The involutive nature of the Littlewood-Richardson commutativity bijection

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Plan

1. Motivation: LR coefficients as structure constants *versus* combinatorial numbers
2. LR tableaux, Gelfand-Tsetlin patterns and LR hives
3. Involution commutators of LR tableaux and LR hives
   - based on the Schützenberger involution
   - our involution commutators

Littlewood-Richardson coefficients as structure constants

- *The basis of Schur polynomials for the ring Λ_n.* Let $x = (x_1, \ldots, x_n)$. Schur polynomials $s_\lambda(x)$ for all partitions with $\ell(\lambda) \leq n$, form a $\mathbb{Z}$-linear basis for the ring $\Lambda_n := \mathbb{Z}[x]^{\mathfrak{S}_n}$ of symmetric polynomials in x,

$$s_\mu s_\nu = \sum_{\substack{\lambda \\ \ell(\lambda) \leq n}} c_{\mu\,\nu}^{\lambda} s_\lambda.$$

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- These numbers $c^\lambda_{\mu,\nu}$ also arise as
  - *tensor product multiplicities*. Schur polynomials $s_\lambda(x)$ may be interpreted as irreducible characters of the general linear group $GL_n(\mathbb{C})$. The decomposition of the tensor product of two irreducible polynomial representations V^μ and V^ν of the general linear group $GL_n(\mathbb{C})$ into irreducible representations of $GL_n(\mathbb{C})$, is given by

$$V^\mu \otimes V^\nu = \bigoplus_{\ell(\lambda) \leq n} V^{\lambda^{\oplus c^\lambda_{\mu\,\nu}}}.$$

Littlewood-Richardson coefficients as structure constants

- *intersection numbers*. Schur polynomials $s_\lambda(x)$ may be interpreted as representatives of Schubert classes σ_λ, with λ inside the rectangle $d \times (d-n)$. Schubert classes σ_λ with λ inside the rectangle $d \times (d-n)$, form a $\mathbb{Z}$-linear basis for the cohomology ring $H^*(G(n,d))$ of the Grassmannian $G(n,d)$ of complex n-dimensional linear subspaces of $\mathbb{C}^d$, and

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq n \times (d-n)} c_{\mu\,\nu}^{\lambda} \sigma_\lambda.$$

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- *Positivity of Littlewood-Richardson coefficients in existence problems*. There exist $n \times n$ non singular matrices A, B and C, over a *local principal ideal domain*, with Smith invariants μ, ν and λ respectively, such that $AB = C$ iff $c_{\mu\,\nu}^{\lambda} > 0$.

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- *The commutativity of Littlewood-Richardson coefficients*

$$c^\lambda_{\mu\nu} = c^\lambda_{\nu\mu}$$

$$c^\lambda_{\mu\nu} > 0 \text{ iff } c^\lambda_{\nu\mu} > 0.$$

Littlewood-Richardson coefficients as numbers which count

- The Littlewood-Richardson (LR) rule (*D.E. Littlewood and A. Richardson 34; M.-P. Schützenberger 77; G.P. Thomas 74*) states that the coefficients appearing in the expansion of a product of Schur polynomials s_μ and s_ν

$$s_\mu(x)\ s_\nu(x) = \sum_\lambda\ c^\lambda_{\mu\nu}\ s_\lambda(x)$$

are given by

$$\begin{aligned} c^\lambda_{\mu\nu} &= \#\{T \in \mathcal{LR}(\lambda/\mu,\nu)\} = \#\mathcal{LR}(\lambda/\mu,\nu) \\ c^\lambda_{\nu,\mu} &= \#\{T \in \mathcal{LR}(\lambda/\nu,\mu)\} = \#\mathcal{LR}(\lambda/\nu,\mu). \end{aligned}$$

$\mathcal{LR}(\lambda/\mu,\nu)$ the set of Littlewood-Richardson tableaux of shape λ/μ and weight ν.

Littlewood-Richardson tableaux (*D.E. Littlewood and A. Richardson, 1934)*

- A Young tableau T of shape λ/μ is said to be a Littlewood-Richardson tableau if
  - it is *semistandard* (SSYT)
    1. the entries in each row of λ/μ are weakly increasing from left to right, and
    2. the entries in each column of λ/μ are strictly increasing from top to bottom,
  - and satisfy the *lattice permutation property*
    1. the number of entries s occurring in the first r rows of T does not exceed the number of entries $s-1$ occurring in the first $r-1$ rows of T, for all $r \geq 1$ and $s \geq 2$.

					1	1	2
				1	2		
1	2	2	2	3			

					1	1	1
				2	3		
1	2	2	3	4			

					1	1	1
				1	2		
1	2	2	2	3			

lattice permutation property $\Rightarrow$ ν is a partition, $\ell(\nu) \leq \ell(\lambda) \underset{SST}{\Rightarrow} \nu \subseteq \lambda$

LR tableaux split into Gelfand-Tsetlin patterns led to hives

- I.M. Gelfand, A.V. Zelevinsky (1986), A.D. Berenstein, A.V.Zelevinsky (1989)

$T =$

							1	1
					1	1	2	2
			1	1	2			
1	2	2	3					
3								

$\mu = 75300 \quad \nu = 75200 \quad \lambda = 99641$

$$G_\mu = \begin{array}{ccccccccc} &&&&0&&&& \\ &&&0&&0&&& \\ &&3&&1&&0&& \\ &5&&5&&3&&1& \\ 7&&7&&6&&4&&1 \end{array}$$

$$G_\nu = \begin{array}{ccccccccc} &&&&7&&&& \\ &&&7&&5&&& \\ &&6&&5&&2&& \\ &4&&3&&1&&0& \\ 2&&2&&0&&0&&0 \end{array}$$

$$G_\lambda = \begin{array}{ccccccccc} &&&&7&&&& \\ &&&7&&5&&& \\ &&9&&6&&2&& \\ &9&&8&&4&&1& \\ 9&&9&&6&&4&&1 \end{array}$$

$T_\mu =$

1	2	2	2	3	3	4
2	3	3	4	4		
4	5	5				

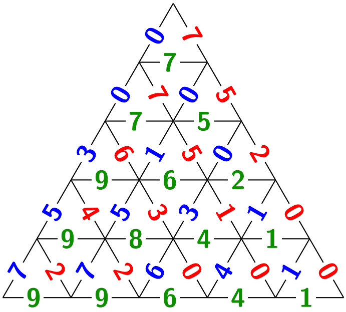
$T_\nu =$

1	1	2	2	3	3	4
2	2	3	4	4		
4	5					

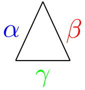
$T_\lambda =$

1	1	1	1	1	1	1	3	3
2	2	2	2	2	3	4	4	5
3	3	4	4	5	5			
4	5	5	5					
5								

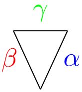
Interlock the three GT patterns



- A *hive in the edge representation form, R.C. King, C. Tollu, F. Toumazet (2006)*, is a labelling of all edges of a planar, equilateral triangular graph satisfying the triangle and the betweeness conditions



$\alpha + \beta = \gamma$



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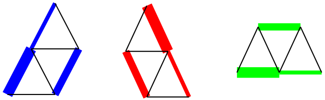
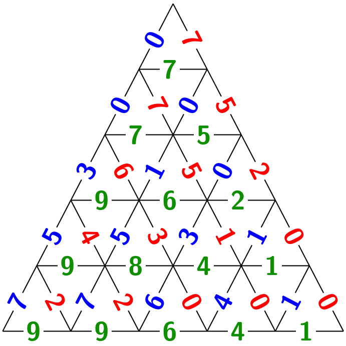
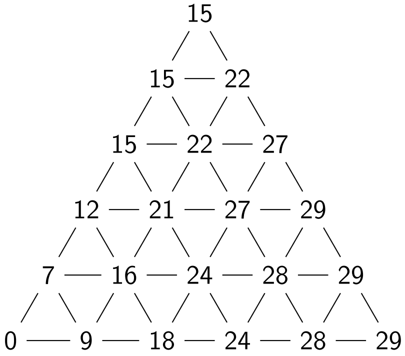


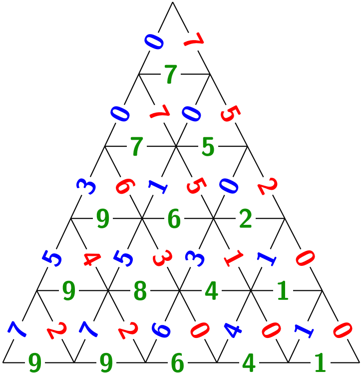
From edge KTT (2006), to vertex representation of a hive A. Knutson, T. Tao (1999), and A.S. Buch (2000)

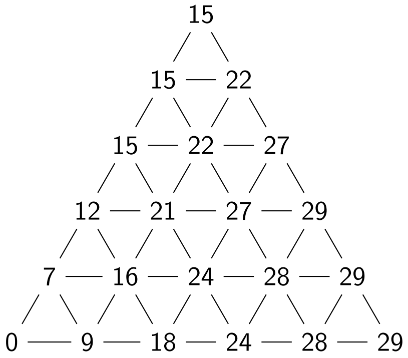
The transformation from edge to vertex labelling starts with the bottom left vertex with label 0, and, inductively, find the label of a vertex as the the sum of a vertex label to its left with the label of the edge connecting these two vertices. The triangle condition ensures a consistent result.

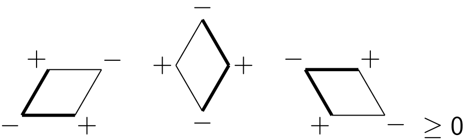




Edge and vertex representation of a hive A. Knutson, T. Tao (1999), and A.S. Buch (2000)

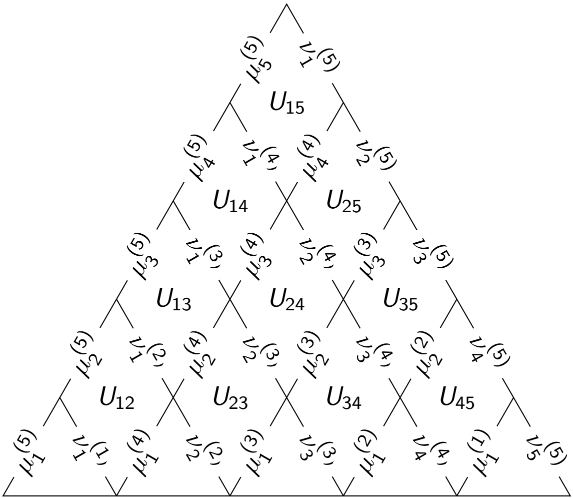


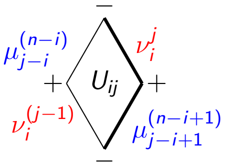




Gradient representation of a hive

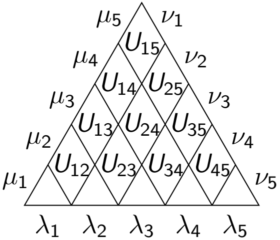
The hive is expressed only in terms of the μ, ν and λ edge labels, and upright rhombus gradients.





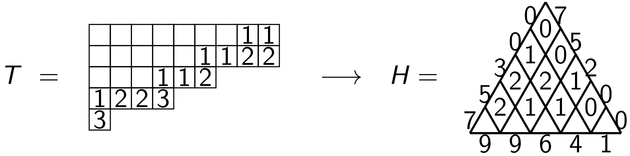
$$U_{ij} = \mu_{j-i}^{(n-i)} - \mu_{j-i+1}^{(n-i+1)} = \nu_i^j - \nu_i^{(j-1)} \geq 0$$

Gradient representation of a hive

- 

$$\mu_{i+1} + \sum_{k=1}^{j} U_{k,i+1} \leq \mu_i + \sum_{k=1}^{j-1} U_{ki},$$
$$\nu_{i+1} - \sum_{k=j+1}^{n} U_{i+1,k} \leq \nu_i - \sum_{k=j}^{n} U_{ik},$$
$$\lambda_k = \left(\mu_k + \sum_{i=1}^{k-1} U_{ik}\right) + \left(\nu_k - \sum_{j=k+1}^{n} U_{kj}\right)$$

- n-hives $\mathcal{H}^{(n)}$ in the gradient representation and n-LR tableaux $\mathcal{LR}^{(n)}$ are in bijection.



$U_{ij} = \#$ of i's in row j of T, $1 \leq i < j \leq n$, add the boundary edge labels λ, μ and ν.

$$c^{\lambda}_{\mu,\nu} = \#\mathcal{H}^{(n)}(\lambda, \mu, \nu) = \#\mathcal{LR}(\lambda/\mu, \nu)$$

The interlocking GT pattern pair (T_μ, T_ν) and the tensor product of $\mathfrak{gl}_n$-crystal bases

- Let B_λ denote the crystal basis of the irreducible representation V_λ of $U_q(\mathfrak{gl}_n)$.
- B_λ can be taken to be the set of all SSYTs of shape λ, in the alphabet $\{1, \dots, n\}$, equipped with crystal operators. The highest weight element is the Yamanouchi tableau Y_λ, and the lowest weight element $\xi(Y_\lambda) =: Y_{rev\lambda}$, with ξ *the Schützenberger involution.*

$$Y_{753} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1\\ \hline 2&2&2&2&2 \\ \cline{1-5} 3&3&3 \\ \cline{1-3}\end{array} \qquad Y_{357} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3\\ \hline 2&2&2&3&3 \\ \cline{1-5} 3&3&3 \\ \cline{1-3}\end{array}$$

- *$\mathfrak{gl}_n$-Littlewood-Richardson rule*. The map $U \otimes V \to (P(U \otimes V), Q(U \otimes V))$ gives the $\mathfrak{gl}_n$-isomorphism

$$B_\mu \otimes B_\nu \cong \bigoplus_{\substack{\lambda \\ T \in \mathcal{LR}(\lambda/\mu,\nu)}} B_\lambda(T),$$

with $B_\lambda(T) = B_\lambda \times \{T\} \cong B_\lambda$. The multiplicity of B_λ in $B_\mu \otimes B_\nu$ is

$$\#\text{highest (lowest) weight elements of weight } \lambda \ (rev\lambda) \text{ in } B_\mu \otimes B_\nu$$
$$= |\mathcal{LR}(\lambda/\mu,\nu)| = c^\lambda_{\mu,\nu}$$

- *The lowest and the highest weight elements of $B_\lambda(T)$ in $B_\mu \otimes B_\nu$*. Each crystal connected component in $B_\mu \otimes B_\nu$, isomorphic to $B_\lambda(T)$, has highest weight element $Y_\mu \otimes T_\nu$, and lowest weight element $T_\mu \otimes Y_{rev\nu}$, where (T_μ, T_ν) is the GT-pattern pair obtained from $T \in \mathcal{LR}(\lambda/\mu,\nu)$.

Involution commutators and the Schützenberger involution

- Henriques-Kamnitzer crystal commutator (arxiv 2004). For each $T \in \mathcal{LR}(\lambda/\mu, \nu)$ there exists $T^* \in \mathcal{LR}(\lambda/\nu, \mu)$ such that the map

$$\begin{array}{ccc} B(\mu) \otimes B(\nu) & \rightarrow & B(\nu) \otimes B(\mu) \\ U \otimes V & \mapsto & \xi(V) \otimes \xi(U) \end{array}$$

  sends

$$\begin{array}{ccccc} Y_\mu \otimes T_\nu & \rightarrow & T^*_\nu \otimes \xi Y_\mu, & & \xi T_\nu = T^*_\nu \\ T_\mu \otimes \xi Y_\nu & \rightarrow & Y_\nu \otimes T^*_\mu, & & \xi T_\mu = T^*_\mu. \end{array}$$

- Henriques-Kamnitzer LR involution commutator

$$\begin{array}{cccl} Com_{HK} : \mathcal{LR}(\lambda/\mu, \nu) & \rightarrow & \mathcal{LR}(\lambda/\nu, \mu) & \\ T & \rightarrow & T^* & : \; T^*_\nu = \xi T_\nu, \, T^*_\mu = \xi T_\mu \end{array}$$

- Pak-Vallejo LR commutators (arxiv 2004): $\rho_2 = \gamma^{-1}\xi\tau$ and $\rho_2' = \tau^{-1}\xi\gamma = \rho_2^{-1}$.

$$\rho_2 : T \in \mathcal{LR}(\lambda/\mu, \nu) \underset{\tau}{\longrightarrow} T_\nu \underset{\xi}{\longrightarrow} \xi(T_\nu) \underset{\gamma^{-1}}{\longrightarrow} Q \in \mathcal{LR}(\lambda/\nu, \mu)$$

$$\rho_2' : T \in \mathcal{LR}(\lambda/\mu, \nu) \underset{\gamma}{\longrightarrow} T_\mu \underset{\xi}{\longrightarrow} \xi(T_\mu) \underset{\tau^{-1}}{\longrightarrow} Q' \in \mathcal{LR}(\lambda/\nu, \mu)$$

  Conjectured $\rho_2 = \rho_2' \Leftrightarrow Q = Q'$.

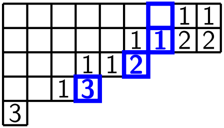
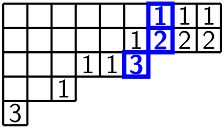
The nature of the LR tableau and LR hive commutators ρ_3 (A. 1999) and $\sigma^{(n)}$ (A., King, Terada 2016)

- The general philosophy of the maps ρ_3 and $\sigma^{(n)}$: an LR tableau $T \in \mathcal{LR}(\lambda/\mu, \nu)$ and an LR hive $H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$ can be completely specified by means of the GT pattern with type ν, and the shape μ.

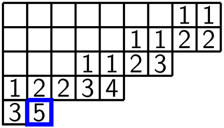
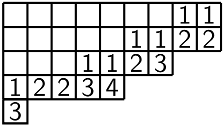
- The nature of the map ρ_3 on $T \in \mathcal{LR}(\lambda/\mu, \nu)$ and of the map $\sigma^{(n)}$ on $H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$ is that it proceeds by providing:
  - a sequence consisting of successively smaller LR tableaux in which the sequence of inner shapes determines the image $S \in \mathcal{LR}(\lambda/\nu, \mu)$; and
  - a sequence consisting of successively smaller LR hives in which the sequence of left-hand boundary edges determine the image $K \in \mathcal{H}^{(n)}(\lambda, \nu, \mu)$.
    They do so by specifying completely a GT pattern with type μ associated with S and K.
  - Both maps have inverses acting in a reverse manner. Their action on $T \in \mathcal{LR}(\lambda/\mu, \nu)$ and $H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$ culminate in the same images S and K. The maps ρ_3 and $\sigma^{(n)}$ are involutions.

ρ_3: The action of deletion operators on LR tableaux

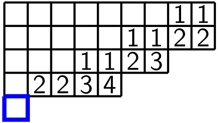
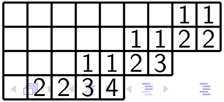
- For any given $T \in \mathcal{LR}(\lambda/\mu, \nu)$, and any corner cell (r, λ_r) the *deletion operator* δ_{r,λ_r} is defined to be such that:
  - $1 \leq T(r, \lambda_r) < r$: the terminating row number is in $\{1, \ldots, r-1\}$

$T =$  $\rightarrow \quad \delta_{4,4} T =$ 

  - $T(r, \lambda_r) = r$: terminating row number is always 0.

$T =$  $\rightarrow \quad \delta_{5,2} T =$ 

  - the cell (r, λ_r) is blank: terminating row number is always r.

$T =$  $\rightarrow \quad \delta_{5,1} T =$ 

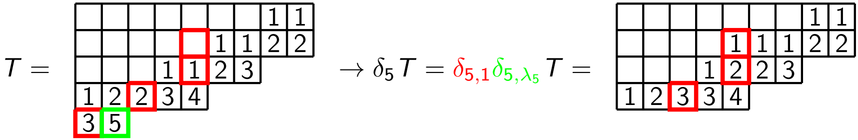
The terminating row numbers of full deletion operators

- For any given $T \in \mathcal{LR}(\lambda/\mu,\nu)$ with $\ell(\lambda) = r$ the *full r-deletion* operator δ_r is defined by

$$\delta_r := \delta_{r,1}\delta_{r,2}\cdots\delta_{r,\lambda_r}$$

  The operator δ_r produces λ_r terminating row numbers in $\{0,1,\dots,r\}$ where the multiplicity of 0 is ν_r and the multiplicity of r is μ_r.

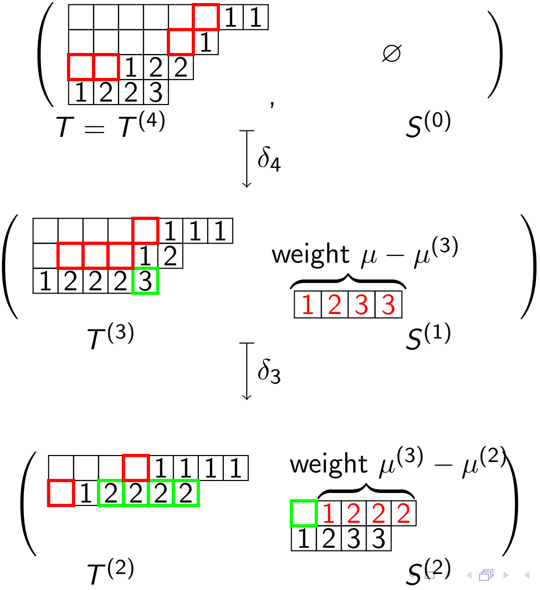
- **Example**

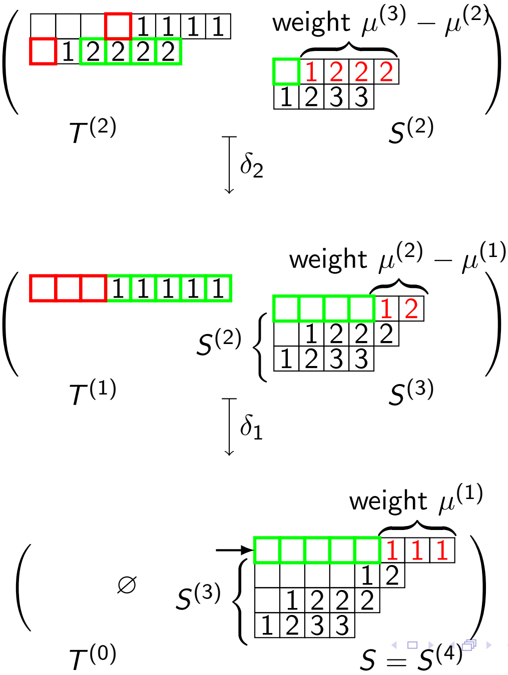


  The full 5-deletion operator δ_5 has terminating row numbers 0, 2, recorded as $\boxed{}\boxed{2}$.

- Goal: piling up the terminating row numbers of δ_r applied T, followed with the terminating row numbers of δ_{i-1} applied to $\delta_i \dots \delta_r T$, $i = r,\dots,2$.

The LR tableau commutator ρ_3




weight $\mu^{(3)} - \mu^{(2)}$
$T^{(2)}$
$S^{(2)}$
δ_2
weight $\mu^{(2)} - \mu^{(1)}$
$S^{(2)}$
$T^{(1)}$
$S^{(3)}$
δ_1
weight $\mu^{(1)}$
$\varnothing$
$S^{(3)}$
$T^{(0)}$
$S = S^{(4)}$

The partner tableau

- The partner tableau of T

$$\rho_3(T) = S = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & & & & & 1 & 1 & 1 \\
\hline
 & & & & 1 & 2 \\
\cline{1-6}
 & 1 & 2 & 2 & 2 \\
\cline{1-5}
1 & 2 & 3 & 3 \\
\cline{1-4}
\end{array} \in LR(\lambda/\nu, \mu)$$

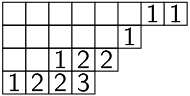
- *The GT pattern of type μ of the partner tableau*. The sequence of inner shapes produced in the deletion procedure gives the GT pattern of type μ of S, the partner tableau of T

$$S_\mu = \begin{array}{ccccccc}
 & & & 3 & & & \\
 & & 4 & & 1 & & \\
 & 5 & & 4 & & 0 & \\
6 & & 5 & & 2 & & 0
\end{array}$$

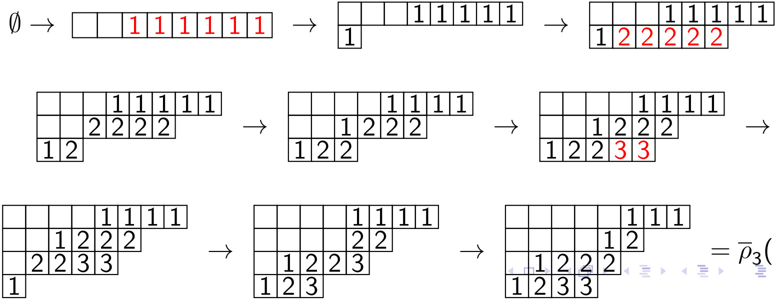
- *The multiplicity of the positive terminating row numbers of operator δ_i is given by $\mu^i - \mu^{(i-1)}$, $i = 1, 2, 3, 4$.*

The inverse $\overline{\rho}_3$

- To prove that ρ_3 is a bijection, we exhibit its inverse $\overline{\rho}_3$.
- Given $T \in \mathcal{LR}(\lambda/\mu, \nu)$ we use the GT pattern T_ν to construct by insertion of blank boxes the inner shape ν of $\overline{\rho}_3 T \in \mathcal{LR}(\lambda/\nu, \mu)$, and *adding, to each row i, μ_i boxes marked with i*.

$T =$ 

$$T_\nu = \begin{array}{ccccccc} & & & 2 & & & \\ & & 3 & & 0 & & \\ & 4 & & 2 & & 0 & \\ 5 & & 4 & & 1 & & 0 \end{array}$$

- 

$= \overline{\rho}_3(T)$

$$\rho_3(T) = \overline{\rho}_3(T)$$

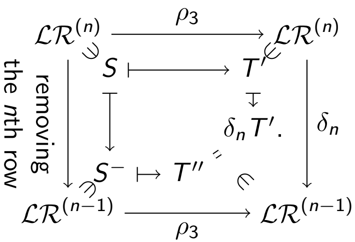
Properties of operators δ_n

Lemma

- $\rho_3(T) = S \Longrightarrow \rho_3(\delta_n T) = S^-$ obtained by removing the nth row of S.
- $T, T' \in \mathcal{LR}(\lambda/\mu, \nu)$, $\delta_n T = \delta_n T' \Longrightarrow T = T'$.

Proposition

(A. 2000); A., King, Terada (2016)



$$\rho_3(S^-) = \delta_n(\rho_3(S)).$$

Theorem

$\rho_3^2 = id$.

Proof. By induction on n.

$$n = 1, \quad T = \boxed{}\boxed{}\boxed{1}\boxed{1}\boxed{1}\boxed{1} \underset{\rho_3}{\rightarrow} S = \boxed{}\boxed{}\boxed{}\boxed{}\boxed{1}\boxed{1} \underset{\rho_3}{\rightarrow} T = \boxed{}\boxed{}\boxed{1}\boxed{1}\boxed{1}\boxed{1}$$

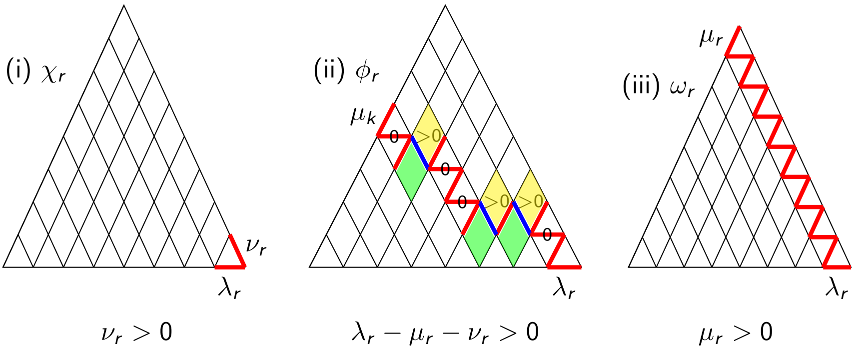
$$n > 1, \quad T \in \mathcal{LR}(\lambda/\mu, \nu), \quad \rho_3(T) = S$$

$$\rho_3(\delta_n T) = S^- \underset{induction}{\Rightarrow} \rho_3^2(\delta_n T) = \delta_n T = \rho_3 S^-$$

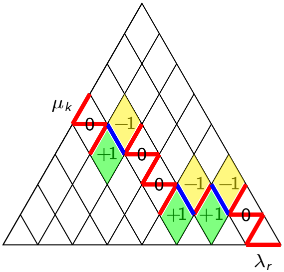
$$\Leftrightarrow \delta_n T = \rho_3 S^- \underset{Proposition}{=} \delta_n(\rho_3(S)) \Rightarrow \rho_3(S) = T.$$

An LR hive commutator (A.,King, Terada, 2016)

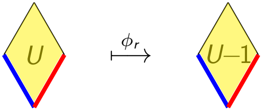
- Path removals in a hive with gradient representation. Three path removal operators χ_r, ϕ_r and ω_r on a hive H. In each case the action on the hive H is to decrease the label of each **red** edge by 1 and to increase that of each **blue** edge by 1, along a path starting from the edge labelled λ_r.



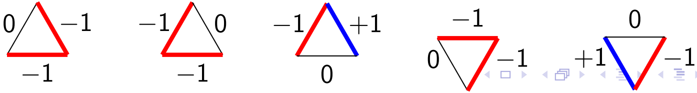
Reduction in gradients of upright rhombi



The only reductions in gradient are of the form:



The triangle conditions are preserved under any of the three path removal procedures



The path removal operator

- For any given hive $H \in \mathcal{H}^{(r)}(\lambda, \mu, \nu)$ with $r = \ell(\lambda)$, the path removal operator θ_{r,λ_r} is defined by

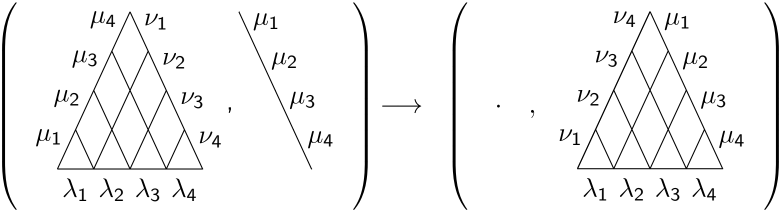
$$\theta_{r,\lambda_r} = \begin{cases} \chi_r & \text{if } \nu_r > 0; \\ \phi_r & \text{if } \nu_r = 0 \text{ and } U_{ir} > 0 \text{ for some } i < r; \\ \omega_r & \text{if } \nu_r = 0,\ U_{ir} = 0 \text{ for all } i < r, \text{ and } \mu_r > 0. \end{cases}$$

- For any given hive $H \in \mathcal{H}^{(r)}(\lambda, \mu, \nu)$ with $\ell(\lambda) \leq r$ the *full r-hive path removal operator* θ_r is defined by

$$\theta_r := \theta_{r,1}\theta_{r,2} \cdots \theta_{r,\lambda_r} ,$$

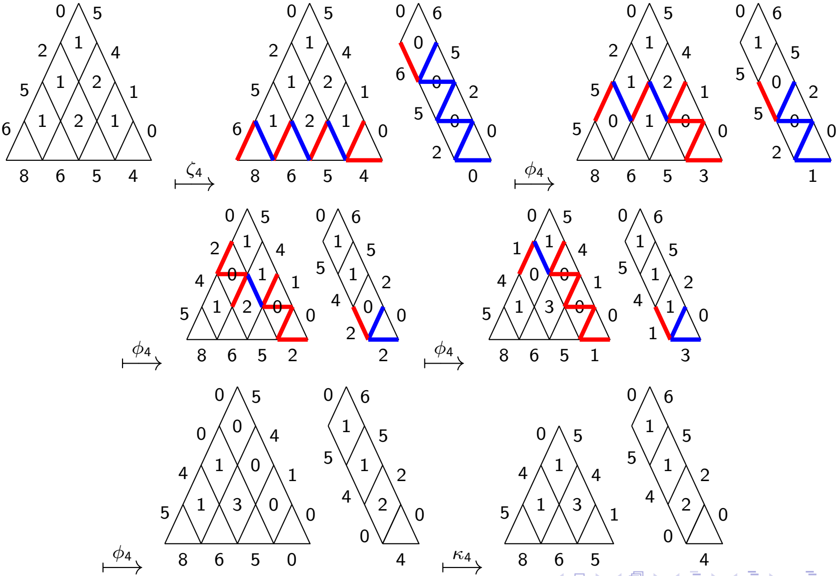
$$\theta_r = \omega_r^{\mu_r} \ \phi_r^{\lambda_r - \mu_r - \nu_r} \ \chi_r^{\nu_r}$$

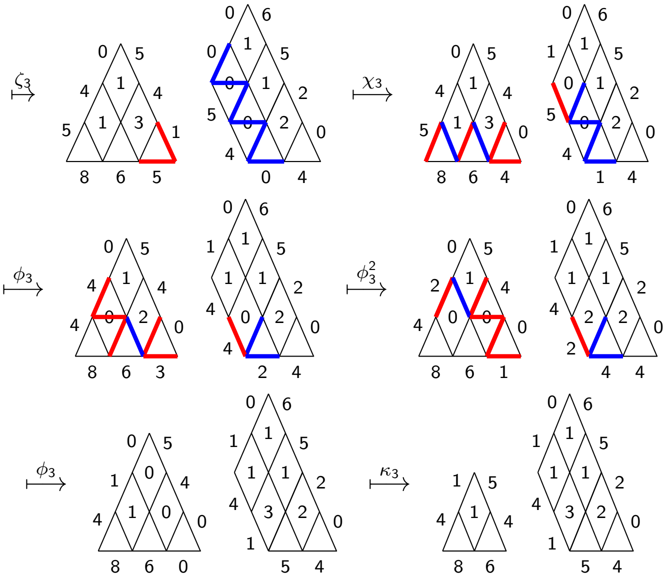
An LR hive commutator (A., King, Terada, 2016)



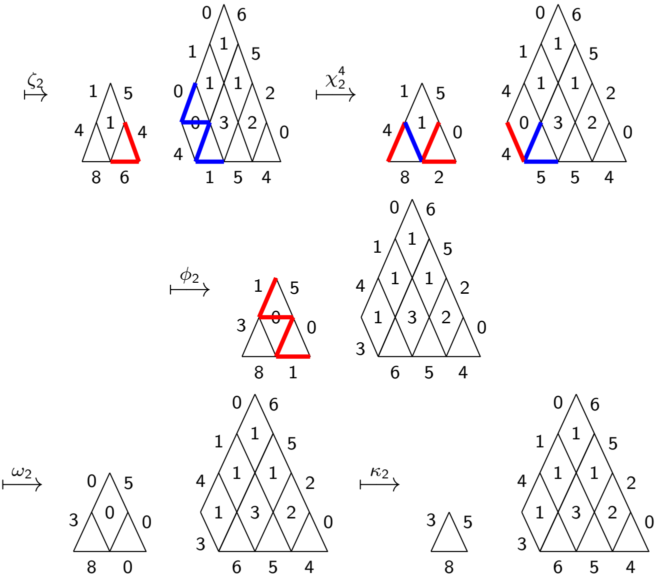
$$(H^{(4)}, K^{(0)}) \longrightarrow (H^{(0)}, K^{(4)})$$

The LR hive commutator $\sigma^{(n)}$ (A., King, Terada, 2016)

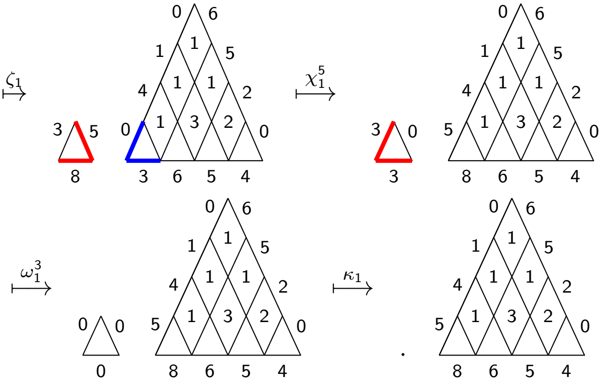




ζ_3
χ_3
ϕ_3
ϕ_3^2
ϕ_3
κ_3



ζ_2
χ_2^4
ϕ_2
ω_2
κ_2



$S =$

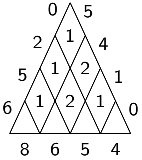
					1	1	1
				1	2		
	1	2	2	2			
1	2	3	3				

$$S_\mu = \begin{array}{ccccccc} & & & 3 & & & \\ & & 4 & & 1 & & \\ & 5 & & 4 & & 0 & \\ 6 & & 5 & & 2 & & 0 \end{array}$$

Example: $Com_{HK} = \gamma^{-1}\xi\tau$

- $\mathcal{LR}(\lambda/\mu,\nu) \to \mathcal{LR}(\lambda/\nu,\mu)$, $\quad \mathcal{H}^{(4)}(\lambda,\mu,\nu) \to \mathcal{H}^{(4)}(\lambda,\nu,\mu)$

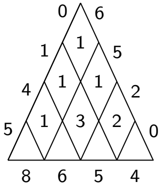
$H =$



$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & & & 1 & 1 \\ \hline & & & & & 1 \\ \cline{1-6} & & 1 & 2 & 2 \\ \cline{1-5} 1 & 2 & 2 & 3 \\ \cline{1-4} \end{array} \underset{\tau}{\to} T_\nu = \begin{array}{ccccccc} & & & 2 & & & \\ & & 3 & & 0 & & \\ & 4 & & 2 & & 0 & \\ 5 & & 4 & & 1 & & 0 \end{array}$$

$$\underset{\xi}{\to} \xi T_\nu = \begin{array}{ccccccc} & & & 4 & & & \\ & & 5 & & 2 & & \\ & 5 & & 2 & & 1 & \\ 5 & & 4 & & 1 & & 0 \end{array} \underset{\gamma^{-1}}{\to} T^* = \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & & 1 & 1 & 1 \\ \hline & & & & 1 & 2 \\ \cline{1-6} & 1 & 2 & 2 & 2 \\ \cline{1-5} 1 & 2 & 3 & 3 \\ \cline{1-4} \end{array}$$

$H^* =$



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Example: $\tau^{-1}\xi\gamma$

- $$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & 1 & 1 \\ \hline \ & \ & \ & \ & \ & 1 \\ \cline{1-6} \ & \ & 1 & 2 & 2 \\ \cline{1-5} 1 & 2 & 2 & 3 \\ \cline{1-4}\end{array} \xrightarrow[\gamma]{} T_\mu = \begin{array}{ccccccc} & & & 4 & & & \\ & & 5 & & 3 & & \\ & 6 & & 3 & & 1 & \\ 6 & & 5 & & 2 & & 0 \end{array} \underset{\xi}{\longmapsto}$$

  $$\xrightarrow[\xi]{} \xi T_\mu = \begin{array}{ccccccc} & & & 3 & & & \\ & & 4 & & 1 & & \\ & 5 & & 4 & & 0 & \\ 6 & & 5 & & 2 & & 0 \end{array} \xrightarrow[\tau^{-1}]{} Q = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & 1 & 2 \\ \cline{1-6} \ & 1 & 2 & 2 & 2 \\ \cline{1-5} 1 & 2 & 3 & 3 \\ \cline{1-4}\end{array}$$