ON SURFACES OF GENERAL TYPE WITH $p_g = 6$ AND $K^2 = 13$

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Abstract. We study nonsingular minimal surfaces of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image is contained in a single nonsingular quadric. We show that under these assumptions, such surfaces are regular and their canonical ideal has codimension 3. Led by Buchsbaum–Eisenbud's structure theorem, we define a map of S into a weighted Grassmannian, which factors through an embedding of the canonical model of S. We achieve this by constructing a bundle on S with appropriate invariants. We deduce that the canonical model of S is a complete intersection of four quasihomogeneous forms of degree 2 in a weighted Grassmannian.

1. Introduction

Let V be an indecomposable Gorenstein Fano 3-fold with at most Gorenstein canonical singularities. Mukai's linear section theorem asserts that if the genus of V is one of 7, 8, 9 or 10, then V is isomorphic to a linear section of an homogeneous space. We know that Fano 3-folds of the type described above have genus $\leqslant 10$ or equal to 12. For genus less than or equal to 6 the anticanonical model, $\operatorname{Proj} R(V, -K_V)$, is a complete intersection in (weighted) projective space or in $\mathrm{G}(2, 5)$; or it is a double-cover over one such variety. Therefore, for these genera, we can describe the anticanonical model of V by a relatively simple computation of $R(V, -K_V)$. This is not true in the case of genus $\geqslant 7$. In this range, the anticanonical ring of V is necessarily a quotient of a polynomial ring by an ideal of codimension greater than or equal to 4. Buchsbaum–Eisenbud's theorem no longer applies, nor do there exist, to our knowledge, any similar structure theorems. On the other hand, the linear section theorem gives a complete description of $\operatorname{Proj} R(V, -K_V)$ without explicitly calculating $R(V, -K_V)$. In fact, the structure of the anticanonical ring is extracted from that of the homogeneous ring of the ambient (homogeneous) space.

The main ingredient in Mukai's proof is called the vector bundle method. It consists in constructing on V (or on a linear section $T \in |-K_V|$) the restriction of the tautological vector bundle of X. In a series of articles [5, 6, 8, 9, 10] dedicated to this result, Mukai uses ladders $C \subset T \subset V$ of linear sections of V to set up the embedding $V \hookrightarrow X$; whether using T to construct a bundle on V or using C to construct a bundle on T, embedding T in X and then extending the embedding to V. Although, as Mukai explains in [5], the genus 12 Fano 3-fold is not a section of an homogeneous space, the vector bundle method still yields a satisfactory description of the anticanonical model. The linear section theorem and the vector bundle method provide a unified way of describing the anticanonical model of Fano 3-folds of genus greater than or equal to 7.

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The work contained in this article is motivated by the situation we have just described. Initially one can try to extend the vector bundle method to Fano 3-Folds, not necessarily Gorenstein (allowing cyclic quotient singularities). As test cases, we find those Fano 3-folds whose anticanonical ring has codimension 3. Despite that Buchsbaum-Eisenbud's theorem already says that these varieties are subvarieties of weighted Grassmannians (or cones over them) if we are to understand how to proceed in higher codimension, it would of interest to investigate exactly how would the vector bundle method yield the same result. The relation of the present work with what we have been discussing is that surfaces of general type, with $p_g = 6$ and $K^2 = 13$, under our assumptions, prove to be members of $|-2K_V|$ of a non-Gorenstein Fano 3-fold.

Our main theorem is the following.

Theorem 1.1. *Let S be a nonsingular minimal surface of general type with $p_g = 6$ and $K^2 = 13$ whose image under the canonical map is contained in a single nonsingular quadric. Then, S is regular and the canonical linear system is free. Additionally, there exists a map $\rho\colon S \to \mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ that factors through the pluricanonical morphism $S \to \operatorname{Proj} R(S, K_S)$ and the image of S under ρ is a complete intersection of four quasihomogeneous forms of degree 2 in $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$.*

In Proposition 3.1, of Section 3, we establish the regularity of S and show that $|K_S|$ is base point free. Using the hyperplane principle we show in Corollary 3.7 that $R(S, K_S)$ is generated by $H^0(K_S)$. Section 4 is devoted to the construction of ρ, via Mukai's vector bundle technique, and also to the proof that the image of S under this map is a complete intersection in $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$.

2. Preliminaries — Weighted Grassmannians

As to displaying skew symmetric matrices, we follow the convention of only writing their upper triangle. If M is skew 5×5 matrix, $\mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5$ denote the five submaximal Pfaffians of M, i.e., Pf_i is the Pfaffian of the matrix obtained by removing the i-th line and i-th column of M. $\mathrm{Pf}\, M$ denotes the ideal generated by the submaximal Pfaffians of M. Let $\mathcal{C}$ be a nonsingular curve. We denote by $\operatorname{gon}(\mathcal{C})$ the least integer r such that there exists a g^1_r on $\mathcal{C}$.

Weighted Grassmannians were defined by Corti and Reid in [4]. Consider a polynomial ring $\mathbb{C}[m_{ij}]$ where $1 \leqslant i < j \leqslant 5$ and suppose that there exist $c_i \in \frac{1}{2}\mathbb{Z}$ such that $\mathrm{wt}(m_{ij}) = c_i + c_j \in \mathbb{N}$. The weighted Grassmannian of weights $c_1, \dots, c_5$ (denoted by $\mathbb{G}(c_1, \dots, c_5)$) is the subscheme of $\mathbb{P}[m_{ij}]$ defined by the ideal generated by the submaximal Pfaffians of the skew matrix:

$$\begin{bmatrix} m_{12} & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \end{bmatrix}. \tag{1}$$

(Notice that despite the terminology, this definition only generalises the notion of Grassmannian of subspaces of dimension 2 of a fixed 5-dimensional vector space.) Let $\mathrm{aG}(2,5)$ denote the affine cone over the Grassmannian $\mathrm{G}(2,5) \subset \mathbb{P}^9$. In the same way as weighted projective space is the quotient of the punctured affine cone over ordinary projective space by a weighted $\mathbb{C}^*$-action, so is the weighted Grassmannian a quotient of punctured $\mathrm{aG}(2,5)$ by a weighted $\mathbb{C}^*$-action. We draw yet another comparison between weighted projective space and weighted Grassmannian, this time within the context of graded rings. Suppose that (X, D) is a pair consisting of an algebraic variety and an ample Weil divisor. Let $x_1, \dots, x_n$ be a choice of generators of the graded ring $R(X, D)$ and consider the epimorphism $\mathrm{ev}\colon \mathbb{C}[x_1, \dots, x_n] \to R(X, D)$. Suppose, additionally, that there exist m quasihomogeneous forms $f_1, \dots, f_m$ in the kernel of the map ev such that $m + \dim X = n$ and such that the image of X under the map $\psi\colon X \longrightarrow \mathbb{P}[X_1, \dots, X_n]$ defined by $p \mapsto (x_1(p), \dots, x_n(p))$ is cut out by $f_1, \dots, f_m$. Then if

$$D^{\dim X} = \deg(\psi(X)) = \frac{\deg(f_1) \cdots \deg(f_m)}{\mathrm{wt}(x_1) \cdots \mathrm{wt}(x_n)}$$

we can conclude that $\psi(X)$ is a complete intersection. Moreover, we deduce that

$$R(X, D) = \frac{\mathbb{C}[x_1, \dots, x_n]}{(f_1, \dots, f_m)}.$$

This argument is tacitly employed when one calculates divisorial rings, when they happen to be complete intersections. One important fact to be borne in mind when applying the previous argument is that weighted projective spaces are arithmetically Cohen–Macaulay schemes. In particular this ensures that a sequence like $f_1, \dots, f_m$, cutting out a "correct-dimensional" subscheme cannot give rise to embedded components. Weighted Grassmannians are suitable varieties in which to draw arguments like the previous, when the ring $R(X, D)$ is no longer a complete intersection, but a quotient by a 5×5 Pfaffian ideal, which is to say, the first nontrivial structure in codimension 3. We recall the main properties of weighted Grassmannians. See [4] for a proof.

Proposition 2.1. *Let $\mathbb{G}$ be the weighted Grassmannian $\mathbb{G}(c_1, \dots, c_5)$. Denote $\sum_{i=1}^5 c_i$ by k. Then*

(i) $\omega_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}}(-2k)$;

(ii) $\deg \mathbb{G} = \frac{\sum \binom{k-c_i}{3} - \sum \binom{k+c_i}{3} + \binom{2k}{3}}{\prod_{i<j}(c_i + c_j)}$;

(iii) $H^i(\mathcal{O}_{\mathbb{G}}(j)) = 0$ *for all* $0 < i < 6$.

Proposition 2.2. *A general complete intersection of four quasihomogeneous forms of degree 2 in $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ is a nonsingular regular surface of general type with $p_g = 6$, $K^2 = 13$ whose canonical image is contained in a single nonsingular quadric.*

Proof. The weighted Grassmannian $X = \mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ is a subscheme of weighted projective space $\mathbb{P}[x_{ij}, y_i]$ where $1 \leqslant i < j \leqslant 5$, $\mathrm{wt}(x_{ij}) = 1$ and $\mathrm{wt}(y_i) = 2$. We claim that X is nonsingular away from $\mathbb{P}[y_1, \dots, y_4]$. Indeed, since X and $\mathrm{G}(2, 5) \subset \mathbb{P}^9$ have the same affine cone and $\mathrm{G}(2, 5)$ is nonsingular, X is nonsingular away from the locus of points of the affine cone with nontrivial $\mathbb{C}^*$-stabiliser. It is clear that if four quadric quasihomogeneous forms are chosen general enough (i.e. so that it is possible to write them as $y_k = q_k(x_{ij})$, with $\deg(q_k) = 2$, for $k = 1, \dots, 4$) the subscheme they cut out in X does not meet $\mathbb{P}[y_1, \dots, y_4]$ and therefore, by an argument of the type of Bertini's theorem this subscheme is nonsingular. In this situation, the intersection of these forms with $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ is a surface, S, with $K_S = \mathcal{O}_X(1)_{|S}$. We calculate its invariants: $p_g = 6$, $q = 0$, $K_S^2 = 2^4 \cdot \frac{13}{2^4} = 13$. Since $2K_S = \mathcal{O}_X(2)_{|S}$ the map $\pi\colon H^0(\mathcal{O}_X(2)) \to H^0(2K_X)$ is surjective. On the other hand, by the Riemann–Roch theorem, $\dim H^0(2K_S) = K_S^2 + \chi(\mathcal{O}_S) = 13 + 7 = 20$. In other words, the kernel of π is generated by the initial 4 quadric quasihomogeneous forms. Since we chose these forms general, so to say, in order to eliminate the variables y_i, the map $\mathrm{S}^2 H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{O}_X(2))$ maps onto a subspace of $H^0(\mathcal{O}_X(2))$ with null intersection with $\mathrm{Ker}\,\pi$ and of dimension $\binom{7}{2} - 1 = 20 = \dim H^0(\mathcal{O}_X(2)) - 4$. Hence the composition

$\mathrm{S}^2 H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{O}_X(2)) \to H^0(2K_S)$ is surjective. From the commutative diagram

$$\begin{array}{ccc} \mathrm{S}^2 H^0(\mathcal{O}_X(1)) & \longrightarrow & \mathrm{S}^2 H^0(K_S) \\ \downarrow & & \downarrow \\ H^0(\mathcal{O}_X(2)) & \longrightarrow & H^0(2K_S) \end{array}$$

we deduce that $\mathrm{S}^2 H^0(K_S) \to H^0(2K_S)$ is also surjective. Finally, this shows that the space of quadrics containing the canonical image of S has dimension $\dim \mathrm{S}^2 H^0(K_S) - \dim H^0(2K_S) = \binom{7}{2} - 20 = 1$, corresponding to the nonsingular quadric given by the fifth maximal Pfaffian. □

3. The canonical map of S

If the canonical image of S is contained in a single nonsingular quadric then, S is not contained in a pencil of quadrics. In this section we work with this more general hypothesis.

Proposition 3.1. *Let S be a nonsingular minimal surface of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Then S is regular and its canonical map is a birational morphism.*

Proof. The canonical image of S is not contained in a pencil of quadrics if and only if the map

$$\mathrm{sym}^2\colon\ \mathrm{S}^2 H^0(K_S) \to H^0(2K_S)$$

has a kernel of dimension $\leqslant 1$. Now, on one hand, by the Riemann–Roch theorem

$$\chi(\mathcal{O}_S(2K_S)) = \chi(\mathcal{O}_S) + K_S^2 \Longleftrightarrow h^0(2K_S) = 20 - q(S).$$

On the other hand, $\dim \mathrm{S}^2 H^0(K_S) = 21$ and accordingly,

$$q(S) \geqslant 1 \implies \dim \mathrm{Ker}(\mathrm{sym}^2) \geqslant 2.$$

Therefore if the canonical image of S is not contained in a pencil of quadrics then $q(S) = 0$. As to the canonical map, a classical result of Bombieri [1] states that for a nonsingular surface of general type with $K^2 \geqslant 5$ the linear system $|2K_S|$ is base point free. Hence, for a surface of general type with $K^2 \geqslant 5$ such that sym^2 is surjective, the canonical linear system is base point free. Since $|K_S|$ is base point free and $K^2 > 0$, the canonical image of S is not a curve. We deduce that φ_{K_S} is a birational morphism onto a surface of degree 13. □

As far as the canonical ring is concerned, if sym^2 is surjective, $R(S, K_S)$ does not need any new generators in degree 2. It is a theorem of Ciliberto [3] that the degree of the generators of the canonical ring is less than or equal to 3. Hence, we should determine whether the map

$$H^0(K_S) \otimes H^0(2K_S) \to H^0(3K_S)$$

is or is not surjective. We will show that $R(S, K_S)$ is generated by $H^0(K_S)$ so the answer will be affirmative. Our proof relies on the hyperplane principle and on the geometry of a general halfcanonical curve $\mathcal{C} \in |K_S|$. We recall the statement of the hyperplane principle.

Proposition 3.2 (Hyperplane principle)**.** *Let R be a graded ring and $x_0 \in R_d$ a nonzero divisor. Denote by $\widehat{R}$ the quotient $R/(x_0)$. Then, there exists an exact sequence:*

$$0 \to R(-d) \xrightarrow{x_0} R \xrightarrow{\pi} \widehat{R} \to 0$$

of graded homomorphisms of degree 0 such that the following hold:

(i) *If $\widehat{x_1}, \ldots, \widehat{x_n} \in \widehat{R}$ generate $\widehat{R}$ then choosing pre-images $x_1, \ldots, x_n$ under π of $\widehat{x_1}, \ldots, \widehat{x_n}$ the elements $x_0, \ldots, x_n \in R$ generate R.*

(ii) *If $\widehat{R} = \mathbb{C}[\widehat{x_1}, \ldots, \widehat{x_n}]/(f_1, \ldots, f_m)$ then there exist $F_1, \ldots, F_m$ in $\mathbb{C}[x_0 \ldots, x_n]$ such that $R = \mathbb{C}[x_0 \ldots, x_m]/(F_1, \ldots, F_m)$.*

Suppose that $S \subset \mathbb{P}^n$ is a canonical surface. Then a general hyperplane section is a curve with a halfcanonical divisor, given by adjunction as the restriction to the curve of the canonical divisor of S. Let $f \in H^0(K_S)$ be an equation of the hyperplane then, if S is regular, the ring $R(S, K_S)/(f)$ is the halfcanonical ring $R(\mathcal{C}, A)$.

Proposition 3.3. *Let S be a nonsingular regular surface of general type. If $\mathcal{C} \in |K_S|$ is a nonsingular curve cut out by $f \in H^0(K_S)$, then*

$$R(\mathcal{C}, A) = R(S, K_S)/(f).$$

Proof. By regularity and the Kawamata-Viehweg vanishing theorem, the cohomology space $H^1(nK_S)$ is null for any integer n. Hence from the restriction exact sequence, we deduce that

$$0 \to H^0((n-1)K_S) \xrightarrow{f} H^0(nK_S) \to H^0(\mathcal{C}, nA) \to 0.$$

The direct sum of these sequences implies that $R(\mathcal{C}, A) \simeq R(S, K_S)/(f)$. □

We return to the case of a nonsingular surface of general type with $p_g = 6$ and $K_S^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Recall from Proposition 3.1, that such a surface is necessarily regular and its canonical map is a birational morphism. By Bertini's theorem, a general member $\mathcal{C} \in |K_S|$ is a nonsingular reduced curve of genus 14 with a halfcanonical divisor $A = K_{S|\mathcal{C}}$ whose space of global sections $H^0(A)$, by regularity of S, is 5-dimensional. Moreover A is free and the map

$$\mathrm{sym}^2\colon\ \mathrm{S}^2 H^0(A) \to H^0(2A), \tag{2}$$

(which we still denote by sym^2) is surjective. The next proposition together with the hyperplane principle shows that $R(S, K_S)$ is generated in degree 1.

Proposition 3.4. *Let $\mathcal{C}$ be nonsingular curve of genus 14 with a halfcanonical divisor A such that $h^0(A) = 5$. Assume that sym^2 is surjective. Then $\mathrm{gon}(\mathcal{C}) \geqslant 6$, $\mathcal{C}$ has no g^2_9 and $R(\mathcal{C}, A)$ is a quotient of a polynomial ring whose variables have weight 1 by an ideal of codimension 3.*

Proof. We start by proving the following lemma.

Lemma 3.5. *Let A and B be two divisors on an algebraic curve $\mathcal{C}$. The extension bundles of $\mathcal{O}_\mathcal{C}(B)$ by $\mathcal{O}_\mathcal{C}(A)$ with maximum number of global sections are parametrised by the cokernel of the multiplication map*

$$H^0(K_\mathcal{C} - A) \otimes H^0(B) \to H^0(K_\mathcal{C} + B - A).$$

Proof. The argument we will use is due to Mukai and is contained in the proof of Lemma 3.6 of [8]. The group classifying extensions of $\mathcal{O}_\mathcal{C}(B)$ by $\mathcal{O}_\mathcal{C}(A)$ is $\mathrm{Ext}^1(B,A)$. By Serre duality we have:

$$\mathrm{Ext}^1(B,A) = \mathrm{Ext}^1(K_\mathcal{C}+B-A, K_\mathcal{C}) \simeq H^0(K_\mathcal{C}+B-A)^\vee.$$

On the other hand, extensions of $\mathcal{O}_\mathcal{C}(B)$ by $\mathcal{O}_\mathcal{C}(A)$ with maximum number of global sections,

$$0 \to \mathcal{O}_\mathcal{C}(A) \to \mathscr{F} \to \mathcal{O}_\mathcal{C}(B) \to 0$$

have zero connecting homomorphism $H^0(B) \to H^1(A)$, i.e., $\mathscr{F}$ has maximum number of global sections if and only if its class $[\mathscr{F}] \in \mathrm{Ext}^1(B,A)$, under the canonical morphism

$$\mathrm{Ext}^1(B,A) \to \mathrm{Hom}(H^0(B), H^1(A))$$

maps to zero. Which is to say, $\mathscr{F}$ has maximum number of global sections if and only if

$$[\mathscr{F}] \in \mathrm{Ker}\left\{\mathrm{Ext}^1(B,A) \to H^0(B)^\vee \otimes H^1(A)\right\}. \tag{3}$$

Again by Serre duality we have

$$H^1(A) \simeq \mathrm{Ext}^0(A, K_\mathcal{C})^\vee = \mathrm{Ext}^0(0, K_\mathcal{C}-A)^\vee = H^0(K_\mathcal{C}-A)^\vee.$$

Finally, dualising statement (3) we see that extension classes corresponding to bundles with maximum number of global sections are in bijection with the cokernel of

$$H^0(K_\mathcal{C}-A) \otimes H^0(B) \to H^0(K_\mathcal{C}+B-A).$$

□

The lemma is used in the remainder of this proof to equate the cokernel of the maps

$$H^0(A) \otimes H^0(nA) \to H^0((n+1)A) \tag{4}$$

with certain extension bundles on $\mathcal{C}$. First, let us deal with the case $n \geqslant 3$. From Lemma 3.5, showing that (4) is surjective is equivalent to showing that all extension bundles of $\mathcal{O}_\mathcal{C}(A)$ by $\mathcal{O}_\mathcal{C}((2-n)A)$ with 5 global sections are split extensions. Let $\mathscr{F}$ be such an extension bundle

$$0 \to \mathcal{O}_\mathcal{C}((2-n)A) \to \mathscr{F} \to \mathcal{O}_\mathcal{C}(A) \to 0. \tag{5}$$

Since $h^0(\mathscr{F}) = 5$, there exists a section of $\mathscr{F}$ with nontrivial divisor of zeros δ. This section yields an embedding $\mathcal{O}_\mathcal{C}(\delta) \hookrightarrow \mathscr{F}$, which upon saturation gives:

$$0 \to \mathcal{O}_\mathcal{C}(\xi) \to \mathscr{F} \to \mathcal{O}_\mathcal{C}((3-n)A-\xi) \to 0.$$

Since ξ is effective and $n \geqslant 3$ we deduce that $h^0(\xi) = h^0(A) = 5$. Besides, as $\mathcal{O}_\mathcal{C}(\xi)$ does not embed into $\mathcal{O}_\mathcal{C}((2-n)A)$, the composition of $\mathcal{O}_\mathcal{C}(\xi) \to \mathscr{F}$ with the map $\mathscr{F} \to \mathcal{O}_\mathcal{C}(A)$ of (5) is injective. Since A is free, we conclude that $\xi \simeq A$ and therefore that $\mathscr{F}$ is isomorphic to the split extension. Let us now show that the map

$$H^0(A) \otimes H^0(2A) \to H^0(3A) \tag{6}$$

is surjective. Let $\mathscr{F}$ be an extension of $\mathcal{O}_\mathcal{C}(A)$ by $\mathcal{O}_\mathcal{C}$ with 6 global sections,

$$0 \to \mathcal{O}_\mathcal{C} \to \mathscr{F} \to \mathcal{O}_\mathcal{C}(A) \to 0. \tag{7}$$

Our aim is to show that $\mathscr{F}$ is split. From the dimension of the space of global sections of $\mathscr{F}$ we deduce that for any two $p, q \in \mathcal{C}$ there exists a section of that bundle vanishing on $p + q$. Denote the divisor of zeros of such a section by δ. Saturating the embedding $\mathcal{O}_\mathcal{C}(\delta) \hookrightarrow \mathscr{F}$ we obtain

$$0 \to \mathcal{O}_\mathcal{C}(\xi) \to \mathscr{F} \to \mathcal{O}_\mathcal{C}(A - \xi) \to 0 \tag{8}$$

where $\xi \supset \delta$ is an effective divisor. Since p, q can be chosen general enough we have $h^0(A-\xi) \leqslant h^0(A)-2$ and accordingly $h^0(\xi) \geqslant 3$. Assume for the time being that $\mathcal{C}$ has no g^2_9. Then $\deg(\xi) \geqslant 10$. But then $h^0(A - \xi) \leqslant 1$ since, if $\mathcal{C}$ has no g^2_9 then is has no g^1_4. By the exact sequence (8) we have $h^0(\xi) \geqslant h^0(\mathscr{F}) - h^0(A - \xi) \geqslant 6 - 1 = h^0(A)$ and together with (7) we get $\xi \subset \mathcal{O}_\mathcal{C}(A)$. From the fact that A is free we conclude that $\xi = A$. In other words, an element of the cokernel of (6) corresponds to the split extension, i.e. the cokernel is null. Hence, provided that we can show that $\mathcal{C}$ has no g^2_9 we can show that $R(\mathcal{C}, A)$ is generated in degree 1. To do so, we need an auxiliary result.

Lemma 3.6. *Let $\mathcal{C}$ be a nonsingular curve and A a halfcanonical divisor for which* sym^2 *is surjective. Let D be an effective divisor on $\mathcal{C}$. Denote by d the dimension of $H^0(A)/H^0(A - D)$. Then:*

$$\deg(D) - h^0(D) \leqslant \frac{d(d+1)}{2} - 1.$$

Proof. Since sym^2 is surjective, the induced map

$$\mathrm{S}^2\left(\frac{H^0(A)}{H^0(A-D)}\right) \to \frac{H^0(K_\mathcal{C})}{H^0(K_\mathcal{C} - D)}$$

is surjective. Hence $h^0(K_\mathcal{C}) - h^0(K_\mathcal{C} - D) \leqslant \frac{d(d+1)}{2}$. On the other hand, by the Riemann-Roch theorem,

$$h^0(K_\mathcal{C}) - h^0(K_\mathcal{C} - D) = \deg(D) - h^0(D) + 1.$$

□

To show that the surjectivity of sym^2 implies the nonexistence of a g^2_9 we begin by showing that $\mathrm{gon}(\mathcal{C}) \geqslant 6$. Suppose there exists a divisor on $\mathcal{C}$ with $h^0(D) = 2$ and $\deg(D) \leqslant 5$. Since

$$h^0(A) - h^0(A - (A - D)) = 3,$$

applying Lemma 3.6, we deduce that $h^0(A - D) \geqslant 8 - \deg(D)$. Hence $h^0(A) - h^0(A - D) \leqslant \deg(D) - 3$. In particular, $\deg(D) = 4$ or 5. By the same lemma, we deduce that if $\deg(D) = 5$, then $\deg(D) - 2 \leqslant 2$ and if $\deg(D) = 4$, that $\deg(D) - 2 \leqslant 0$. A contradiction in both cases. We have shown that $\mathrm{gon}(\mathcal{C}) \geqslant 6$. Finally, assume that for some $d \leqslant 9$, there exists a free g^2_d on $\mathcal{C}$ and let us denote the divisor of the associated complete linear system by D. Since $h^0(A) - h^0(A - (A - D)) \leqslant 2$, from Lemma 3.6 we deduce that $\deg(A - D) - h^0(A - D) \leqslant 2$, i.e., $h^0(A - D) \geqslant 2$, which is a contradiction, since $\mathrm{gon}(\mathcal{C}) \geqslant 6$. We have finished the proof of Proposition 3.4. □

Corollary 3.7. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Then the canonical ring $R(S, K_S)$ is generated in degree* 1*. Consequently, the canonical morphism factors through $S \to \mathrm{Proj}\, R(S, K_S)$, the pluricanonical morphism, and moreover, the canonical image of S is a surface with at most Du Val singularities.*

4. The embedding of $\operatorname{Proj} R(S, K_S)$

In the previous section we have shown that a surface satisfying the hypothesis of Theorem 1.1 is necessarily regular, its canonical linear system is free and moreover, the canonical morphism factors through the pluricanonical morphism. In particular, this implies that the image of the canonical map has at most Du Val singularities. To define an embedding of $\operatorname{Proj} R(S, K_S)$ as complete intersection in a weighted Grassmannian we apply Mukai's vector bundle method. Accordingly, we need to construct a vector bundle of rank 2 on S, whose global sections, plus a section of $H^0(\mathscr{E}(K_S))$, yield the map we seek. A priori, we need that its determinant be K_S, that the dimension of $H^0(\mathscr{E})$ be 4 and that the product map $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$ have a one-dimensional cokernel.

We have shown in the proof of Proposition 3.4, that a general member of $|K_S|$ is a nonsingular curve $\mathcal{C}$ with $\operatorname{gon}(\mathcal{C}) \geqslant 6$. Suppose that there exists a nonsingular member of $|K_S|$ with a (free and complete) g^1_6. Let us denote it by ξ. The evaluation morphism $2\mathcal{O}_S \to 2\mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{C}(\xi)$ has a locally free kernel of rank 2. Its dual, $\mathscr{E}$, is a locally free sheaf of rank 2 (equivalently a vector bundle of rank 2) that fits into the exact sequence:

$$0 \to 2\mathcal{O}_S \to \mathscr{E} \to \mathscr{E}xt^1(\mathcal{O}_\mathcal{C}(\xi), \mathcal{O}_\mathcal{C}) \to 0. \tag{9}$$

This is the bundle we use to write the embedding of S into $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$. Before we enumerate its properties, we show that there exists a nonsingular curve $\mathcal{C} \in |K_S|$ with a g^1_6.

Proposition 4.1. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K_S^2 = 13$ whose canonical image is contained in a single nonsingular quadric. Then there exists a nonsingular curve in $|K_S|$ with a g^1_6.*

Proof. Let $\Sigma \subset \mathbb{P}^5$ be the canonical image of S. From Corollary 3.7, Σ has at most isolated Du Val singularities. To show that there exists a nonsingular curve $\mathcal{C} \in |K_S|$ having a pencil of degree 6, it is enough to show that the hyperplane section of Σ determining $\mathcal{C}$ is contained in a quadric of rank $\leqslant 4$, as the ruling of a quadric of rank 3 induces on the curve a pencil of degree 6 and the rulings of a quadric of rank 4, pencils of degree 6 and 7. By assumption, Σ is contained in a single nonsingular quadric, hence it is enough to show that there exists a tangent hyperplane to Q which does not contain tangent planes to Σ at nonsingular points and does not meet any singularity of Σ. Since Σ has only isolated singularities, the last condition is satisfied if we show that the set of tangent hyperplanes to Q, not containing tangent planes to Σ at nonsingular points, contains an open set of Q^*, the dual variety of Q in dual projective space. To this end, it suffices to show that $Q^* \setminus \Sigma^* \neq \emptyset$. Suppose that $\Sigma^* \supset Q^*$. Then, since Σ^* is irreducible and $\dim Q^* = 4 \geqslant \dim \Sigma^*$ we would deduce that $\Sigma^* = Q^*$, which is not true, since otherwise $\Sigma = Q$. □

Remark 4.2. As we mentioned before, $\mathcal{C} \in |K_S|$ has a halfcanonical divisor A with $h^0(A) = 5$, given by the restriction of the canonical divisor of S. Notice that by Proposition 3.3, we deduce that the map $\operatorname{sym}^2 \colon \mathrm{S}^2 H^0(A) \to H^0(K_\mathcal{C})$ is surjective. Hence, from Proposition 3.4, for the curve $\mathcal{C}$ constructed above, we deduce that $\operatorname{gon}(\mathcal{C}) = 6$ and that $\mathcal{C}$ has no g^2_9.

Proposition 4.3. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image is contained in a single nonsingular quadric. Then, there exists a bundle $\mathscr{E}$ of rank 2 with determinant K_S such that:*

(i) $\dim H^0(\mathscr{E}) = 4$;
(ii) $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$ *is an isomorphism;*
(iii) $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$ *has a* 1*-dimensional cokernel.*

Proof. Let $\mathcal{C} \in |K_S|$ be a nonsingular curve with a g^1_6. Since the gonality of $\mathcal{C}$ is 6 such linear system is necessarily free and complete. Denote a g^1_6 on $\mathcal{C}$ by ξ. In what follows we make no distinction between ξ and its associated divisor. From Castelnuovo's free-pencil trick we obtain

$$0 \to \mathcal{O}_{\mathcal{C}}(A - \xi) \to 2\mathcal{O}_{\mathcal{C}}(A) \to \mathcal{O}_{\mathcal{C}}(A + \xi) \to 0.$$

computing dimension of the spaces of global sections, $2h^0(A) \leqslant h^0(A-\xi) + h^0(A+\xi)$, which by Serre duality and the Riemann–Roch theorem yields $h^0(A - \xi) \geqslant h^0(A) - \frac{\deg(\xi)}{2} = 5 - 3 = 2$. On the other hand, using Lemma 3.6, with $D = \xi$, we get $h^0(A - \xi) \leqslant 2$. Hence $|A - \xi|$ is a g^1_7. Let us denote it by η. Recall, from the beginning of the present section, that we can define a vector bundle of rank 2, using the space of global sections of ξ:

$$0 \to \mathscr{E}^\vee \to 2\mathcal{O}_S \to \mathcal{O}_{\mathcal{C}}(\xi) \to 0. \tag{10}$$

Since $\mathscr{E}xt^1_{\mathcal{O}_S}(\mathcal{O}_{\mathcal{C}}(\xi), \mathcal{O}_S) = \mathcal{O}_{\mathcal{C}}(A - \xi)$, the dual of the above sequence is

$$0 \to 2\mathcal{O}_S \to \mathscr{E} \to \mathcal{O}_{\mathcal{C}}(\eta) \to 0. \tag{11}$$

The fact that $\mathcal{C} \in |K_S|$ together with (11) implies $\det(\mathscr{E}) = K_S$. By regularity of S, $\dim H^1(2\mathcal{O}_S) = 0$. Taking global sections of (11) we get $h^0(\mathscr{E}) = h^0(2\mathcal{O}_S) + h^0(\eta) = 4$. This proves (i).

To prove (iii) we start by tensoring (11) with $\mathcal{O}_S(K_S)$, obtaining:

$$0 \to 2\mathcal{O}_S(K_S) \to \mathscr{E}(K_S) \to \mathcal{O}_{\mathcal{C}}(A + \eta) \to 0.$$

By the Riemann–Roch theorem and Serre duality on $\mathcal{C}$, and using the regularity of S, we deduce that

$$h^0(\mathscr{E}(K_S)) = 2h^0(K_S) + h^0(A + \eta) = 2h^0(K_S) + h^1(\xi) = 21.$$

Accordingly, to prove (iii) it is enough to show that the kernel of:

$$H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S)) \tag{12}$$

is 4-dimensional. To see this, we identify this map with the map on global sections of a map of sheaves. We choose a 2-dimensional subspace of $H^0(\mathscr{E})$ projecting down to $H^0(\eta)$ and write evaluation maps in the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & 2\mathcal{O}_S & \xrightarrow{\sim} & 2\mathcal{O}_S \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathscr{N} & \longrightarrow & 2\mathcal{O}_S \bigoplus 2\mathcal{O}_S & \longrightarrow & \mathscr{E} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 2\mathcal{O}_S & \longrightarrow & \eta
\end{array}$$

Tensoring the middle sequence with $\mathcal{O}_S(K_S)$, we deduce that the map on global sections of the morphism $2\mathcal{O}_S(K_S) \oplus 2\mathcal{O}_S(K_S) \longrightarrow \mathscr{E}(K_S)$ is exactly that of (12). Therefore its kernel is isomorphic to $H^0(\mathscr{N}(K_S))$. The snake lemma applies to the first two rows, giving

$$0 \to \mathscr{N} \to 2\mathcal{O}_S \to \eta. \tag{13}$$

Hence we need to show that $\dim H^0(\mathscr{N}(K_S)) = 4$. By Castelnuovo's free-pencil trick, we have

$$0 \to \mathcal{O}_{\mathcal{C}}(B-\eta) \to 2\mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(\eta - B) \to 0,$$

where B denotes the base locus of η; and hence the map

$$H^0(\eta) \otimes H^0(A) \to H^0(A+\eta) \tag{14}$$

has a kernel isomorphic to $H^0(\xi + B)$. Since $\deg B \leqslant 1$ and the curve $\mathcal{C}$ has no g^2_9 we deduce that $h^0(\xi + B) = 2$. By (13) the dimension of $\mathscr{N}(K_S)$ equals the dimension of the kernel of the map $H^0(\eta) \otimes H^0(K_S) \to H^0(\mathcal{C}, A+\eta)$ given by restriction and multiplication of the two linearly independent sections of $H^0(\eta)$. This map factors through the map $H^0(\eta) \otimes H^0(K_S) \to H^0(\eta) \otimes H^0(A)$ (whose kernel is 2-dimensional) and the map of (14). We conclude that $\dim H^0(\mathscr{N}(K_S)) = 4$. This shows that the kernel of the map (12) is 4-dimensional, which proves (iii).

To prove (ii), we use an argument involving the restriction of $\mathscr{E}$ to the curve $\mathcal{C} \in |K_S|$ used to construct $\mathscr{E}$. Tensoring the sequence (11) with $\mathcal{O}_{\mathcal{C}}$ we obtain:

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{E}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(\eta) \to 0. \tag{15}$$

Since $\mathscr{E}(-K_S) \simeq \mathscr{E}^\vee$ and from (10), $H^0(\mathscr{E}^\vee) = 0$, we deduce that the restriction $H^0(\mathscr{E}) \to H^0(\mathscr{E}_{\mathcal{C}})$ is injective. Since $h^0(\mathscr{E}_{\mathcal{C}}) \leqslant h^0(\xi) + h^0(\eta) = 4$ we have $H^0(\mathscr{E}_{\mathcal{C}}) \simeq H^0(\mathscr{E})$. Additionally, as the map $H^0(K_S) \to H^0(A)$ is surjective and has a 1-dimensional kernel, to prove (ii) we start by showing that

$$\bigwedge^2 H^0(\mathscr{E}_{\mathcal{C}}) \to H^0(A)$$

has a kernel of dimension $\leqslant 3$. Our argument follows the ideas of Castelnuovo's trick. Let us denote by W the kernel of this map. The projective space $\mathbb{P}[W]$ is a linear subspace of $\mathbb{P}(\bigwedge^2 H^0(\mathscr{E}_{\mathcal{C}}))$ that also contains the variety of skew tensors of rank 2, $\mathrm{G}(2, H^0(\mathscr{E}_{\mathcal{C}}))$. Let $a \wedge b$ be an element of $\mathbb{P}[W] \cap \mathrm{G}(2, H^0(\mathscr{E}_{\mathcal{C}}))$. This means that $a, b \in H^0(\mathscr{E}_{\mathcal{C}})$ span a (torsion free) subsheaf of $\mathscr{E}$ of rank 1, that is given by $\mathcal{O}_{\mathcal{C}} \cdot a + \mathcal{O}_{\mathcal{C}} \cdot b$. Since $\mathcal{C}$ is nonsingular this sheaf is invertible. The saturation of $\mathcal{O}_{\mathcal{C}} \cdot a + \mathcal{O}_{\mathcal{C}} \cdot b \subset \mathscr{E}_{\mathcal{C}}$ yields

$$0 \to \mathcal{O}_{\mathcal{C}}(\delta) \to \mathscr{E}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(A - \delta) \to 0,$$

where $\mathcal{O}_{\mathcal{C}} \cdot a + \mathcal{O}_{\mathcal{C}} \cdot b \subset \mathcal{O}_{\mathcal{C}}(\delta)$. Since $\mathrm{gon}(\mathcal{C}) = 6$ and $h^0(\delta) \geqslant 2$ we deduce that $\deg(\delta) \geqslant 6$. From (15) we deduce that $\mathcal{O}_{\mathcal{C}}(\delta) \hookrightarrow \xi$ or $\mathcal{O}_{\mathcal{C}}(\delta) \hookrightarrow \eta$. Therefore either $\delta = \xi$ or $\delta = \eta$ or $\delta = \eta - B$ where B is the base locus of η (if there is exists one). We conclude that an element of $\mathbb{P}[W] \cap \mathrm{G}(2, H^0(\mathscr{E}_{\mathcal{C}}))$ yields an embedding of one of three bundles line bundles on $\mathcal{C}$ into $\mathscr{E}_{\mathcal{C}}$. Each embedding is parametrised by $\mathrm{Hom}(\mathcal{O}_{\mathcal{C}}(\delta), \mathscr{E}_{\mathcal{C}}) \simeq H^0(\mathscr{E}_{\mathcal{C}}(-\delta))$, (where δ is one of the above), whose dimension, by (15) is $\leqslant 2$. We conclude that $\dim \mathbb{P}[W] \cap \mathrm{G}(2, H^0(\mathscr{E}_{\mathcal{C}})) \leqslant 1$. Therefore we must have $\dim W \leqslant 3$, and accordingly, $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$ has a kernel of dimension $\leqslant 2$. This implies that

$$\mathrm{Pf}\begin{bmatrix} s_1 \wedge s_2 & s_1 \wedge s_3 & s_1 \wedge s_4 \\ & s_2 \wedge s_3 & s_2 \wedge s_4 \\ & & s_3 \wedge s_4 \end{bmatrix} = 0,$$

where $\{s_1, s_2, s_3, s_4\}$ is a basis of $H^0(\mathscr{E})$, yields a nontrivial quadric relation in $R(S, K_S)$, of rank at least 3. However by our assumption the rank of this quadric should be 6, hence we conclude that the kernel of $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$ must be trivial. Computing dimensions we deduce that this map is an isomorphism. $\square$

Proposition 4.4. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K_S^2 = 13$ whose canonical image is contained in a single nonsingular quadric. Then there exists a map ρ of S into the weighted Grassmannian $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$, factoring through the pluricanonical morphism $S \to \operatorname{Proj} R(S, K_S)$ which is an embedding away from -2-cycles. The image of S under this map is contained in the intersection of four quasihomogeneous forms of degree 2.*

Proof. Let X denote the weighted Grassmannian $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$. By definition, X is a projectively Gorenstein subscheme of $\mathbb{P}(1^6, 2^4)$. We fix notation for the variables of $\mathbb{P}(1^6, 2^4)$ as m_{ij} and n_i, respectively, for $1 \leqslant i < j \leqslant 4$, with $\operatorname{wt}(m_{ij}) = 1$ and $\operatorname{wt}(n_i) = 2$. Fix a basis of $H^0(\mathscr{E})$ and denote it by s_1, s_2, s_3, s_4. Then, fix a choice of a section $t \in H^0(\mathscr{E}(K_S))$ which is not in the image of the multiplication map $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$. Define a map $\rho\colon S \to \mathbb{P}(1^6, 2^4)$ in the following way:

$$p \mapsto [(s_i \wedge s_j)(p), (s_i \wedge t)(p)] \quad \text{where } 1 \leqslant i < j \leqslant 4.$$

Denote the image of S under ρ by Γ. Since s_1, s_2, s_3, s_4, t are sections of a vector bundle of rank 2 on S, the displayed wedges satisfy the defining equations of X and thus $\Gamma \subset X$. If we compose ρ with a projection onto the linear subspace $\mathbb{P}[m_{ij}] \subset \mathbb{P}(1^6, 2^4)$ we obtain the canonical map of S. By Corollary 3.7 the canonical map is an embedding away from -2-cycles. Therefore we deduce that ρ is an embedding away from -2-cycles and factors through the pluricanonical morphism. Since $s_i \wedge t$ is an element of $H^0(2K_S)$ and the map $\operatorname{sym}^2\colon \operatorname{S}^2 H^0(K_S) \to H^0(2K_S)$ is surjective we deduce that there exist $q_1, q_2, q_3, q_4 \in \operatorname{S}^2 \langle m_{ij} \rangle$ such that the four quasihomogeneous forms of degree 2, $n_i - q_i$, vanish on Γ. □

In what follows, we fix the choice of q_i, the quadratic polynomials in the variables m_{ij}, mentioned in the proof of the previous proposition.

Proposition 4.5. *Let $\Upsilon \subset \mathbb{P}^5 = \mathbb{P}[m_{ij}]$ be the scheme defined by 5 submaximal Pfaffians of*

$$\begin{bmatrix} m_{12} & m_{13} & m_{14} & q_1 \\ & m_{23} & m_{24} & q_2 \\ & & m_{34} & q_3 \\ & & & q_4 \end{bmatrix}$$

Then, Υ is of pure dimension 2.

Proof. Denote by Pf_i the i-th Pfaffian of the matrix in the above display. We start by proving the following lemma.

Lemma 4.6. *The two cubic polynomials Pf_1 and Pf_2 a linearly independent modulo $\langle m_{ij} \rangle \operatorname{Pf}_5$. In other words, if $\alpha \operatorname{Pf}_1 + \beta \operatorname{Pf}_2 + M \operatorname{Pf}_5 = 0$ with $\alpha, \beta \in \mathbb{C}$ and $M \in \langle m_{ij} \rangle$, then $\alpha = \beta = M = 0$.*

Proof. Suppose that

$$\alpha \operatorname{Pf} \begin{bmatrix} m_{23} & m_{24} & q_2 \\ & m_{34} & q_3 \\ & & q_4 \end{bmatrix} + \beta \operatorname{Pf} \begin{bmatrix} m_{13} & m_{14} & q_1 \\ & m_{34} & q_3 \\ & & q_4 \end{bmatrix} + M \operatorname{Pf} \begin{bmatrix} m_{12} & m_{13} & m_{14} \\ & m_{23} & m_{24} \\ & & m_{34} \end{bmatrix} = 0 \tag{16}$$

and $\alpha=\beta=0$, then clearly $M=0$. We may, therefore, assume that one of α,β is nonzero. Suppose that $\alpha\neq 0$. Then, from (16), we deduce

$$\alpha\,\mathrm{Pf}\begin{bmatrix} m_{23} & m_{24} & (1/\alpha)Mm_{12}+q_2 \\ & m_{34} & (1/\alpha)Mm_{13}+q_3 \\ & & (1/\alpha)Mm_{14}+q_4 \end{bmatrix} + \beta\,\mathrm{Pf}\begin{bmatrix} m_{13} & m_{14} & q_1 \\ & m_{34} & (1/\alpha)Mm_{13}+q_3 \\ & & (1/\alpha)Mm_{14}+q_4 \end{bmatrix} = 0,$$

which is equivalent to

$$\mathrm{Pf}\begin{bmatrix} \alpha m_{23}+\beta m_{13} & \alpha m_{24}+\beta m_{14} & \alpha((1/\alpha)Mm_{12}+q_2)+\beta q_1 \\ & m_{34} & (1/\alpha)Mm_{13}+q_3 \\ & & (1/\alpha)Mm_{14}+q_4 \end{bmatrix} = 0.$$

Since $\alpha\neq 0$ the sequence of linear forms $m_{34}, \alpha m_{23}+\beta m_{13}, \alpha m_{24}+\beta m_{34}$ is regular in $\mathbb{C}[m_{ij}]$. Accordingly, if the Pfaffian of the skew matrix above is zero, we deduce that there exist linear forms $A,B\in\langle m_{ij}\rangle$ such that

$$(1/\alpha)Mm_{13}+q_3 = Am_{34}+B(\alpha m_{23}+\beta m_{13}).$$

The equality above restricted to image of S by ρ becomes:

$$\begin{aligned} q_3 &= \left(-(1/\alpha)M+\beta B\right)m_{13}+Am_{34}+\alpha Bm_{23} \\ \Longrightarrow s_3\wedge t &= \left(-(1/\alpha)M+\beta B\right)s_1\wedge s_3+As_3\wedge s_4+\alpha Bs_2\wedge s_3 \\ \Longleftrightarrow 0 &= s_3\wedge\left(t-\left((1/\alpha)M+\beta B\right)s_1-As_4+\alpha Bs_2\right). \end{aligned}$$

Let us denote the section $t-\left((1/\alpha)M+\beta B\right)s_1-As_4+\alpha Bs_2\in H^0(\mathscr{E}(K_S))$ by u. By our assumptions $u\neq 0$. Consider the map $\mathscr{E}\to\mathcal{O}_S(2K_S)$ given by wedging sections of $\mathscr{E}$ with u. Its image is a torsion free sheaf $\mathcal{O}_S(2K_S-D)\otimes\mathcal{I}_\delta$, with D a divisor and δ a 0-dimensional subscheme of S. Since $\mathscr{E}$ is globally spanned except possibly at a 0-dimension locus of S, the divisor D coincides with the divisor of zeros u. Additionally, we have

$$0\to\mathcal{O}_S(D-K_S)\otimes\mathcal{I}_{\delta'}\to\mathscr{E}\to\mathcal{O}_S(2K_S-D)\otimes\mathcal{I}_\delta\to 0$$

for some 0-dimensional subscheme δ' of S. By the computation above, the kernel of the map on global sections, $H^0(\mathscr{E})\to H^0(2K_S)$ is non-zero. This means that the dimension of $H^0(\mathcal{O}_S(D-K_S)\otimes\mathcal{I}_{\delta'})$ is greater than zero. Then, $h^0(D-K_S)>0$ and therefore $u\in H^0(\mathscr{E}(K_S)\otimes\mathcal{O}_S(-K_S))$. In turn, this means that $u\in H^0(K_S)\otimes H^0(\mathscr{E})$, which not true. Hence, we must have $\alpha=\beta=M=0$. □

Let us now return to the proof of the proposition. To show that the components of Υ have dimension $\leqslant 2$ we argue by contradiction. Suppose that Υ has a component Z of dimension $\geqslant 3$. Since

$$Z\subset\mathrm{G}(2,4)=(\mathrm{Pf}_5=0)\subset\mathbb{P}^5[m_{ij}]$$

there are only two possibilities. Either Z is 4-dimensional or it is 3-dimensional. In the first instance $Z=\mathrm{G}(2,4)$ and then there would not be any cubic hypersurfaces through Z, which were not multiples of Pf_5, this contradicts Lemma 4.6. We deduce that we must have $\dim Z=3$. Since $\mathrm{G}(2,4)$ is a nonsingular 4-dimensional hypersurface of $\mathbb{P}^5$, its Picard group is free of rank 1. This implies that there exists $d\geqslant 1$ such that Z is the complete intersection of $\mathrm{G}(2,4)$ and a hypersurface of degree d. Since Z is already contained in a cubic hypersurface which does not vanish on $\mathrm{G}(2,4)$, d has to equal 3. But then Z is a complete intersection of type $(2,3)$ in $\mathbb{P}^5$ and as such is not contained in two cubic hypersurfaces whose equations are linearly independent modulo the quadric equation. By Lemma 4.6

this is a contradiction. We deduce that Υ has components of dimension $\leqslant 2$. However, notice that Υ is also the intersection of $X = \mathrm{G}(\frac{1}{2}^4, \frac{3}{2})$ with four quasihomogeneous forms $y_1 - q_1, \ldots, y_4 - q_4$ and since X has dimension 6, this implies that the components of Υ have dimension $\geqslant 2$. □

We can now finish proving Theorem 1.1. Namely, we end by showing that the image of the surface S under the map ρ is a complete intersection in X.

Proposition 4.7. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image is contained in a single nonsingular quadric. Let $\rho\colon S \to X$ be the map of S into a weighted Grassmannian of Proposition 4.4. Then, Γ, the image of S by this map, is a complete intersection in X of four quasihomogeneous forms of degree 2.*

Proof. We know from Proposition 4.4 that $\Gamma \subset X$ is contained in the intersection of four quasihomogeneous forms of degree 2, $n_1 - q_1$, $n_2 - q_2$, $n_3 - q_3$, $n_4 - q_4$. Consider $\mathbb{C}[X]$ the homogeneous ring of $X = \mathrm{G}(\frac{1}{2}^4, \frac{3}{2})$ in $\mathbb{P}(1^6, 2^4)$ and, contained in it, the homogeneous ideal of Γ, which we denote by I. Saying that Γ is the complete intersection of 4 quasihomogeneous forms in X is the same as saying that $J = (\pi(n_1 - q_1), \pi(n_2 - q_2), \pi(n_3 - q_3), \pi(n_4 - q_4))$, where $\pi\colon \mathbb{C}[m_{ij}, n_i] \to \mathbb{C}[X]$ is the quotient homomorphism. We know from Proposition 4.5 that J defines a scheme Υ of dimension 2. In particular Γ, is one of its components. To show that J is the homogeneous ideal of Γ it will suffice to show that J is prime.

Lemma 4.8. *The ideal J is prime.*

Proof. By Proposition 4.5, Υ has pure dimension 2, which is to say that $J \subset \mathbb{C}[X]$ has codimension 4. Since X is arithmetically Cohen-Macaulay and J is generated by 4 elements, by Unmixedness, the radical ideal of each primary ideal in a primary decomposition $J = P_1 \cap \cdots \cap P_n$, is minimal over J. This means that $\operatorname{Rad} P_1, \ldots, \operatorname{Rad} P_n$ are the homogeneous ideals of the irreducible components of Υ and in particular for some integer i, $\operatorname{Rad} P_i$ is the homogeneous ideal of Γ. In particular this means that the degree of $\operatorname{Rad} P_i$ is 13. Since J is generated by 4 quasihomogeneous forms of degree 2 we deduce that $\deg J = 2^4 \deg(X)$. By Proposition 2.1, $\deg(X) = \frac{13}{2^4}$ and accordingly $\deg J = 13$. We have:

$$\deg J = 13 = \sum \deg P_i \geqslant \sum \deg \operatorname{Rad} P_i \geqslant 13.$$

Hence $\deg P_i = \deg \operatorname{Rad} P_i$ (therefore P_i equals $\operatorname{Rad} P_i$) and $n = 1$. In other words, J is prime. □

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