Linear time equivalence of Littlewood-Richardson coefficient symmetry maps

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FPSAC09 Hagenberg, Österreich, July 20-24, 2009

- Pak and Vallejo have introduced in [Reductions of Young tableau bijections, *SIAM J. Discrete Math.*] the notion of linear reduction of Young tableau bijections. In [Combinatorics and geometry of Littlewood-Richardson cones, *Europ. J. Combinat*] they have shown that the S_3-symmetries of Littlewood-Richardson coefficients, defined by the action of the subgroup of index 2, can be given by maps of linear cost, and, therefore, the commutative symmetries are given by linearly equivalent maps. As a follow-up, the $\mathbb{Z}_2 \times S_3$-symmetries are analysed: the symmetries defined by the action of a subgroup of index 2 can be given by maps of linear cost, thus commutative symmetry maps, conjugation symmetry maps and Schützenberger involution are linearly reducible to each other; three known Young tableau conjugation symmetry maps are shown to be identical. The *difficulty* to exhibit commutative and conjugation symmetries *seems to be* a common feature of the universe of Littlewood-Richardson rules.

Littlewood-Richardson coefficients: $c_{\mu\,\nu}^{\lambda}$

- Schur functions form a basis for the algebra of symmetric functions

$$s_\mu s_\nu = \sum_{\lambda} c_{\mu\,\nu}^{\lambda} s_\lambda.$$

- Decomposition of the tensor product of two irreducible polynomial representations V^μ and V^ν of the general linear group $GL_d(\mathbb{C})$ into irreducible representations of $GL_d(\mathbb{C})$

$$V^\mu \otimes V^\nu = \sum_{l(\lambda)\leq d} c_{\mu\,\nu}^{\lambda} V^\lambda.$$

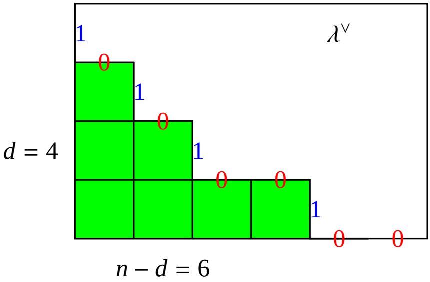
- Schubert classes σ_λ form a linear basis for $H^*(G(d,n))$, the cohomology ring of the Grassmannian $G(d,n)$ of complex d-dimensional linear subspaces of $\mathbb{C}^n$,

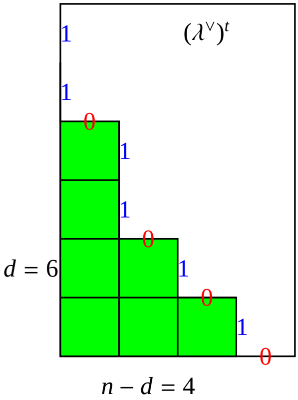
$$\sigma_\mu \sigma_\nu = \sum_{\lambda\subseteq d\times(n-d)} c_{\mu\,\nu}^{\lambda} \sigma_\lambda.$$

- There exist $d \times d$ non singular matrices A, B and C, over a *pid*, with Smith invariants μ, ν and λ respectively, such that $AB = C$ iff $c_{\mu\,\nu}^{\lambda} > 0$.

Conjugate partitions /mirror reflections of 01-strings & 0's and 1's swapped

$n = 10$





$\lambda = (4, 2, 1, 0) \leftrightarrow$ 0010010101

$\lambda^t = (3, 2, 1, 1, 0, 0)$ 0101011011

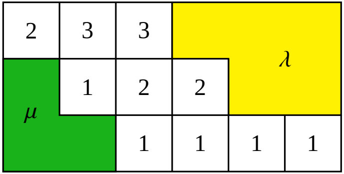
$\lambda^\vee = (6, 5, 4, 2) \leftrightarrow$ 1010100100

$(\lambda^\vee)^t = (4, 4, 3, 3, 2, 1)$ 1101101010

I: Littlewood-Richardson tableaux

- $c_{\mu\,\nu\,\lambda}$ is the number of semistandard Young tableaux with shape $\lambda^\vee/\mu$ and content ν, with the following property:
  - If one reads the labeled entries in reverse reading order, that is, from right to left across rows taken in turn from bottom to top, at any stage, the number of i's encountered is at least as large as the number of $(i+1)$'s encountered, $\#1's \geq \#2's \ldots$.

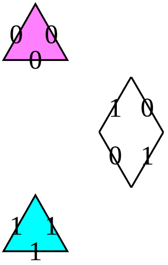
$$c_{210,532,320} = c^{643}_{210,532} = c_{000010101\ 010010100\ 000101001}$$

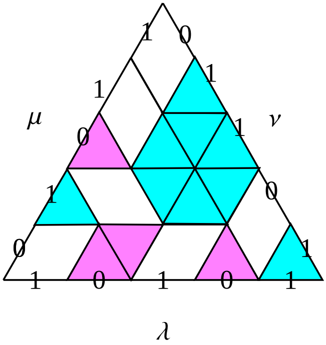


$\nu = (5, 3, 2)$

II: Knutson-Tao-Woodward puzzle rule

- A puzzle of size n is a tiling of an equilateral triangle of side length n with puzzle pieces each of unit side length.
- Puzzle pieces may be rotated in any orientation *but not reflected*, and wherever two pieces share an edge, the numbers on the edge must agree.
- (Knutson-Tao-Woodward) $c_{\mu\,\nu\,\lambda}$ is the number of puzzles with μ, ν and λ appearing clockwise as 01-strings along the boundary.





Littlewood-Richardson number $\mathbb{Z}_2 \times S_3$-symmetries

- Littlewood-Richardson coefficients $c_{\mu\,\nu\,\lambda}$ are invariant under the action of $\mathbb{Z}_2 \times S_3$ as follows: the non–identity element of $\mathbb{Z}_2$ transposes simultaneously μ, ν and λ, and S_3 permutes μ, ν and λ
- S_3-symmetries

$$c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu}$$

$$c_{\mu\,\nu\,\lambda} = c_{\nu\,\mu\,\lambda}$$
$$c_{\mu\,\nu\,\lambda} = c_{\mu\,\lambda\,\nu}$$
$$c_{\mu\,\nu\,\lambda} = c_{\lambda\,\nu\,\mu}$$

- $\mathbb{Z}_2 \times S_3$-symmetries

$$c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu}$$
$$c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\mu^t\,\lambda^t} = c_{\mu^t\,\lambda^t\,\nu^t} = c_{\lambda^t\,\nu^t\,\mu^t}$$

$$c_{\mu\,\nu\,\lambda} = c_{\nu\,\mu\,\lambda} \qquad c_{\mu\,\nu\,\lambda} = c_{\mu^t\,\nu^t\,\lambda^t}$$
$$c_{\mu\,\nu\,\lambda} = c_{\mu\,\lambda\,\nu} \qquad c_{\mu\,\nu\,\lambda} = c_{\lambda^t\,\mu^t\,\nu^t}$$
$$c_{\mu\,\nu\,\lambda} = c_{\lambda\,\nu\,\mu} \qquad c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\lambda^t\,\mu^t}$$

Littlewood-Richardson rules and $\mathbb{Z}_2 \times S_3$-symmetries

- Six of the twelve $\mathbb{Z}_2 \times S_3$-symmetries, in particular, three of the six S_3-symmetries, can be *easily exhibited* in the Littlewood-Richardson rules

$$c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu}$$
$$c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\mu^t\,\lambda^t} = c_{\mu^t\,\lambda^t\,\nu^t} = c_{\lambda^t\,\nu^t\,\mu^t}$$

Either the conjugation symmetry or the commutativity are difficult to exhibit in the Littlewood-Richardson rules.

$$c_{\mu\,\nu\,\lambda} = c_{\nu\,\mu\,\lambda} \qquad c_{\mu\,\nu\,\lambda} = c_{\mu^t\,\nu^t\,\lambda^t}$$
$$c_{\mu\,\nu\,\lambda} = c_{\mu\,\lambda\,\nu} \qquad c_{\mu\,\nu\,\lambda} = c_{\lambda^t\,\mu^t\,\nu^t}$$
$$c_{\mu\,\nu\,\lambda} = c_{\lambda\,\nu\,\mu} \qquad c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\lambda^t\,\mu^t}$$

♦ Involution

- $LR(\mu,\ \nu,\ \lambda) \xrightarrow{\blacklozenge} LR(\lambda^t,\ \nu^t,\ \mu^t)$
- $c_{\mu\ \nu\ \lambda} = c_{\lambda^t\ \nu^t\ \mu^t}$

T =

1	1	3	3		
		2	2	2	
			1	1	1

$\xrightarrow{\blacklozenge}$

T♦ =

5		
4		
2	3	
1	2	3
	1	2
		1

1112223311 $\xrightarrow{\lozenge}$ 1231231245

♠ Involution

- $LR(\mu, \nu, \lambda) \xrightarrow{\spadesuit} LR(\nu^t, \mu^t, \lambda^t)$
- $c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\mu^t\,\lambda^t}$

$T =$

1	3					
	2	2	3			
		1	2	2		
				1	1	1

$\longrightarrow$

1	3					
a	2	2	3			
a	b	1	2	2		
a	b	c	d	1	1	1

$\longrightarrow$

a	b					
a	3	c	3			
a	2	2	2	2		
1	b	1	d	1	1	1

$\longrightarrow$

a	b					
3	3	a	c			
2	2	2	2	a		
1	1	1	1	1	b	d

$\longrightarrow$

d			
b			
	a		
		c	
		a	
			b
			a

$= T^{\spadesuit}$

♠♦♠ involution

- $LR(\mu, \nu, \lambda) \xrightarrow{♠♦♠} LR(\mu^t, \lambda^t, \nu^t)$
- $c_{\mu\,\nu\,\lambda} = c_{\mu^t\,\lambda^t\,\nu^t}$

T =

1	3					
	2	2	3			
		1	2	2		
				1	1	1

$\longrightarrow$

1	3	a	b	c	d	e
	2	2	3	a	b	c
		1	2	2	a	b
				1	1	1

$\longrightarrow$

a	b	1	c	d	3	e
	a	b	2	2	c	3
		a	1	b	2	2
				1	1	1

$\longrightarrow$

a	b	1	2	2	3	3
	a	b	1	1	2	2
		a	c	d	1	1
				b	c	e

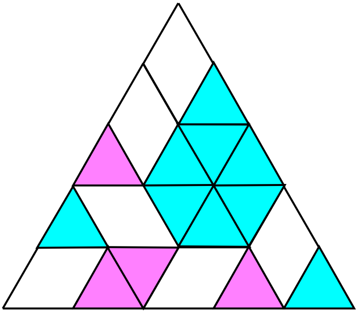
$\longrightarrow$

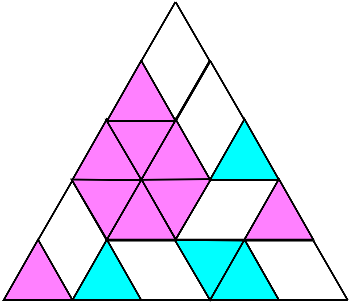
e			
c			
b	d		
	c		
	a	b	
		a	b
			a

$= T^{♠♦♠}$

Puzzle mirror reflections with 0's and 1's swapped

- $c_{\mu\ \nu\ \lambda} = c_{\nu^t\ \mu^t\ \lambda^t}$ ♠
- $c_{\mu\ \nu\ \lambda} = c_{\lambda^t\ \nu^t\ \mu^t}$ ♦
- $c_{\mu\ \nu\ \lambda} = c_{\mu^t\ \lambda^t\ \nu^t}$ ♠♦♠ = ♦♠♦

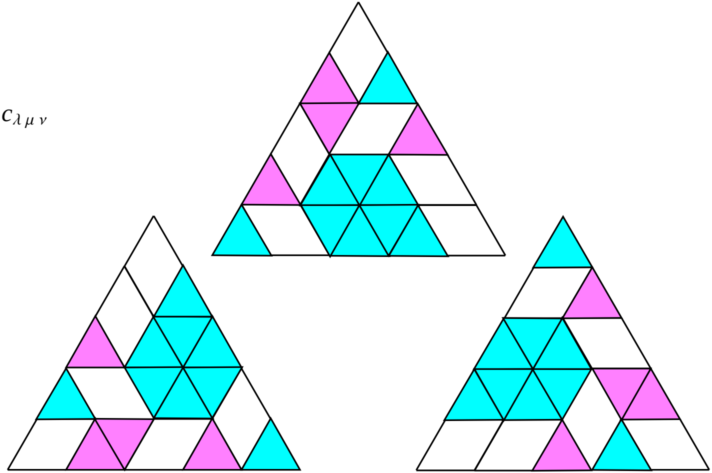




Puzzle $2\pi/3$–rotations

- $c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu}$ ♦♠
- $c_{\mu\,\nu\,\lambda} = c_{\nu\,\lambda\,\mu}$ ♠♦ = ♦♠♦♠

$c_{\mu\,\nu\,\lambda} = c_{\nu\,\lambda\,\mu} = c_{\lambda\,\mu\,\nu}$



An index 2 subgroup of $\mathbb{Z}_2 \times S_3$-symmetries easy to exhibit

- The group generated by the puzzle *mirror reflections with the 0's and 1's swapped* /*simple involutions* ♠, ♦ form a subgroup of index 2 of $\mathbb{Z}_2 \times S_3$

$$< \text{puzzle mirror reflections } \& \ 0 \leftrightarrow 1 >$$
$$\|$$
$$< \spadesuit, \blacklozenge > = \{\mathbf{1}, \spadesuit, \blacklozenge, \spadesuit\blacklozenge\spadesuit, \spadesuit\blacklozenge, \blacklozenge\spadesuit\}$$

- ♦ and ♠ are involutions of linear cost
- Conjugation and commutative symmetry maps are linearly reducible to each other
- *Commutativity symmetry is as difficult as transposition symmetry to be exhibited*

Linear reduction of LR-symmetry maps and Pak-Vallejo's question

- **Pak-Vallejo Theorem** The following maps are linearly equivalent:
  (1) RSK correspondence.
  (2) *Jeu de taquin* map.
  (3) Littlewood–Robinson map.
  (4) Tableau–switching map.
  (5) Schützenberger involution E for normal shapes.
  (6) Reversal e.

  (7) (Fundamental) commutative symmetry map $\rho : LR(\mu,\nu,\lambda) \to LR(\nu,\mu,\lambda)$.
- **Pak-Vallejo's question**: Conjugation symmetry maps
  $\varrho : LR(\mu,\nu,\lambda) \to LR(\mu^t,\nu^t,\lambda^t)$:
  - White-Hanlon-Sundaram bijection ϱ^{WHS} (1992)
  - Benkart-Sottile-Stroomer bijection ϱ^{BSS} (1996)
  - ϱ^{AZ} (1999)
- Are ϱ^{WHS}, ϱ^{BSS} and ϱ^{AZ} identical and linearly equivalent to a map already in the list?
- **Theorem** ϱ^{BSS}, ϱ^{WHS} and ϱ^{AZ} are identical, and linearly equivalent to the Schützenberger involution E,

$$\begin{array}{ccccc} T & \overset{e\,\bullet}{\longleftrightarrow} & T^{e\bullet} & \overset{\blacklozenge}{\longleftrightarrow} & T^{e\bullet\blacklozenge} \\ \tau \updownarrow & & \tau \updownarrow & & \\ P & \underset{E}{\overset{\text{evacuation}}{\longleftrightarrow}} & P^{E}. & & \end{array}$$

ϱ^{BSS} bijection

- Benkart-Sottile-Stroomer bijection ϱ^{BSS}

$$\begin{array}{rcl} \varrho^{BSS} : LR(\mu,\nu,\lambda) & \longrightarrow & LR(\mu^t,\nu^t,\lambda^t) \\ T & \mapsto & \varrho(T) = [Y(\nu^t)]_K \cap [\widehat{T}^t]_{dK} \end{array}$$

- $LR(\mu\ \nu\ \lambda) \mapsto LR(\mu^t\ \lambda^t\ \nu^t)$ standardization + tableau-switching

$T =$

4		
1	3	
	2	
		1

$\rightarrow \quad \widehat{T} =$

5		
1	4	
	3	
		2

$\rightarrow \quad \widehat{T}^t =$

2			
	3	4	
		1	5

$\rightarrow$

2	a	a	b
	3	4	a
		1	5

$\rightarrow$

2	3	4	5
	a	b	1
		a	a

$\rightarrow$

	a	b	
		a	a

- $\rho : LR(\mu^t\ \lambda^t\ \nu^t) \mapsto LR(\mu^t\ \nu^t\ \lambda^t)$ tableau-switching

1	1	1	2
	a	b	1
		a	a

$\rightarrow$

1	a	a	b
	1	2	a
		1	1

$\rightarrow$

1			
	1	2	
		1	1

$= \quad \varrho^{BSS}(T)$

Linear reduction of ϱ^{BSS} bijection

- $LR(\mu\ \nu\ \lambda) \xrightarrow{\spadesuit\blacklozenge\spadesuit} LR(\mu^t\ \lambda^t\ \nu^t) \xrightarrow{\rho} LR(\mu^t\ \nu^t\ \lambda^t)$
- ϱ^{BSS} = puzzle mirror reflection+ commutative symmetry
- $\spadesuit\blacklozenge\spadesuit$: $T =$

4		
1	3	
	2	
		1

$\rightarrow$

4	a	b
1	3	a
	2	a
		1

$\rightarrow$

a	b	4
a	1	3
	a	2
		1

$\rightarrow$

a	1	4
a	b	3
	a	2
		1

$\rightarrow$

	a	b	
		a	a

$= T^{\spadesuit\blacklozenge\spadesuit}$

- $\rho =$ Tableau-switching

1	1	1	2
	a	b	1
		a	a

$\rightarrow$

1	a	a	b
	1	2	a
		1	1

$\rightarrow$

1			
	1	2	
		1	1

$= \varrho^{BSS}(T)$

ϱ^{AZ} bijection

ϱ^{AZ} = puzzle mirror reflection+ commutative symmetry map

- $\rho^{AZ} : LR(\mu\ \nu\ \lambda) \overset{\rho = \bullet e}{\longrightarrow} LR(\lambda\ \nu\ \mu) \overset{\blacklozenge}{\longrightarrow} LR(\mu^t, \nu^t, \lambda^t)$

$$\rho : LR(\mu,\nu,\lambda) \overset{e}{\longrightarrow} LR(\mu,\nu^*,\lambda) \underset{\pi\text{-rotation}}{\overset{\bullet}{\longrightarrow}} LR(\lambda,\nu,\mu)$$

- $\rho^{AZ} = (\blacklozenge \bullet)\, e = \blacklozenge\, \rho$

2	3	3			
	1	2	2		
		1	1	1	1

1111221332

$\overset{e}{\longrightarrow}$ reversal

3	3	3			
	2	2	2		
		1	1	3	3

$\longrightarrow$ 3311222333

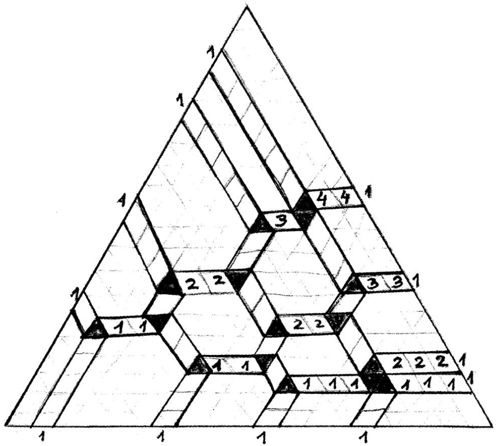
$\overset{\blacklozenge\bullet}{\longrightarrow}$

5		
4		
2	3	
1	2	3
	1	2
		1

$\longrightarrow$ 1231231245

$$11(1(12)2)(1332) \longrightarrow 22(1(12)2)(1332) \longrightarrow 2211(2(213)3)2 \longrightarrow 3311(2(213)3)3$$
$$\longrightarrow 33(1(12)2)1333 \longrightarrow 3311222333$$
$$\overset{*}{\longrightarrow} 1112223311 \overset{\diamondsuit}{\longrightarrow} 1231231245$$

A bijection between puzzles and LR tableaux: Tao's bijection



	1	1	2	2	3	4	4								
					1	1	2	2	3	3					
								1	1	1	2	2	2		
											1	1	1		

Purbhoo mosaics are in bijection with puzzles and LR tableaux

A mosaic is a tiling of an hexagon, which has angles and side lengths as below, with unitary triangles, unitary squares, and unitary rhombi with angles 30^o and 150^o all packed into the three 150^o nests.