SCHUR FUNCTIONS, PAIRING OF PARENTHESES, *JEU DE TAQUIN* AND INVARIANT FACTORS

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For Eduardo Marques de Sá on his 60th birthday

ABSTRACT. We translate the coplactic operation by Lascoux and Schützenberger which, based on the standard pairing of parentheses, transforms a word on a two-letter alphabet into one of reversed weight ([18, 22]), to the action of the *jeu de taquin* on two neighbor columns of a semi-standard Young tableau of skew shape. That enables to extend the action of the symmetric group on frank words to arbitrary k-column words, in a Knuth class. On the other hand, considering variants of the *jeu de taquin* on two-column words, that allows us to introduce variants of the mentioned Lascoux-Schützenberger operation on words, based on nonstandard pairing of parentheses, and to give a combinatorial description of the invariant factors associated with certain types of sequences of product of matrices, over a local principal ideal domain.

1. INTRODUCTION

It is well-known that there is a remarkable relationship between the combinatorics of semi-standard Young tableaux and Schur functions [14, 23, 24]. Not so well-known is the relationship between those combinatorial objects with the invariant factors of matrices over a local principal ideal domain. Indeed a highlight in this analogy is the fact that the Littlewood-Richardson rule describes the invariant factors of a product of matrices over a local principal ideal domain, as well as the product of two Schur functions as a linear combination of the same functions [14, 15, 16, 24]. In this paper we go further and some important combinatorial operations on semi-standard Young tableaux like Bender-Knuth involution [10], Lascoux-Schützenberger operators based on standard pairing of parentheses

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[18, 22], and Schützenberger *jeu de taquin*, are interpreted in the context of the invariant factors of matrices over a local principal ideal domain. On the way, we extend the action of the symmetric group defined by Lascoux-Schützenberger on frank words [19, 14] to arbitrary k-column words in a plactic class. This action on k-column words is everywhere defined and in turn extends the one in [19], on k-column words, not everywhere defined. On the other hand, in the free algebra, that is, on all words, this action is translated into to the Lascoux-Schützenberger action of the symmetric group based on the standard pairing of parentheses [18, 22].

In section 2, the combinatorial definition of Schur function where the Bender-Knuth involution plays an important role is reviewed as well as its relationship with other involutions on semi-standard Young tableaux [17, 21, 24]. In section 3, we extend the action of the symmetric group defined by Lascoux-Schützenberger on frank words [19] to arbitrary k-column words in a plactic class; the translation to the action of the symmetric group in the free algebra *via* a variant of the dual RSK-correspondence is explained. In the last section, the previous combinatorial operations are interpreted in the context of the invariant factors of matrices over a local principal ideal domain, and, in this context, some interesting generalizations arise. The combinatorics of the invariant factors and its relationship with Yamanouchi tableaux (Littlewood-Richardson tableaux of partition shape) has been developed earlier by several authors, like J. A. Green, T. Klein and R. C. Thompson *et al* in [12, 25, 2, 3, 6, 1], with key-tableaux and frank words in [3, 4, 5] and, more recently, in [7, 8, 9], with R. Mamede.

2. Schur functions and semi-standard Young tableaux

2.1. **Symmetric functions.** Let $\mathbb{N}$ be the set of nonnegative integers. Let $x = (x_1, x_2, \dots)$ be a vector of indeterminates and $n \in \mathbb{N}$. A homogeneous symmetric function of degree n over a commutative ring R with identity is a formal power series $f(x) = \sum_\alpha c_\alpha x^\alpha$ where α runs over all weak compositions $\alpha = (\alpha_1, \alpha_2, \cdots)$ of n, $c_\alpha \in R$ and x^α stands for the monomial $x^{\alpha_1} x^{\alpha_2} \cdots$, such that $f(x_{\pi(1)}, x_{\pi(2)}, \dots) = f(x_1, x_2, \dots)$ for every permutation π of the positive integers [24]. The set of all homogeneous symmetric functions of degree n over R is an R-module and a vector space when $R = \mathbb{Q}$. Different bases for this vector space are known. An important one is given by the monomial symmetric functions

$$m_\lambda = \sum_\alpha x^\alpha \tag{2.1}$$

with λ a partition of n, written $\lambda | - n$, and α running over all distinct permutations of the entries of λ.

2.2. **Semi-standard Young tableaux.** Given $t \geq 1$, we put $[t] := \{1, \dots, t\}$ and denote by $[t]^*$ the free monoid in the alphabet $[t]$ and by ϵ the empty word. The word $\omega = x_1 \cdots x_r$ over the alphabet $[t]$ is called a column word if $x_1 > \cdots > x_r$, and a row word if $x_1 \leq x_2 \leq \cdots \leq x_r$. The set of columns of $[t]^*$ is denoted by V. The length r of ω is written $|\omega|$. We define the weight of $\omega \in [t]^*$ as $\mathbf{wt}\omega \in \mathbb{N}^t$, where $(\mathbf{wt}\omega)_i$ counts the number of letters i in ω.

Given a partition λ, the number of its nonzero parts $l(\lambda)$ is the length of λ. A partition is identified with the diagram of the boxes arranged in left justified rows at the bottom. (The French convention is adopted, that is, the longest row of the partition is in the bottom.) The empty partition is denoted by $\varnothing$. A semi-standard Young tableau of shape λ is an array $T = (t_{ij})$, $1 \leq i \leq l(\lambda)$, $1 \leq j \leq \lambda_i$, of positive integers of shape λ, weakly increasing in every row and strictly decreasing down in every column. Let μ be a partition such that $\mu \subseteq \lambda$, that is, $\mu_i \leq \lambda_i$. The diagram λ/μ, obtained from λ by removing μ, is called a skew-diagram. Similarly, we define a semi-standard Young tableau of shape λ/μ as an a array $T = (t_{ij})$, $1 \leq i \leq l(\lambda)$, $\mu_i < j \leq \lambda_i$. The partitions λ and μ are called, respectively, the outer and the inner shape of the semi-standard Young tableau of shape λ/μ. When $\mu = \varnothing$ we get a semi-standard Young tableau of partition shape. Examples of semi-standard Young tableaux of shape $(5,4,2)$ and $(4,4,2,1)/(3,1)$ are

$$T = \begin{array}{ccccc} 5 & 5 & & & \\ 2 & 4 & 4 & 4 & \\ 1 & 1 & 1 & 3 & 4 \end{array} \quad \text{and} \quad H = \begin{array}{cccc} 6 & & & \\ 5 & 5 & & \\ & 4 & 4 & 5 \\ & & & 3 \end{array} . \tag{2.2}$$

The reading word of a semi-standard Young tableau T is the sequence of entries of T obtained by concatenating the column words of T left to right. For instance the reading word of T is $521\,541\,41\,43\,4$ and of H is $65\,54\,4\,53$. The weight of a SSYT is the weight of its reading word. T has weight $(3,1,1,4,2)$. A SSYT of partition shape is identified with its reading word.

A SSYT T of shape λ/μ and weight α may also be represented by a nested sequence of partitions $T = (\lambda^0, \lambda^1, \ldots, \lambda^t)$, where $\mu = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^t = \lambda$, such that for $k = 1, \ldots, t$, the skew diagram λ^k/λ^{k-1} is labeled by k, with $\alpha_k = |\lambda^k| - |\lambda^{k-1}|$ [24].

Cyclic permutations on words induce an operation on SSYT's of partition shape, called *cyclage*, providing a rank-poset structure on the of SSYT's of given weight α; the row word $1^{\alpha_1}2^{\alpha_2}\ldots$ is the unique minimal element [18, 20]. For instance, the cyclage chain of the SSYT's with weight $(3,2)$ is the interval $[1^3 2^2,\ Y = 21211]$, defined by $Y = \begin{array}{ccc} 2 & 2 & \\ 1 & 1 & 1 \end{array} \rightarrow \begin{array}{cccc} 2 & & & \\ 1 & 1 & 1 & 2 \end{array} \rightarrow \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2. \end{array}$

2.3. **Schur functions.** There are several ways to define Schur functions [24]. For our purposes, we adopt the one in terms of m_λ (2.1) which exhibits the relationship with semi-standard Young tableaux. For any semi-standard Young tableau T of weight $\alpha = (\alpha_1, \alpha_2, \cdots)$, we define the monomial $x^T = x_1^{\alpha_1} x_2^{\alpha_2} \ldots$. For instance for the SSYT's above (2.2) we have $x^T = x_1^3 x_2 x_3 x_4^4 x_5^2$ and $x^H = x_3 x_4^2 x_5^3 x_6$.

Given $\mu \subseteq \lambda$, the skew-Schur function $s_{\lambda/\mu}$ of shape λ/μ in the variables $x_1, x_2, \ldots$ is the formal power series $S_{\lambda/\mu}(x) = \sum_T x^T$ where T runs over all SSYT's of shape λ/μ. If $\mu = \varnothing$, $\lambda/\mu = \lambda$ and we call $s_\lambda(x)$ the Schur function of shape λ.

Theorem 2.1. [10] *The number of SSYT's of shape λ/μ with weight α is independent of the permutations of the entries of α.*

Proof. The Bender-Knuth involution t_k, for short B-K, on the set SSYT's of shape λ/μ and weight α, introduced in [10], performs an interchange of the contiguous components α_k and α_{k+1} in the weight α and leaves the shape of T unchanged. Let T be a SSYT of shape λ/μ and weight α. All entries $t_{ij} \neq k, k+1$ remain unchanged. A portion of T with parts equal to k or $k+1$ has the form

(2.3)
$$\begin{array}{cccccccccc} k+1 & k+1 & & & & & & & & \\ k & k & k & k & k & k+1 & k+1 & k+1 & k+1 & k+1 \\ & & & & & & & k & k & k. \end{array}$$

Ignoring the columns with both parts k and $k+1$, we obtain in each row of T a word of the form $k^r(k+1)^s$, for some $r,\ s \geq 0$, which we replace with $k^s(k+1)^r$. The transformation t_k acts on T by performing this interchange, independently for each row, leaving the column words $k+1\ k$ unchanged. □

From this combinatorial result it follows

Corollary 2.2. *The skew-Schur functions are symmetric functions.*

Clearly the B-K involution t_k satisfies $t_k t_i = t_i t_k$ for $|i-k| > 2$ but does not give rise to an action of the symmetric group on SSYT's. As one may check, the application of $t_1 t_2 t_1$ and $t_2 t_1 t_2$ to T above (2.2) leads to different results, although both have the same weight $(1,1,3,4,2,1,0,1)$. In the proof of theorem 2.1 we might have used the Lascoux-Schützenberger involution θ_k, for short L-S involution, which satisfies the Moore-Coxeter relations of the symmetric group. The involution θ_k, based on the standard parenthesis pairing procedure, is described as follows. Let $\omega \in [t]^*$. First we extract from ω a subword ω' containing the letters k and $k+1$ only. Second, in the word ω' we remove consecutively all factors $k+1\,k$. As a result we obtain a subword of the form $k^r\ k+1^s$. We replace it with the word $k^s\ k+1^r$ and, after this, recover all the removed pairs and all the letters which differ from k and $k+1$. We arrive to a new word, denoted by $\theta_k\omega$ in $[t]^*$ whose weight is the interchange of $(\mathbf{wt}\omega)_k$ with $(\mathbf{wt}\omega)_{k+1}$. In general, $t_k \neq \theta_k$, as may be easily checked applying θ_2 and t_2 to T (2.2).

Nevertheless it is worth to point out that over the key tableaux [14, 19], that is SSYT's of partition shape whose weight is a permutation of the shape, the B-K involution t_k and the pairing L-S involution θ_k agree [19, 22]. The array (2.3) either is always of the form $\begin{matrix} k+1 & k+1 & & & \\ k & k & k & k & k \end{matrix}$ or $\begin{matrix} k+1 & k+1 & & & \\ k & k & k+1 & k+1 & k+1. \end{matrix}$

The Schützenberger involution *evac*, based on the evacuation operation, is an involution on SSYT's which preserves the shape and reverses the weight. It is shown in [17] that it can be defined as $evac = t_1\ t_2t_1\ t_3t_2t_1 \dots t_{n-1} \dots t_1,$. Thus evacuation equals the involution $\tau := \theta_1\ \theta_2\theta_1\ \theta_3\theta_2\theta_1 \dots \theta_{n-1} \dots \theta_1$ on key tableaux. These agreements do not follow for general SSYT's (see also [17, 21]). Another advantage of the involutions θ_i is their application to prove that the Littlewood-Richardson number $c^{\lambda}_{\mu\,\nu}$, the number of LR tableaux (tableaux whose rectification is a Yamanouchi tableau) of shape λ/μ and weight ν, is independent of the permutation of ν. In particular, the involution τ can be used to exhibit the symmetry $c^{\lambda'}_{\mu'\,\nu'} = c^{\lambda}_{\mu\,\nu}$ where μ', ν', λ' denote the conjugate partitions (see [6]).

3. Action of the symmetric group on the set of k-column words and on the free algebra

3.1. **Column words and SSYT's.** Given $k \geq 0$, and $u_i \in V$, $1 \leq i \leq k$, we define a k-column word as $u = (u_1, \dots, u_k) \in V^k$. Two k-columns words $(u_1, \dots, u_k)$ and $(v_1, \dots, v_k)$ are equal if $u_i = v_i$, $1 \leq i \leq k$. The shape of the k-column word $u = (u_1, \dots, u_k) \in V^k$ is $(|u_1|, \dots, |u_k|)$. (When the column word is a SSYT of partition shape, it will be clear from the context whether the column or row shape is considered.) We shall write often $u = u_1 \cdots u_k$ keeping in mind that the u_i's are in V. A k-column word u may be identified with a unique SSYT, with k columns of lengths $|u_1|, \dots, |u_k|$ and reading word $u_1 \cdots u_k$, as follows. For $k = 0$, the SSYT is $\varnothing$, and, for $k = 1$, is u_1. For $k = 2$, the pair (u_1, u_2) is aligned to form a SSYT of two columns with maximal row overlaping q. This SSYT with reading word u_1u_2 has shape $(q + r_1 + s_2, q + r_1)/(r_1, 0)$, where $q + r_1 = |u_2|$ and $q + s_2 = |u_1|$ for some $r_1, s_2 \geq 0$. For $k > 2$, each pair (u_{i-1}, u_i) of successive columns of u is aligned to form a SSYT with two columns as in case $k = 2$. A *frank word* is a distinguished column word whose shape is a permutation of the column shape of the unique SSYT in its plactic class. In particular, if $u_{i-1}u_i$ is a frank word [19, 14], the pair is aligned at: the top whenever the right column u_i is longer, and, in this case, $s_i = 0$; and at the bottom whenever the left column u_{i-1} is longer, and, in this case, $r_{i-1} = 0$. Note that the outcome SSYT has inner shape $(\sum_{j=1}^{k-1} r_j, \dots, r_{k-1}, 0)$. A SSYT satisfying this property for each successive pair of columns is said to be in the *compact form.* In particular, a SSYT of partition shape is always in the compact form.

The k-column word of a SSYT with k columns is defined as $(u_1, \ldots, u_k) \in V^k$ with u_i the i-th column. Two SSYT's with k columns are said to be *equivalent* if they have the same k-column word. This means that they have the same compact form in the sense that every SSYT is *jeu de taquin equivalent, by vertical slides*, to its compact form. However, not every pair of SSYT's with k columns, *jeu de taquin* equivalent, has the same compact form. For instance (for convenience we represent the inner shape by X) $\begin{array}{cc} 6 & \\ 5 & \\ X & 4 \end{array}$ is in the compact form while $\begin{array}{cc} 6 & \\ 5 & \\ X & 4 \\ X & X \end{array}$ is not, but it is *equivalent* to the first one; and $\begin{array}{cc} 6 & \\ 5 & \\ 4 & X \end{array}$ is in the compact form but it is not *equivalent* to the previous ones, although *jeu de taquin* equivalent. The *six*-column word $53\,52\,76\,\epsilon\,5\,53$ is identified with the six-column SSYT, $\begin{array}{cccccc} 5 & & & & & \\ 3 & 5 & 7 & & & \\ X & 2 & 6 & & & \\ X & X & X & X & 5 & 5 \\ X & X & X & X & X & 3 \end{array}$, with shape $(5, 4, 4, 2, 2, 2)/(3, 2, 2, 2, 1, 0)$.

3.2. **A variant of the dual RSK-correspondence.** We consider a variant of the dual Robinson-Schensted-Knuth correspondence [13], [14], Appendix A.4.3, to establish a bijection between tableau-pairs (P, Q) of conjugate shapes and SSYT's in the compact form. Let $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_1 & \cdots & u_k \\ v_1 & \cdots & v_k \end{pmatrix}$ be a biword without repeated biletters, where $u_1, \ldots, u_k \in [n]$ and $v_1, \ldots, v_k \in [t]$. Sorting the biletters of $\begin{pmatrix} u \\ v \end{pmatrix}$ by weakly increasing rearrangement for the anti-lexicographic order with priority on the first row, we get $\Sigma = \begin{pmatrix} 1^{f_1} & \cdots & n^{f_n} \\ w_1 & \cdots & w_n \end{pmatrix}$, where $\mathbf{wt}u = (|w_1|, \ldots, |w_n|)$ and $\omega = w_1 \cdots w_n \in V^n$ (here V is the set of columns of $[t]^*$); and by weakly decreasing rearrangement of the biletters of $\begin{pmatrix} u \\ v \end{pmatrix}$ for the lexicographic order with priority on the second row, we get $\Sigma' = \begin{pmatrix} J_t & \cdots & J_1 \\ t^{m_t} & \cdots & 1^{m_1} \end{pmatrix}$, where $\mathbf{wt}v = (|J_1|, \ldots, |J_t|)$ and $J = J_t \cdots J_1 \in V^t$ (here V is the set of columns of $[n]^*$). As usual, given a word ω, $P(\omega)$ denotes the unique SSYT of partition shape in the Knuth class of ω, and $Q(\omega)$ the corresponding Q-symbol [22]. From Greene's theorem [11], we have

Lemma 3.1. (a) *The transformation* $\Sigma \leftrightarrow \Sigma'$*, as above, establishes a bijective correspondence between the* k*-tuples of disjoint weakly increasing subwords of* $J = J_t \cdots J_1$ *and those of decreasing subwords of* $\omega = w_1 \cdots w_n$*.*

(b) *The SSYT's* $P(\omega)$ *and* $P(J)$ *have conjugate shapes with* $\mathbf{wt}\omega$ *the reverse shape of the* t*-column word* $J_t \cdots J_1$*, and* $\mathbf{wt}J$ *the shape of the* n*-column word* $w_1 \cdots w_n$*.*

We consider the variant of the dual RSK-correspondence, for short RSK*, defined and denoted by $\begin{pmatrix} u \\ v \end{pmatrix} \overset{RSK^*}{\longrightarrow} (P,Q)$, where $P = P(w)$ and $Q = P(J)$. The SSYT's in this pair are related as follows, where $v \downarrow$ $(u \uparrow)$ denotes v by weakly decreasing (u by weakly increasing) order

Proposition 3.2. *Let* $\Sigma = \begin{pmatrix} u \uparrow \\ \omega \end{pmatrix}$ *and* $\Sigma' = \begin{pmatrix} J \\ v \downarrow \end{pmatrix}$ *correspond by RSK** *to* (P,Q)*. Then*

(a) Q *is the unique SSYT of weight* $(|w_1|, \ldots, |w_n|)$ *such that* $Q(\omega) = std(Q)^T$*, where std stands for standardization.*

(b) P *is the unique SSYT of evaluation* $(|J_1|, \ldots, |J_t|)$ *such that* $Q(J) = std(evac\, P)^T$*.*

The RSK* correspondence establishes a bijection between the biwords Σ (Σ') over the alphabet $[n] \times [t]$ and tableau-pairs (P,Q) of conjugate shapes, with $P \in [t]^*$ and $Q \in [n]^*$. As we have seen in the previous section there is a bijection between column words and SSYT's in the compact form. We have therefore the following bijections

$$\{SSYT's \text{ with } n\text{-column word } \omega = w_1 \ldots w_n\} \leftrightarrow \begin{pmatrix} 1^{f_1} & \cdots & n^{f_n} \\ w_1 & \cdots & w_n \end{pmatrix} \leftrightarrow$$

$$\leftrightarrow \begin{pmatrix} J_t & \cdots & J_1 \\ t^{m_t} & \cdots & 1^{m_1} \end{pmatrix} \leftrightarrow (P,Q) \text{ SSYT's of conjugate shapes.} \tag{3.4}$$

Identify a column word with its underlying set. The set $J_i \subseteq [n]$ is the set of the column indices of the letter i in any SSYT with n-column word $\omega = w_1 \ldots w_n$. A SSYT defines a unique pair (ω, J), in the conditions of proposition 3.2, and $J = J_t \ldots J_1$ is called the *indexing set column word.*

Let $\sigma \in S_t$ and s_i the transposition of the integers i and $i+1$. Put $\sigma^{\#} := rev\, \sigma$ with rev the reverse permutation in S_t. In particular, $s_i^{\#} = s_{t-i}\, rev$. If m is a weak composition, put σm for the usual action of σ on m. In particular, put $m^{\#} := rev\, m$. If J is a frank word, σJ denotes the frank word congruent with J whose shape is the action of σ on the shape of J. In particular, put $J^{\#} := rev\, J$. Given the sequence of nonnegative integers $u = (u_1, \cdots, u_r)$, we define the r-column word $Mu = u_1 \cdots 1\ u_1 + u_2 \cdots u_1 + 1 \cdots\ u_r + \cdots + u_1 \cdots u_{r-1} + \cdots + u_1 + 1$ [19]

whose shape is u. The tableau-pair (K, Q) of conjugate shapes with K a key and the frank words in Q are related as follows. From proposition 3.2 we have

Proposition 3.3. [9] *Let (K, Q) be a tableau-pair of conjugate shapes such that the conjugate shape of Q is m. Let $\sigma \in S_t$. Let $\begin{pmatrix} J \\ K\downarrow \end{pmatrix}$ correspond by RSK* to the pair (K, Q). The following statements are equivalent*
(a) K is the key with weight σm.
(b) J is the frank word of shape $\sigma^{\#}m$ in the Knuth class of Q.
(c) $Q(J) = M(\sigma^{\#}m)$.

3.3. **Jeu de taquin on two-column words.** The *jeu de taquin* on consecutive columns $t-i$, $t-i+1$ of a t-column SSYT, in the compact form, exchanging the shape of these columns, is translated, by RSK*-correspondence, into the operation θ_i on all words over the alphabet $[t]$. In particular, *jeu de taquin* on frank words is translated into the operation θ_i on words congruent with keys.

Define the operation Θ on a two-column SSYT in the compact form, with column lengths $(q+s, q+r)$ and inner shape (r), for some $q, r, s \geq 0$, as follows. If $r > s$ $(r < s)$, perform *jeu de taquin* slides on the first $|r-s|$ inside (outside) corners until they become outside (inside) corners in the second (first) column. In other words, we slide down (up) the first (second) column, maximally up to $|r-s|$ positions; then we exchange the east (west) neighbors with these corners. Then $\Theta\, T = T'$ is a two-column SSYT, in the compact form, with column lengths $(q+r, q+s)$ and with inner shape (s), if $s \geq r$, and (r), otherwise. In particular, when $r = 0$ or $s = 0$, Θ is the *jeu de taquin* on frank words. For instance, the *jeu de taquin* slides with respect to the corner $\blacksquare$ as below define the operation Θ on T and T'

$$T = \begin{array}{cc} 7 & \\ 3 & 6 \\ 2 & 5 \\ 1 & 4 \\ \blacksquare & 3 \\ & 2 \end{array} \quad \overset{\Theta}{\longleftrightarrow} \quad T' = \begin{array}{cc} 7 & \\ 6 & \blacksquare \\ 3 & 5 \\ 2 & 4 \\ 1 & 3 \\ & 2 \end{array}. \tag{3.5}$$

Let T be a t-column SSYT, in the compact form. Define the operation Θ_i on T as follows: apply Θ to the columns i and $i+1$ of T and put the outcome t-column SSYT in the compact form. As *jeu de taquin* preserves Knuth equivalence, we have $\Theta_i T \equiv T$. The operations θ_i on words over the alphabet $[t]$, and Θ_{t-i} on t-column semi-standard Young tableaux in the compact form (equivalence classes of SSYT's) are a translation of each other in the sense of the diagram below

$$\begin{array}{ccc}\Sigma = \begin{pmatrix} T\uparrow \\ w \end{pmatrix} & \longleftrightarrow & \Sigma' = \begin{pmatrix} T \\ \cdots i+1^{q+s} i^{q+r} \cdots \end{pmatrix} \\ \updownarrow & & \updownarrow \\ \widetilde{\Sigma} = \begin{pmatrix} T'\uparrow \\ \theta_i w \end{pmatrix} & \longleftrightarrow & \widetilde{\Sigma}' = \begin{pmatrix} \Theta_{t-i}T \\ \cdots i+1^{q+r} i^{q+s} \cdots \end{pmatrix}\end{array}. \tag{3.6}$$

If the columns $t-i$ and $t-i+1$ of $\Theta_{t-i}T$ are of the form $\begin{array}{cc} A & \\ B & C \\ & D \end{array}$, where $A \cup B$, $C \cup D$ are columns such that $B \leq C$, $|B| = |C| = q$, $|A| = s$, and $|D| = r$, then θ_i is based on the pairing of parentheses defined by the increasing injections $j : B \to C \cup D$. For instance, for the two-column SSYT T (3.5), we have the following diagram

$$\begin{array}{ccc}\Sigma = \begin{pmatrix} 1\,2\,2\,3\,3\,4\,5\,6\,7 \\ 2\,2\,1\,2\,1\,1\,1\,1\,2 \end{pmatrix} & \longleftrightarrow & \Sigma' = \begin{pmatrix} T \\ 2^{3+1}\ 1^{3+2} \end{pmatrix} \\ \updownarrow & & \updownarrow \\ \widetilde{\Sigma} = \begin{pmatrix} 1\,2\,2\,3\,3\,4\,5\,6\,7 \\ 2\,2\,1\,2\,1\,1\,1\,2\,2 \end{pmatrix} & \longleftrightarrow & \widetilde{\Sigma}' = \begin{pmatrix} \Theta_1 T \\ 2^{3+2} 1^{3+1} \end{pmatrix}\end{array},$$

where $\omega = (2(21)(21)1)1^2 2 \to \theta_1\omega = (2(21)(21)1)12^2$.

As $\Theta_i T \equiv T$, from proposition 3.2, the operator θ_i preserves the Q-symbol, $Q(\omega) = Q(\theta_i\omega)$. For a two-column SSYT T with $r > s$, we have,

$$Q(T) = \begin{pmatrix} q+s+1 & \cdots & 2q+s & & & & \\ 1 & \cdots & q & \cdots q+s & 2q+s+1 & \cdots & 2q+s+r \end{pmatrix}^T$$

and

$$Q(\Theta_1 T) = \begin{pmatrix} q+r+1 & \cdots & 2q+r & & & & \\ 1 & \cdots & q & \cdots q+r & 2q+r+1 & \cdots & 2q+s+r \end{pmatrix}^T.$$

From proposition 3.2, $Q(T) = std(evac\, P(\omega))^T$ and $Q(\Theta_{t-i}T) = std(evac\, P(\theta_i\omega))^T$. Thus $P(\theta_i\omega) = \theta_i P(\omega)$.

Theorem 3.4. *Given Q a SSYT, let J be a t-column word congruent with Q. Let P be a SSYT of weight the reverse shape of J. If $\begin{pmatrix} J \\ P\downarrow \end{pmatrix}$ and $\begin{pmatrix} Q\uparrow \\ w \end{pmatrix}$ are biwords in RSK* correspondence with (P, Q), then $\begin{pmatrix} \Theta_{t-i}J \\ \theta_i P\downarrow \end{pmatrix}$ and $\begin{pmatrix} Q\uparrow \\ \theta_i w \end{pmatrix}$ correspond by RSK* to $(\theta_i P, Q)$.*

Corollary 3.5. [9] *Given Q a SSYT and $\sigma \in S_t$, let K be the key tableau of weight the shape of σQ. If $\begin{pmatrix} \sigma^{\#}Q \\ K\downarrow \end{pmatrix}$ and $\begin{pmatrix} Q\uparrow \\ \omega \end{pmatrix}$ are biwords in RSK* correspondence with (K, Q), then $\begin{pmatrix} s_{t-i}\sigma^{\#}Q \\ \theta_i K\downarrow \end{pmatrix}$ and $\begin{pmatrix} Q\uparrow \\ \theta_i\omega \end{pmatrix}$ correspond by RSK* to $(\theta_i K, Q)$.*

Remark 3.1. *If $J = J_2\, J_1$ is a two-column frank word with $J_2 = y+q+r+s\dots y+q+r+1\dots y+q+1\dots q+1$ and $J_1 = y+q+r+1\dots y+q+1\dots y+1$, we have $\omega = P(\omega)$ a key tableau and $\theta_1 P(\omega) = t_1 P(\omega)$* (see also [5]).

Corollary 3.6. *The following statements are equivalent:*

(a) *The operations Θ_i, $1 \le i \le t-1$, define an action of the symmetric group S_t on the set of t-column words, equivalently, on the t-column SSYT's in the compact form. Moreover, $\Theta_i T \equiv T$, $1 \le i \le t-1$.*

(b) *The operations θ_i, $1 \le i \le t-1$, defines an action of the symmetric group on all words over the alphabet $[t]$. These operations preserve the Q-symbol and $\omega \equiv P$ iff $\theta_i\omega \equiv \theta_i P$.*

Example 3.1. *An action of S_3 on three-column SSYT's in the compact form*

$$\begin{tikzcd}
& \begin{array}{ccc} 9 & & \\ 7 & 7 & \\ 2 & 6 & 6 \\ 1 & 4 & 5 \\ & 3 & 4 \\ & 2 & 3 \\ & 1 & 2 \end{array} \arrow[r, leftrightarrow, "\Theta_2"] & \begin{array}{ccc} 9 & & \\ 7 & & \\ 2 & 6 & 7 \\ 1 & 4 & 6 \\ & 3 & 5 \\ & 2 & 4 \\ & 1 & 3 \\ & & 2 \end{array} \arrow[dr, leftrightarrow, "\Theta_1"] & \\
\begin{array}{ccc} 9 & & \\ 7 & & \\ 6 & & \\ 4 & 7 & \\ 2 & 3 & 6 \\ 1 & 2 & 5 \\ & 1 & 4 \\ & & 3 \\ & & 2 \end{array} \arrow[ur, leftrightarrow, "\Theta_1"] \arrow[dr, leftrightarrow, "\Theta_2"'] & & & \begin{array}{ccc} 9 & & \\ 7 & & \\ 6 & & \\ 2 & 4 & 7 \\ 1 & 3 & 6 \\ & 2 & 5 \\ & 1 & 4 \\ & & 3 \\ & & 2 \end{array} \\
& \begin{array}{ccc} 9 & & \\ 7 & & \\ 6 & 7 & \\ 4 & 6 & \\ 2 & 3 & 5 \\ 1 & 2 & 4 \\ & 1 & 3 \\ & & 2 \end{array} \arrow[r, leftrightarrow, "\Theta_1"] & \begin{array}{ccc} 9 & & \\ 7 & 7 & \\ 6 & 6 & \\ 2 & 4 & 5 \\ 1 & 3 & 4 \\ & 2 & 3 \\ & 1 & 2 \end{array} \arrow[ur, leftrightarrow, "\Theta_2"'] &
\end{tikzcd}$$

4. Invariant factors and semi-standard tableaux

Given an n by n non-singular matrix A, with entries in a local principal ideal domain with prime p, by Gaußian elimination one can reduce A to a diagonal matrix Δ_μ with diagonal entries $p^{\mu_1}, \dots, p^{\mu_n}$, for unique nonnegative integers $\mu_1 \geq \dots \geq \mu_n$, called the *Smith normal form* of A. The sequence $p^{\mu_1}, \dots, p^{\mu_n}$ defines the invariant factors of A, and $(\mu_1, \dots, \mu_n)$ is the invariant partition of A. It is known that μ, β, γ are invariant partitions of nonsingular matrices A, B, and C such that $AB = C$ if and only if there exists a Littlewood-Richardson tableau T of type (μ, β, γ), that is, a SSYT of shape γ/μ whose word is in the Knuth class of the key tableau of weight β (Yamanouchi tableau of weight β). Apart from other approaches [12, 25, 15, 16], this result can be derived in a purely matrix context when one introduces the following

Definition 4.1. [3] *Let $T = (\mu^0, \mu^1, ..., \mu^t)$ and $F = (0, \delta^1, ..., \delta^t)$ be SSYT's both of weight $(m_1, ..., m_t)$. We say that a sequence of n by n nonsingular matrices $A_0, B_1, ..., B_t$ is a matrix realization of the pair (T, F) of SSYT's if:*

I. *For each $r \in \{1, ..., t\}$, the matrix B_r has invariant partition $(1^{m_r}, 0^{n-m_r})$.*

II. *For each $r \in \{0, 1, ..., t\}$, the matrix $A_r := A_0 B_1 ... B_r$ has invariant partition the conjugate of μ^r.*

III. *For each $r \in \{1, ..., t\}$, the matrix $B_1 ... B_r$ has invariant partition the conjugate of δ^r.*

Given $J \subseteq [n]$, we write D_J for the diagonal matrix having the ith diagonal entry equal to p whenever $i \in J$ and 1 otherwise. For the Bender-Knuth involution we have the following interpretation.

Theorem 4.1. [3, 5] *Let T be a SSYT with inner shape μ, indexing sets J_1, J_2, and word ω. Let $\mu = (a_1^{l_1}, a_2^{l_2}, \dots, a_k^{l_k})$, $a_1 > \dots > a_k$, $l_i > 0$, $1 \leq i \leq k$, and $X_q = \{\sum_{j=0}^{q-1} l_j + 1, \dots, \sum_{j=0}^{q} l_j\}$ with $l_0 = 0$ and $1 \leq q \leq k$. If T' is the SSYT realized by $\Delta_\mu, D_{J_2}, D_{J_1}$ with indexing sets J_1', J_2' and word ω', then*

(a) $(J_2' \cap X_q)(J_1' \cap X_q) = \Theta_1(J_2 \cap X_q)(J_1 \cap X_q)$, $1 \leq q \leq k$.

(b) $\omega_q' = t_1 \omega_q$, *where ω_q and ω_q' are the subwords, respectively, of ω and ω' restricted to the positions in X_q, $1 \leq q \leq k$.*

T' is the result of the application of the evacuation to T [5, 17]. In fact we have $\Delta_\mu D_{J_1} D_{J_2} = \bigoplus_{q=1}^k \Delta_{a_q} D_{J_1 \cap X_q} D_{J_2 \cap X_q}$, and $\Delta_\mu D_{J_2} D_{J_1} = \bigoplus_{q=1}^k \Delta_{a_q} D_{J_2 \cap X_q} D_{J_1 \cap X_q}$.

Since $\Delta_{a_q} = p^{a_q} I$, it suffices to analyze the following situation. Let $m = |J_1 \cap J_2|$, we have

$$D_{J_1}, D_{J_2} \sim D_{[|J_1|]} D_{[m]} (I_{|J_1|} \oplus D_{|J_2|-m}) \leftrightarrow F = \begin{array}{lll} 2 \dots 2 & & \\ 1 \dots 1 & 1^{|J_1|-m} & 2^{|J_2|-m} \end{array}$$

and

$$D_{J_2}, D_{J_1} \sim D_{[|J_2|]} D_{[m]} (I_{|J_2|} \oplus D_{|J_1|-m}) \leftrightarrow t_1 F = \begin{array}{lll} 2 \dots 2 & & \\ 1 \dots 1 & 1^{|J_2|-m} & 2^{|J_1|-m} \end{array} = evac\, F,$$

where $\sim$ denotes unimodular equivalence.

Recall the operation Θ on a two-column semistandard Young tableau T in the compact form, defined in the previous section, and denote by $\tilde{\Theta}$, a variant of Θ, based on a variant of the *jeu de taquin* on T, running as follows: (1) If $r-s=r$ ($s-r=s$), add q extra vacant positions to the top (bottom) of the first (second) column and slide the labeled entries in the first (second) column along the enlarged column such that the row weak increasing order is preserved and a common label to the two columns never has a vacant west (east) neighbor; then mark r vacant positions in the first (second) column with labeled east (west) neighbors and exchange with them. When the labeled entries of the first (second) column are slided down (up) maximally such that the row weakly order is preserved, we get the *jeu de taquin* operation Θ on frank words. (2) If $0 \leq r-s < r$ ($0 \leq s-r < s$), add q vacant positions to the top (bottom) of the first (second) column and slide the $q+s$ labeled entries along the enlarged first column with $q+r$ vacant positions such that the previous restrictions are attained; then mark $r-s$ vacant positions in the first (second) column with east (west) labeled neighbors and exchange with them. When the labeled entries of the first (second) column are slided down (up) maximally at most $r-s$ positions such that the row weakly order is preserved, we get the operation *jeu de taquin* Θ on T. Unless, $\tilde{\Theta} = \Theta$, $\tilde{\Theta} T \not\equiv T$. For instance,

$$\begin{array}{cc} 5 & \\ \blacksquare & 4 \\ 1 & 3 \\ & 2 \end{array} \overset{\Theta}{\longleftrightarrow} \begin{array}{cc} 5 & \\ 4 & \blacksquare \\ 1 & 3 \\ & 2 \end{array}. \tag{4.7}$$

$$\tilde{\Theta}: \begin{array}{cc} 5 & \\ 1 & 4 \\ & 3 \\ \blacksquare & 2 \end{array} \longrightarrow \begin{array}{cc} 5 & \\ 2 & 4 \\ 1 & 3 \\ & \blacksquare \end{array} \quad \tilde{\Theta}: \begin{array}{cc} 5 & \\ 1 & 4 \\ \blacksquare & 3 \\ & 2 \end{array} \longrightarrow \begin{array}{cc} 5 & \\ 1 & 4 \\ 3 & \blacksquare \\ & 2 \end{array} \longrightarrow \begin{array}{cc} 5 & \\ 3 & 4 \\ 1 & 2 \\ & \blacksquare \end{array}. \tag{4.8}$$

(4.9)
$$\tilde{\Theta}: \begin{array}{cc} 5 & \\ 3 & \blacksquare \\ 1 & 4 \\ & 2 \end{array} \longrightarrow \begin{array}{cc} 5 & \\ \blacksquare & 3 \\ 1 & 4 \\ & 2 \end{array} \longrightarrow \begin{array}{cc} 5 & \\ 1 & 4 \\ \blacksquare & 3 \\ & 2 \end{array} \quad \tilde{\Theta}: \begin{array}{cc} 5 & \\ 3 & 4 \\ 1 & \blacksquare \\ & 2 \end{array} \longrightarrow \begin{array}{cc} 5 & \\ 3 & 4 \\ \blacksquare & 1 \\ & 2 \end{array} \longrightarrow \begin{array}{cc} 5 & \\ 3 & 4 \\ \blacksquare & 2 \\ & 1 \end{array}.$$

$$\tilde{\Theta}: \begin{array}{cc} 6 & \\ 5 & \blacksquare \\ 3 & \blacksquare \\ 1 & 4 \\ & 2 \end{array} \longrightarrow \begin{array}{cc} 6 & \\ \blacksquare & 5 \\ \blacksquare & 3 \\ 1 & 4 \\ & 2 \end{array} \longrightarrow \begin{array}{cc} 6 & \\ 1 & 5 \\ \blacksquare & 4 \\ \blacksquare & 3 \\ & 2 \end{array}. \tag{4.10}$$

$$(4.11)\qquad \begin{matrix} \blacksquare & 4 \\ 3 & 3 \\ 1 & 2 \end{matrix} \overset{\Theta}{\longrightarrow} \begin{matrix} 4 & \blacksquare \\ 3 & 3 \\ 1 & 2 \end{matrix} \qquad \tilde{\Theta}: \begin{matrix} 3 & 4 \\ 1 & 3 \\ \blacksquare & 2 \end{matrix} \longrightarrow \begin{matrix} 3 & 4 \\ 1 & 3 \\ 2 & \blacksquare \end{matrix} \longrightarrow \begin{matrix} 3 & \blacksquare \\ 2 & 4 \\ 1 & 3 \end{matrix}.$$

Using RSK*-correspondence, as explained in section 3, we define the operation $\tilde{\theta}$ as a translation of the operation $\tilde{\Theta}$. Recall the operation θ_i, described at the end of section 2. Then $\tilde{\theta}_i$ denotes a variant of θ_i based on a non standard pairing of parentheses procedure. Considering ω restricted to the two-letter alphabet $\{i, i+1\}$, written $\omega_{|\{i,i+1\}}$, we remove consecutively subwords $i+1\,i$ and as a result we obtain a subword of the form:

(1) $i^{r-l}(i+1\,i)^l$, with $0 \le l \le r$, $[(i+1\,i)^l i+1^{s-l}$, $0 \le l \le s]$ which is replaced with $i+1^{r+l}$ $[i^{s+l}]$. In this case, $P(\tilde{\theta}_i\omega_{|\{i,i+1\}}) \le P(\theta_i\omega_{|\{i,i+1\}})$ with respect to the cyclage order, and, in particular, $P(\tilde{\theta}_i\omega_{|\{i,i+1\}}) = P(\theta_i\omega_{|\{i,i+1\}})$, if $l = 0$.

(2) $i^{r-l}(i+1\,i)^l i+1^s$, $0 \le l \le r-s$, $[i^r(i+1\,i)^l i+1^{s-l}$, $0 \le l \le s-r,]$ which is replaced by $i^{s+l}i+1^r$ $[i^s i+1^{r+l}]$. In this case, we may have either $P(\tilde{\theta}_i\omega_{|\{i,i+1\}}) \le P(\theta_i\omega_{|\{i,i+1\}})$ or $P(\tilde{\theta}_i\omega_{|\{i,i+1\}}) \ge P(\theta_i\omega_{|\{i,i+1\}})$ but if $l = 0$, we have always $P(\tilde{\theta}_i\omega_{|\{i,i+1\}}) \ge P(\theta_i\omega_{|\{i,i+1\}})$ with respect to the cyclage order.

For instance, from (4.7), $\theta_1(21112) = 21122$ is based on the standard pairing procedure; from (4.8), $\tilde{\theta}_1(21112) = 21212$, based on the nonstandard pairing (2111)2 which leaves $1^2\,2$ free and is replaced with $1\,2^2$, and $\tilde{\theta}_1(21112) = 22112$ is based on the nonstandard pairing (211)12 which leaves $1^2 2$ free and is replaced with 212; from (4.9) $\tilde{\theta}_1(21212) = 21112$ is based on the pairing (2121)2, or (21)212, which leaves $1^0(21)^1 2^1$ free and is replaced with $1^{1+1}2^{0+1}$, and $\tilde{\theta}_1(21212) = 11212$ is based on the pairing 21(21)2 which leaves $1^0(21)^1 2^1$ free and is replaced with $1^{1+1}2^{0+1}$; from (4.10) we have $\tilde{\theta}_1(212122) = 211112$ is based on the pairing (21)2122 which leaves $1^0(21)^1 2^2$ free and is replaced with $1^{2+1}2^{0+1}$.

We reduce the study of the invariant factors associated with the sequence of matrices $A_0, B_1, \cdots, B_t$, satisfying conditions $(I)-(III)$ of definition 4.1, to the cases Δ_μ, UD_K and Δ_μ, UD_F where U is an n by n unimodular matrix, D_K denotes the sequence $(D_{[m_1]}, \cdots, D_{[m_t]})$ which realizes the key K of weight $(m_1, \cdots, m_t)$, and D_F denotes the sequence $(D_{F_1}, \cdots, D_{F_t})$ which realizes the SSYT F with indexing sets $F_1, \cdots, F_t$. The combinatorics associated with the sequences Δ_μ, UD_K involves frank words and words congruent with keys and has been developed in [2, 3, 4, 5, 6] and more recently with R. Mamede in [7, 8, 9]. In this case, the matrix interpretation of the operations $\tilde{\theta}_i$ and $\tilde{\Theta}_{t-i}$ and their relationship is as follows.

Theorem 4.2. [4, 7, 8] *Let T and T' be the SSYT's realized by the sequences Δ_μ, UD_K and $\Delta_\mu, UD_{\theta_1 K}$, with K the key of weight (m_1, m_2). Let J_2J_1, $J_2'J_1'$ be the two-column indexing sets, and ω, ω' the words of T and T' respectively. Then,*

(a) J_2J_1, $J_2'J_1'$ are frank words such that $\tilde{\Theta}J_2J_1 = J_2'J_1'$.

(b) $\omega \equiv K$ and $\omega' = \tilde{\theta}_1\omega \equiv \theta_1 K$.

(c) there exists an unimodular matrix U' such that $\Delta_\mu, U'D_K$ and $\Delta_\mu, U'D_{\theta_1 K}$ realize the SSYT's T and T'' such that the two-column word of indexing sets $J_2''J_1''$ and the word ω'' of T'' satisfy $J_2''J_1'' = \Theta J_2J_1$ and $\omega'' = \theta_1\omega$.

Theorem 4.3. [8] *Let T be the SSYT realized by $\Delta_\mu, U\ D_K$, with word ω and J the column word of indexing sets. Then $P(\omega) = K$ and J is a frank word of shape $m^{\#}$.*

Theorem 4.4. [7, 9] *Let $\sigma \in < s_1, s_2 >$ and $\theta \in < \theta_1, \theta_2 >$ with the same reduced word. Let σT be the tableau realized by Δ_μ, $UD_{\theta K}$ with word $\sigma\omega$ and indexing set column word σJ. Then $\{\sigma T : \sigma \in < s_1, s_2 >\}$ are the vertices of a hexagon such that*

(a) $s_i\omega = \tilde{\theta}_i w \equiv \theta_i\omega$, $1 \le i \le 2$, where $< \tilde{\theta}_1, \tilde{\theta}_2 >$ satisfy the Moore-Coxeter relations of the symmetric group S_3.

(b) $s_iJ = \tilde{\Theta}_i J$, $1 \le i \le 2$, where $< \tilde{\Theta}_1, \tilde{\Theta}_2 >$ satisfy the Moore-Coxeter relations of the symmetric group S_3.

A complete description of the hexagons defined in (a) and (b) is given in [7, 9]. This family of hexagons contains in particular the hexagon defined by the operators θ_i and its description is based on the characterization of the Knuth class of a key tableau, over a three-letter alphabet, as the set of the shuffles of its columns [8].

Example 4.1. [9] *Let $U = P_{4321}T_{14}(p)$, where P_{4321} is the permutation matrix associated with $4321 \in S_4$ and $T_{14}(p)$ is the elementary matrix obtained from the identity by placing the prime p in position $(1,4)$. With $\mu = (2,1)$ the sequences $\Delta_\mu, UD_{[3]}, D_{[2]}$ and $\Delta_\mu, UD_{[2]}, D_{[3]}$ are, respectively, matrix realizations for $T = \begin{array}{cccc} 2 & & & \\ \bullet & 1 & 2 & \\ \bullet & \bullet & 1 & 1 \end{array}$ and $T' = \begin{array}{cccc} 2 & & & \\ \bullet & 2 & 2 & \\ \bullet & \bullet & 1 & 1 \end{array}$. The words $w = 21211$ of T and $w' = 22211$ of T' satisfy $\tilde{\theta}_1 w = w' \equiv \theta_i\omega$, where $\tilde{\theta}_1$ is the operation based on the parentheses matching $(21(21)1)$. However, if we choose $U' = P_{3241}T_{24}(p)$, with P_{3241} the permutation matrix associated with 3241, the sequences $\Delta_\mu, U'\ D_{[3]}$, $D_{[2]}$ and $\Delta_\mu, U'\ D_{[2]}$, $D_{[3]}$ are, respectively, matrix realizations for T and $T'' = \begin{array}{cccc} 2 & & & \\ \bullet & 1 & 2 & \\ \bullet & \bullet & 1 & 2 \end{array}$. In this case, the word w'' of T'' satisfy $\theta_1 w = w''$. The corresponding operations on the indexing set words (frank words) are displayed in (4.11).*

Given F a two-letter SSYT of partition shape, its indexing sets (F_1, F_2) are $F_1 = [q] \cup [q+1, q+r]$ and $F_2 = [q] \cup [q+r+1, q+r+s]$ for some $q, r, s \geq 0$. The indexing sets of $\theta_1 F$ are $F_1' = [q] \cup [q+1, q+s]$ and $F_2' = [q] \cup [q+s+1, q+s+r]$. When either r or $s = 0$, F is a key tableau K.

Theorem 4.5. *Let Δ_μ, UD_F be a matrix realization of T with word ω and two-column word of indexing sets $J = J_2 J_1$. Then*

(a) $\omega \equiv G$ with G in the interval $[F, K]$ of the chain of cyclages on the SSYT's of weight $(q+r, q+s)$.

(b) $J_2 J_1$ is a two-column SSYT, in the compact form, with column lengths $(q+s, q+r)$ and inner shape $(r-f)$, for some $0 \leq f \leq min\{r, s\}$.

Example 4.2. *Let $\mu = (3,2,1)$ and $F = 11122$. The sequences Δ_μ, UD_F, with U running over the unimodular matrices of order 5, give rise to SSYT's of inner shape μ with words congruent with P running over $\{11122;\ 21\,112;\ 21\,21\,1\}$. For instance, (a) $U = I$,* $\begin{array}{lllll} 1 & & & & \\ \bullet & 1 & & & \\ \bullet & \bullet & 1 & & \\ \bullet & \bullet & \bullet & 2 & 2 \end{array}$, *$\omega = 11122$; (b) $U = P_{4321}$,* $\begin{array}{lllll} 2 & & & & \\ \bullet & 1 & & & \\ \bullet & \bullet & 1 & & \\ \bullet & \bullet & \bullet & 1 & 2 \end{array}$,

$\omega = 21112$; and (c) $U = P_{12543}P_{4321}$, $\begin{array}{lllll} 2 & & & & \\ \bullet & 1 & & & \\ \bullet & \bullet & 2 & & \\ \bullet & \bullet & \bullet & 1 & 2 \end{array}$, *$\omega = 21211$.*

Theorem 4.6. *Let Δ_μ, UD_F be a matrix realization of T with word ω and indexing set word $J_2 J_1$, and let $\Delta_\mu, UD_{\theta_1 F}$ be a matrix realization of T' with word ω' and indexing set word $J_2' J_1'$. Then*

(a) $\omega \equiv G$ and $\omega' = \tilde{\theta}_1 \omega \equiv \theta_1 H$ with G and H in the interval $[F, K]$ in the chain of cyclages on the SSYT's of weight $(q+r, q+s)$.

(b) $J_2' J_1' = \tilde{\Theta} J_2 J_1$.

When $F = K$, we have $G = K = H$ and we recover theorem 4.2.

Example 4.3. *Let* $F = \begin{array}{llll} 2 & & & \\ 1 & 1 & 1 & 2 \end{array} \leftarrow \quad Y = \begin{array}{lll} 2 & 2 & \\ 1 & 1 & 1 \end{array}$ *and*

$\theta_1 F = \begin{array}{llll} 2 & & & \\ 1 & 1 & 2 & 2 \end{array} \leftarrow \quad \theta_1 Y = \begin{array}{lll} 2 & 2 & \\ 1 & 1 & 2 \end{array}$ *be cyclage chains. The indexing sets of F and $\theta_1 F$ are, respectively, $F_1 = \{1,2,3\}$, $F_2 = \{1,4\}$ and $F_1' = \{1,2\}$, $F_2' = \{1,3,4\}$. Let $U = P_{54321}T_{15}(p)$, $V = P_{(15)}T_{15}(p)$, and $W = P_{32541}T_{15}(p)$ unimodular matrices, where P_{54321} and P_{32541} are the permutation matrices associated with 54321 and $32541 \in S_5$ respectively, and $T_{15}(p)$ is the elementary matrix obtained from the identity by placing the prime p in position $(3,5)$. Let $\mu = (2,1,1)$.* (1) *The sequences Δ_μ, UD_F and $\Delta_\mu, UD_{\theta_1 F}$ are, respectively, matrix realizations* *for* $T = \begin{array}{lllll} 2 & & & & \\ \bullet & 1 & 2 & & \\ \bullet & \bullet & \bullet & 1 & 1 \end{array}$ *and* $T' = \begin{array}{lllll} 2 & & & & \\ \bullet & 2 & 2 & & \\ \bullet & \bullet & \bullet & 1 & 1 \end{array}$. *The words* w *of* T *and* w' *of* T' *satisfy* $w = 21211 \equiv Y$ *and* $w' = 22211 \equiv \theta_1 Y$ *satisfy* $\tilde{\theta}_1 w = w'$, *where* $\tilde{\theta}_1$ *is the operation based on the parentheses matching* (21(21)1). (2) *The sequences* Δ_μ, VD_F *and* $\Delta_\mu, VD_{\theta_1 F}$ *are, respectively, matrix realizations for* $T = \begin{array}{lllll} 2 & & & & \\ \bullet & 1 & 1 & & \\ \bullet & \bullet & \bullet & 1 & 2 \end{array}$ *and* $T' = \begin{array}{lllll} 2 & & & & \\ \bullet & 1 & 2 & & \\ \bullet & \bullet & \bullet & 1 & 2 \end{array}$. *We have for the words* w *of* T *and* w' *of* T', $w = 21112 \equiv F$ *and* $w' = 21212 = \theta_1 Y = \tilde{\theta}_1 \omega \not\equiv \theta\omega$ *based on the parentheses matching* (2111)2. (3) *The sequences* Δ_μ, WD_F *and* $\Delta_\mu, WD_{\theta_1 F}$ *are, respectively, matrix realizations for* $T = \begin{array}{lllll} 2 & & & & \\ \bullet & 1 & 1 & & \\ \bullet & \bullet & \bullet & 1 & 2 \end{array}$ *and* $T' = \begin{array}{lllll} 2 & & & & \\ \bullet & 1 & 1 & & \\ \bullet & \bullet & \bullet & 2 & 2 \end{array}$. *We have* $w = 21112 = F$ *and* $w' = 21122 = \theta_1 F$. (4) $G = 11122$, $\theta_1 G = 11222$. $U = P_{(23)}$, $T = \begin{array}{lllll} 1 & & & & \\ \bullet & 1 & 1 & & \\ \bullet & \bullet & \bullet & 2 & 2 \end{array}$ *and* $T' = \begin{array}{lllll} 1 & & & & \\ \bullet & 2 & 1 & & \\ \bullet & \bullet & \bullet & 2 & 2 \end{array}$. *The corresponding operations on the indexing set words are displayed in examples (4.7), (4.9).*

Open Problem: *Let* $\sigma \in < s_1, s_2 >$ *and* $\theta \in < \theta_1, \theta_2 >$ *with the same reduced word. Let* F *be a three-letter SSYT of partition shape , and* σT *the SSYT realized by* Δ_μ, UD_F. *Describe the hexagon* $\{\sigma T : \sigma \in < s_1, s_2 >\}$.

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