-dimensional cokernel. Plus Castelnuovo's free-pencil trick shows that σ_2 is surjective. We deduce that the cokernel of (2.3) is at most 1-dimensional. By RR and Serre duality the dimension of $H^0(\mathscr{E}(A))$ equals

$$\dim H^0(\mathscr{E}) - 2\deg(A) + 3\deg(A) = 12.$$

Thus all we need to show is that there exist a nontrivial kernel to (2.3). It is easily checked that

$$(s_2 \wedge s_3) \otimes s_1 - (s_1 \wedge s_3) \otimes s_2 + (s_1 \wedge s_2) \otimes s_3$$

is an element of $H^0(A) \otimes H^0(\mathscr{E})$ that gets mapped to zero. □

We do not go into too much detail in what follows. Partly because this is an example of some conjectural nature. Let $\langle s_1, s_2, s_3 \rangle = H^0(\mathscr{E})$ be a choice of basis for the space of global sections of $\mathscr{E}$ and $t \in H^0(\mathscr{E}(A))$ be a section spanning a complementary space to the image of (2.3). Then, since $\mathscr{E}$ is a vector bundle of rank 2 the skew matrix

$$\begin{pmatrix} s_1 \wedge s_2 & s_1 \wedge s_3 & s_1 \wedge t \\ & s_2 \wedge s_3 & s_2 \wedge t \\ & & s_3 \wedge t \end{pmatrix}$$

has rank 2 and therefore its Pfaffian vanishes. Regarding the entries of this matrix as elements of $R(\mathcal{C}, A)$ this Pfaffian is expressing a relation between the generators of this ring. Indeed we obtain a cubic relation. As we have chosen the section t complementary to the product $H^0(A) \otimes H^0(\mathscr{E})$ this guaranties that we have not a trivial relation. We do not show this here (see the sections below for similar computations). But we can show that replacing t with any section in the image of (2.3) produces a trivial relation. Let $\sum_i f_i s_i$ be such an element; where $f_i \in H^0(A)$. Then

$$\mathrm{Pf}(s_1, s_2, s_3, \sum_i f_i s_i) = \sum_i f_i \,\mathrm{Pf}(s_1, s_2, s_3, s_i) = \sum_i f_i \cdot 0 \equiv 0,$$

where we are using the notation $\mathrm{Pf}(a_1, a_2, a_3, a_4)$ to mean the Pfaffian of the skew matrix $(a_i \wedge a_j)_{ij}$.

CONJECTURE IV.12. *Let $\mathcal{C}$ be a nonsingular curve of genus 10 with a halfcanonical divisor with $\dim H^0(A) = 4$. Assume that $\mathrm{gon}(\mathcal{C}) = 6$. Fix a choice of a g^1_6 on $\mathcal{C}$ and let $\mathscr{E}_1$ and $\mathscr{E}_2$ be the two stable bundles of rank 2 with $\det(\mathscr{E}_1) = \det(\mathscr{E}_2) = A$ and $\dim H^0(\mathscr{E}_1) = \dim H^0(\mathscr{E}_2) = 3$ as in Proposition IV.10. Consider the cubic relation F_i yielded by $\mathscr{E}_i$. Then F_1 and F_2 are linearly independent. In other words I_A is generated by F_1 and F_2.*

IV.3. Genus 8 and $h^0(A) = 3$

As result of independent work, Mukai and Ide have recently proved the main theorem of this section. See [**IM**]. Their point of view is fundamentally different. We are chiefly concerned with the geometrical aspects of (Gorenstein) codimension 3 ring theory. They focus on a complete description of curves of genus 8. Curiously, as in the following section, in the proof of the main theorem, we will use Mukai's vector bundle method. This is not the case in [**IM**]. We refer the reader to [**IM**] for an excellent treatment of curves of genus 8 and to compare both approaches.

This is the main result of this section.

THEOREM IV.13. *Let $\mathcal{C}$ be a nonsingular curve of genus 8. Then $\mathcal{C}$ has a free halfcanonical net if and only if $\mathcal{C}$ is isomorphic to a complete intersection of a quasi-homogeneous form of degree 3 and four quasihomogeneous forms of degree 2 in the weighted Grassmannian $\mathbb{G}(\frac{1}{2}^3, \frac{3}{2}^2)$.*

Proof. In this paragraph we only prove the easy direction. Let X denote the weighted Grassmannian $\mathbb{G}(\frac{1}{2}^3, \frac{3}{2}^2)$. Recall from Section IV.1 that X is the subscheme of $\mathbb{P}[m_{ij}, n_{lk}, z]$, where

$$\mathrm{wt}(M) = \mathrm{wt}\begin{pmatrix} m_{12} & m_{13} & n_{11} & n_{12} \\ & m_{23} & n_{21} & n_{22} \\ & & n_{31} & n_{23} \\ & & & z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 2 & 2 \\ & & & 3 \end{pmatrix}$$

defined by the 5 submaximal Pfaffians of the skew matrix M. Let $\mathcal{C}$ be the complete intersection of a cubic quasihomogeneous form, F_3 and four quadric quasihomogeneous forms F_2^i:

$$\mathcal{C} = X \cap \bigcap_{i=1}^{4} F_2^i \cap F_3.$$

LEMMA IV.14. *$\mathcal{C}$ is a nonsingular curve of genus 8 with a free halfcanonical net.*

Proof. The affine cone of the singular locus of X, which we denote by a(Sing X) is contained in the cone over

$$\operatorname{rank} \begin{bmatrix} n_{11} & n_{21} & n_{31} \\ n_{12} & n_{22} & n_{32} \end{bmatrix} \leq 1$$

with vertex at z. The rank condition defines the affine cone of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$; therefore a(Sing X) has dimension ≤ 5. Since we are including a cubic form in the section yielding $\mathcal{C}$, this eliminates the cone variable. Together the five general quasihomogeneous forms cut out the empty set on a(Sing X), which is to say $\mathcal{C}$ does not meet Sing X. By Bertini's theorem $\mathcal{C}$ is nonsingular. The dualising sheaf on X is given by $\mathcal{O}(-9)$ (Proposition IV.5). Thus, by adjunction $A = \mathcal{O}(1)_{|\mathcal{C}}$ is a halfcanonical divisor. If follows from the vanishing of the cohomology $H^i(\mathcal{O}_X(i))$ for each $0 < i < 6$ that the dimension of $H^0(A)$ is 3 and since by what we said above,

$$\mathcal{C} \cap (m_{ij} = 0) = \emptyset,$$

the net $|A|$ is free. Further, $\deg X = \frac{7}{3 \cdot 2^4}$ and so $\deg(A) = 7$. □

IV.3.1. The ring $R(\mathcal{C}, A)$.

PROPOSITION IV.15. *Let $\mathcal{C}$ be a nonsingular curve of genus 8 with a halfcanonical net A. Assume that A is free. Then $R(\mathcal{C}, A)$ is generated in degree 1 by x_1, x_2, x_3 and in degree 2 by y_1, y_2. The ideal I_A has no generators in degree ≤ 2, the dimension of $I_{A,3}$ is 2 and the dimension of the quotient $I_{A,4}/I'_{A,4}$ is 3.*

Proof. Let us show that $\operatorname{gon}(\mathcal{C}) \geq 4$. Denote by ξ a complete free linear system of degree ≤ 3. Then by Proposition III.9 there exists a symmetric tensor of rank 3 in the kernel of the map

$$\operatorname{sym}^2\colon \mathrm{S}^2 H^0(A) \to H^0(2A).$$

We deduce that the morphism φ_A maps $\mathcal{C}$ onto a plane conic. But then $|A|$ cannot be free. We conclude that $\operatorname{gon}(\mathcal{C}) \geq 4$. Hence from Proposition III.5 we get that

$R(\mathcal{C}, A)$ is generated in degree 2. By the reasoning above there are no generators of I_A in degree ≤ 2. In particular this implies that $R(\mathcal{C}, A)$ is generated by x_1, x_2, x_3 in degree 1 and by a further two generators y_1, y_2 in degree 2. In degree 3 we have:

$$H^0(3A) \supset \mathrm{S}^3(x_1, x_2, x_3) + \langle x_i y_j \rangle ;$$

and thus we will necessarily have two cubic relations. Not more than two, since the ring $R(\mathcal{C}, A)$ is generated in degree 1 and 2. We write these two cubics in the form:

$$\begin{cases} L_1 y_1 + L_2 y_2 = F(x_1, x_2, x_3) \\ L_3 y_1 + L_4 y_4 = G(x_1, x_2, x_3) \end{cases}$$

The following is an obvious but useful remark.

Lemma IV.16. $\det \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \neq 0$.

Proof of the lemma. Suppose that $\det \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} = 0$. Then there exist α and β constants, not simultaneously zero, such that $\alpha F + \beta G = 0$. Since A is free and thus there is no cubic through x_1, x_2, x_3, this leads to a $\alpha = \beta = 0$ which is not true. $\square$

In degree 4 we have

$$H^0(4A) \supset \mathrm{S}^4(x_1, x_2, x_3) + \mathrm{S}^2(x_1, x_2, x_3) \cdot \langle y_1, y_2 \rangle + \mathrm{S}^2 \langle y_i \rangle .$$

The number of generators in each summand is 15, 12 and 3, respectively. Subtracting 6 to account for the multiples of the two cubic relations (notice that there are no syzygies in degree 4 holding between the two cubics) we conclude that there exist an extra 3 relations in degree 4. In other words the dimension of $I_{A,4}/I'_{A,4}$ is 3. $\square$

IV.3.2. The bundle.

Proposition IV.17. *Let $\mathcal{C}$ be a nonsingular curve of genus 8 with a halfcanonical net A. Assume that A is free. Then, as $\mathscr{E}$ ranges over all stable bundles of rank 2 and determinant A over $\mathcal{C}$ the maximum value of $h^0(\mathscr{E})$ is 3 Moreover, up to isomorphism there exists a unique stable bundle attaining the value $h^0(\mathscr{E}) = 3$.*

Proof. Fix a general point $p \in \mathcal{C}$. Since A is free, the linear system $\xi_p = |A - p|$ is a free g^1_6.

Lemma IV.18. *The map*

$$H^0(A + \xi_p) \otimes H^0(\xi_p) \to H^0(A + 2\xi_p)$$

has a 1*-dimensional cokernel.*

Proof of the lemma. By Castelnuovo's free-pencil trick, the kernel of this map is isomorphic to $H^0(A)$ and thus is 3-dimensional. The dimension of the tensor product is 14 and the dimension of the target space is 12. $\square$

By this and Lemma III.7 we conclude that there exists a unique nondecomposable extension

$$0 \to A - \xi_p \to \mathscr{E}_p \to \xi_p \to 0 \tag{3.1}$$

having $h^0(\mathscr{E}_p) = 3$. Let us show that the bundle $\mathscr{E}_p$ is stable. If η is any subbundle we have

$$0 \to \eta \to \mathscr{E}_p \to A - \eta \to 0. \tag{3.2}$$

We have to show that $\deg(\eta) \leq 3$. Suppose that $\deg(\eta) \geq 4$, then $\deg(A - \eta) \leq 3$ and since $\mathcal{C}$ is not trigonal this implies that $h^0(A - \eta) \leq 1$. We deduce that in this situation we must have $h^0(\eta) \geq 2$. In particular this shows that $\eta \subset \xi_p$ by composing (3.1) with the exact sequence above. But since ξ_p is free we obtain $\eta = \xi_p$ implying that $\mathscr{E}_p$ is decomposable. Such is not true. Hence $\deg(\eta) \leq 3$ and $\mathscr{E}_p$ is a stable bundle. We conclude that

$$\max\left\{h^0(\mathscr{E}) \mid \mathscr{E} \text{ stable, } \operatorname{rank}\mathscr{E} = 2 \text{ and } \det\mathscr{E} = A\right\} \geq 3.$$

Consider now a stable bundle $\mathscr{E}$ of rank 2 and determinant A having $h^0(A) \geq 3$. There exists at least one section of $\mathscr{E}$ vanishing at p. We denote by $\mathcal{O}_{\mathcal{C}}(\delta)$ the saturation of $\mathcal{O}_{\mathcal{C}}(p) \hookrightarrow \mathscr{E}$. From this we obtain

$$0 \to \mathcal{O}_{\mathcal{C}}(\delta) \to \mathscr{E} \to \mathcal{O}_{\mathcal{C}}(A - \delta) \to 0.$$

If $\deg(\delta) \geq 2$ then $h^0(A - \delta) \leq h^0(\xi_p) - 1 = 1$ and therefore $h^0(\delta) \geq 2$. But this is impossible since by stability $\deg(\delta) \leq 3$ and $\mathcal{C}$ is not trigonal. Hence $\deg(\delta) = 1$, i.e. $\delta = p$, and $\mathscr{E} \simeq \mathscr{E}_p$. It follows that

$$\max\left\{h^0(\mathscr{E}) \mid \mathscr{E} \text{ stable, } \operatorname{rank}\mathscr{E} = 2 \text{ and } \det\mathscr{E} = A\right\} = 3,$$

and that there exists a unique stable bundle of rank 2 for which $h^0(\mathscr{E}) = 3$. $\square$

Proposition IV.19. *Let $\mathcal{C}$ be a nonsingular curve of genus 8 with a halfcanonical net A. Assume that A is free. Let $\mathscr{E}$ be the unique stable bundle of rank 2 and determinant A on $\mathcal{C}$ having $h^0(A) = 3$. Then*

(i) *the map $\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$ is an isomorphism and*

(ii) *the cokernel of $H^0(A) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(A))$ is 2-dimensional.*

Proof. Suppose there exist two sections $s_1, s_2 \in H^0(\mathscr{E})$ such that $s_1 \wedge s_2 = 0$. Then s_1 and s_2 generate a line bundle $\mathcal{L}$. Consider the composition $\mathcal{L} \to \mathscr{E} \to \mathcal{O}_\mathcal{C}(\xi_p)$ with $\mathscr{E} \to \mathcal{O}_\mathcal{C}(\xi_p)$ and ξ_p as in (3.1). Since $\langle s_1, s_2 \rangle \subset H^0(\mathcal{L})$ we have $h^0(\mathcal{L}) \geq 2$. This shows that the line bundle $\mathcal{L}$ is not contained in the kernel of $\mathscr{E} \to \mathcal{O}_\mathcal{C}(\xi_p)$ and therefore the composition map has to be injective. But since the linear system ξ_p is base point free, we conclude that $\mathcal{L} \simeq \mathcal{O}_\mathcal{C}(\xi_p)$ and this way $\mathscr{E}$ would have to split. This is a contradiction. We have shown (i).

To prove item (ii) we start by writing down the exact sequence

$$0 \to \mathcal{O}_\mathcal{C}(-A) \to 3\mathcal{O}_\mathcal{C} \to \mathscr{E} \to 0 \tag{3.3}$$

that we obtain considering the evaluation epimorphism: $3\mathcal{O}_\mathcal{C} \twoheadrightarrow \mathscr{E}$ given by a choice of a basis of $H^0(\mathscr{E})$. Notice that by (i) and the fact that A is free, we deduce that the bundle $\mathscr{E}$ is spanned by its global sections. Tensoring (3.3) with A and taking global sections we conclude that the map $H^0(A) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(A))$ has a 1-dimensional kernel. Furthermore, by Riemann–Roch, $h^0(\mathscr{E}(A)) = h^0(\mathscr{E}) + \deg(A) = 10$. Therefore the dimensional of its cokernel is $10 - 8 = 2$. $\square$

Proposition IV.20. *Let $\mathcal{C}$ be a nonsingular curve of genus 8 with a halfcanonical divisor net A. Assume that A is free. Let $\mathscr{E}$ be the unique stable bundle of rank 2 and determinant A on $\mathcal{C}$ having $h^0(A) = 3$. Denote by $\langle s_1, s_2, s_3 \rangle$ a choice of basis of $H^0(\mathscr{E})$ and by $\langle t_1, t_2 \rangle \subset H^0(\mathscr{E}(A))$ a complementary space to $H^0(A) \otimes H^0(\mathscr{E})$. Consider a polynomial ring $\mathbb{C}[m_{ij}, n_{kl}, z]$ where*

$$\mathrm{wt}(M) = \mathrm{wt}\begin{pmatrix} m_{12} & m_{13} & n_{11} & n_{12} \\ & m_{23} & n_{21} & n_{22} \\ & & n_{31} & n_{32} \\ & & & z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 2 & 2 \\ & & & 3 \end{pmatrix}$$

and define a map $\mathrm{ev}^+ \colon \mathbb{C}[m_{ij}, n_{kl}, z] \to R(\mathcal{C}, A)$ by setting $\mathrm{ev}^+(m_{ij}) = s_i \wedge s_j$, $\mathrm{ev}^+(n_{kl}) = s_l \wedge t_k$ and $\mathrm{ev}^+(z) = t_1 \wedge t_2$. Then

(i) ev^+ *is surjective.*

Additionally, consider $\mathrm{ev}\colon \mathbb{C}[x_1, x_2, x_3, y_1, y_2] \to R(\mathcal{C}, A)$*, a minimal surjection. Denote by* $\mathrm{Pf}_1, \ldots, \mathrm{Pf}_5$ *the* 5 *submaximal Pfaffians of* M*. Then, for any choice of* $\langle s_1, s_2, s_3\rangle$*,* $\langle t_1, t_2\rangle$ *and of a surjection* $\lambda\colon \mathbb{C}[m_{ij}, n_{kl}, z] \to \mathbb{C}[x_1, x_2, x_3, y_1, y_2]$ *such that* $\mathrm{ev}^+ = \lambda \circ \mathrm{ev}$*, we have:*

(ii) $\lambda\{\mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5\} \subset I_A$

(iii) $\lambda(\mathrm{Pf}_4), \lambda(\mathrm{Pf}_5)$ *form a basis of* $I_{A,3}$.

(iv) $\lambda(\mathrm{Pf}_1), \lambda(\mathrm{Pf}_2), \lambda(\mathrm{Pf}_3)$ *project to a basis of* $I_{A,4}/I'_{A,4}$.

Proof. The matrix $\mathrm{ev}^+(M)$ has rank 2 on $\mathcal{C}$ since $\mathscr{E}$ is a vector bundle of rank 2. Hence $\mathrm{ev}^+(\mathrm{Pf}_i) = 0$. In particular $\lambda(\mathrm{Pf}_i) \in I_A$. Item (i) follows from (i) of Proposition IV.19, Lemma IV.16 and item (iii) below.

Proof of (iii). Suppose there exist $\alpha, \beta \in \mathbb{C}$ such that $\alpha\lambda(\mathrm{Pf}_4) + \beta\lambda(\mathrm{Pf}_5) = 0$. Consider $u = \beta t_1 + \alpha t_2$ and define $\widetilde{\mathrm{ev}}^+$ with respect to $\langle s_1, s_2, s_3\rangle$ and $\langle u, t_2\rangle$. Also, define a surjection $\widetilde{\lambda}$ by

$$\widetilde{\lambda}(m_{ij}) = \lambda(m_{ij}), \quad \widetilde{\lambda}(n_{l1}) = \beta\lambda(n_{l1}) + \beta\lambda(n_{l1}),$$
$$\widetilde{\lambda}(n_{l2}) = \lambda(n_{l2}), \quad \text{and} \quad \widetilde{\lambda}(z) = \beta\lambda(z).$$

We see that $\widetilde{\lambda}$ is still a surjective homomorphism and moreover we have

$$\widetilde{\lambda}(\mathrm{Pf}_5) = \beta\lambda(\mathrm{Pf}_5) + \alpha\lambda(\mathrm{Pf}_4) = 0. \tag{3.4}$$

Hence we could have assumed from start that $\lambda(\mathrm{Pf}_5) = 0$. By Proposition IV.19, the map

$$\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$$

is an isomorphism. Thus $\{\lambda(m_{ij})\}$ is a regular sequence in $\mathbb{C}[x_1, x_2, x_3, y_1, y_2]$, and so if $\lambda(\mathrm{Pf}_5) = \lambda(m_{12})\lambda(n_{13}) - \lambda(m_{13})\lambda(n_{12}) + \lambda(m_{23})\lambda(n_{11}) = 0$ then there exist $A, B, C \in \langle x_1, x_2, x_3\rangle$ such that

$$\begin{pmatrix} \lambda(n_{11}) \\ \lambda(n_{12}) \\ \lambda(n_{13}) \end{pmatrix} = \begin{pmatrix} A\lambda(m_{12}) + B\lambda(m_{13}) \\ C\lambda(m_{12}) + B\lambda(m_{23}) \\ C\lambda(m_{13}) - A\lambda(m_{23}) \end{pmatrix}$$

and in particular $s_1 \wedge t_1 = \mathrm{ev}(A)s_1 \wedge s_2 + \mathrm{ev}(B)s_1 \wedge s_3$, which means that

$$s_1 \wedge (t_1 - \mathrm{ev}(A)s_2 - \mathrm{ev}(B)s_3) = 0.$$

The proof of (iii) finishes with the following lemma.

LEMMA IV.21. *Let $t \in H^0(\mathscr{E}(A))$ be such that $t \notin H^0(A) \otimes H^0(\mathscr{E})$. Then the map $H^0(\mathscr{E}) \xrightarrow{\wedge t} H^0(2A)$ is injective.*

Proof of the lemma. Let us write δ for the divisor of zeros of t. Since $\mathscr{E}$ is spanned by its global sections, wedging with t yields the following sequence:

$$0 \to \mathcal{O}_{\mathcal{C}}(\delta - A) \to \mathscr{E} \xrightarrow{\wedge t} \mathcal{O}_{\mathcal{C}}(2A - \delta) \to 0.$$

If the dimension of $H^0(\delta - A)$ is > 0 then $t \in f \cdot H^0(\mathscr{E}) \subset H^0(\mathscr{E}(A))$ where $f \in H^0(A)$, which is false by assumption. Therefore the dimension of $H^0(\delta - A)$ is zero and accordingly the map $H^0(\mathscr{E}) \xrightarrow{\wedge t} H^0(2A)$ is injective. $\square$

Proof of (iv). Suppose that there exist $\alpha, \beta, \gamma \in \mathbb{C}$ and $f, g \in \langle x_1, x_2, x_3 \rangle$ such that

$$\alpha\lambda(\mathrm{Pf}_1) + \beta\lambda(\mathrm{Pf}_2) + \gamma\lambda(\mathrm{Pf}_3) = f\lambda(\mathrm{Pf}_4) + g\lambda(\mathrm{Pf}_5).$$

We argue by contradiction. Without loss in generality we may assume that $\alpha = 1$. Consider the sections:

$$\begin{aligned} a = s_2 + \beta s_1, \quad b = s_3 + \gamma s_3 \quad &\in H^0(\mathscr{E}) \quad \text{and} \\ c = t_1 - f s_1, \quad d = t_2 - g s_2 \quad &\in H^0(\mathscr{E}(A)). \end{aligned}$$

Then by the skew multilinearity of Pf,

$$\mathrm{Pf}\begin{pmatrix} a \wedge b & a \wedge c & a \wedge d \\ & b \wedge c & b \wedge d \\ & & d \wedge d \end{pmatrix} = \mathrm{ev}^+(\mathrm{Pf}_1 + \beta\,\mathrm{Pf}_2 + \gamma\,\mathrm{Pf}_3 - f\,\mathrm{Pf}_4 - g\,\mathrm{Pf}_5) = 0.$$

By defining $\widetilde{\mathrm{ev}}^+$ with respect to these new bases and $\widetilde{\lambda}$ accordingly, we deduce that it enough to treat the case $\lambda(\mathrm{Pf}_1) = 0$. Let us write

$$\lambda(\mathrm{Pf}_1) = \mathrm{Pf}\begin{pmatrix} x_3 & q_1 + Ax_3 & q_3 + Cx_3 \\ & q_2 + Bx_3 & q_4 + Dx_3 \\ & & p_3 \end{pmatrix}$$

where $q_i \in \langle y_1, y_2 \rangle \oplus \mathrm{S}^2\langle x_1, x_2, x_3 \rangle$ do not contain any multiples of x_3 and p_3 is a quasihomogeneous form of degree 3 in the variables x_1, x_2, x_3, y_1, y_2. Consider the change of basis:

$$t_1 \mapsto t_1 - As_3 + Bs_2 \quad t_2 \mapsto t_2 - Cs_3 + Ds_2.$$

Define ev^+ with respect to these t_1, t_2 and $\widetilde{\lambda}$ by setting

$$n_{21} \mapsto q_1, \quad n_{22} \mapsto q_3, \quad n_{31} \mapsto q_2 \quad \text{and} \quad n_{32} \mapsto q_4.$$

Then $\mathrm{ev}^+ = \mathrm{ev} \circ \widetilde{\lambda}$. Therefore we can assume that $A = B = C = D = 0$. But then $\lambda(\mathrm{Pf}_1) = 0$ implies that $p_3 = 0$. In particular $t_1 \wedge t_2 = 0$. This implies that $u_1 \wedge u_2 = 0$ for every two distinct points u_1, u_2 on the line $\mathbb{P}[\langle t_1, t_2\rangle]$.

Consider the exact sequence:

$$0 \to 2A - \xi_p \to \mathscr{E}(A) \to A + \xi_p \to 0$$

obtained by twisting (3.2) with $\mathcal{O}_{\mathcal{C}}(A)$. Let Δ be a divisor on $\mathcal{C}$ consisting of two general points. By RR and Serre duality, $h^0(A + p) = 3$, hence $h^0(A - \Delta + p) \leq 1$. On the other hand $h^0(A + \xi_p - \Delta) = h^0(A - \xi_p + \Delta) + 4 \leq 5$ since $\mathcal{C}$ is not trigonal. We deduce that $h^0(\mathscr{E}(A - \Delta)) \leq 6$. This implies that the subvariety $\mathcal{V}$ of $\mathbb{P}[H^0(\mathscr{E}(A))]$ consisting of sections which vanish at least at two points has dimension $\leq 6 - 1 + 2 = 7$. This variety contains $\mathbb{P}[H^0(A) \otimes H^0(\mathscr{E})]$ which by Proposition IV.19 is 7-dimensional. We conclude that for every choice of sections $\langle t_1, t_2\rangle \subset H^0(\mathscr{E}(A))$ spanning a complementary set to $H^0(A) \otimes H^0(\mathscr{E}) \subset H^0(\mathscr{E}(A))$, the line $\mathbb{P}[\langle t_1, t_2\rangle]$ is not contained in $\mathcal{V}$. Let $u \in \mathbb{P}[\langle t_1, t_2\rangle]$ be such that $u \notin \mathcal{V}$. Consider the exact sequence:

$$0 \to \mathcal{O}_{\mathcal{C}}(\delta) \to \mathscr{E}(A) \xrightarrow{\wedge u} \mathcal{O}_{\mathcal{C}}(3A - \delta) \to 0$$

where we have denoted by δ the divisor of zeros of u. (Notice that $\mathscr{E}(A)$ is spanned by its global sections). By assumption $\deg(\delta) \leq 1$ and thus $h^0(\delta) = 1$. But if $t_1 \wedge t_2 = 0$ as above then $h^0(\delta) \geq 2$. This is a contradiction. $\square$

IV.3.3. Proof of Theorem IV.13.

Proposition IV.22. *Let $\mathcal{C}$ be a nonsingular curve of genus 8 with a free half-canonical net, A. Let $\mathscr{E}$ be the unique stable bundle of rank 2 and determinant A with $h^0(A) = 3$. Let $\langle s_1, s_2, s_3\rangle$ be a choice of basis of $H^0(\mathscr{E})$ and $\langle t_1, t_2\rangle$ a choice of a complementary space to the image of $H^0(A) \otimes H^0(\mathscr{E})$ in $H^0(\mathscr{E}(A))$. Let X denote denote the weighted Grassmannian $\mathbb{G}(\frac{1}{2}^3, \frac{3}{2}^2)$. Define a map*

$$\eta\colon \mathcal{C} \to X$$

by setting $p \mapsto (s_i \wedge s_j(p), s_l \wedge t_k(p), t_1 \wedge t_2(p))$. Then

(i) *η is an embedding*

(ii) *$\eta(\mathcal{C}) \subset X$ is cut out by a quasihomogeneous form of degree 3 and four quasihomogeneous forms of degree 2.*

Proof. The map η fits in a commutative diagram:

$$\begin{tikzcd} C \arrow[r, "\eta"] \arrow[d, "\varphi_{A^+}"'] & X \arrow[d] \\ \mathbb{P}[x_1, x_2, x_2, y_2, y_3] \arrow[r, "\mathbb{P}(\lambda)"'] & \mathbb{P}[m_{ij}, n_{kl}, z] \end{tikzcd}$$

where $\mathbb{P}(\lambda)$ is the projectivised of the surjective homomorphism

$$\lambda\colon \mathbb{C}[m_{ij}, n_{kl}, z] \to \mathbb{C}[x_1, x_2, x_3, y_1, y_2].$$

Since $\mathbb{P}(\lambda)$ is an embedding and so is φ_{A^+} we deduce that

$$\eta = \mathbb{P}(\mathrm{ev}^+) = \mathbb{P}(\lambda) \circ \varphi_{A^+}$$

is also an embedding. We have shown in the proof of Proposition IV.15 that the curve $\mathcal{C}$ is not trigonal. Clearly $\mathcal{C}$ is not isomorphic to a plane quintic. We deduce from Theorem III.14 that the image of $\mathcal{C}$ under the embedding of φ_{A^+} is cut out set-theoretically by forms of degree ≤ 4. Therefore the image of $\mathcal{C}$ under $\mathbb{P}(\lambda) \circ \varphi_{A^+}$ is cut out by $\operatorname{Ker} \lambda + \lambda^{-1}(I_{A,3} \oplus I_{A,4})$. By Proposition IV.20, $\lambda^{-1}(I_{A,3} \oplus I_{A,4})$ is generated $\langle \mathrm{Pf}_4, \mathrm{Pf}_5 \rangle \oplus \langle \mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3 \rangle$ which cut out $X \subset \mathbb{P}[m_{ij}, n_{kl}, z]$. Hence we conclude that the image of $\mathcal{C}$ under the embedding η is cut out by $\operatorname{Ker} \lambda$, which is generated by a cubic quasihomogeneous form and four quadric quasihomogeneous forms. $\square$

Proof of Theorem IV.13. The intersection of a cubic quasihomogeneous form and four quadric quasihomogeneous forms on X has degree 7 (see Proposition IV.5), which is equal to the degree of $\varphi_{A^+}(\mathcal{C}) \subset \mathbb{P}[x_1, x_2, x_3, y_1, y_2]$. Therefore by Proposition IV.2, $\mathcal{C}$ is a complete intersection of these forms in X. $\square$

Corollary IV.23. *Let $\mathcal{C}$ be a nonsingular curve of genus 8 with a free halfcanonical net. Then the ring $R(\mathcal{C}, A)$ has codimension 3 and is given as*

$$\mathbb{C}[x_1, x_2, x_3, y_1, y_2]/I_A$$

where the ideal I_A is generated by 2 *cubic and* 3 *quartic quasihomogeneous forms in the variables x_1, x_2, x_3, y_1, y_2, that are the* 5 *submaximal Pfaffians of a skew matrix*

$$M = \begin{pmatrix} x_1 & x_2 & q_1 & q_2 \\ & x_2 & q_3 & q_4 \\ & & q_5 & q_6 \\ & & & p_3 \end{pmatrix}$$

where $q_i \in \langle y_1, y_2 \rangle \oplus \mathrm{S}^2 \langle x_1, x_2, x_3 \rangle$ and p_3 is a quasihomogeneous form of degree 3 *in the variables x_1, x_2, x_3, y_1, y_2.* $\square$

IV.4. Genus 14 and $h^0(A) = 5$

What is special about this case when compared against the previous case is that the ideal I_A is generated in degree 3. Theorem III.14 says that the image of $\mathcal{C}$ by the $\varphi_A \colon \mathcal{C} \to \mathbb{P}^4$ is cut out by forms of degree ≤ 4 and for our purposes we need to know that it is, in fact, cut out by forms of degree ≤ 3. Below we prove that $\varphi_A(\mathcal{C})$ is a component of the intersection of all cubics through $\varphi_A(\mathcal{C})$ and use for the first time to its full strength Proposition IV.2. The aim of this section is to prove the following theorem.

Theorem IV.24. *Let $\mathcal{C}$ be a nonsingular curve of genus* 14*. Assume that $\mathcal{C}$ has no g^2_9. Then $\mathcal{C}$ has a halfcanonical divisor A such that $h^0(A) = 5$ if and only if $\mathcal{C}$ is a complete intersection of two quasihomogeneous forms of degree* 1 *and four quasihomogeneous forms of degree* 2 *in the generalised weighted Grassmannian $\mathbb{P}(1)\ltimes\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$.*

Proof. Denote by X the generalised weighted Grassmannian $\mathbb{P}(1)\ltimes\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$. Recall from Section IV.1 that X is the projectively Gorenstein subscheme of $\mathbb{P}[v, m_{ij}, n_i]$ defined by the ideal generated by the five submaximal Pfaffians of the skew matrix:

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & n_1 \\ & m_{23} & m_{24} & n_2 \\ & & m_{34} & n_3 \\ & & & n_4 \end{pmatrix}.$$

Let $\mathcal{C}$ be the 1-dimensional subscheme of X given by the complete intersection of two general quasihomogeneous forms of degree 1 and four general quasihomogeneous forms of degree 2:

$$\mathcal{C} = X \cap \bigcap_{i=1}^{4} F_2^i \cap \bigcap_{j=1}^{2} F_1^j.$$

The singularities of X lie on the set defined by $m_{ij} = 0$ which includes the vertex of the cone, $\mathbb{P}(1)$. Let us denote this locus by S. The affine cone of S is of dimension 5. Therefore, choosing the forms F_2^j and F_1^i general, a$\mathcal{C}$, the affine cone of $\mathcal{C}$ will not meet aS, the affine cone of S. Thus by Bertini's theorem $\mathcal{C}$ is nonsingular. By Proposition IV.9, the dualising sheaf of X is given by $\mathcal{O}_X(-8)$ and therefore, by Gorenstein adjunction,

$$K_{\mathcal{C}} = \mathcal{O}_X(10-8)_{|\mathcal{C}}.$$

Hence $A = \mathcal{O}_X(1)_{|\mathcal{C}}$ is a halfcanonical divisor on $\mathcal{C}$. Additionally the vanishing of cohomology $H^i(\mathcal{O}_X(j))$ for $0 < i < 7$ gives $\dim H^0(A) = 5$. By Proposition IV.9 the degree of X is $\frac{13}{2^4}$. Therefore $\mathcal{C}$ is a curve of genus 14. Moreover, the map sym^2 is surjective. We will see below in Proposition IV.25 that this implies that $W_9^2(\mathcal{C}) = \emptyset$. $\square$

The assumption that $W_9^2(\mathcal{C}) = \emptyset$ is convenient from the point of view of moduli of curves and Brill–Noether theory. However, especially when considering $\mathcal{C} \in |K_S|$, a curve in the canonical linear system of a surface of general type with $p_g = 6$ and $K^2 = 13$, or, as above, when working with quasilinear sections, it is better to have on $\mathcal{C}$ equivalent assumptions on sym^2 and/or on the linear system $|A|$.

IV.4.1. The linear system $|A|$.

Proposition IV.25. *Let $\mathcal{C}$ be a nonsingular curve of genus 14. Let A be a half-canonical divisor on $\mathcal{C}$ with dim $H^0(A) = 5$. Then $\mathcal{C}$ has no g_9^2 if and only if* sym^2 *is surjective. If either is true then $|A|$ is very ample.*

Proof. Suppose that $\mathcal{C}$ has no g_9^2. We deduce that $\mathrm{gon}(\mathcal{C}) \geq 4$. In fact we can show that $\mathrm{gon}(\mathcal{C}) \geq 6$. If ξ is a pencil of degree 5 then ξ is free and by Proposition III.2, the dimension of $H^0(A - \xi)$ is ≥ 3. Therefore $|A - \xi|$ would yield a g_8^2.

Let $|B| \subset |A|$ denote the free part of $|A|$. The morphism φ_B has degree ≤ 3. If this degree is 3 then φ_B maps $\mathcal{C}$ onto a rational curve and this would imply that $\mathcal{C}$ is trigonal. If the degree of φ_B is 2 and consequently the degree of $\varphi_B(\mathcal{C})$ is ≤ 6; the projection off a secant line to $\varphi_B(\mathcal{C})$ composed with φ_B yields a g_d^2 where $d \leq 8$. This is impossible. Hence $\deg(\varphi_B) = 1$. To show that $|A|$ is free and that sym^2 is surjective we must divide the proof into two cases.

Firstly observe that by Proposition III.2 the gonality of $\mathcal{C}$ cannot be 7. Hence either $\text{gon}(\mathcal{C}) = 6$ or $\text{gon}(\mathcal{C}) = 8$. If $\text{gon}(\mathcal{C}) = 8$ the proof that sym^2 is surjective follows easily from Proposition III.9. Let us show that in this case we also have $|A|$ free. Let η be a free g^1_8 on $\mathcal{C}$ and let p be any point on $\mathcal{C}$. By Castelnuovo's free-pencil trick we have

$$0 \to \mathcal{O}_{\mathcal{C}}(A - p - \eta) \to 2\mathcal{O}_{\mathcal{C}}(A - p) \to \mathcal{O}_{\mathcal{C}}(A - p + \eta) \to 0.$$

Therefore $2h^0(A - p) \leq h^0(A - p - \eta) + h^0(A - p + \eta)$. The degree of $A - p - \eta$ is 4 hence $h^0(A - p - \eta) \leq 1$. By RR and Serre duality, $h^0(A - p + \eta) = h^0(A + p - \eta) + 7$. Since the degree of $A + p - \eta$ is 6 we have $h^0(A + p - \eta) \leq 1$. Adding up we have $h^0(A - p) \leq \frac{9}{2}$ which means that $h^0(A - p) \leq 4$. In other words $|A|$ is free.

Let us show that if $\mathcal{C}$ has no g^2_9 and $\text{gon}(\mathcal{C}) = 6$ then sym^2 is surjective. The proof of this case is quite lengthy and will involve a case-by-case analysis of the intersection of quadrics through $\varphi_A(\mathcal{C})$. First off, let us show that $|A|$ is very ample. Let ξ be any gonality divisor and let p, q be any two points of $\mathcal{C}$. By Castelnuovo's free-pencil trick, we have

$$0 \to \mathcal{O}_{\mathcal{C}}(A - p - q - \xi) \to 2\mathcal{O}_{\mathcal{C}}(A - p - q) \to \mathcal{O}_{\mathcal{C}}(A - p - q + \xi) \to 0.$$

Taking global sections we get

$$2h^0(A - p - q) \leq h^0(A - p - q - \xi) + h^0(A - p - q + \xi).$$

The divisor $A - p - q - \xi$ has degree 5, hence $h^0(A - p - q - \xi) \leq 1$. On the other hand, by RR and Serre duality, $h^0(A - p - q + \xi) = h^0(A + p + q - \xi) + \deg(\xi) - 2$. Hence $2h^0(A - p - q) \leq h^0(A + p + q - \xi) + 5$. The divisor $A + p + q - \xi$ has degree 9 and since by assumption there are no g^2_9 we have $h^0(A + p + q - \xi) \leq 2$. Altogether this implies that $h^0(A - p - q) \leq \frac{7}{2}$ which is to say $h^0(A - p - q) \leq h^0(A) - 2$ and thus $|A|$ is very ample.

Denote the image of $\mathcal{C}$ under the embedding φ_A by C_{13}. To show that sym^2 is surjective we argue by contradiction. Assume that $\text{Ker}\,\text{sym}^2$ contains two linearly independent quadrics, Q_1, Q_2. The proof now breaks into two cases.

Suppose that $Q_1 \cap Q_2 = S$ is irreducible.

LEMMA IV.26. *The surface S has at most du Val singularities.*

Proof of the lemma. Suppose that S is a cone with vertex p over some curve of degree 4 in a hyperplane of $\mathbb{P}^4$. (This happens, for example, if Q_1 and Q_2 are two quadrics of rank 4 with common singular locus). Let us denote this space curve by C_4. The genus of C_4 is either 0 or 1. The projection off p restricts to a map of degree ≤ 3 on the curve C_{13}. In particular C_{13} must go through the vertex. If the genus of C_4 is 0 then we deduce that C_{13} is trigonal or hyperelliptic. This however, is not true if we assume that there are no g^2_9. If C_4 is a nonsingular elliptic curve then the isomorphism $C_4 \simeq E_3 \subset \mathbb{P}^2$ composed with the projection from p yields a g^2_9 on C_{13}. Again this is a contradiction. Hence we can assume that S is not a cone.

Suppose now that S has a positive dimensional singular locus. By reasoning on the degree of S the positive dimensional part of the singular locus is either a line or a conic. Let q be a point on $S \backslash \operatorname{Sing} S$. Then the projection maps S to a surface of degree 3 in $\mathbb{P}^3$. (Recall that we know that S is not a cone). Let us denote the cubic surface by Σ_3. This cubic must have a positive dimensional singular locus and it has to be a line of double points. In particular away from the plane section of S determined by this line and q, the projection is an embedding. By choosing q not on C_{13} the projection of S onto Σ_3 restricts to C_{13} has a birational morphism which is an embedding away from where C_{13} meets the plane section determined by $\operatorname{Sing} \Sigma_3$ and q. Denote the birational transform of C_{13} on Σ_3 by D_{13}. Finally the projection from a point on $\operatorname{Sing} \Sigma_3 \backslash D_{13}$ determines a birational morphism onto the projective plane which is an embedding away from $\operatorname{Sing} \Sigma_3$. We deduce that C_{13} has a plane birational transform of degree 13 and geometric genus 14. The composite birational map is an embedding of C_{13} away from a finite set of points that get mapped to a single point of the plane (where the plane meets the line $\operatorname{Sing} \Sigma_3$). Let μ be the multiplicity of the birational transform of C_{13} at this point. By the genus formula we have $14 = \frac{1}{2} 12 \cdot 11 - \frac{1}{2}\mu(\mu - 1)$ which has no integer solution.

We have shown that S has only isolated singularities. All that remains to be seen is that these are ordinary double points. The surface S is a complete intersection of a pencil $\mathcal{L}$ of quadrics in $\mathbb{P}^4$. Let $p \in S$ be a singular point. Suppose that

$$p \in \bigcap_{Q \in \mathcal{L}} \operatorname{Sing} Q.$$

Then any secant line to S passing through p must be contained in Q for every $Q \in \mathcal{L}$ and hence is contained in S. But we have shown that S is not a cone. Thus we can assume that the general quadric $Q \in \mathcal{L}$ is nonsingular at p. Let Q_1 and Q_2 be two nonsingular quadrics at p in $\mathcal{L}$. Write their equations locally as:

$$Q_1 = (x + q_1(x, y, z, w) = 0) \quad \text{and} \quad Q_2 = (x + q_2(x, y, z, w) = 0)$$

where q_i are quadratic forms. By the implicit function theorem on an analytic neibourhood of p in Q_1 the variable x is a implicit function of y, z and w. Let us denote this function by ϕ. Using ϕ and Q_2 we deduce that, locally analytically at p, S is given by

$$\phi(y, z, w) + q_2(\phi(y, z, w), y, z, w) = 0. \tag{4.1}$$

Since p is not a smooth point, the power series ϕ has no linear terms. Additionally since q_2 is not divisible by x there is a nontrivial term of order 2 in (4.1). Since p is an isolated singularity this must be a nondegenerate quadratic form in y, z, w. This shows that p is analytically isomorphic to an isolated hypersurface singularity of multiplicity 2 in $\mathbb{C}^3$. In other words p is an ordinary double point. This finishes the proof of the lemma. $\square$

LEMMA IV.27. *S is the birational transform of the projective plane by a linear system of plane cubics through 5 (including possibly infinitely near) points.*

Proof. If S is nonsingular this is well-known. See for example [**Beau**, p. 52]. In this case we know that S is isomorphic to $\mathbb{P}^2$ blown up at 5 general points and embedded in $\mathbb{P}^4$ via the linear system of plane cubics through these points.

If S is singular, there exists a birational map $\pi \colon S \to \mathbb{P}^2$ given by projecting from a secant line to S at a singular point. Its inverse $\pi^{-1} \colon \mathbb{P}^2 \to S \subset \mathbb{P}^4$ is given by a linear system $\mathcal{H} \subset |nL|$ where L is the class of a line in $\mathbb{P}^2$. We resolve this linear system by blowing up the plane at a set of (possibly including infinitely near) points. Let E_i with $i = 1, \ldots, t$, be the birational transforms of the exceptional divisors of these blow-ups and denote the resulting blown-up plane and composite morphism by

$\sigma_1 \colon \widehat{\mathbb{P}^2} \to \mathbb{P}^2$. Furthermore, let us denote by $\sigma_2 \colon \widehat{\mathbb{P}^2} \to S$ the morphism yielded by $\widehat{\mathcal{H}}$, the resolved linear system. Since S has only du Val singularities, by adjunction we know that $K_S = -H_S$ and thus,

$$\widehat{\mathcal{H}} = \sigma_2^*(H_S) = \sigma_2^*(-K_S) = -K_{\widehat{\mathbb{P}^2}} + \sum \alpha_j E_j^{'} = 3\sigma_1^*(L) - \sum_{i=1}^{t} E_i + \sum \alpha_j E_j^{'}$$

where $E_j^{'}$ are curves on $\widehat{\mathbb{P}^2}$ contracted by σ_2. This shows that $n = 3$. Additionally notice that if we blow up more than 6 points in the plane the degree of $\widehat{\mathcal{H}}$ is ≤ 3 which if false. However it possible that that $\mathcal{H}$ consists of cubics through 4 points of the projective plane with a fixed tangent direction at one of them. $\square$

Since $C_{13} \subset S$ is nonsingular, its birational transform in $\widehat{\mathbb{P}^2}$ is also nonsingular. We use the same notation for the transform. Let us write

$$C_{13} = mL - \sum_{i=1}^{5} m_i E_i.$$

with m and m_i positive integers. The curve C_{13} is not rational so we can assume that $m \neq 0$. The linear system $|2L - \sum_{i=1}^{5} E_i|$ is at least 0-dimensional and its members are lines in $\widehat{\mathbb{P}^2}$ since

$$\left(3L - \sum_{i=1}^{5} E_i\right)\left(2L - \sum_{i=1}^{5} E_i\right) = 6 - \left(\sum_{i=1}^{5} E_i\right)^2 = 6 - 5 = 1.$$

(Notice that from $\deg \widehat{\mathcal{H}}$ we can deduce that $\left(\sum_{i=1}^{5} E_i\right)^2 = 5$). By the nonexistence of 4-secant lines,

$$C_{13}\left(2L - \sum_{i=1}^{5} E_i\right) = 2m + \left(\sum_{i=1}^{5} m_i E_i\right)\left(\sum_{i=1}^{5} E_i\right) \leq 3.$$

Combining this with the formula for the degree

$$13 = 3m + \left(\sum_{i=1}^{5} m_i E_i\right)\left(\sum_{i=1}^{5} E_i\right)$$

we get $m \geq 10$. From the genus formula,

$$26 = \left(mL - \sum_{i=1}^{5} m_i E_i\right)^2 - \deg(C_{13}) = m^2 + \left(\sum_{i=1}^{5} m_i E_i\right)^2 - 13$$

which implies that

$$m^2 - 39 = -\left(\sum_{i=1}^{5} m_i E_i\right)^2 \implies -\left(\sum_{i=1}^{5} m_i E_i\right)^2 \geq 61.$$

For each j the line E_j cannot meet C_{13} more than 3 times, thus

$$\left(-\sum_{i=1}^{5} m_i E_i\right) E_j \leq 3 \implies 61 \leq -\left(\sum_{i=1}^{5} m_i E_i\right)^2 \leq 3\sum_{i=1}^{5} m_i \implies \sum_{i=1}^{5} m_i \geq 21$$

and this implies that $m_{j_0} = \max\{m_i\} \geq 4$. Since the curve E_{j_0} is not contracted by $|3L - \sum_{i=1}^{5} E_i|$ we must have

$$\left(-\sum_{i=1}^{5} E_i\right) E_{j_0} > 0 \implies \left(-\sum_{i=1}^{5} m_i E_i\right) E_{j_0} \geq m_j \left(-\sum_{i=1}^{5} E_i\right) E_{j_0} \geq m_j \geq 4$$

which is a contradiction. Hence we have shown that S cannot be a complete intersection of the two quadrics Q_1, Q_2.

Suppose $Q_1 \cap Q_2 = S$ is reducible.

Then the component of S containing C_{13} is an irreducible non degenerate surface of degree ≤ 3. Therefore C_{13} is contained in either $\mathbb{F}(3,0)$ or $\mathbb{F}(1,2)$, the cone over a rational normal curve of degree 3, or the cubic surface scroll, respectively. In the first case, by projecting off the vertex we obtain a pencil of degree ≤ 4 and this is not possible.

Assume that $C_{13} \subset \mathbb{F}(1,2)$. Write H for the hyperplane section of $\mathbb{F}(1,2)$ and L for the class of the ruling of $\mathbb{F}(1,2)$. We have $C_{13} = aH + bL$. By adjunction we get

$$3a^2 + 2ab - 5a - 2b = 26; \tag{4.2}$$

and since $C_{13}H = 13$, $b = 13 - 3a$. Substituting we see that (4.2) has no integer solutions. We have proved

$$\mathcal{C} \text{ has no } g^2_9 \implies \text{sym}^2 \text{ is surjective.}$$

Conversely, suppose that sym^2 is surjective. Let us start by showing that this implies that $\text{gon}(\mathcal{C}) \geq 6$. We will make use of Lemma III.41 on page 70. Suppose there exists a divisor on $\mathcal{C}$ with $h^0(D) = 2$ and $\deg(D) \leq 5$. Since

$$\dim H^0(A) - \dim H^0(A - (A - D)) - 1 = 2,$$

applying Lemma III.41, we deduce that $h^0(A - D) \geq 8 - \deg(D)$. Hence

$$\dim H^0(A) - \dim H^0(A - D) - 1 \leq \deg(D) - 4.$$

In particular $\deg(D) = 4$ or 5. By the same lemma, we deduce that $\deg(D) - 2 \leq 2$, if $\deg(D) = 5$, or that $\deg(D) - 2 \leq 0$, if $\deg(D) = 4$. A contradiction in both cases. We have shown that $\mathrm{gon}(\mathcal{C}) \geq 6$.

Assume that for some $d \leq 9$, there exists a free g^2_d on $\mathcal{C}$ and let us denote it by η. Since $\mathrm{gon}(\mathcal{C}) \geq 6$ we have $8 \leq d \leq 9$. By Proposition III.2 we have $\dim H^0(A-\eta) > 0$. In fact since $\deg(A - \eta) \leq 5$ we actually have $\dim H^0(A - \eta) = 1$. On the other hand, let D be a divisor of degree d such that $\eta \subset H^0(D)$. Then $\dim H^0(D) \geq 3$ and therefore,

$$\dim H^0(A) - \dim H^0(A - (A - D)) - 1 \leq 1,$$

thus, from Lemma III.41, we deduce that

$$\deg(A - D) - \dim H^0(A - D) \leq 2 \Longleftrightarrow \dim H^0(A - D) \geq 2$$

which is a contradiction. We have proved

$$\mathrm{sym}^2 \text{ surjective} \implies \text{there exists no } g^2_9$$

and so finished the proof of Proposition IV.25. $\square$

IV.4.2. The ring $R(\mathcal{C}, A)$.

PROPOSITION IV.28. *Let $\mathcal{C}$ be nonsingular curve of genus 14 with a halfcanonical divisor A such that $h^0(A) = 5$. Assume that sym^2 is surjective. Then $R(\mathcal{C}, A)$ is a codimension 3 ring generated in degree 1. The ideal I_A has a single generator in degree 2 and the quotient $I_{A,3}/I'_{A,3}$ has dimension 4.*

Proof. Since sym^2 is surjective, by Proposition IV.25 and its proof, the linear system $|A|$ is free and the gonality of $\mathcal{C}$ is ≥ 6. From Proposition III.5 we deduce that $R(\mathcal{C}, A)$ is generated in degree ≤ 3. Let us show that the map

$$H^0(A) \otimes H^0(2A) \to H^0(3A) \tag{4.3}$$

is surjective. According to Lemma III.7 to an element of the cokernel of (4.3) corresponds an extension

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A) \to 0 \tag{4.4}$$

with $h^0(\mathscr{F}) = 1+5 = 6$. In particular for any two $p, q \in \mathcal{C}$ there exists a section of $\mathscr{F}$ vanishing on $p+q$. Denote the divisor of zeros of such a section by $\delta \supset p+q$. Then saturating the embedding $\mathcal{O}_{\mathcal{C}}(\delta) \hookrightarrow \mathscr{F}$ we obtain

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A-\xi) \to 0$$

where $\xi \supset \delta$ is an effective divisor. Since p, q can be chosen general enough we have $h^0(A-\xi) \leq h^0(A)-2$ and accordingly $h^0(\xi) \geq 3$. Since $\mathcal{C}$ has no g^2_9 (Proposition IV.25) we deduce that $\deg(\xi) \geq 10$. But then $h^0(A-\xi) \leq 1$ since $\mathrm{gon}(\mathcal{C}) \geq 6$. Therefore $h^0(\xi) \geq h^0(A)$ and we must have $\xi \subset \mathcal{O}_{\mathcal{C}}(A)$. By the fact that A is free we conclude that $\xi = A$. In other words, an element of the cokernel of (4.4) corresponds to the split extension, i.e. the cokernel is null. We conclude that $R(\mathcal{C}, A)$ is generated in degree 1.

By our assumptions it is clear that there is exactly one generator of I_A in degree 2. In degree 3 the space $\mathrm{S}^3\langle x_1, \ldots, x_5\rangle$ must surject onto $H^0(3A)$. The former has dimension $\binom{7}{3} = 35$ and to this we have to subtract the 5 multiples of the quadratic generator of I_A. We still get an excess of 4 generators, since by RR, the dimension of $H^0(3A)$ is 26. Thus there will be 4 new cubic forms in $I_{A,3}$. $\square$

IV.4.3. Remark. This proposition fails to conclude the description of $R(\mathcal{C}, A)$ in degree ≤ 4. This means we will not be able to use Theorem III.14 on page 40 as we have been doing so far. One way to deal with this problem would be to try to run a Petri-style analysis on the generators of $R(\mathcal{C}, A)$. All attempts in this direction were unfruitful. Notice that such analysis would have to be specific to the genus 14 case. (Although the assumption that $W^2_9(\mathcal{C}) = \emptyset$ may point us to the right direction). This is supported by the fact that there are nontrigonal curves whose ring $R(\mathcal{C}, A)$ is generated in degree 1 and yet whose ideal I_A needs generators in degree 4. Consider the case of curves of genus 9 with a halfcanonical divisor A with $\dim H^0(A) = 4$. (See page 59). On the other hand, that the ideal I_A is indeed generated by $I_{A,2} + I_{A,3}$ is true by an application of Buchsbaum–Eisenbud's theorem.

IV.4.4. The bundle.

Proposition IV.29. *Let $\mathcal{C}$ be a nonsingular curve curve of genus* 14 *with a half-canonical divisor such that $h^0(A) = 5$. Assume that $W^2_9(\mathcal{C}) = \emptyset$. Then there exists a bundle $\mathscr{E}$ on $\mathcal{C}$ of rank* 2 *and determinant A with the following properties.*

(i) $\dim H^0(\mathscr{E}) = 4$.

(ii) *The map $H^0(A) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(A))$ has a cokernel of dimension* 1*. Let $t \in H^0(\mathscr{E}(A))$ span a complementary space to the image of this map.*

(iii) *The image of the map $H^0(\mathscr{E}) \xrightarrow{\wedge t} H^0(2A)$ is* 4*-dimensional.*

(iv) *The kernel of the map $\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$ is at most* 2*-dimensional.*

Proof. The first part of this proof works out quite differently for $\mathrm{gon}(\mathcal{C}) = 6$ than it does for $\mathrm{gon}(\mathcal{C}) = 8$ and we will have to break it into two cases. (Recall from the proof of Proposition IV.25 that if $\mathcal{C}$ has no g^2_9 either $\mathrm{gon}(\mathcal{C}) = 6$ or $\mathrm{gon}(\mathcal{C}) = 8$).

IV.4.5. Existence for $\mathrm{gon}(\mathcal{C}) = 8$**.** Suppose that $\mathcal{C}$ has $\mathrm{gon}(\mathcal{C}) = 8$. Then the map φ_A is an embedding onto a curve of $\mathbb{P}^4$ contained in a quadric of rank 5 (Proposition III.9). In other words,

$$\varphi_{|A|} \colon \mathcal{C} \to C_{13} \subset \mathrm{G}(2,4) \cap H \subset \mathbb{P}^5 \cap H.$$

View $\mathcal{C}$ as a subvariety of $\mathrm{G}(2,4)$. Consider the universal bundles of $\mathrm{G}(2,4)$.

$$0 \to \mathscr{F}_{\mathrm{G}(2,4)} \to 4\mathcal{O}_{\mathrm{G}(2,4)} \to \mathscr{E}_{\mathrm{G}(2,4)} \to 0.$$

The bundle $\mathscr{F}_{\mathrm{G}(2,4)}$ is the tautological subbundle of $\mathrm{G}(2,4)$. Its fibre over a point $[L] \in \mathrm{G}(2,4)$ is the vector space L. The quotient of $4\mathcal{O}_{\mathrm{G}(2,4)}$ by $\mathscr{F}_{\mathrm{G}(2,4)}$ is a bundle of rank 2 for which $\dim H^0(\mathscr{E}_{\mathrm{G}(2,4)}) = 4$. Let $\mathscr{E}$ be its restriction to $\mathcal{C}$. Consider the restriction sequence:

$$0 \to \mathscr{E}_{\mathrm{G}(2,4)} \otimes \mathcal{I}_{\mathcal{C}} \to \mathscr{E}_{\mathrm{G}(2,4)} \to \mathscr{E} \to 0$$

If $h^0(\mathscr{E}_{\mathrm{G}(2,4)} \otimes \mathcal{I}_{\mathcal{C}}) > 0$ then there exists a section of $\mathscr{E}_{\mathrm{G}(2,4)}$ vanishing on $\mathcal{C}$. This implies that $\mathcal{C} \subset \mathrm{G}(2,4)$ is contained in the corresponding Schubert cycle, which is a 3-dimensional linear space. This is impossible since $\varphi_A(\mathcal{C}) \subset \mathbb{P}^4$ is nondegenerate. Hence $h^0(\mathscr{E}) \geq 4$ and we also have $\det(\mathscr{E}) = \mathcal{O}(1)_{|\mathcal{C}} = A$. Consider

$$\bigwedge^2 H^0(\mathscr{E}) \to H^0(A).$$

If $h^0(\mathscr{E}) > 4$ then by Castelnuovo's linear-bilinear principle there exist two sections $s_1, s_2 \in H^0(\mathscr{E})$ spanning a line bundle $\mathcal{L}$, subsheaf of $\mathscr{E}$. Saturating this line bundle we get a sequence

$$0 \to \mathcal{O}_{\mathcal{C}}(\delta) \to \mathscr{E} \to \mathcal{O}_{\mathcal{C}}(A - \delta) \to 0.$$

Since $h^0(\delta) \geq 2$ and $\mathrm{gon}(\mathcal{C}) = 8$ we deduce that $\deg(\delta) \geq 8$ and thus $\deg(A - \delta) \leq 5$ which implies that $h^0(A - \delta) \leq 1$. But then $h^0(\delta) \geq 4$. Since $\mathcal{O}_{\mathcal{C}}(\delta) \subset \det \mathscr{E} = \mathcal{O}_{\mathcal{C}}(A)$ there are two cases to consider. Either $h^0(\delta) = 4$ or $h^0(\delta) = 5$. Suppose that $h^0(\delta) = 4$, then $|A - \delta| = |p|$ and the map

$$H^0(2A - \delta) \otimes H^0(A - \delta) \to H^0(3A - 2\delta)$$

is surjective, which means that the only extension of $\mathcal{O}_{\mathcal{C}}(A - \delta)$ by $\mathcal{O}_{\mathcal{C}}(\delta)$ with 5 global section splits. But then $\bigwedge^2 H^0(\mathscr{E})$ only spans a linear space of dimension 4 in $H^0(A)$. This is a contradiction since $\varphi_A(\mathcal{C})$ is nondegenerate and $H^0(\mathscr{E})$ contains the restriction of the global sections of $\mathscr{E}_{\mathrm{G}(2,4)}$. In the other case, when $h^0(A - \delta) = 5$ and consequently $\delta = A$ we argue exactly in the same way. We have shown that $h^0(\mathscr{E}) = 4$. Which settles (i) in the case of $\mathrm{gon}(\mathcal{C}) = 8$. Additionally, notice that in this case $\mathscr{E}$ is generated by its global sections.

Existence for $\mathrm{gon}(\mathcal{C}) = 6$**.** From now on, let ξ denote a fixed g^1_6 on $\mathcal{C}$.

Lemma IV.30. *The map*

$$H^0(2A - \xi) \otimes H^0(A - \xi) \to H^0(3A - 2\xi) \tag{4.5}$$

has a 1*-dimensional cokernel.*

Proof of the lemma. Denote $A - \xi$ by η. Since $\deg(\eta) = 7$, the dimension of $H^0(\eta)$ is ≤ 2. Furthermore, by Proposition III.2 indeed $h^0(A - \xi) \geq 2$. We have two possibilities. Either η is free, in which case the kernel of the map (4.5) is, by Castelnuovo's free-pencil trick, isomorphic to $H^0(A)$, or η has a (single) base point that we denote by p and the kernel of (4.5) is then isomorphic to $H^0(A + p)$. But as A is free $h^0(A - p) = 4$, so that by RR and Serre duality, $h^0(A + p) = 5$. Since

$$\dim H^0(3A - 2\xi) = 2(\deg(A) - \deg(\xi)) = 14,$$

$\dim H^0(2A - \xi) \otimes H^0(A - \xi) = 18$, we deduce that the cokernel of (4.5) has dimension $14 - 18 + 5 = 1$. $\square$

From this and Lemma III.7, it follows that there exists a single nonsplit extension

$$0 \to \xi \to \mathscr{E} \to \eta \to 0 \tag{4.6}$$

with maximum number of global sections, that is, $h^0(\mathscr{E}) = 4$. The vector bundle $\mathscr{E}$ is of rank 2 and determinant A. Notice that $\mathscr{E}$ is generated by its global sections if and only if η is free.

Proof of (ii) for $\operatorname{gon}(\mathcal{C}) = 8$**.** Since in this case the bundle $\mathscr{E}$ is spanned by its global section we have

$$0 \to \mathscr{F} \to 4\mathcal{O}_{\mathcal{C}} \to \mathscr{E} \to 0 \tag{4.7}$$

where $\mathscr{F}$ is a rank 2 vector bundle of determinant $-A$. In particular $\mathscr{F}(A)$, whose global sections give the kernel of

$$H^0(A) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(A)),$$

has determinant A. As such, by repeating the argument played on $\mathscr{E}$ when proving that $h^0(\mathscr{E}) \leq 4$ we show that $h^0(\mathscr{F}(A)) \leq 4$. Hence to show (ii) we need to find 4 linearly independent tensors in the kernel of (4.7). For each set of 3 elements of $\langle s_1, s_2, s_3, s_4 \rangle = H^0(A)$, say for example s_1, s_2, s_3, the tensor

$$(s_2 \wedge s_3) \otimes s_1 - (s_1 \wedge s_3) \otimes s_2 + (s_1 \wedge s_2) \otimes s_3$$

maps to zero.

Proof of (ii) for $\operatorname{gon}(\mathcal{C}) = 6$**.** By (4.6) it is enough to analyse the maps:

$$\begin{aligned} \sigma_1 \colon H^0(\xi) \otimes H^0(A) \to H^0(A + \xi) \\ \sigma_2 \colon H^0(\eta) \otimes H^0(A) \to H^0(A + \eta). \end{aligned} \tag{4.8}$$

Recall that ξ is a free pencil, hence $\dim(\operatorname{Ker} \sigma_1) = h^0(A - \xi) = 2$. Since

$$h^0(A + \xi) = h^0(\eta) + \deg(\xi) = 8$$

(by RR and Serre duality) it follows that σ_1 has no cokernel. So the cokernel must come from σ_2. The kernel of this map is isomorphic to $H^0(A - \eta)$, in the case when η is free, and is isomorphic to $H^0(A - \eta + p)$ in the case when η has a base point. In the free case it is 2-dimensional, but it is 2-dimensional as well in the non free case since $A - \eta + p$ has degree 7 and by assumption there are no g^2_9 on $\mathcal{C}$. Finally, as the dimension of $H^0(A+\eta)$ equals to 9 we deduce that the cokernel of σ_2 has dimension $9 - 10 + 2 = 1$. This shows item (ii). Notice that by Serre duality and RR,

$$h^0(\mathscr{E}_{\mathcal{C}}(A)) = h^1(\mathscr{E}_{\mathcal{C}}) = 4 + \deg(A) = 17,$$

and on the other hand the dimension of $H^0(A) \otimes H^0(\mathscr{E})$ is 20. Thus there are 4 linearly independent tensors in the kernel of the map of item (ii). As before we can write them explicitly.

Proof of (iii). If $\mathrm{gon}(\mathcal{C}) = 6$ then from (4.6) we see that

$$0 \to A + \xi \to \mathscr{E}(A) \to A + \eta \to 0$$

and since, by RR, both $A+\xi$ and $A+\eta$ are free we deduce that $\mathscr{E}(A)$ is spanned by its global sections. If $\mathrm{gon}(\mathcal{C}) = 8$ the vector bundle $\mathscr{E}$ is globally spanned and therefore so is $\mathscr{E}(A)$. Let us denote the divisor of zeros of $t \in H^0(\mathscr{E}(A))$ by δ. Wedging sections of $\mathscr{E}$ with t produces the following surjective morphism:

$$\mathscr{E} \xrightarrow{\wedge t} \mathcal{O}_{\mathcal{C}}(2A - \delta) \to 0$$

The kernel is a sheaf of rank 1 and therefore, as we are working on a nonsingular curve, must be invertible. Taking determinants we deduce that the kernel of the morphism above is $\mathcal{O}_{\mathcal{C}}(\delta - A)$. If $h^0(\delta - A) > 0$, then $t \in H^0(A) \otimes H^0(\mathscr{E})$ and this is not true. Hence $h^0(\delta - A) = 0$. We conclude that the space $H^0(\mathscr{E}) \wedge t \subset H^0(2A)$ has dimension 4.

Proof of (iv). Let us denote the kernel of the map

$$\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$$

by W. If $\mathrm{gon}(\mathcal{C}) = 8$ then we know that $\dim W = 1$ and we are done. Thus we can assume that $\mathrm{gon}(\mathcal{C}) = 6$. $\mathbb{P}[W]$, the projectivised of W is a linear subspace of $\mathbb{P}(\bigwedge^2 H^0(\mathscr{E}))$, which also contains $\mathrm{G}(2, H^0(\mathscr{E}))$, the variety of skew tensors of rank 2.

Lemma IV.31. *There exists a injective map*

$$\mathbb{P}[W] \cap \mathrm{G}(2, H^0(\mathscr{E})) \to W^1_6(\mathcal{C}).$$

Proof of the lemma. Let $a \wedge b$ represent an element of $\mathbb{P}[W] \cap \mathrm{G}(2, H^0(\mathscr{E}))$. By definition $a, b \in H^0(\mathscr{E})$ span a subsheaf of $\mathscr{E}$ which after saturated yields a line subbundle:

$$0 \to \mathcal{O}_{\mathcal{C}}(\delta) \to \mathscr{E} \to \mathcal{O}_{\mathcal{C}}(A - \delta) \to 0.$$

Since $\mathrm{gon}(\mathcal{C}) = 6$ we deduce that $\deg(\delta) \geq 6$. But if $h^0(\mathscr{E}(-\delta)) > 0$ (as we see from above) then from (4.6) we deduce that $\deg(\delta) \leq 7$. Suppose that $\deg(\delta) = 7$. Then $A - \delta$ is a free pencil and since

$$h^0(\mathscr{E}(-\delta)) > 0 \implies \delta \subset \eta \implies \xi \subset A - \delta,$$

if follows that $A - \delta = \xi$ and thus $\mathscr{E}$ would be split. As this is not the case δ has degree 6 and is therefore a g^1_6. This establishes a map

$$\mathbb{P}[W] \cap \mathrm{G}(2, H^0(\mathscr{E})) \to W^1_6(\mathcal{C}). \tag{4.9}$$

Let us show now that this map is injective. Take $a \wedge b$ and $c \wedge d$, two distinct elements of $\mathbb{P}[W] \cap \mathrm{G}(2, H^0(\mathscr{E}))$ giving rise to the same g^1_6, which we denote by ξ_1. This means that there are two distinct embeddings of ξ_1 into $\mathscr{E}$. In other other words $\mathscr{E}(-\xi_1) \geq 2$. But then, from (4.6) we deduce that $\xi_1 = \xi$ (the g^1_6 we have fixed from the beginning) and furthermore, $\xi \subset \eta$. Summing up, we have

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathscr{E}(-\xi) \to \mathcal{O}_{\mathcal{C}}(p) \to 0.$$

To see this cannot happen, notice that

$$H^0(2A) \otimes H^0(p) \to H^0(2A + p)$$

is surjective, and hence any extension like $\mathscr{E}(-\xi)$ having 2 global sections must split. Therefore the map (4.9) is injective. $\square$

We use this lemma to proof (iv). Suppose that $\dim \mathbb{P}[W] \geq 2$. Then, the intersection $\mathbb{P}[W] \cap \mathrm{G}(2, H^0(\mathscr{E}))$ is positive dimensional and so $W^1_6(\mathcal{C})$ is also positive dimensional. It follows from Proposition III.9 that $\mathbb{P}[\mathrm{Ker}\,\mathrm{sym}^2]$ must as well, be positive dimensional (one quadric yields at most two distinct g^1_6). However, by Proposition IV.25 the map sym^2 is surjective and consequently $\mathbb{P}[\mathrm{Ker}\,\mathrm{sym}^2]$ is a point. Therefore, $\mathbb{P}[W]$ has dimension strictly less that 2, that is to say

$$\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$$

has a kernel of dimension at most 2. $\square$

Proposition IV.32. *Let $\mathcal{C}$ be a nonsingular curve of genus 14 with a halfcanonical divisor such that $h^0(A) = 5$. Assume that $W_9^2(\mathcal{C}) = \emptyset$. Let $\mathscr{E}$ be a bundle on $\mathcal{C}$ given as in Proposition IV.29. Consider a polynomial ring $\mathbb{C}[v, m_{ij}, n_i]$ with*

$$\mathrm{wt}(M) = \mathrm{wt}\begin{pmatrix} m_{12} & m_{13} & m_{14} & n_1 \\ & m_{23} & m_{24} & n_2 \\ & & m_{34} & n_3 \\ & & & n_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{pmatrix} \tag{4.10}$$

and $\mathrm{wt}(v) = 1$. Let $\langle s_1, s_2, s_3, s_4\rangle$ be a choice of basis of $H^0(\mathscr{E})$ and $t \in H^0(\mathscr{E}(A))$ be an element spanning a complementary space to the image of the map of (ii) *of Proposition IV.29. Choose $\langle u\rangle \subset H^0(A)$ a complementary space to the image of the map in* (iv) *of Proposition IV.29. Let ev^+ be the map $\mathrm{ev}^+ : \mathbb{C}[v, m_{ij}, n_i] \to R(\mathcal{C}, A)$ defined by*

$$v \mapsto u, \quad m_{ij} \mapsto s_i \wedge s_j \quad \textit{and} \quad n_i \mapsto s_i \wedge t.$$

Denote by $\mathrm{Pf}_1, \dots, \mathrm{Pf}_5$ the 5 submaximal Pfaffians of M. Then,

(i) ev^+ *is surjective.*

Let ev be a minimal surjection $\mathbb{C}[x_1, \dots, x_5] \to R(\mathcal{C}, A)$. Then, there exists a surjection, $\lambda : \mathbb{C}[v, m_{ij}, n_i] \twoheadrightarrow \mathbb{C}[x_1, \dots, x_5]$, such that $\mathrm{ev}^+ = \mathrm{ev} \circ \lambda$ and

(ii) $\lambda\{\mathrm{Pf}_1, \cdots, \mathrm{Pf}_5\} \subset I_A$.

(iii) $\lambda^{-1} I_{A,3} \subset (\mathrm{Pf}_1, \dots, \mathrm{Pf}_5)$.

Proof. Item (i) is straightforward. Likewise item (ii) is a consequence of the fact that $\mathscr{E}$ is a rank 2 bundle on $\mathcal{C}$ and therefore the matrix in (4.10) has rank 2 on $\mathcal{C}$. Finally the crucial point of this Proposition is item (iii).

Let us start with a few remarks. Notice that the map λ is not unique. Since by Proposition IV.28, $R(\mathcal{C}, A)$ has a quadric relation, a minimal surjection like ev is uniquely determined (up to linear isomorphism) only in degree 1, simply by setting $\lambda(c) = \mathrm{ev}^{-1} \circ \mathrm{ev}^+(c)$ for any element $c \in \mathbb{C}[v, m_{ij}, n_i]$ of degree 1. However, λ_2 is unique only up to $\mathrm{Ker}\,\mathrm{ev}_2$. Nevertheless let us make the following observation. Since by item (iii) of Proposition IV.29 the space $\langle s_1, s_2, s_3, s_4\rangle \wedge t \subset H^0(2A)$ is 4-dimensional, we have, in particular that $\mathrm{ev}^+(n_i) \neq 0$ and therefore $\lambda(n_i) \neq 0$ for any choice of λ. We should bear this mind as we will use it in the end of this proof. To show (iii), by Proposition IV.28, it will be enough to prove the following lemma.

LEMMA IV.33. *Given a surjective homomorphism* $\lambda\colon \mathbb{C}[v, m_{ij}, n_i] \to \mathbb{C}[x_1, \dots, x_5]$, *such that* $\mathrm{ev} \circ \lambda = \mathrm{ev}^+$, *we have*

(i) $\lambda(\mathrm{Pf}_5) \neq 0$.

(ii) *The set* $\lambda\{\mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4\}$ *is linearly independent modulo*

$$I'_{A,3} = \langle x_1, \dots, x_5 \rangle \cdot \lambda(\mathrm{Pf}_5).$$

Proof of the lemma. The form Pf_5 is a quadric of rank 6 in the variables m_{ij}. By item (iv) of Proposition IV.29, the restriction of the map λ to the linear space $\langle m_{ij} \rangle$ has at most a 2-dimensional kernel but since Pf_5 is a quadric of rank 6 this implies that $\lambda(\mathrm{Pf}_5) \neq 0$. In fact, as quadric in the variables $x_1, \dots, x_5$ the quadric $\lambda(\mathrm{Pf}_5)$ has rank ≥ 3. From this we deduce that $\lambda(\mathrm{Pf}_5)$ can be taken as a generator of the space $I_{A,2}$. In particular the assertion of item (ii) is independent of the ambiguity of the definition of λ in degree 2. Suppose that there exist $\alpha_i \in \mathbb{C}$ and $L \in \langle x_1, \dots, x_5 \rangle$ such that

$$\sum_{i=1}^{4} \alpha_i \lambda(\mathrm{Pf}_i) + L\lambda(\mathrm{Pf}_5) = 0.$$

In particular

$$\mathrm{ev}\left(\sum_{i=1}^{4} \alpha_i \lambda(\mathrm{Pf}_i) + L\lambda(\mathrm{Pf}_5)\right) = \sum_{i=1}^{4} \alpha_i \,\mathrm{ev}^+(\mathrm{Pf}_i) + \mathrm{ev}(L)\,\mathrm{ev}^+(\mathrm{Pf}_5) = 0$$

By (i) of this lemma, we can assume that $\alpha_1 \neq 0$. And indeed to ease notation we set it to 1. Consider the following set of sections of $\mathscr{E}$ and $\mathscr{E}(A)$:

$$a_i = s_i + \alpha_i s_1 \quad \text{and} \quad b = t + \mathrm{ev}(L)s_1$$

where $i = 2, 3$ or 4. Let us also rename the section s_1 by a_1. Define a new surjection $\widetilde{\mathrm{ev}}^+$, using the same procedure as for ev^+ but using $\langle a_1, a_2, a_3, a_4 \rangle$ and $b \in H^0(\mathscr{E}(A))$, and maintaining the choice of $u \in H^0(A)$. Also choose a surjection $\widetilde{\lambda}$ defined by

$$c \mapsto \mathrm{ev}^{-1}(\widetilde{\mathrm{ev}}^+(c)) \quad \text{for} \quad c \in \langle v, m_{ij} \rangle \quad \text{and} \quad \widetilde{\lambda}(n_i) = \lambda(n_i).$$

Then, $\widetilde{\lambda}(m_{ij}) = \lambda(m_{ij}) - \alpha_i\lambda(m_{1j}) - \alpha_j\lambda(m_{1i})$ where m_{11} is interpreted as 0. Therefore, we have that

$$\widetilde{\lambda}(\mathrm{Pf}_1) = \sum_{i=1}^{4} \alpha_i \lambda(\mathrm{Pf}_i) + L\lambda(\mathrm{Pf}_5) = 0.$$

Hence we have reduced the question to assuming that $\lambda(\mathrm{Pf}_1) = 0$. We argue by contradiction. Suppose that

$$\lambda(\mathrm{Pf}_1) = \lambda(m_{23})\lambda(n_4) - \lambda(m_{24})\lambda(n_3) + \lambda(m_{34})\lambda(n_2) = 0.$$

Since by item (iv) of Proposition IV.29, ev^+ restricted to $\langle m_{23}, m_{24}, m_{34}\rangle$ has at most a 2-dimensional kernel, and therefore λ restricted to $\langle m_{23}, m_{24}, m_{34}\rangle$ has at most a 2-dimensional kernel, we can assume that $\lambda(m_{23}) \neq 0$. Furthermore, proceeding as we did before in the proof of this lemma, we can change the basis $\langle s_2, s_3, s_4\rangle$, the section t modulo $H^0(A) \otimes H^0(\mathscr{E})$ and the map λ to assume, without loss of generality, that on the right hand side of

$$\lambda(m_{23})\lambda(n_4) = \lambda(m_{24})\lambda(n_3) - \lambda(m_{34})\lambda(n_2)$$

we have polynomial of zero degree in the linear form $\lambda(m_{23})$. But then necessarily $\lambda(n_4) = 0$, which is not true. $\square$

IV.4.6. Proof of Theorem IV.24.

PROPOSITION IV.34. *Let $\mathcal{C}$ be a nonsingular curve of genus 14 with a halfcanonical divisor A such that $h^0(A) = 5$. Assume that $W^2_9(\mathcal{C}) = \emptyset$. Let $\mathscr{E}$ be any bundle of rank 2 and determinant A as in this Proposition IV.29. Choose a basis $\langle s_1, s_2, s_3, s_4\rangle \subset H^0(\mathscr{E})$, a section $t \in H^0(\mathscr{E}(A))$ spanning a complementary space to $H^0(A) \otimes H^0(\mathscr{E}) \subset H^0(\mathscr{E}(A))$ and a section $u \in H^0(A)$, spanning a complementary space to the image of $\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$. Denote by X the generalised weighted Grassmannian $\mathbb{P}(1)\ltimes\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$. Define a map η, from $\mathcal{C}$ into X by*

$$\mathcal{C} \ni p \mapsto (u(p), s_i \wedge s_j(p), s_i \wedge t(p)) \in X \subset \mathbb{P}(1, 1^6, 2^4).$$

Then

(i) *η is an embedding.*

(ii) *Let* $\lambda\colon \mathbb{C}[v, m_{ij}, n_i] \to \mathbb{C}[x_1, \dots, x_5]$ *be the as in Proposition IV.32. The forms generating the ideal* $\operatorname{Ker}\lambda$ *cut out in* X *a* 1*-dimensional scheme containing* $\eta(\mathcal{C})$*.*

Proof. We have the following commutative diagram.

$$\begin{tikzcd} C \arrow[r, "\eta"] \arrow[d, "\varphi_A"'] & X \arrow[d] \\ \mathbb{P}[H^0(A)] \arrow[r, "\mathbb{P}(\lambda)"] & \mathbb{P}(1, 1^6, 2^4) \end{tikzcd} \tag{4.11}$$

where $\mathbb{P}(\lambda)$ is the projectivised of the homomorphism λ. By Proposition IV.28, $R(\mathcal{C}, A)$ is generated in degree 1 hence the map φ_A is an embedding. The homomorphism λ is surjective, thus $\mathbb{P}(\lambda)$ is an embedding. Since $\eta = \varphi_A \circ \mathbb{P}(\lambda)$ this map must also be an embedding.

Since $\lambda(\mathrm{Pf}_1), \dots, \lambda(\mathrm{Pf}_5) \subset I_A$ and the forms of I_A vanish on $\varphi_A(\mathcal{C}) \subset \mathbb{P}^4$, we deduce that the intersection of

$$\lambda(\mathrm{Pf}_1) = \cdots = \lambda(\mathrm{Pf}_5) = 0 \tag{4.12}$$

contains $\varphi_A(\mathcal{C})$. The subscheme of $\mathbb{P}[H^0(A)]$ defined by (4.12) is the preimage of the subscheme of X cut out by the forms of $\operatorname{Ker}\lambda$. Therefore to prove (ii) is equivalent to showing that the equations of (4.12) define a subscheme of $\mathbb{P}[H^0(A)]$ of dimension 1. Let us the denote by S the irreducible component of (4.12) containing $\varphi_A(\mathcal{C})$. We write $\lambda(n_i) = q_i \in \mathrm{S}^2 \langle x_1, \dots, x_5\rangle$. Recall from Lemma IV.33 that $\lambda(\mathrm{Pf}_5)$ is a nonzero quadratic polynomial in the variables $x_1, \dots x_5$ and that $\lambda(\mathrm{Pf}_1), \dots \lambda(\mathrm{Pf}_4)$ are cubic forms in the same variables such that no linear combination of $\lambda(\mathrm{Pf}_1), \dots \lambda(\mathrm{Pf}_4)$ is a multiple of $\lambda(\mathrm{Pf}_5)$.

To make the argument clear we split the proof in two cases.

Suppose that map $\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$ is surjective.

In this case $\operatorname{Ker}\lambda$ has a single generator in degree 1, in other words we do not need to take a cone; the variety X is the weighted Grassmannian $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$. This can be made consistent with our notation for $v \in \mathbb{C}[v, m_{ij}, n_i]$ and $u \in H^0(A)$ by setting

$u = 0$ and $\mathrm{ev}^+(v) = 0$. Let us write (4.12) in a matrix format.

$$\mathrm{Pf}\begin{pmatrix} \lambda(m_{12}) & \lambda(m_{13}) & \lambda(m_{14}) & q_1 \\ & \lambda(m_{23}) & \lambda(m_{24}) & q_2 \\ & & \lambda(m_{34}) & q_3 \\ & & & q_4 \end{pmatrix} = 0 \tag{4.13}$$

where $\langle \lambda(m_{ij}) \rangle = \langle x_1, \dots, x_5 \rangle$. We deduce that the locus of $\mathbb{P}[x_1, \dots, x_5]$ defined by (4.13) is isomorphic to a hyperplane section of

$$\mathrm{Pf}\begin{pmatrix} z_1 & z_2 & z_3 & q_1 \\ & z_4 & z_5 & q_2 \\ & & z_6 & q_3 \\ & & & q_4 \end{pmatrix} = 0. \tag{4.14}$$

(All we did was to change variables $\langle x_1, \dots, x_5 \rangle \subset \langle z_1, \dots, z_6 \rangle$, in particular notice that we are not saying that $z_6 = 0$ is the hyperplane section). Let us denote the linear form defining the hyperplane section by $L = L(z_1, \dots, z_6)$ so that $\mathbb{P}[H^0(A)]$ is given by $L = 0$. Let $\widehat{S}$ denote the intersection of the five Pfaffians of (4.14). Hence $S \subset \widehat{S} \cap \mathbb{P}[H^0(A)]$. We want to show that $\dim \widehat{S} = 2$ and that $\widehat{S}$ is not contained in $\mathbb{P}[H^0(A)]$ for this will show that $\dim S = 1$.

Claim. *If the dimension of $\widehat{S}$ is 2 then $\widehat{S}$ is not contained in $\mathbb{P}[H^0(A)]$.*

Proof of the claim. Since $\widehat{S}$ inside $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ is given as the intersection of 4 hypersurfaces of degree 2, the dimension of $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ is 6 and this variety is arithmetically Cohen–Macaulay, if the dimension of $\widehat{S}$ is 2 we deduce that these 4 quasihomogeneous forms form a regular sequence in the weighted Grassmannian's homogeneous ring. (Proposition II.15). Since $H^i(\mathcal{O}_{\mathbb{G}}(j)) = 0$ for $0 < i < 6$, (see Proposition IV.5) and we are taking a regular sequence of quadratic forms, we deduce that

$$H^0(\mathcal{O}_{\mathbb{G}}(1)) \twoheadrightarrow H^0(\mathcal{O}_{\widehat{S}}(1)).$$

From the definition of $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ we know that

$$H^0(\mathcal{O}_{\mathbb{P}}(1)) \twoheadrightarrow H^0(\mathcal{O}_{\mathbb{G}}(1)),$$

where $\mathbb{P} = \mathbb{P}[m_{ij}, n_i]$. Hence $\widehat{S}$ is not contained in any hyperplane. In particular it is not contained in $\mathbb{P}[H^0(A)]$. $\square$

To show $\dim \widehat{S} = 2$ we argue by contradiction. From now on we assume that the component of $\widehat{S}$ containing $\varphi_A(\mathcal{C})$ has dimension ≥ 3. Let us denote this component by $\overline{S} \subset \widehat{S}$.

Claim. *There is only one quadric through $\overline{S}$, namely the quadratic Pfaffian of (4.14); and the cubic Pfaffians of (4.14) are linearly independent modulo the quadratic Pfaffian. In particular* $\dim \overline{S} \leq 3$.

Proof of the claim. Suppose that there is a second quadric through $\overline{S}$. Denote its equation by Q. Also, denote by Q_6 the quadratic Pfaffian:

$$z_1 z_6 - z_2 z_5 + z_4 z_3.$$

Recall that we are assuming in the main statement of this proposition that $W_9^2(\mathcal{C}) = \emptyset$. As a consequence, by Proposition IV.25, there is only one quadric of $\mathbb{P}[H^0(A)]$ through $\varphi_A(\mathcal{C})$. Since $\varphi_A(\mathcal{C}) \subset \overline{S} \cap \mathbb{P}[H^0(A)]$ this implies that there is only one quadric of $\mathbb{P}[H^0(A)]$ through $\overline{S} \cap \mathbb{P}[H^0(A)]$. Thus we have

$$Q_6 - Q = LA$$

with A another linear form in the variables $z_1, \ldots, z_6$. In particular this means that $\overline{S} \subset \mathbb{P}[H^0(A)]$. Since we are assuming that $\dim \overline{S} \geq 3$ and $\mathbb{P}[H^0(A)]$ is 4-dimensional, this implies that $\overline{S} \cap \mathbb{P}[H^0(A)] = Q_6 \cap \mathbb{P}[H^0(A)]$. In particular all the cubic Pfaffians of (4.13) are multiples of the quadratic Pfaffian. This is a contradiction. Furthermore if there exists a linear combination of the cubic Pfaffians of (4.14) then its restriction to $\mathbb{P}[H^0(A)]$ givens a linear combination between the cubic Pfaffians of (4.13), which has to be trivial. $\square$

Let us summarise our argument up to now. We are assuming that $\overline{S}$, the component of $\widehat{S} \subset \mathbb{P}[z_1, \ldots, z_6]$ containing $\varphi_A(\mathcal{C}) \subset \mathbb{P}[H^0(A)] \subset \mathbb{P}[z_1, \ldots, z_6]$ has dimension ≥ 3, that there is only one quadric though it, Q_6 and that the cubic Pfaffians of (4.14) are linearly independent modulo Q_6.

The variety $\overline{S}_{\text{red}}$ is a Weil divisor in the smooth quadric 4-fold, V, given by $Q_6 = 0$. By Lefschetz's hyperplane theorem $\text{Pic}(V) = \mathbb{Z}H_V$. Hence $\overline{S} = nH_V$. In particular $\overline{S}$ is the complete intersection of V with some hypersurface of degree n. Since $\overline{S}$ is also contained in a cubic hypersurface whose defining equation is not a multiple of Q_6 we deduce that $n \leq 3$. If $n = 3$ then $\overline{S}$ is the complete intersection of V and the hypersurface cut out by one of the cubic Pfaffians. But then, there cannot be 4 of them linearly independent through $\overline{S}$. If $n = 2$ then there are two linearly independent quadrics through $\overline{S}$, which again is not true. If $n = 1$ then $\overline{S}$ is contained in a hyperplane section, since $\varphi_A(\mathcal{C}) \subset \overline{S}$ and $\varphi_A(\mathcal{C})$ is nondegenerate in $\mathbb{P}[H^0(A)]$ this hyperplane can only be $\mathbb{P}[H^0(A)]$. But then repeating the argument of above on dimension, we see that this leads to a contradiction.

We have shown that the variety $\overline{S}$, the component of $\widehat{S}$ containing $\varphi_A(\mathcal{C})$, has dimension ≤ 2. This finishes the proof of (ii) in the case when $\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$ is surjective.

Suppose that $\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$ is not surjective.

Let us start by writing $\lambda(\mathrm{Pf}_i)$ in the form:

$$\mathrm{Pf}\begin{pmatrix} z_1 & z_2 & z_3 & q_1 \\ & z_4 & z_5 & q_2 \\ & & z_6 & q_3 \\ & & & q_4 \end{pmatrix} = 0 \quad \text{plus} \quad L_1 = L_2 = 0 \tag{4.15}$$

where the forms q_i are quasihomogeneous polynomials of degree 2 in the variables $z_1, \ldots, z_6, v$ and L_1, L_2 are linear forms in $z_1, \ldots, z_6$. Notice that $\mathbb{P}[H^0(A)]$ is the linear section $L_1 = L_2 = 0$.

Recall from the proof of item (iv) in Proposition IV.29 on page 110 that there are at most two elements of the kernel of $\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$ meeting $\mathrm{G}(2, H^0(\mathscr{E}))$, the variety of skew tensors of rank 2. This means that there exists an element of the kernel of this map that is a skew tensor of rank 4. In other words in the pencil $\langle L_1, L_2 \rangle$ there exists a linear form defining a hyperplane that is not tangent to Q_6, the quadratic Pfaffian of (4.15). Say $L_1 = 0$ is not tangent. Let us show that

$$\mathrm{Pf}\begin{pmatrix} z_1 & z_2 & z_3 & q_1 \\ & z_4 & z_5 & q_2 \\ & & z_6 & q_3 \\ & & & q_4 \end{pmatrix} = 0 \quad \text{plus} \quad L_1 = 0 \tag{4.16}$$

defines a variety of dimension ≤ 2 not contained in any hyperplane other than $L_1 = 0$. This will be enough to prove item (ii). Denote the component of the intersection (4.16) containing $\varphi_A(\mathcal{C})$ by $\widehat{S}$.

Claim. *If* $\dim \widehat{S} = 2$ *then* $\widehat{S}$ *is not contained in any hyperplane other than the given by* $L_1 = 0$.

Proof. We repeat the argument already used before. If $\dim \widehat{S} = 2$ then $\widehat{S}$ is the intersection of a regular sequence in X made of one quasihomogeneous form of degree 1 and 4 quasihomogeneous forms of degree 2. By the vanishing of cohomology of X (see Proposition IV.9) we deduce that $\widehat{S}$ is contained in no hyperplane other than that given by $L_1 = 0$. $\square$

To show that $\dim \widehat{S} = 2$ we argue by contradiction. Suppose that $\dim \widehat{S} \geq 3$.

Claim. *The variety* $\widehat{S}$ *is contained in a single quadric, namely that given by that quadratic Pfaffian of (4.16); and the cubic Pfaffians of (4.16) are linearly independent modulo the quadratic Pfaffian. In particular* $\dim \widehat{S} \leq 3$.

Proof of the claim. We repeat a previous argument. If there is another quadric through $\widehat{S}$ then $\widehat{S} \subset \mathbb{P}[H^0(A)]$ and this means that the cubic Pfaffians of (4.15) are linearly dependent modulo the quadric Pfaffian. Likewise any linear combination of the cubic Pfaffians of (4.16) as a multiple of the quadratic Pfaffian of (4.16) restricts to $\mathbb{P}[H^0(A)]$. $\square$

Set $v = L(z_1, \ldots, z_6)$ a general linear form. Denote by

$$\overline{S} = \widehat{S} \cap (v = L) \subset (v - L = L_1 = 0) \simeq \mathbb{P}^4,$$

the intersection $\widehat{S} \cap (v = L)$. By generality we can assume that $\overline{S}$ is of dimension 2, is contained in a single quadric Q_5 (the smooth quadric of rank 5, restriction to $\mathbb{P}^4 \simeq (v - L = L_1 = 0)$ of the quadratic Pfaffian, Q_6) and that the cubic Pfaffians when restricted to $v = L$ are linearly independent modulo Q_5.

If $\dim \overline{S} = 3$ then $\overline{S}_{\text{red}}$ is a Weil divisor in Q_5. By Lefschetz's hyperplane theorem the Picard group of Q_5 is generated by the hyperplane section. As in the previous case we deduce that $\overline{S}$ is cut out by a single hypersurface in Q_5. From here the proof follows exactly in the same way as before.

We have finished the proof of (ii) of Proposition IV.34. $\square$

Proof of Theorem IV.24. Consider the ideal $\operatorname{Ker} \lambda \subset \mathbb{C}[x_1, \ldots, x_5]$. Proposition IV.34 says that the image of $\mathcal{C}$ under the embedding η is contained in $Z(I) \cap X$.

It also says that the dimension of $Z(I) \cap X$ is 1. Moreover, since the degree of X is $\frac{13}{2^4}$ (see Proposition IV.9) we deduce that $\deg(\eta(\mathcal{C})) = \deg(X) \cdot 2^4$. Applying Proposition IV.2 we conclude that $I(\eta(\mathcal{C})) = I + (\mathrm{Pf}_1, \dots, \mathrm{Pf}_5)$. Which implies that $\eta(\mathcal{C})$ is a complete intersection in X. $\square$

Since $I_A = \lambda I(\eta(\mathcal{C})) = (\lambda(\mathrm{Pf}_1), \dots, \lambda(\mathrm{Pf}_5))$ we also have the following characterisation of $R(\mathcal{C}, A)$.

COROLLARY IV.35. *Let $\mathcal{C}$ be a nonsingular curve of genus 14 having a halfcanonical divisor A such that $h^0(A) = 5$. Assume that $W^2_9(\mathcal{C}) = \emptyset$. Then, the ring $R(\mathcal{C}, A)$ is a codimension 3 ring, generated in degree one. Moreover the ideal I_A is generated by the 5 submaximal Pfaffians of a skew matrix*

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & q_1 \\ & m_{23} & m_{24} & q_2 \\ & & m_{34} & q_3 \\ & & & q_4 \end{pmatrix}$$

where $m_{ij} \in \langle x_1, \dots x_5 \rangle \simeq H^0(A)$, span a subspace of $H^0(A)$ of dimension ≥ 4 (not necessarily the whole of $H^0(A)$) and q_i are general quadrics in the variables $x_1, \dots x_5$.

CHAPTER V

Applications to surfaces of general type

In this chapter we give two applications of the results of Chapter IV. We construct the canonical model of a surface of general type in a family of each of the following birational classes:

$$q = 0, \quad p_g = 4 \quad \text{and} \quad K^2 = 7,$$
$$q = 0, \quad p_g = 6 \quad \text{and} \quad K^2 = 13.$$

In both examples the basic idea is to explore the canonical linear system $|K_S|$ and derive our results from the previous chapters. Using a general member of $|K_S|$ and the hyperplane principle we obtain an initial description of the canonical ring $R(S, K_S)$. Next, we construct a vector bundle $\mathscr{E}$ on S which yields and embedding of the canonical model of the surface into a generalised weighted Grassmannian.

THEOREM V.1. *Let S be a nonsingular regular surface of general type with $p_g = 4$ and $K^2 = 7$. Assume that $|K_S|$ yields a birational morphism onto a surface with at most ordinary singularitites. Then the canonical model* $\operatorname{Proj} R(S, K_S)$ *is a complete intersection of four quasihomogeneous forms of degree* 2 *and one quasihomogeneous form of degree* 3 *in the generalised weighted Grassmannian* $\mathbb{P}(1)\ltimes\mathbb{G}(\tfrac{1}{2}^3, \tfrac{3}{2}^2)$.

THEOREM V.2. *Let S be a nonsingular regular surface of general type with $p_g = 6$ and $K^2 = 13$. Assume that the map* sym^2 *is surjective. Then the canonical model* $\operatorname{Proj} R(S, K_S)$ *is a complete intersection of four quasihomogeneous forms of degree* 2 *and one quasihomogeneous form of degree* 1 *in the generalised weighted Grassmannian* $\mathbb{P}(1)\ltimes\mathbb{G}(\tfrac{1}{2}^4, \tfrac{3}{2})$.

Conventions. Let S be a smooth surface. Following Kodaira's classification, the surface S is of general type if and only if for some $n \gg 0$ the map φ_{nK_S} is birational. In the modern terminology we say that *K_S is big.* Following the philosophy of the Minimal Model Programme, the assumption that K_S is *nef* implies that S is minimal:

if C is a -1-curve on S then by adjunction $K_S C < 0$. In this work we adopt the following definition.

Definition V.3. A surface S is of general type if and only if the canonical divisor K_S is nef and big.

For a nef divisor L on a projective reduced variety, L is big if and only if $L^n > 0$. Hence an equivalent definition of surface of general type is given by K_S nef and $K_S^2 > 0$. Accordingly, in the theorems above we can replace "of general type" by "with K_S nef."

V.1. The hyperplane principle

In this section we describe an important result that makes it possible to reduce the analysis of a graded ring R to a lower dimensional ring $\bar{R}$. This idea comes from geometry and corresponds to the basic operation of taking a hyperplane section of an embedded variety $X \subset \mathbb{P}^n$.

Theorem V.4. *Let R be a graded ring and $x_0 \in R_d$ a nonzero-divisor. There exists an exact sequence:*

$$0 \to R(-d) \xrightarrow{x_0} R \xrightarrow{\pi} \widehat{R} = R/(x_0) \to 0;$$

and we have:

(i) *If $\widehat{x_1}, \dots, \widehat{x_n} \in \widehat{R}$ generate the ring $\widehat{R}$ then choosing preimages $x_1, \dots, x_n$ under π of $\widehat{x_1}, \dots, \widehat{x_n}$ the elements $x_0, \dots, x_n \in R$ generate R.*

(ii) *If $f_1, \dots, f_m$ are relations between $\widehat{x_1}, \dots, \widehat{x_n}$, i.e. if*

$$\widehat{R} = \mathbb{C}[\widehat{x_1}, \dots, \widehat{x_n}]/(f_1, \dots, f_m)$$

then there exist $F_1, \dots, F_m$ between $x_0, \dots, x_n$ such that $\pi(F_i) = f_i$ and F_i generate the ideal $\operatorname{Ker} \operatorname{ev}$*, where*

$$\operatorname{ev}\colon \mathbb{C}[x_0, \dots, x_n] \to R.$$

Proof. Consider the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{J} \arrow[d] & \mathcal{I} \arrow[d] & \\
0 \arrow[r] & (x_0) \arrow[r] \arrow[d, dashed] & \mathbb{C}[x_0, \dots x_n] \arrow[r] \arrow[d] & \mathbb{C}[\widehat{x_1}, \dots, \widehat{x_n}] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R(-d) \arrow[r, "x_0"] & R \arrow[r, "\pi"] & \widehat{R} \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

where the two horizontal sequences are exact and the dashed morphism is unique morphism making the diagram commute. Let $fx_0 \in (x_0)$. Then under the dashed morphism fx_0 maps to $f \in R_{\deg(f)} = R(-d)_{\deg(fx_0)}$. This morphism is well defined since x_0 is a nonzero-divisor (in other words $(x_0) \subset R$ is a free R-module). Moreover, it is an isomorphism. Hence, by the snake lemma, we deduce that

$$\mathbb{C}[x_0, \dots, x_n] \to R$$

is surjective. In other words R is generated by x_0 and any choice of preimages of $\widehat{x_1}, \dots, \widehat{x_n}$ under π. At the same time we deduce that $\mathcal{J} \simeq \mathcal{I}$ as modules over $\mathbb{C}[x_0, \dots x_n]$. Suppose that $f_1, \dots, f_m$ are generators of $\mathcal{I}$ over $\mathbb{C}[\widehat{x_1}, \dots, \widehat{x_n}]$ then $f_1, \dots, f_m$ are generators of $\mathcal{I}$ over $\mathbb{C}[x_0, \dots, x_n]$. Therefore there exist $F_1, \dots F_m$ that generate $\mathcal{J}$ over $\mathbb{C}[x_0, \dots, x_n]$. Finally notice that the isomorphism $\mathcal{J} \xrightarrow{\sim} \mathcal{I}$ is simply the restriction of $\mathbb{C}[x_0, \dots, x_n] \to \mathbb{C}[\widehat{x_1}, \dots, \widehat{x_n}]$, which is uniquely defined by $x_i \mapsto \widehat{x_i}$ for $i \geq 1$ and $x_0 \mapsto 0$. $\square$

If S is a surface of general type and $|K_S|$ is not empty then we can take a "hyperplane section" corresponding to a nonzero-divisor, $f \in H^0(K_S)$ of degree 1 in the ring $R(S, K_S)$. If the corresponding section $(f = 0) = \mathcal{C} \in |K_S|$ is a nonsingular curve, we have automatically on $\mathcal{C}$ a halfcanonical divisor A given by adjunction as $K_{S|\mathcal{C}}$. The point is that under some assumptions, the ring $R(S, K_S)/(f)$ is the halfcanonical ring $R(\mathcal{C}, A)$.

Proposition V.5. *Let S be a nonsingular regular surface of general type. If $\mathcal{C} \in |K_S|$ is a nonsingular curve cut out by $f \in |K_S|$, then $R(\mathcal{C}, A) = R(S, K_S)/(f)$.*

Proof. By regularity and Kodaira vanishing the cohomology space $H^1(nK_S)$ is null for any integer n. Hence from the restriction exact sequence we deduce that

$$0 \to H^0((n-1)K_S) \xrightarrow{f} H^0(nK_S) \to H^0(\mathcal{C}, nA) \to 0;$$

and hence taking the direct sum of these sequences:

$$0 \to R(S, K_S)(-1) \to R(S, K_S) \to R(\mathcal{C}, A) \to 0$$

in other words, $R(\mathcal{C}, A) \simeq R(S, K_S)/(f)$. $\square$

This illustrates the role of the regularity of S in this work. There will be one more place where regularity of S is important. This is when we apply the vector bundle method to S.

V.2. Indecomposability

The notion of indecomposable linear system will be used for the first birational class of surfaces we study in this chapter, and will appear again for the surfaces with $p_g = 6$ and $K^2 = 13$. However indecomposability of $|K_S|$ for surfaces with $p_g = 4$ is almost automatic. Indeed it follows from the assumption that $|K_S|$ is base-point free. The latter is a classical case assumption in the study of a particular birational class of surfaces of general type. The point we wish to make is that the assumption of an indecomposable canonical linear system in the study of a birational class of surfaces of general type, when not itself a case assumption (distinguishing, for example, canonical rings of different codimension) is a generality assumption. This is not clear in the $p_g = 4$ and $K^2 = 7$ case but *it is* in the case $p_g = 6$ and $K^2 = 13$, where indecomposability translates to a condition on the rank of the quadric containing the canonical image of S. The motivation to introduce this notion comes from the vector bundle method. This is, off course, already clear in Mukai's work [**M95a**] and has appeared recently in Takagi's work of classification of $\mathbb{Q}$-Fano 3-folds [**T03**].

Definition V.6. Let X be a variety and D a divisor on X. We say that a complete linear system $|D|$, or simply D, is *decomposable* if there exist A and B Weil divisors on X such that A and B are mobile, i.e. $h^0(A)$, $h^0(B) \geq 2$ and

$$D = A + B.$$

We refer to A, B as a *mobile decomposition* of D. The linear system $|D|$ or simply D is called *indecomposable* if it has no mobile decompositions.

Even by requiring that $|K_S|$ be free we can still have $|K_S|$ decomposable. This can be checked against the case of regular surfaces of general type with $p_g = 6$ and $K^2 = 13$ below.

Proposition V.7. *Let X be a variety and D an effective divisor on X. Then D is decomposable if and only if there exists a symmetric tensor on rank 3 or 4 in the kernel of the map*

$$\mathrm{S}^2 H^0(D) \to H^0(2D). \tag{2.1}$$

Proof. The main idea has already showed up in the proof of Proposition III.9 on page 36. Suppose D is decomposable. Then there exist A and B Weil divisors such that

$$D = A + B$$

and $h^0(A)$, $h^0(B) \geq 2$. Let $V_1 \subset H^0(A)$ and $V_2 \subset H^0(B)$ be two subspaces of dimension 2. There is map

$$V_1 \otimes V_2 \to H^0(D) \tag{2.2}$$

defined by sending $(u, v) \in V_1 \times V_2$ to $vu \in H^0(D)$. If the kernel of this map is of dimension ≥ 2 its projectivised kernel in $\mathbb{P}[V_1 \otimes V_2]$, being at least one dimensional, must meet the hypersurface $\mathbb{P}[V_1] \times \mathbb{P}[V_2]$ and therefore* there exists $0 \neq u \in V_1$ and $0 \neq v \in V_2$ such that $uv = 0$, which is false. Thus the kernel has dimension ≤ 1. In other words, denoting by $\{u_1, u_2\}$ and $\{v_1, v_2\}$ bases for V_1 and V_2, respectively, there is a single linear relation holding between

$$u_1v_1,\ u_1v_2,\ u_2v_1,\ u_2v_2,\ \in H^0(D).$$

*This idea is Castelnuovo's linear-bilinear principle.

There is a relation holding between these generators:

$$(u_1v_1)(u_2v_2) - (u_2v_1)(u_1v_2) = 0$$

and by what we have showed, this tensor corresponds to an element of the kernel of (2.1) of rank 4 if the kernel of (2.2) is trivial, or of rank 3 in case that kernel is one dimensional.

Conversely if the image of the map $\varphi_{|D|}\colon X \to \Sigma \subset \mathbb{P}^n$ is contained in a quadric of rank 3 or 4 then the hyperplane section has a mobile decomposition (this is still true if $n = 2$ and X maps to a plane conic) and therefore $D = \varphi_*^{-1}(H)$ is decomposable. $\square$

V.3. Surfaces with $p_g = 4$ and $K_S^2 = 7$

Regular surfaces of general type with $p_g = 4$ and birational canonical map were systematically studied by Enriques in [**En**]. In the article [**C81**] Ciliberto constructs a family of regular surfaces of general type with $p_g = 4$, birational canonical morphism and K^2 in the set $\{5, 6, 7, 8, 9, 10\}$. See our introductory discussion on page 2 for general background.

If the canonical linear system $|K_S|$ is free then the canonical map φ_{K_S} maps S to a surface Σ of degree 7 in $\mathbb{P}^3$. In particular, there are no quadrics through the image of S and consequently $|K_S|$ is indecomposable. By Bertini's theorem, a general member $\mathcal{C} \in |K_S|$, is a nonsingular irreducible curve. By adjunction, it comes with a halfcanonical divisor $A = K_{S|\mathcal{C}}$. By the regularity of S the dimension of $|A|$ is 2. The canonical map φ_{K_S} restricts to $\mathcal{C}$ as the birational morphism

$$\varphi_A\colon \mathcal{C} \to C_7 \subset \mathbb{P}^2,$$

mapping $\mathcal{C}$ to a septic plane curve. Since the genus of $\mathcal{C}$ is 8, and by the genus formula $p_a(C_7) = 15$, the septic C_7 must be singular. A general curve $\mathcal{C} \in |K_S|$ like above, moves in a positive dimensional family. We conclude that Σ, the image of S by φ_{K_S}, has a singular locus of positive dimension. In particular there exists a curve $\gamma \subset \Sigma$ on which Σ is singular. We add to this the assumption that Σ has at most ordinary singularities. This means that only a finite number of points of γ are not ordinary double points of Σ and those that are not double points are triple points of Σ and are the origins of three linear branches of γ with non coplanar tangent lines. (See [**C81**]).

The following proposition gives a lower bound for the gonality of a nonsingular curve in $|K_S|$. It is clear that for most surfaces of general type a curve $\mathcal{C}$ in the canonical linear system cannot be Brill–Noether general. In fact if $p_g > 2$ then $\mathcal{C}$ has a halfcanonical linear system of positive dimension and then by Gieseker's result (see [**Gie**]) $\mathcal{C}$ cannot be Petri general. However, in the case of regular surfaces, the hyperplane principle imposes on the halfcanonical ring the same codimension as the canonical ring. Therefore we expect to get a lower bound for, say, the gonality of a nonsingular member of the canonical linear system.

Proposition V.8. *Let S be a nonsingular regular surface of general type with $p_g = 4$ and $K_S^2 = 7$. Assume that $|K_S|$ yields a birational morphism onto a surface with at most ordinary singularities. Then a nonsingular curve $\mathcal{C}$ in the canonical linear system $|K_S|$ is nontrigonal.*

Proof. A nonsingular member of $|K_S|$ is a curve of genus 8. The canonical divisor of S restricts to $\mathcal{C}$ as a halfcanonical divisor A. Since K_S is free so is A. Suppose that $\mathcal{C}$ has a g^1_3. Denote it by ξ. Then by Proposition III.2 on page 31 the space $H^0(A - 2\xi)$ is positive dimensional. Since $h^0(2\xi) \geq 3$ and $h^0(A) = 3$, we deduce that A cannot be free. This is a contradiction. Hence $\mathcal{C}$ is nontrigonal. $\square$

Proposition V.9. *Let S be a regular nonsingular surface of general type with $p_g = 4$ and $K^2 = 7$. Assume that $|K_S|$ yields a birational morphism onto a surface with at most ordinary singularities. Then there exists a nonsingular curve $\mathcal{C} \in |K_S|$ with* $\operatorname{gon}(\mathcal{C}) = 4$.

Proof. Let Σ denote the image of the canonical morphism. Since by assumption Σ has at most ordinary singularities, it follows that the number of triple points of γ (which will be triple points of Σ) equals $\binom{7-4}{3} = 1$. (The general formula for the number of triple points of the double curve of a surface in $\mathbb{P}^3$ with only ordinary singularities is $\binom{d-4}{3}$; where d is the degree of the surface. This is classical result by Enriques. Proofs are given in the original text by Enriques [**En**, p 174] and in Griffith and Harris' textbook [**GH**]). We deduce that a generic plane section of Σ through the triple point of γ is an irreducible curve with a triple point and double points at every other point of intersection with γ. Therefore, the inverse image under φ_{K_S} is a nonsingular curve in $|K_S|$ with a free g_4^1. $\square$

V.3.1. The ring $R(S, K_S)$. For a general curve $\mathcal{C} \in |K_S|$, $\mathcal{C}$ is a nonsingular curve of genus 8 with a free halfcanonical linear system A of dimension 2. Therefore by Corollary IV.23 the ring $R(\mathcal{C}, A)$ is a codimension 3 ring generated by elements of degree 1 and 2 and the ideal $\mathcal{I}_A$ is generated by two cubics and three quartics. Thus the following proposition is a straightforward application of the hyperplane principle, Theorem V.4.

Proposition V.10. *Let S be a nonsingular regular surface of general type with $p_g = 4$ and $K^2 = 7$. Assume that $|K_S|$ yields a birational morphism onto a surface with at most ordinary singularities. Then the ring $R(S, K_S)$ is a codimension 3 ring generated by elements of degree 1 and 2. Furthermore, the canonical ideal $\mathcal{I}_{K_S}$ is generated by two cubics and three quartics.* $\square$

What we show in the next paragraphs is that the Pfaffian format of $R(\mathcal{C}, A)$ can, as well, be pulled back to $R(S, K_S)$.

V.3.2. The bundle.

Proposition V.11. *Let S be a nonsingular regular surface of general type with $p_g = 4$ and $K^2 = 7$. Assume that $|K_S|$ yields a birational morphism onto a surface with at most ordinary singularities. Then there exists a vector bundle $\mathscr{E}$ on S of rank 2 and determinant K_S with the following properties:*

(i) $\dim H^0(\mathscr{E}) = 3$.
(ii) *The map $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$ is injective.*
(iii) *The map $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$ has a 2-dimensional cokernel.*

Proof. Let $\mathcal{C}$ be as given by Proposition V.9 and let ξ be a free g_4^1 on $\mathcal{C}$. The bundle $\mathcal{O}_{\mathcal{C}}(\xi)$ is globally generated and accordingly there exists a surjective morphism

$$2\mathcal{O}_S \twoheadrightarrow 2\mathcal{O}_{\mathcal{C}} \twoheadrightarrow \mathcal{O}_{\mathcal{C}}(\xi).$$

Let us denote the kernel of the map $2\mathcal{O}_S \to \mathcal{O}_{\mathcal{C}}(\xi)$ by $\mathscr{F}$. We have

$$0 \to \mathscr{F} \to 2\mathcal{O}_S \to \mathcal{O}_{\mathcal{C}}(\xi) \to 0 \tag{3.1}$$

The rank 2 sheaf $\mathscr{F}$ is locally free and is called an elementary modification of $2\mathcal{O}_S$ along $\mathcal{O}_\mathcal{C}(\xi)$. See [**HL**, pag. 129]. Denote the restriction of K_S to $\mathcal{C}$ by A and by $\mathscr{E}$ the dual of $\mathscr{F}$.

LEMMA V.12. *The dual sequence to (3.1) is*

$$0 \to 2\mathcal{O}_S \to \mathscr{E} \to \mathcal{O}_\mathcal{C}(A-\xi) \to 0. \tag{3.2}$$

In other words, $\mathscr{E}xt^1_S(\mathcal{O}_\mathcal{C}(\xi), \mathcal{O}_S) = \mathcal{O}_\mathcal{C}(A-\xi)$.

Proof of the lemma. From the exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_\mathcal{C} \to 0$$

applying the functor $\mathscr{H}om(\cdot, \mathcal{O}_S)$ we obtain $\mathscr{E}xt^1_S(\mathcal{O}_\mathcal{C}, \mathcal{O}_S) = \mathcal{O}_\mathcal{C}(A)$. Let $\delta \in |\xi|$ be an effective divisor. From the exact sequence of sheaves on S

$$0 \to \mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{C}(\delta) \to \mathcal{O}_\delta \to 0,$$

applying the functor $\mathscr{H}om(\cdot, \mathcal{O}_S)$ we obtain

$$0 \to \mathscr{E}xt^1_S(\mathcal{O}_\mathcal{C}(\delta), \mathcal{O}_S) \to \mathcal{O}_\mathcal{C}(A) \to \mathscr{E}xt^2_S(\mathcal{O}_\delta, \mathcal{O}_S) \to \mathscr{E}xt^2_S(\mathcal{O}_\mathcal{C}(\delta), \mathcal{O}_S) \to 0. \tag{3.3}$$

Applying $\mathscr{H}om(\cdot, \mathcal{O}_S)$ to the exact sequence

$$0 \to \mathscr{F} \to 2\mathcal{O}_S \to \mathcal{O}_\mathcal{C}(\delta) \to 0$$

we see that

$$0 \to 2\mathcal{O}_S \to \mathscr{E} \to \mathscr{E}xt^1_S(\mathcal{O}_\mathcal{C}(\xi), \mathcal{O}_S) \to 0$$

and additionally we deduce that $\mathscr{E}xt^2_S(\mathcal{O}_\mathcal{C}(\delta), \mathcal{O}_S) = 0$. In the exact sequence of (3.3) the sheaf $\mathscr{E}xt^1_S(\mathcal{O}_\mathcal{C}(\delta), \mathcal{O}_\mathcal{C})$ is supported on $\mathcal{C}$ (see [**HL**, pag. 5]) and as a sheaf on $\mathcal{C}$ is torsion free, therefore, since $\mathcal{C}$ is nonsingular, it is a locally free sheaf on $\mathcal{C}$. Since $\mathscr{E}xt^2_S(\mathcal{O}_\mathcal{C}(\delta), \mathcal{O}_S) = 0$ and $\mathscr{E}xt^2_S(\mathcal{O}_\delta, \mathcal{O}_S)$ is supported on δ, by taking determinants on $\mathcal{C}$ we deduce that

$$\mathscr{E}xt^1_S(\mathcal{O}_\mathcal{C}(\delta), \mathcal{O}_S) = \mathcal{O}_\mathcal{C}(A-\delta). \quad \square$$

From the sequence (3.2) we get $\det \mathscr{E} = K_S$. By invoking the regularity of S, item (i) is likewise straightforward. (Recall that ξ corresponds to the projection from a singular point of multiplicity 3 on the plane septic $C_7 = \varphi_A(\mathcal{C}) \subset \pi \simeq \mathbb{P}^2$).

Proof of (ii). To show (ii) we argue by contradiction. Suppose that there exists a nontrivial skew tensor in the kernel of

$$\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S).$$

Since $H^0(\mathscr{E})$ is 3-dimensional, such tensor is necessarily of rank 2. Which is to say that there exist two sections $s_1, s_2 \in H^0(\mathscr{E})$ such that $\mathcal{O}_S \cdot s_1 + \mathcal{O}_S \cdot s_2$ span a (torsion free) subsheaf of $\mathscr{E}$ of rank 1. Denote the saturation of $\mathcal{O}_S \cdot s_1 + \mathcal{O}_S \cdot s_2 \hookrightarrow \mathscr{E}$ by $\mathcal{L}_1$ and the torsion free quotient $\mathscr{E}/\mathcal{L}_1$ by $\mathcal{L}_2$. In virtue of

$$0 \to \mathcal{L}_1 \to \mathscr{E} \to \mathcal{L}_2 \to 0$$

we deduce that $K_S = \mathrm{c}_1(\mathcal{L}_1) + \mathrm{c}_2(\mathcal{L}_2)$. The sheaf $\mathcal{L}_1$ is locally free, whereas for some subscheme Z of codimension 2, the sheaf $\mathcal{L}_2$ equals $\mathscr{I}_Z \otimes \mathcal{L}_2^{\vee\vee}$. Since $s_1, s_2 \in H^0(\mathcal{L}_1)$ the system $|\,\mathrm{c}_1(\mathcal{L}_1)|$ is mobile. The existence of a nonzero morphism $\mathscr{E} \to \mathcal{L}_2$ implies that $h^0(\mathscr{E}^\vee \otimes \mathcal{L}_2) > 0$. As a consequence of $\mathscr{E}^\vee = \mathscr{F} \subset 2\mathcal{O}_S$ we get $\mathscr{E}^\vee \otimes \mathcal{L}_2 \subset 2\mathcal{L}_2$. Therefore we deduce that $h^0(\mathcal{L}_2) > 0$. Suppose that $\mathcal{L}_2 \simeq \mathcal{O}_S$. Then $\mathscr{E}$ is an extension of $\mathcal{O}_S$ by $\mathcal{O}_S(K_S)$. The group $\mathrm{Ext}^1(\mathcal{O}_S, \mathcal{O}_S(K_S))$ classifies all such extensions and is easily seen to be trivial, by the regularity of S. Hence $\mathcal{L}_2 \simeq \mathcal{O}_S$ implies that $\mathscr{E} \simeq \mathcal{O}_S \oplus \mathcal{O}_S(K_S)$ which is not true. (Incidentally the case of $\mathcal{L}_2 \simeq \mathcal{O}_S$ is also ruled out by the fact that $\mathcal{L}_1$ being invertible has no second Chern class, whereas $\mathscr{E}$ has nonzero Chern class coming from the degree of ξ on the curve). If $h^0(\mathcal{L}_2) > 0$ and $\mathcal{L}_2 \not\simeq \mathcal{O}_S$ then either $\mathrm{c}_1(\mathcal{L}_2)$ moves or the linear system it spans consists of a (fixed) effective divisor. But from (3.2) we see that $\mathscr{E}$ is spanned by its global sections outside the support of the divisor $|A - \xi|$ on $\mathcal{C}$. Therefore $\mathrm{c}_1(\mathcal{L}_2)$ must be a mobile linear system. This is a contradiction since $|K_S|$ is not decomposable.

Proof of (iii). As we mention above, by (3.2) the bundle $\mathscr{E}$ is globally generated everywhere except at the support of $|A-\xi|$. Thus, the evaluation morphism associated to a choice of basis of $H^0(\mathscr{E})$ yields

$$0 \to \mathcal{O}_S(-K_S) \otimes \mathscr{I}_Z \to 3\mathcal{O}_S \to \widetilde{\mathscr{E}} \to 0 \tag{3.4}$$

where $\mathscr{I}_Z$ is the ideal sheaf of a subscheme Z of codimension 2 and $\widetilde{\mathscr{E}} \subset \mathscr{E}$ is the image of $3\mathcal{O}_S \to \mathscr{E}$. Even though the sections $H^0(\mathscr{E})$ do not generate $\mathscr{E}$ at $\delta = \mathrm{supp}(A - \xi)$

they still span a a vector space of dimension 1, hence $\widetilde{\mathscr{E}}$ is a little larger than $\mathscr{E} \otimes \mathscr{I}_\delta$, in fact it fits in the exact sequence

$$0 \to \widetilde{\mathscr{E}} \to \mathscr{E} \to \mathcal{O}_\delta \to 0$$

and consequently $\mathrm{c_t}\,\widetilde{\mathscr{E}} = 1 + K_S + 0[\mathrm{pt}]$. This implies that $\mathrm{c}_2(\mathcal{O}_S(-K_S) \otimes \mathcal{I}_Z) = 0$, by taking total Chern classes of (3.4). Consequently, $Z = \emptyset$. The map

$$H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\widetilde{\mathscr{E}}(K_S)) \subset H^0(\mathscr{E}(K_S)) \tag{3.5}$$

can now be obtained by tensoring (3.4) with $\mathcal{O}_S(K_S)$ and taking global sections. We see that it has a kernel isomorphic to $H^0(\mathcal{O}_S)$. The fact that $H^0(\mathscr{E}(K_S)) = 13$, which we require to finish this proof, can be deduced from (3.2) using the regularity of S and Riemann–Roch on $\mathcal{C}$

$$h^0(\mathcal{C}, 2A - \xi) = h^0(\mathcal{C}, \xi) + \deg(A) - \deg(\xi) = 5. \quad \square$$

Notice that we can easily find a generator of the kernel of the map of (iii). It corresponds to the tautology:

$$(s_2 \wedge s_3) \cdot s_1 - (s_1 \wedge s_3) \cdot s_2 + (s_1 \wedge s_2) \cdot s_3 = 0.$$

As we shall see, $\mathscr{E}$ is the restriction of the tautological orbi-bundle of a generalised weighted Grassmannian. The above relation is one of the ten relations between the five generators of the Serre module of $\mathscr{E}$. (See our introduction on page 9).

V.3.3. $R(S, K_S)$ and the bundle $\mathscr{E}$.

Proposition V.13. *Let S be a regular nonsingular surface of general type with $p_g = 4$ and $K^2 = 7$. Assume that $|K_S|$ yields a birational morphism onto a surface with at most ordinary singularities. Let $\mathscr{E}$ be a vector bundle on S of rank 2 and canonical determinant satisfying items* (i), (ii) *and* (iii) *of Proposition V.11. Fix $\langle s_1, s_2, s_3 \rangle$, a choice of basis of $H^0(\mathscr{E})$ and $\langle t_1, t_2 \rangle \subset H^0(\mathscr{E}(K_S))$ an orthogonal to $H^0(K_S) \otimes H^0(\mathscr{E})$. Let $u \in H^0(K_S)$ span a complementary space to the image of the map $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$. Consider the polynomial ring $\mathbb{C}[x, m_{ij}, n_{kl}, z]$ where*

$$\mathrm{wt}(M) = \mathrm{wt}\begin{pmatrix} m_{12} & m_{13} & n_{11} & n_{12} \\ & m_{23} & n_{21} & n_{22} \\ & & n_{31} & n_{32} \\ & & & z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 2 & 2 \\ & & & 3 \end{pmatrix}$$

and $\mathrm{wt}(x) = 1$. *Define a map* $\mathrm{ev}^+ : \mathbb{C}[x, m_{ij}, n_{kl}, z] \to R(S, K_S)$ *by setting* $\mathrm{ev}^+(x) = u$, $\mathrm{ev}^+(m_{ij}) = s_i \wedge s_j$, $\mathrm{ev}^+(n_{kl}) = s_l \wedge t_k$ *and* $\mathrm{ev}^+(z) = t_1 \wedge t_2$. *Then*

(i) ev^+ *is surjective.*

Let $\mathrm{ev} : \mathbb{C}[x_1, x_2, x_3, x_4, y_1, y_2] \to R(S, K_S)$ *be a minimal surjective morphism. Denote by* $\mathrm{Pf}_1, \dots, \mathrm{Pf}_5$ *the* 5 *submaximal Pfaffians of the matrix* M *above. Then, for any choice of basis* $\langle s_1, s_2, s_3 \rangle$, $\langle t_1, t_2 \rangle$, u *and of a surjection*

$$\lambda : \mathbb{C}[x, m_{ij}, n_{kl}, z] \to \mathbb{C}[x_1, x_2, x_3, x_4, y_1, y_2]$$

such that $\mathrm{ev}^+ = \lambda \circ \mathrm{ev}$, *we have:*

(ii) $\lambda \{\mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5\} \in I_{K_S}$;

(iii) $\lambda(\mathrm{Pf}_4), \lambda(\mathrm{Pf}_5)$ *are a basis of* $I_{K_S,3}$;

(iv) $\lambda(\mathrm{Pf}_1), \lambda(\mathrm{Pf}_2), \lambda(\mathrm{Pf}_3)$ *are a basis of* $I_{K_S,4}/I'_{K_S,4}$.

Proof. We reduce the proof of this result to an earlier result. (Proposition IV.20).

Lemma V.14. *There exists nonsingular curve* $\mathcal{C} \in |K_S|$ *such that* $A = K_{S|\mathcal{C}}$ *is base-point free and* $\mathscr{E}_{\mathcal{C}}$ *is a stable bundle on* $\mathcal{C}$ *of rank* 2 *and determinant* A *with* $\dim H^0(\mathscr{E}_{\mathcal{C}}) = 3$.

Proof of the lemma. Since $|K_S|$ is free, by Bertini's theorem we know that a general member $\mathcal{C}$ of $|K_S|$, is a nonsingular curve with a free halfcanonical net. The restriction of $\mathscr{E}$ to $\mathcal{C}$ is such that

$$0 \to \mathscr{E}(-K_S) \to \mathscr{E} \to \mathscr{E}_{\mathcal{C}} \to 0$$

From (3.2) we deduce that $h^0(\mathscr{E}(-K_S)) = 0$. Since $\mathscr{E}(-K_S) = \mathscr{F}$, by (3.1) we also have $h^1(\mathscr{E}(-K_S)) = 0$. Thus $h^0(\mathscr{E}_{\mathcal{C}}) = 3$. We are left with showing that $\mathscr{E}_{\mathcal{C}}$ is a stable bundle. We will show that $\mathscr{E}_{\mathcal{C}}$ is isomorphic to the unique indecomposable extension

$$0 \to \mathcal{O}_{\mathcal{C}}(A - \xi_p) \to \mathscr{E}_p \to \mathcal{O}_{\mathcal{C}}(\xi_p) \to 0$$

having $h^0(\mathscr{E}_p) = 3$ (where p is a general point of $\mathcal{C}$ and $\xi_p = |A - p|$ is a free g^1_6) since, as we saw in the proof of Proposition IV.20, $\mathscr{E}_p$ is a stable bundle. We proceed in a similar way to the proof of that proposition. From $h^0(\mathscr{E}_{\mathcal{C}}) = 3$ we deduce that there exist a section vanishing at p. We saturate the embedding $\mathcal{O}_{\mathcal{C}}(p) \hookrightarrow \mathscr{E}_{\mathcal{C}}$ to obtain

$$0 \to \mathcal{O}_{\mathcal{C}}(\delta) \to \mathscr{E}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(A - \delta) \to 0.$$

This argument was used in the proof of Proposition IV.20 to prove uniqueness. There we assumed stability of $\mathscr{E}_{\mathcal{C}}$ to show that $h^0(\delta) \leq 1$. Here we cannot use stability but we can use what we know of $\mathscr{E}$ on S. Suppose that $h^0(\delta) > 1$. Then the map

$$\bigwedge^2 H^0(\mathscr{E}_{\mathcal{C}}) \to H^0(A) \tag{3.6}$$

has a nontrivial kernel. Let V be the image of the map $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$ as in (ii) of Proposition V.11. Then (3.6) above implies that $\mathcal{C} \in \mathbb{P}[V] \subset \mathbb{P}[H^0(K_S)]$ which is not true for a general $\mathcal{C} \in |K_S|$ since $\mathbb{P}[V]$ has codimension 1 in $\mathbb{P}[K_S]$. Hence $h^0(\delta) \leq 1$. We draw together with the proof of Proposition IV.20 once more. If $\deg \delta > 1$ then surely $h^0(A - \delta) < h^0(\xi_p) < 2$ and this means that $h^0(\mathscr{E}_{\mathcal{C}}) < 3$, which is a contradiction. Therefore $\delta = p$. $\square$

Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}[x, m_{ij}, n_{kl}, z] \arrow[r, "\lambda"] \arrow[d] & \mathbb{C}[x_1, \ldots, x_4, y_1, y_2] \arrow[r, "\mathrm{ev}"] \arrow[d] & R(S, K_S) \arrow[d] \\
\mathbb{C}[m_{ij}, n_{kl}, z] \arrow[r, "\lambda_{\mathcal{C}}"] & \mathbb{C}[x_1, \ldots, x_3, y_1, y_2] \arrow[r, "\mathrm{ev}_{\mathcal{C}}"] & R(\mathcal{C}, A)
\end{tikzcd}$$

given by restriction to $\mathcal{C} \in |K_S|$ whose equation, by the lemma, we may take to be $\mathrm{ev}^{-1}(u) = x_4 = 0$, without any loss in generality. The proof follows from an application of Proposition IV.20 on page 93 $\square$

V.3.4. Proof of Theorem V.1.

Proposition V.15. *Let S be a regular nonsingular surface of general type with $p_g = 4$ and $K^2 = 7$. Assume that $|K_S|$ yields a birational morphism onto a surface with at most ordinary singularities. Let $\mathscr{E}$ be a vector bundle on S of rank 2 and canonical determinant satisfying items (i), (ii) and (iii) of Proposition V.11. Fix $\langle s_1, s_2, s_3 \rangle$, a choice of basis of $H^0(\mathscr{E})$ and $\langle t_1, t_2 \rangle \subset H^0(\mathscr{E}(K_S))$ a complementary space to $H^0(K_S) \otimes H^0(\mathscr{E})$. Let $u \in H^0(K_S)$ span a complementary space to the image of the map $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$. Let X denote the generalised weighted Grassmannian $\mathbb{P}(1) \ltimes \mathbb{G}(\frac{1}{2}^3, \frac{3}{2}^2)$. Define a map*

$$\eta \colon S \to X \tag{3.7}$$

by setting $p \mapsto (u(p), s_i \wedge s_j(p), s_l \wedge t_k(p), t_1 \wedge t_2(p))$. Then

(i) *η factors through the pluricanonical map $S \to \operatorname{Proj} R(S, K_S)$.*

(ii) *The canonical model is cut out on X by 4 quasihomogeneous forms of degree 2 and one quasihomogeneous form of degree 3 in X.*

Proof. The map η fits into the commutative diagram:

$$\begin{tikzcd}
S \arrow[rr] \arrow[d, "\varphi"'] & & X \arrow[d] \\
\mathbb{P}[x_i, y_k] \arrow[rr, "\mathbb{P}(\lambda)"] & & \mathbb{P}[x, m_{ij}, n_{lk}, z]
\end{tikzcd} \tag{3.8}$$

where $\varphi\colon S \to \operatorname{Proj} R(S, K_S) \subset \mathbb{P}[x_1, \ldots, x_4, y_1, y_4]$ is the projectivised of the map $\operatorname{ev}\colon \mathbb{C}[x_1, \ldots, x_4, y_1, y_2] \to R(S, K_S)$ and the map $\mathbb{P}(\lambda)\colon R \to \mathbb{P}[x, m_{ij}, n_{lk}, z]$ is the projectivised of the surjection:

$$\lambda\colon \mathbb{C}[x, m_{ij}, n_{lk}, z] \to \mathbb{C}[x_1, \ldots, x_4, y_1, y_2]$$

as in Proposition V.13. The vertical arrow is the embedding of X into weighted projective space. Since $\eta = \mathbb{P}(\lambda) \circ \varphi$ item (i) is proved.

The image of S under φ is cut out by the generators of $\mathcal{I}_{K_S}$ which we denote by A_4, B_4, C_4, and F_3, G_3 (three quartics and two cubics) and consequently the image of S under $\eta = \mathbb{P}(\lambda) \circ \varphi$ is cut out by $\lambda^{-1}\{A_4, B_4, C_4, F_3, G_3\}$ plus four quasihomogeneous forms of degree 2 and one quasihomogeneous form of degree 3 that span $\operatorname{Ker}\operatorname{ev}$. But from Proposition V.13 and Proposition V.10 it follows that

$$\lambda^{-1}(A_4, B_4, C_4, F_3, G_3) = (\mathrm{Pf}_1, \ldots, \mathrm{Pf}_5)$$

Where $(\mathrm{Pf}_1, \ldots, \mathrm{Pf}_5)$ is the homogeneous ideal of X in $\mathbb{P}[x, m_{ij}, n_{lk}, z]$, Therefore $\operatorname{Proj}(S)$ is set-theoretically cut out in X by four quasihomogeneous forms of degree 2 and one quasihomogeneous form of degree 3. $\square$

Proof of Theorem V.1. This is straightforward from Proposition V.15. We use the computation of the degree of X, which by Proposition IV.9 is $\frac{7}{2^4 \cdot 3} \cdot 2^4 \cdot 3$ and apply Proposition IV.2. $\square$

V.4. Surfaces with $p_g = 6$ and $K_S^2 = 13$

Assume that S is a nonsingular regular surface of general type with $p_g = 6$ and $K_S^2 = 13$. If $|K_S|$ is free then, since 13 is prime, the canonical morphism is automatically birational. Suppose additionally that sym^2 is surjective. A general member $\mathcal{C}$ of the canonical linear system $|K_S|$ is a nonsingular curve of genus 14 with a halfcanonical divisor $A = K_{S|\mathcal{C}}$ such that $\dim H^0(A) = 5$. By Corollary IV.35 on page 120, the halfcanonical ring $R(\mathcal{C}, A)$ is a codimension 3 ring. We deduce that $R(S, K_S)$ is a codimension 3 ring. Evidently, if sym^2 is not surjective then $R(S, K_S)$ has codimension ≥ 4, since it requires generators in degree 2.

Proposition V.16. *Let S be a nonsingular regular surface of general type with $p_g = 6$ and $K^2 = 13$. Then if* sym^2 *is surjective the linear system $|K_S|$ is free.*

Proof. Suppose that sym^2 is surjective and that $|K_S|$ is not spanned by its global sections. Then $|2K_S|$ is not spanned by its global sections. However, by a Theorem of Bombieri (see [**Bom**]) the linear system $|2K_S|$ is spanned by its global sections. □

Corollary V.17. *Let S be a nonsingular regular surface of general type with $p_g = 6$ and $K^2 = 13$. Then*

$$\textit{sym}^2 \textit{ is surjective} \iff R(S, K_S) \textit{ has codimension } 3. \quad \square$$

We want to show that the canonical model $\mathrm{Proj}\, R(S, K_S)$ is a complete intersection in a generalised weighted Grassmannian. We will use the vector bundle method. Before we need some preliminary results concerning the elements of the canonical linear system.

Proposition V.18. *Let S be a nonsingular regular surface of general type with $p_g = 6$ and $K_S^2 = 13$. Assume that* sym^2 *is surjective. Then a nonsingular curve of the canonical linear system has no g^1_5. Additionally, there exists a nonsingular curve in $|K_S|$ with a (free) g^1_6.*

Proof. Let $\mathcal{C} \in |K_S|$ be a nonsingular curve. Then $\mathcal{C}$ has genus 14 and has a halfcanonical divisor $A = K_{S|\mathcal{C}}$ such that $\dim H^0(A) = 5$, by the regularity of S.

Additionally

$$\dim \operatorname{Ker}\left\{\mathrm{S}^2 H^0(\mathcal{C}, A) \to H^0(\mathcal{C}, 2A)\right\} = \dim \operatorname{Ker}\left\{\mathrm{S}^2 H^0(K_S) \to H^0(2K_S)\right\} = 1$$

and accordingly, $\mathrm{S}^2 H^0(\mathcal{C}, A) \to \mathrm{S}^2 H^0(\mathcal{C}, 2A)$ is surjective. Thus, as in the proof of Proposition IV.25, we deduce from Lemma III.41 that $\mathcal{C}$ has no g_5^1. Notice that since $R(S, K_S)$ is generated in degree 1 the map

$$\varphi_{K_S}\colon S \to \Sigma \subset \mathbb{P}^5$$

is an embedding away from -2-cycles. The image of S is the canonical model of S and therefore it has at most Du Val singularities. By a numerical argument Σ is contained in a quadric of rank Q of rank ≥ 3. To show that a smooth $\mathcal{C} \in |K_S|$ has a pencil of degree 6, in view of Proposition III.9 on page 36 it is enough to show that

$$\operatorname{sym}^2\colon\ \mathrm{S}^2 H^0(\mathcal{C}, A) \to H^0(\mathcal{C}, 2A)$$

contains in its kernel a symmetric tensor of rank ≤ 4. In other words, it is enough to show that the hyperplane section of Σ determining $\mathcal{C}$ is contained in a quadric of rank ≤ 4. If $\operatorname{rank} Q \leq 4$ then this is obvious: all hyperplane sections satisfy this requirement. In the remaining cases we must show that there exists a hyperplane H such that $Q \cap H$ is a quadric of rank ≤ 4 and $\Sigma \cap H$ is a nonsingular curve, that is to say, H is not tangent to S at any point of $\Sigma \cap H$. Suppose that $\operatorname{rank} Q \geq 5$. One way to make $\operatorname{rank} Q \cap H \leq 4$ is to take a tangent hyperplane to Q. The variety parametrising tangent hyperplanes to Q (the dual of Q) is a quadric in dual projective space. Therefore its dimension is 4. The dual variety of Σ (containing an open subset parametrising hyperplanes containing tangent planes at nonsingular points) has dimension $\leq 2+2 = 4$. These two varieties cannot be equal. Hence there exists a hyperplane tangent to Q and not containing any tangent plane to a point of Σ. The curve $\mathcal{C} \in |K_S|$ determined in this way is a nonsingular curve whose image by φ_{K_S} is contained in quadric of rank ≤ 4. $\square$

Proposition V.19. *Let S be a nonsingular regular surface of general type with $p_g = 6$ and $K^2 = 13$. Assume that* sym^2 *is surjective. Then the canonical ideal is generated by a quadric and four cubic forms.*

Proof. Let $\mathcal{C} \in |K_S|$ be general nonsingular curve in the canonical linear system. Then as we have shown in the proof of the previous proposition, the map

$$\mathrm{S}^2 H^0(A) \to H^0(2A)$$

is surjective. Therefore by Corollary IV.35, the ideal I_A is generated by 4 cubics and one quadric. Hence, this result follows from the hyperplane principle, Theorem V.4. $\square$

V.4.1. The bundle.

PROPOSITION V.20. *Let S be a nonsingular regular surface of general type with $p_g = 6$ and $K^2 = 13$. Assume that* sym^2 *is surjective. Then there exists a bundle $\mathscr{E}$ of rank 2 such that:*

(i) $\mathrm{c}_t \mathscr{E} = 1 + K_S + 7[\mathrm{pt}]$;
(ii) $\dim H^0(\mathscr{E}) = 4$;
(iii) *the map* $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$ *has a kernel of dimension* ≤ 1.
(iv) *the map* $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$ *has a 1-dimensional cokernel.*

Proof. Let $\mathcal{C} \in |K_S|$ be a nonsingular curve in the canonical linear system having a g^1_6. (See Proposition V.18).

Proof of existence. Denote by ξ a g^1_6 on $\mathcal{C}$. Since $\mathcal{C}$ has no g^1_5 the linear system ξ is base-point free. Hence the evaluation morphism $2\mathcal{O}_{\mathcal{C}} \to \xi$ given by $(f, g) \mapsto fs_1 + gs_2$ for a choice of basis $\langle s_1, s_2 \rangle$ of $H^0(\xi)$, is surjective. Let us denote by $\mathscr{E}^\vee$ the kernel of $2\mathcal{O}_S \to \mathcal{O}_{\mathcal{C}}(\xi)$. We have

$$0 \to \mathscr{E}^\vee \to 2\mathcal{O}_S \to \mathcal{O}_{\mathcal{C}}(\xi) \to 0. \tag{4.1}$$

The sheaf $\mathscr{E}^\vee$ is an elementary modification of $2\mathcal{O}_S$ and therefore it is locally free. Applying the functor $\mathscr{H}\!om(\cdot, \mathcal{O}_S)$ to (4.1)

$$0 \to 2\mathcal{O}_S \to \mathscr{E} \to \mathscr{E}\!xt^1_{\mathcal{O}_S}(\xi, \mathcal{O}_S) \to 0.$$

Since $\mathscr{E}\!xt^1(\xi, \mathcal{O}_S) = \mathcal{O}_{\mathcal{C}}(A - \xi)$ (see Lemma V.12) and denoting $A - \xi$ by η, we deduce that there exists a vector bundle $\mathscr{E}$ of rank 2 fitting into the exact sequence

$$0 \to 2\mathcal{O}_S \to \mathscr{E} \to \mathcal{O}_{\mathcal{C}}(\eta) \to 0 \tag{4.2}$$

Proof of (i). Taking total Chern class in (4.2)

$$\mathrm{c}_t\,\mathscr{E} = (\mathrm{c}_t\,2\mathcal{O}_S)(\mathrm{c}_t\,\mathcal{O}_{\mathcal{C}}(\eta)) = \mathrm{c}_t\,\mathcal{O}_{\mathcal{C}}(\eta) = 1 + K_S + c_2\mathcal{O}_{\mathcal{C}}(\eta)$$

From the sequence

$$0 \to \mathcal{O}_S(-K_S) \to \mathcal{O}_S \to \mathcal{O}_{\mathcal{C}} \to 0$$

we have $1 = \mathrm{c}_t\,\mathcal{O}_S = (1 - K_S)\,\mathrm{c}_t\,\mathcal{O}_{\mathcal{C}}$ so that $\mathrm{c}_t\,\mathcal{O}_{\mathcal{C}} = 1 + K_S$. From

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(\eta) \to \mathcal{O}_\eta \to 0$$

we have $c_t\mathcal{O}_{\mathcal{C}}(\eta) = (1 + K_S)(1 + \deg(\eta)[\mathrm{pt}]) = 1 + K_S + 7[\mathrm{pt}]$.

Proof of (ii). By regularity of S, $\dim H^1(2\mathcal{O}_S) = 0$. Therefore after taking global sections of the exact sequence (4.2)

$$0 \to H^0(2\mathcal{O}_S) \to H^0(\mathscr{E}) \to H^0(\mathcal{O}_{\mathcal{C}}(\eta)) \to 0.$$

By Proposition III.2 we have $\dim H^0(\mathcal{O}_{\mathcal{C}}(\eta)) \geq 2$. Since sym^2 is surjective, by Proposition IV.25, $\mathcal{C}$ has no g_9^2. We deduce that $\dim H^0(\mathcal{O}_{\mathcal{C}}(\eta)) = 2$. Thus $\dim H^0(\mathscr{E}) = 4$.

Restriction of $\mathscr{E}$ to $\mathcal{C}$. In the proof of the following item it will be convenient to know the restriction of the bundle $\mathscr{E}$ to the curve $\mathcal{C}$.

Lemma V.21. *The restriction of $\mathscr{E}$ to $\mathcal{C}$ is the unique nonsplit extension*

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{E}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(\eta) \to 0$$

with $\dim H^0(\mathscr{E}) = 2$.

Proof of the lemma. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}^\vee \arrow[r] \arrow[d] & 2\mathcal{O}_S \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{C}}(\xi) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathscr{E} \arrow[r] \arrow[d] & 0 & \\
 & \mathscr{E}_C & \mathcal{O}_{\mathcal{C}}(\eta) & &
\end{tikzcd}$$

and apply the snake lemma. $\square$

Proof of (iii). We use the restriction of $\mathscr{E}$ to $\mathcal{C}$ to prove this item. However, there exists a different proof when $|K_S|$ is slightly more general then we are assuming here.

LEMMA V.22. *In addition to the assumptions of this proposition, suppose that $|K_S|$ is indecomposable. Then the kernel of the map in item (iii) is at most 1-dimensional.*

Proof of the lemma. We refer the reader to the proof of item (ii) of Proposition V.11 for a more detailed proof of a similar result. Suppose that the kernel of the map

$$\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$$

has dimension ≥ 2. Then its projectivised contains a line and thus must intersect $\mathrm{G}(2, H^0(\mathscr{E}))$, the variety of skew tensors of rank 2. A point of this intersection yields a splitting of $\mathscr{E}$

$$0 \to \mathcal{L} \to \mathscr{E} \to \mathcal{M} \to 0 \tag{4.3}$$

where $\mathcal{L}$ is a reflexive sheaf of rank 1 (thus locally free) and $\mathcal{M}$ is a torsion free sheaf of rank 1. Moreover $\dim H^0(\mathcal{L}) \geq 2$. Taking first Chern classes,

$$K_S = \mathrm{c}_1(\mathcal{L}) + \mathrm{c}_1(\mathcal{M}).$$

We want to show that $\dim H^0(\mathcal{M}) \geq 2$ and this way obtaining a contradiction to indecomposability. From the sequence (4.2) we see that $\mathscr{E}$ fails to be spanned by its global sections eventually at the base points of the linear system η. Therefore $\mathcal{M}$ is spanned by global sections except at a finite set of (base) points. From (4.3) and (4.2) since $\dim H^0(\mathscr{E}^\vee \otimes \mathcal{M}) > 0$ we deduce that $\dim H^0(\mathcal{M}) > 0$. Therefore, either $\mathcal{M} = \mathcal{O}_S$ or $\dim H^0(\mathcal{M}) \geq 2$. Notice that

$$\mathcal{M} \text{ torsion free } \implies \mathcal{M} \simeq \mathcal{O}_S(\mathrm{c}_1\,\mathcal{M}) \otimes \mathcal{O}_Z$$

from some subscheme $Z \subset S$ of codimension 2. If $\mathcal{M} \simeq \mathcal{O}_S$ then

$$\dim \mathrm{Hom}(\mathscr{E}, \mathcal{O}_S) = \dim H^0(\mathscr{E}^\vee) > 0$$

and this is false as one checks from (4.1). $\square$

Remark. If the linear system is indecomposable then the kernel of sym^2 contains no quadrics of rank ≤ 4 (Proposition V.7). In other words the rank of the quadric through $\Sigma \subset \mathbb{P}^5$ (the image of S be the canonical map) is ≥ 5. By the argument of the proof of the previous lemma, when nonempty, the projectivised kernel of the map $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$ does not lie in $\mathrm{G}(2, H^0(\mathscr{E}))$.

In the general case, the argument relies on item (iv) of Proposition IV.29. Recall that we showed that the kernel of the map

$$\bigwedge^2 H^0(\mathscr{E}_{\mathcal{C}}) \to H^0(A)$$

had dimension ≤ 2. Suppose that $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$ has a kernel of dimension ≥ 2. Then, composing with the restriction map $H^0(K_S) \to H^0(A)$ we deduce that

$$\bigwedge^2 H^0(\mathscr{E}) \to H^0(A)$$

has a kernel of dimension ≥ 3, since we can see from (4.2) that there exist two sections s_1, s_2 such that $s_1 \wedge s_2 \neq 0$ on S but $s_1 \wedge s_2 = 0$ on $\mathcal{C}$. This is a contradiction.

Proof of (iv). Tensoring (4.2) with $\mathcal{O}_S(K_S)$ we have

$$0 \to 2\mathcal{O}_S(K_S) \to \mathscr{E}(K_S) \to \mathcal{O}_{\mathcal{C}}(A+\eta) \to 0.$$

So that, by RR and Serre duality on $\mathcal{C}$, and by regularity of S,

$$\dim H^0(\mathscr{E}(K_S)) = 2 \cdot \dim H^0(K_S) + \dim H^0(\mathcal{O}_{\mathcal{C}}(A+\eta)) = 21.$$

Since $\dim H^0(K_S) \otimes H^0(\mathscr{E}) = 24$ we are left to prove that the kernel of the map:

$$H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S)) \tag{4.4}$$

is 4-dimensional. To see this, we identify this map with the map on global sections of a map of sheaves. We choose a 2-dimensional subspace of $H^0(\mathscr{E})$ projecting down to

$H^0(\eta)$ and write evaluation maps in the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 2\mathcal{O}_S \arrow[r, "\sim"] \arrow[d] & 2\mathcal{O}_S \arrow[d] \\
0 \arrow[r] & \mathscr{N} \arrow[r] & 2\mathcal{O}_S \bigoplus 2\mathcal{O}_S \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\
 & & 2\mathcal{O}_S \arrow[r] & \eta
\end{tikzcd}$$

Tensoring the middle sequence with $\mathcal{O}_S(K_S)$, we see that the global sections map of

$$2\mathcal{O}_S(K_S) \oplus 2\mathcal{O}_S(K_S) \to \mathscr{E}(K_S)$$

is exactly that of (4.4). Thus its kernel is isomorphic to $H^0(\mathscr{N}(K_S))$. Notice that none of the evaluation maps needs to be surjective, as η and consequently $\mathscr{E}$ might have base points. The snake lemma applies to the first two rows, giving

$$0 \to \mathscr{N} \to 2\mathcal{O}_S \to \eta.$$

Lemma V.23. $\dim H^0(\mathscr{N}(K_S)) = 4$.

Proof. By Castelnuovo's free-pencil trick, the map

$$H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(\eta)) \otimes H^0(\mathcal{C}, A) \to H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(A + \eta))$$

has a kernel isomorphic to $H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(\xi + B))$, where B denotes the base locus of η. As the curve C has no g_5^1, the base locus of η can consist of at most one point. However, since by Proposition IV.25 the curve $\mathcal{C}$ has no g_9^2 and in particular no g_7^2, whether B consists of a point or is empty is irrelevant for always $h^0(\xi + B) = 2$. Finally, since the restriction map

$$2H^0(K_S) \to 2H^0(\mathcal{C}, A)$$

has a 2-dimensional kernel and is surjective, we deduce that

$$\dim H^0(\mathscr{N}(K_S)) = 4. \quad \square$$

Remark. As we have pointed out earlier in this work one can easily write 4 linearly independent tensors in the kernel of $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$:

$$\begin{cases} (s_2 \wedge s_3)s_1 - (s_1 \wedge s_3)s_2 + (s_1 \wedge s_2)s_3 \\ (s_2 \wedge s_4)s_1 - (s_1 \wedge s_4)s_2 + (s_1 \wedge s_2)s_4 \\ (s_3 \wedge s_4)s_1 - (s_1 \wedge s_4)s_3 + (s_1 \wedge s_3)s_4 \\ (s_3 \wedge s_4)s_2 - (s_2 \wedge s_4)s_3 + (s_2 \wedge s_3)s_4 \end{cases}$$

which, gives a straightforward proof that the kernel has dimension ≥ 4. Looking ahead for the next proposition, take each of these tensors and wedge it with $t \in H^0(\mathscr{E}(K_S))$, a generator of the cokernel of $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E})$. In this way we obtain four cubic Pfaffians that together with the quadric tensor in the kernel of sym^2 generate the canonical ideal I_{K_S}.

Proposition V.24. *Let S be a nonsingular regular surface of general type with $p_g = 6$ and $K^2 = 13$. Assume that* sym^2 *is surjective. Let $\mathscr{E}$ be a bundle on S given as in Proposition V.20. Consider a polynomial ring $\mathbb{C}[v, m_{ij}, n_i]$ with*

$$\mathrm{wt}\begin{pmatrix} m_{12} & m_{13} & m_{14} & n_1 \\ & m_{23} & m_{24} & n_2 \\ & & m_{34} & n_3 \\ & & & n_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{pmatrix} \tag{4.5}$$

and $\mathrm{wt}(v) = 1$. *Let $\langle s_1, s_2, s_3, s_4\rangle$ be a choice of basis of $H^0(\mathscr{E})$ and $t \in H^0(\mathscr{E}(K_S))$ be an element spanning a complement to the image of the map in* (iv) *of Proposition V.20. Choose $u \in H^0(K_S)$ such that $\langle u\rangle$ is a complement to the image of the map in* (iii) *of Proposition V.20. Let* ev^+ *be the map* $\mathrm{ev}^+ \colon \mathbb{C}[v, m_{ij}, n_i] \to R(S, K_S)$ *defined by*

$$v \mapsto u, \quad m_{ij} \mapsto s_i \wedge s_j \quad \text{and} \quad n_i \mapsto s_i \wedge t.$$

Denote by $\mathrm{Pf}_1, \dots, \mathrm{Pf}_5$ *the* 5 *submaximal Pfaffians of the skew matrix in (4.5). Then,*

(i) ev^+ *is surjective.*

Let ev *be a minimal surjection $\mathbb{C}[x_1, \dots, x_6] \to R(S, K_S)$. Then, there exists a surjection, $\lambda \colon \mathbb{C}[v_1, v_2, m_{ij}, n_i] \twoheadrightarrow \mathbb{C}[x_1, \dots, x_6]$, such that* $\mathrm{ev}^+ = \mathrm{ev} \circ \lambda$ *and*

(ii) $\lambda\{\mathrm{Pf}_1, \dots, \mathrm{Pf}_5\} \subset I_{K_S}$.

(iii) $\lambda^{-1} I_{K_S,3} \subset (\mathrm{Pf}_1, \dots, \mathrm{Pf}_5)$ *thus, since the ideal I_{K_S} is generated in degree* 3, *we have* $\lambda^{-1} I_{K_S} = (\mathrm{Pf}_1, \dots, \mathrm{Pf}_5)$.

Proof. Consider a commutative diagram:

$$\begin{tikzcd}
\mathbb{C}[v, m_{ij}, n_i] \arrow[r, "\lambda"] \arrow[d] & \mathbb{C}[x_1, \dots, x_6] \arrow[r, "\mathrm{ev}"] \arrow[d] & R(S, K_S) \arrow[d] \\
\mathbb{C}[v, m_{ij}, n_i] \arrow[r, "\lambda_{\mathcal{C}}"] & \mathbb{C}[x_1, \dots, x_5] \arrow[r, "\mathrm{ev}_{\mathcal{C}}"] & R(\widehat{\mathcal{C}}, A)
\end{tikzcd}$$

given by restriction to the curve $\mathcal{C}$, whose equation we can take to be given by $x_6 = 0$. Thus, we can reduce the reduce the proof of this proposition to Proposition IV.32. $\square$

V.4.2. Proof of Theorem V.2.

Proposition V.25. *Let S be a nonsingular regular surface of general type with $p_g = 6$ and $K^2 = 13$. Assume that* sym^2 *is surjective. Let $\mathscr{E}$, $\langle s_1, s_2, s_3, s_4\rangle \subset H^0(\mathscr{E})$, $t \in H^0(\mathscr{E}(K_S))$ and $u \in H^0(K_S)$ be as in Proposition V.24. Let X denote the generalised Grassmannian $\mathbb{P}(1)\ltimes\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$. Define a map from S to X by*

$$S \ni p \mapsto [v(p), s_i \wedge s_j(p), s_i \wedge t(p)] \in X.$$

Then,

(i) *η factors through the pluricanonical morphism $S \to \operatorname{Proj} R(S, K_S)$ and*
(ii) *the canonical model $\operatorname{Proj} R(S, K_S)$ is cut out by four quasihomogeneous forms of degree 2 and one quasihomogeneous forms of degree 1 in X.*

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
S \arrow[r, "\eta"] \arrow[d, "\varphi_{K_S}"'] & X \arrow[d] \\
\mathbb{P}[x_1, \dots, x_5] \arrow[r, "\mathbb{P}(\lambda)"] & \mathbb{P}(1, 1^6, 2^4)
\end{tikzcd}$$

where $\mathbb{P}(\lambda)$ is the projectivised of the homomorphism

$$\lambda\colon \mathbb{C}[v, m_{ij}, n_i] \to \mathbb{C}[x_1, \dots, x_6]$$

and where the unlabeled vertical morphism is the embedding $X \subset \mathbb{P}(1, 1^6, 2^4)$. From this and the fact that $R(S, K_S)$ is generated in degree 1, item (i) is clear. Additionally, as the image of S by φ_{K_S} is cut out by I_{K_S} we deduce that the transform of

$\operatorname{Proj} R(S, K_S)$ by λ is cut out in $\mathbb{P}(1, 1^6, 2^4)$ by

$$\operatorname{Ker}\lambda + \lambda^{-1} I_{K_S} = \operatorname{Ker}\lambda + (\mathrm{Pf}_1, \dots, \mathrm{Pf}_5).$$

The ideal $(\mathrm{Pf}_1, \dots, \mathrm{Pf}_5) \subset \mathbb{C}[v, m_{ij}, n_i]$ is the homogeneous ideal of X in $\mathbb{P}(1^2, 1^6, 2^4)$. Therefore we deduce that $\eta(S)$ is cut out in X by $\operatorname{Ker}\lambda$. Since this ideal is generated by four quasihomogeneous forms of degree 2 and one quasihomogeneous forms of degree 1 we have proved item (ii). $\square$

Proof of Theorem V.2. We apply Proposition IV.2 using the result of the previous proposition and the computation of the degree of X, which is $\frac{13}{2^4}$ (Proposition IV.9). $\square$

Index of Notation

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