Involutions for symmetries of Littlewood-Richardson (LR) coefficients

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based on a joint work with A. Conflitti and R. Mamede

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Plan

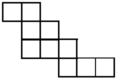
1. The algebra of symmetric functions and LR symmetries
2. LR symmetries and combinatorial models
3. The action of the group $\mathbb{Z}_2 \times \mathfrak{S}_3$ on LR coefficients

The algebra of symmetric functions

- Given $x = (x_1, x_2, \dots)$, let $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \dots$ be the $\mathbb{Q}$-algebra of symmetric functions on x. For all partitions λ with $|\lambda| = n$, the the Schur functions s_λ are homogeneous symmetric functions of degree $|\lambda|$ and form a linear basis for Λ^n,

$$s_\lambda(x) = \sum_T x^T,$$

summed over all semistandard tableaux T of shape λ with $|\lambda| = n$ on the alphabet $\{1, 2, \dots\}$. For all partitions λ, the Schur functions s_λ form a linear basis for Λ.

- Given the skew diagram λ/μ, $\mu \subseteq \lambda$,  the skew Schur function

$$s_{\lambda/\mu}(x) = \sum_T x^T,$$

summed over all tableaux T of shape λ/μ is also a symmetric function in Λ.

The structure coefficients: Littlewood-Richardson coefficients

- For $\mu \subseteq \lambda$,

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\,\nu} s_{\nu},$$

$$s_{\lambda/\nu} = \sum_{\mu} c^{\lambda}_{\mu\,\nu} s_{\mu},$$

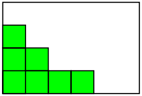
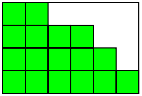
$$s_{\mu} s_{\nu} = \sum_{\lambda} c^{\lambda}_{\mu\,\nu} s_{\lambda},$$

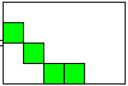
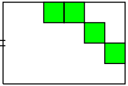
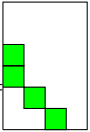
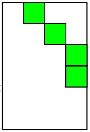
where $c^{\lambda}_{\mu\,\nu}$ are nonnegative integers, called Littlewood-Richardson coefficients.

$$c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$$

- The Schur basis is stable under the reduction of variables, that is, under the specialization $x_i = 0$ for $i > n$, $\{s_\lambda : \ell(\lambda) \leq n\}$ is a basis for $\Lambda_n := \mathbb{Q}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$.

The algebra Λ: transformations on shapes: rotation $\bullet$, transposition $\mathbf{t}$, orthogonal transposition $\blacklozenge$

$\lambda =$  $\lambda^{\vee} =$ 

$\lambda/\mu =$  $(\lambda/\mu)^{\bullet} =$  $(\lambda/\mu)^{t} =$  $(\lambda/\mu)^{\blacklozenge} =$ .

$$\lambda = 4210,\ \mu = 2100,\ \ (\lambda/\mu)^{\bullet} = \mu^{\vee}/\lambda^{\vee},\ \ (\lambda/\mu)^{\blacklozenge} = (\lambda/\mu)^{\bullet\, t} = (\lambda/\mu)^{t\, \bullet}$$

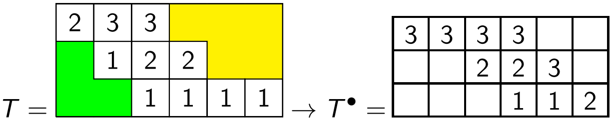
$$\lambda^{\bullet} = 0124,\ \ \lambda^{t} = 321100,\ \ \lambda^{\blacklozenge} = 001123$$

The algebra Λ: rotation • of skew shapes

- $s_{\lambda/\mu} = s_{(\lambda/\mu)^\bullet}$

$$s_{\lambda/\mu} = \sum_T x^T = \sum_{T^\bullet} x^{T^\bullet} = s_{(\lambda/\mu)^\bullet}$$

- $\bullet : T \to T^\bullet : i \mapsto N - i + a$



$$\alpha(T^\bullet) = 235 = rev\,\alpha(T) = 532$$

- linear algebra argument

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu = s_{(\lambda/\mu)^\bullet} = s_{\mu^\vee/\lambda^\vee} = \sum_\nu c^{\mu^\vee}_{\lambda^\vee\nu} s_\nu \Rightarrow c^\lambda_{\mu\nu} = c^{\mu^\vee}_{\lambda^\vee\nu}$$

- notation: $c^\lambda_{\mu\nu} := c_{\mu\nu\lambda^\vee}$

$$c_{\mu\nu\lambda^\vee} = c_{\lambda^\vee\nu\mu}.$$

The algebra Λ: transposition $\mathbf{t}$ and orthogonal transposition of skew shapes ♦

- $\mathbb{Q}$-algebra involution ω in Λ: $\omega(s_\lambda) = s_{\lambda^t}$.
- $\omega(s_{\lambda/\mu}) = s_{(\lambda/\mu)^t} = s_{\lambda^t/\mu^t}$

$$\omega(s_{\lambda/\mu}) = \sum_\nu c^\lambda_{\mu\nu}\omega(s_\nu) = \sum_\nu c^\lambda_{\mu\nu}s_{\nu^t} = s_{\lambda^t/\mu^t} = \sum_\lambda c^{\lambda^t}_{\mu^t\nu^t}s_{\nu^t}$$

- $c_{\mu\nu\lambda^\vee} = c_{\mu^t\nu^t\lambda^{\vee t}}$.

The algebra Λ: transposition **t** and orthogonal transposition of skew shapes ♦

- $\mathbb{Q}$-algebra involution ω in Λ: $\omega(s_\lambda) = s_{\lambda^t}$.
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- $c_{\mu\nu\lambda^\vee} = c_{\mu^t\nu^t\lambda^{\vee t}}$.
- $s_{(\lambda/\mu)^\blacklozenge} = s_{(\lambda/\mu)^{t\bullet}} = s_{(\lambda/\mu)^t} \Rightarrow c_{\mu\nu\lambda^\vee} = c_{\lambda^{\vee t}\nu^t\mu^t}$.

The Littlewood-Richardson (LR) rule

- $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\,\nu} s_\nu$,
- *D.E. Littlewood and A. Richardson 34; M.-P. Schützenberger 77; G.P. Thomas 74,*

$$LR(\mu,\nu,\lambda^\vee) := \{\text{ballot SSYT of shape } \lambda/\mu \text{ and content } \nu\}.$$

$$c^\lambda_{\mu\nu} = c_{\mu\nu\lambda^\vee} = \#L(\mu,\nu,\lambda^\vee).$$

The ballot SSYT's are also known as Littlewood-Richardson tableaux.

- LR tableau, ballot tableau The content of each initial segment of the reading word, right to left across rows and bottom to top, is a partition.

$$T = \begin{array}{|c|c|c|c|c|c|}\hline 2 & 3 & 3 & & & \\ \hline & 1 & 2 & 2 & & \\ \hline & & 1 & 1 & 1 & 1 \\ \hline \end{array},\ 1111221332 \qquad U = \begin{array}{|c|c|c|c|c|c|}\hline 2 & 3 & 3 & & & \\ \hline & 1 & 1 & 2 & & \\ \hline & & 1 & 1 & 1 & 2 \\ \hline \end{array},\ 2111211332$$

one has 3 candidates 1,2,3 each receiving 5,3,2 votes respectively. A particular ordering of the votes is then a *ballot sequence* of length 10 where at any stage candidate 1 has at least many votes as candidate 2, and candidate 2 has at least many votes as candidate 3.

Rotation • on LR tableaux

- $$\bullet : LR(\mu, \nu, \lambda^{\vee}) \longrightarrow LR(\lambda^{\vee}, \nu^{\bullet}, \mu)$$

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 2 & 3 & 3 & & & \\ \hline & 1 & 2 & 2 & & \\ \hline & & 1 & 1 & 1 & 1 \\ \hline \end{array} \mapsto T^{\bullet} = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 3 & 3 & 3 & & \\ \hline & & 2 & 2 & 3 & \\ \hline & & & 1 & 1 & 2 \\ \hline \end{array}$$

$$w = 1111221332 \mapsto w^{*} = 2113223333$$

$$\nu = 532 \qquad \nu^{\bullet} = 235$$

- The map • involution on the union of LR tableaux with dual LR tableaux.

$$c_{\mu\nu\lambda^{\vee}} = c_{\lambda^{\vee}\nu^{\bullet}\mu}$$

♦ orthogonal transposition on LR tableaux

- $$\blacklozenge : \mathrm{LR}(\mu,\nu,\lambda^{\vee})(\mathrm{LR}(\mu,\nu^{\bullet},\lambda)) \longrightarrow \mathrm{LR}(\lambda^{\vee t},\nu^{t},\mu^{t})(\mathrm{LR}(\lambda^{\vee t},\nu^{\bullet\, t},\mu^{t})),$$

$$T = \begin{array}{|c|c|c|c|c|c|}\hline 2 & 3 & 3 & & & \\ \hline & 1 & 2 & 2 & & \\ \hline & & 1 & 1 & 1 & 1 \\ \hline \end{array} \quad \mapsto \quad \begin{array}{|c|c|c|}\hline 3 & & \\ \hline 2 & 5 & \\ \hline 1 & 2 & \\ \hline & 1 & 4 \\ \hline & & 3 \\ \hline & & 2 \\ \hline & & 1 \\ \hline \end{array} = T^{\blacklozenge}$$

$$w = 1111221332 \quad \mapsto \quad w_{col} = 1234125123$$

- $c_{\mu\nu\lambda^{\vee}} = c_{\lambda^{\vee t}\nu^{t}\mu^{t}}$.

Knutson-Tao puzzles

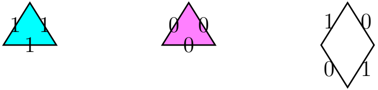
- A puzzle of size n is a tiling of an equilateral triangle of side length n with puzzle pieces each of unit side length:

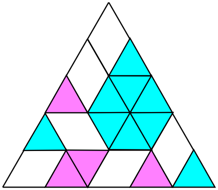
  (a) unit equilateral triangles with all edges labeled 1 (in blue colour);

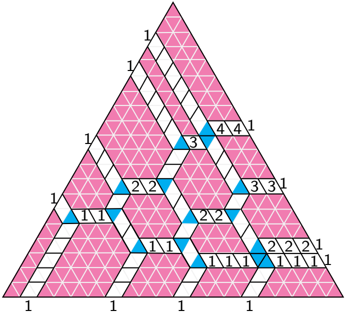
  (b) unit equilateral triangles with all edges labeled 0 (in pink colour); and

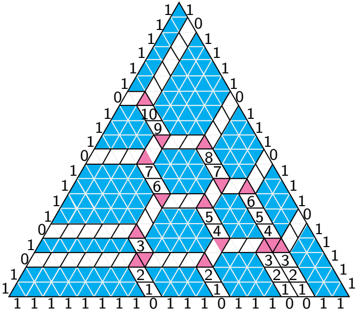
  (c) unit rhombi (two equilateral triangles joined together) with the two edges clockwise of acute vertices labeled 0, and the other two labeled 1.
- Puzzle pieces may be rotated in any orientation *but rhombi can not be reflected*, and wherever two pieces share an edge, the numbers on the edge must agree.
- (Knutson-Tao 04) $c_{\mu\nu\lambda}$ is the number of puzzles with μ, ν and λ appearing clockwise as 01-strings along the boundary.

  $\mu = 01011 = (100)$, $\nu = 01101 = (110)$ and $\lambda = 10101 = (210)$









10			
9			
7			
6			
3	8		
2	7		
1	5		
	4	6	
	2	5	
	1	4	
		3	3
		2	2
		1	1

	1	1	2	2	3	4	4								
					1	1	2	2	3	3					
								1	1	1	2	2	2		
											1	1	1		

The action of the group $\mathbb{Z}_2 \times \mathfrak{S}_3$ on $c_{\mu\nu\lambda}$

- $\mathbb{Z}_2 \times \mathfrak{S}_3 = \langle \tau, \tau\varsigma_1, \tau\varsigma_2 : \tau^2 = \varsigma_1^2 = \varsigma_2^2 = 1, \ (\tau\varsigma_1)^2 = (\tau\varsigma_2)^2 = 1 \rangle$.
- The LR coefficients $c_{\mu\nu\lambda}$ are invariant under the action of the dihedral group $\mathbb{Z}_2 \times \mathfrak{S}_3$ on the triple (μ, ν, λ):

$$c_{\varsigma(\mu,\nu,\lambda)} = c_{\mu\nu\lambda}, \ \varsigma \in \mathbb{Z}_2 \times \mathfrak{S}_3,$$

-the non–identity element τ of $\mathbb{Z}_2$,

$$\tau(\mu, \nu, \lambda) = (\mu^t, \nu^t, \lambda^t)$$

-$\mathfrak{S}_3 = \langle \varsigma_1, \varsigma_2 \rangle$ shuffles freely the three partitions μ, ν and λ,

$$\varsigma_1(\mu, \nu, \lambda) = (\nu, \mu, \lambda), \ \varsigma_2(\mu, \nu, \lambda) = (\mu, \lambda, \nu), \ \varsigma_1\varsigma_2\varsigma_1(\mu, \nu, \lambda) = (\lambda, \nu, \mu).$$

- The action of the index two subgroup $H = \langle \tau\varsigma_1, \tau\varsigma_2 \rangle = \langle \tau\varsigma_1, \tau\varsigma_1\varsigma_2\varsigma_1 = \blacklozenge \rangle$ on puzzles and LR tableaux consists of symmetries exhibited by simple bijections,

$$(\mathbb{Z}_2 \times \mathfrak{S}_3) / H = \{H, \varsigma_1 H = \varsigma_2 H = \varsigma_1\varsigma_2\varsigma_1 H = \tau H\}.$$

$c_{\mu\nu\lambda^\vee} = c_{\varsigma_1\varsigma_2\varsigma_1(\mu\nu\lambda^\vee)} = c_{\lambda^\vee\nu\mu}$

- The reversal map $e : LR(\mu,\nu,\lambda^\vee) \longrightarrow LR(\mu,\nu^\bullet,\lambda^\vee),\ T \mapsto T^e$

$$\mathrm{T} = \begin{array}{|c|c|c|c|c|c|}\hline 2 & 3 & 3 & & & \\ \hline & 1 & 2 & 2 & & \\ \hline & & 1 & 1 & 1 & 1 \\ \hline \end{array} \quad \overset{e}{\rightarrow} \quad \mathrm{T}^e = \begin{array}{|c|c|c|c|c|c|}\hline 3 & 3 & 3 & & & \\ \hline & 2 & 2 & 2 & & \\ \hline & & 1 & 1 & 3 & 3 \\ \hline \end{array}$$

$$w = 1111221332 \quad \rightarrow \quad \sigma_0 w = 3311222333 \equiv w^*$$

$$w \rightarrow 11(1(12)2)(12)\overset{\sigma_1}{\rightarrow}22112212 \rightarrow 2211221332\overset{\sigma_2}{\rightarrow}22(2(23)3)2$$

$$3322333 \rightarrow 3311221333\overset{\sigma_1}{\rightarrow}(1(12)2)1 \rightarrow (1(12)2)2 \rightarrow 3311222333 = \sigma_0 w$$

$c_{\mu\nu\lambda^\vee} = c_{\varsigma_1\varsigma_2\varsigma_1(\mu\nu\lambda^\vee)} = c_{\lambda^\vee\nu\mu}$

- The reversal map $e : LR(\mu,\nu,\lambda^\vee) \longrightarrow LR(\mu,\nu^\bullet,\lambda^\vee)$, $T \mapsto T^e$

$\mathrm{T} =$

2	3	3			
	1	2	2		
		1	1	1	1

$\overset{e}{\to}$ $\mathrm{T}^e =$

3	3	3			
	2	2	2		
		1	1	3	3

$$w = 1111221332 \quad \to \quad \sigma_0 w = 3311222333 \equiv w^*$$

$$w \to 11(1(12)2)(12)\overset{\sigma_1}{\to}22112212 \to 2211221332\overset{\sigma_2}{\to}22(2(23)3)2$$

$$3322333 \to 3311221333\overset{\sigma_1}{\to}(1(12)2)1 \to (1(12)2)2 \to 3311222333 = \sigma_0 w$$

- $\rho : LR(\mu,\nu,\lambda^\vee) \longrightarrow LR(\lambda^\vee,\nu,\mu)$, $T \mapsto \rho(T) = T^{e\bullet} = T^{\bullet e}$

$\mathrm{T} \overset{e}{\to} \mathrm{T}^e =$

3	3	3			
	2	2	2		
		1	1	3	3

$\overset{\bullet}{\to}$ $\mathrm{T}^{e\bullet} =$

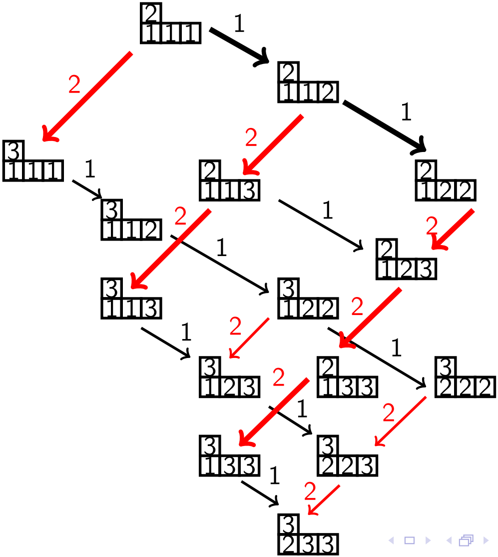
1	1	3	3		
		2	2	2	
			1	1	1

$$w = 1111221332 \quad \to \quad \sigma_0 w = 3311222333 \quad \overset{\bullet}{\to} (\sigma_0 w)^\bullet = 1112223311$$

- a bijective proof $c_{\lambda^\vee,\nu,\mu} = c_{\mu\nu\lambda^\vee} \Rightarrow$ a bijective proof $s_{\lambda/\mu} = s_{(\lambda/\mu)^\bullet}$.

T^e as the lowest weight element of the crystal $B_{\lambda/\mu}$

The crystal basis B_λ of the irreducible representation V_λ of $U_q(\mathfrak{gl}_n)$ can be taken to be the set of all SSYTs of shape λ, in the alphabet $[n]$, equipped with crystal operators.



Knuth and Haiman dual Knuth equivalence

- **Theorem (Haiman 92)**

  Let $\mathcal{D}$ be a dual Knuth equivalence class and $\mathcal{K}$ be a Knuth equivalence class, both corresponding to the same normal shape. Then, there is a unique tableau in $\mathcal{D} \cap \mathcal{K}$.

  Tableau switching **s** allows to compute $\mathcal{D} \cap \mathcal{K}$.

- **Algorithm (Benkart-Sotille-Stroomer 96)**

  Computation of $\mathcal{D} \cap \mathcal{K}$: Let $Q \in \mathcal{D}$ and let $V \in \mathcal{K}$ be the only tableau with normal shape in this class, and W any tableau that Q extends:

  Step 1. Compute

$$\begin{array}{ccc} W \cup Q & & W \cup X \\ \mathbf{s}\downarrow & & \uparrow\mathbf{s} \\ Q^{\mathrm{n}} \cup Z & \rightarrow & V \cup Z. \end{array}$$

  Step 2. $X \overset{d}{\equiv} Q$, $X \equiv V$, and $\mathcal{D} \cap \mathcal{K} = [V]_K \cap [Q]_{dK} = X$.

- Let $T \in \mathrm{LR}(\mu, \nu, \lambda^\vee)$. Then $T^{\mathrm{n}} = Y(\nu)$ and $T^e = [Y(\nu)^E]_K \cap [T]_{dK}$ is the only dual LR tableau in $LR(\mu, \nu^\bullet, \lambda^\vee)$ dual Knuth equivalent to T.

The crystals $B(\lambda/\mu)$ and $B((\lambda/\mu)^\bullet)$

- The crystal skew Littlewood-Richardson rule gives
$$B(\lambda/\mu) \simeq \bigoplus_{\substack{\nu \\ T\in LR(\mu,\nu,\lambda^\vee)}} B(T),$$
$B(T) \simeq B(\nu)$ has highest weight element $T \in LR(\mu,\nu,\lambda^\vee)$ and lowest weight element $T^e \in LR(\mu,\nu^\bullet,\lambda^\vee)$.

- $$B((\lambda/\mu)^\bullet) \simeq \bigoplus_{\substack{\nu \\ T\in LR(\lambda^\vee,\nu,\mu)}} B(T^{e\bullet}).$$
$B(T^{e\bullet}) \simeq B(\nu^\bullet)$ has highest weight $T^{e\bullet} \in LR(\lambda^\vee,\nu,\mu)$ and lowest weight element $T^\bullet \in LR(\lambda^\vee \nu^\bullet \mu)$. It is obtained by applying $\bullet$ to every vertex of $B(T)$ and then flipping the resulting graph (the color edge i is replaced with color i^*). Hence
$$T^{\bullet e} = T^{e\bullet}.$$

- Then $T^{\mathrm{n}} = Y(\nu)$ and $T^e = [Y(\nu)^E]_K \cap [T]_{dK}$
- $T^{\bullet\mathrm{n}} = Y(\nu)^E$, $\rho(T) = T^{e\bullet} = T^{\bullet\, e} = [Y(\nu)]_K \cap [T^\bullet]_{dK}$

$\tau H = \varsigma_1\varsigma_2\varsigma_1 H$

- $LR(\mu,\nu,\lambda^\vee) \to LR(\mu^t,\nu^t,\lambda^{\vee t}),\ T \mapsto \varrho(T) = \blacklozenge\rho(T) = T^{e\bullet\blacklozenge}$

$$\mathrm{T} = \begin{array}{|c|c|c|c|c|c|}\hline 2&3&3&&&\\\hline &1&2&2&&\\\hline &&1&1&1&1\\\hline\end{array} \xrightarrow{\rho} \rho(T) = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&3&3&&\\\hline &&2&2&2&\\\hline &&&1&1&1\\\hline\end{array} \xrightarrow{\blacklozenge} \mathrm{T}^{e\bullet\blacklozenge} = \begin{array}{|c|c|c|}\hline 5&&\\\hline 4&&\\\hline 2&3&\\\hline 1&2&3\\\hline &1&2\\\hline &&1\\\hline\end{array}$$

- $c_{\mu\nu\lambda^\vee} = c_{\mu^t\nu^t\lambda^{\vee t}}$
- Benkart-Sottille-Stroomer LR transposer

$$\varrho^{BSS}(T) = [Y(\nu^t)]_K \cap [\widehat{T}^t]_{dK} = \varrho(T)$$

- $\varrho = \blacklozenge\rho \Rightarrow \varrho H = \rho H.$

LR tableaux split into Gelfand-Tsetlin patterns led to hives

- I.M. Gelfand, A.V. Zelevinsky (1986), A.D. Berenstein, A.V.Zelevinsky (1989)

$T =$

							1	1
					1	1	2	2
			1	1	2			
1	2	2	3					
3								

$\mu = 75300 \quad \nu = 75200 \quad \lambda = 99641$

$$G_\mu = \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 3 & & 1 & & 0 & & \\ & 5 & & 5 & & 3 & & 1 & \\ 7 & & 7 & & 6 & & 4 & & 1 \end{array} \qquad G_\nu = \begin{array}{ccccccccc} & & & & 7 & & & & \\ & & & 7 & & 5 & & & \\ & & 6 & & 5 & & 2 & & \\ & 4 & & 3 & & 1 & & 0 & \\ 2 & & 2 & & 0 & & 0 & & 0 \end{array} \qquad G_\lambda = \begin{array}{ccccccccc} & & & & 7 & & & & \\ & & & 7 & & 5 & & & \\ & & 9 & & 6 & & 2 & & \\ & 9 & & 8 & & 4 & & 1 & \\ 9 & & 9 & & 6 & & 4 & & 1 \end{array}$$

$T_\mu =$

1	2	2	2	3	3	4
2	3	3	4	4		
4	5	5				

$T_\nu =$

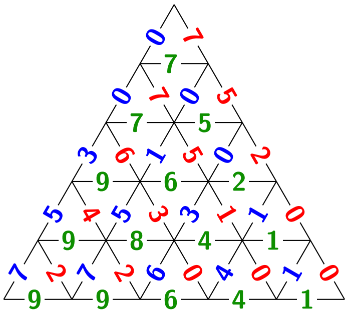
1	1	2	2	3	3	4
2	2	3	4	4		
4	5					

$T_\lambda =$

1	1	1	1	1	1	1	3	3
2	2	2	2	2	3	4	4	5
3	3	4	4	5	5			
4	5	5	5					
5								

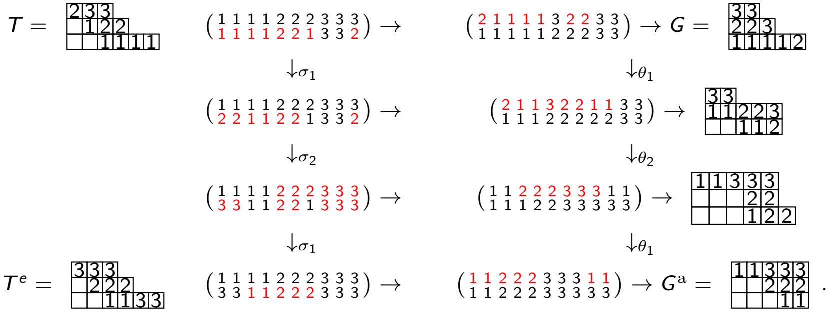
$2 \subseteq 42 \subseteq 630 \subseteq$

Interlock the three GT patterns



GT patterns companions or LR companions

- Lascoux's bicrystal graph on biwords [02]



- **Theorem**
  Let T be a LR tableau with shape λ/μ and companion tableau G. Then

$$\begin{array}{ccccccc}
T & \overset{e}{\longleftrightarrow} & T^e & \overset{\bullet}{\longleftrightarrow} & \rho(T) = T^{e\bullet} & \overset{\blacklozenge}{\longleftrightarrow} & \varrho(T) = T^{e\bullet\blacklozenge} \\
\iota\updownarrow & & \iota\updownarrow & & \iota\updownarrow & & \iota\updownarrow \\
G & \underset{\mathrm{a}}{\longleftrightarrow} & G^{\mathrm{a}} & \underset{\bullet}{\longleftrightarrow} & G^{\mathrm{a}\bullet} = G^E & \underset{\blacklozenge}{\longleftrightarrow} & G^{E\blacklozenge}.
\end{array}$$

$\rho_1 H = \rho_2 H = \rho H = \varrho H$

- $H = \langle \spadesuit, \blacklozenge \rangle = \langle \clubsuit, \blacklozenge \rangle = \langle \spadesuit, \clubsuit \rangle = \{\mathbf{1}, \spadesuit, \clubsuit, \spadesuit\clubsuit, \clubsuit\spadesuit, \blacklozenge = \clubsuit\spadesuit\clubsuit = \spadesuit\clubsuit\spadesuit\}$
- $\rho_1 : LR(\mu, \nu, \lambda) \to LR(\nu, \mu, \lambda)$, $\rho_2 : LR(\mu, \nu, \lambda) \to LR(\mu, \lambda, \nu)$

$$\varrho = \spadesuit\, \rho_1 = \blacklozenge\, \rho = \clubsuit \rho_2.$$