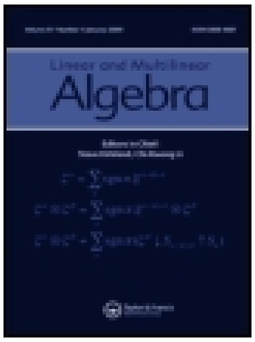

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The Admissible Interval for the Invariant Factors of a Product of Matrices

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It is well known that Littlewood–Richardson sequences give a combinatorial characterization for the invariant factors of a product of two matrices over a principal ideal domain. Given partitions a and c, let $LR(a, c)$ be the set of partitions b for which at least one Littlewood–Richardson sequence of type (a, b, c) exists. I. Zaballa has shown in [20] that $LR(a, c)$ has a minimal element $\mathbf{w}$ and a maximal element $\mathbf{n}$, with respect to the order of majorization, depending only on a and c. In general, $LR(a, c)$ is not the whole interval $[\mathbf{w}, \mathbf{n}]$. Here a combinatorial algorithm is provided for constructing all the elements of $LR(a, c)$. This algorithm consists in starting with the minimal Littlewood–Richardson sequence of shape c/a and successively modifying it until the maximal Littlewood–Richardson sequence of shape c/a is achieved. Also explicit bijections between Littlewood–Richardson sequences of conjugate shape and weight and between Littlewood–Richardson sequences of dual shape and equal weight are presented. The bijections are defined by means of permutations of Littlewood–Richardson sequences.

Keywords: Invariant factors; Young tableaux; Littlewood–Richardson sequences

AMS Subject Classifications: 15A23, 15A33, 05A17, 05E10

1. INTRODUCTION

Several papers [1–3, 17, 19, 20] have been written to show that Littlewood–Richardson (for short LR) sequences give a complete combinatorial characterization for the invariant factors of a product of two matrices over a principal ideal domain. More precisely, necessary and sufficient conditions for the existence of nonsingular matrices A, B and C, with prescribed invariant factors and such that $AB = C$ were given in terms of Littlewood–Richardson sequences.

Littlewood–Richardson sequences arose in the representation theory of the symmetric group $\mathcal{S}_n$ as a combinatorial tool for determining the coefficients in the Schur functions expansion of the product of two given Schur functions [10, 11, 12].

P. Hall, J. A. Green and T. Klein [4, 8] have shown that Littlewood–Richardson sequences also give a complete solution of the existence problem of finitely generated torsion modules $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ over a principal ideal domain, with prescribed invariant factors and such that $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{B} = \mathcal{C}/\mathcal{A}$.

In [17] it is shown that the matrix product and the module extension problems are equivalent. The analysis of these two problems is reducible to the local case (for the matrix product problem, see [1] or [17]), *i.e.*, to the case where a local principal domain with maximal ideal (p) is considered. The localization of these two equivalent problems means that we shall be working essentially with powers of a prime p. Thus, the invariant factors of a nonsingular matrix A over a local principal domain with maximal ideal (p) will be identified with the partition of integers defined by the exponents in decreasing order of the invariant factors of A which are powers of p. We call this partition the invariant partition of A.

Therefore, given partitions a, b and c there are nonsingular matrices A, B and C with invariant partitions a, b and c respectively, and such that $AB = C$ if and only if there is a Littlewood–Richardson sequence of type (a, b, c).

Littlewood–Richardson sequences give an implicit solution to the matrix product and extension module problems. Although, several explicit conditions are already known in terms of divisibility relations involving the invariant factors of A, B and C (see, for example, [13, 15, 17, 18]), the problem, as far as we know, is not completely solved.

Let $LR(a, b, c)$ be the set of Littlewood–Richardson sequences of type (a, b, c). It is now clear that the problem of characterizing the invariant factors of the product of two matrices (or of the module extension) is equivalent to the problem of characterizing the partitions a, b and c for which $LR(a, b, c) \neq \emptyset$. That is, given partitions a, b and c, we would like to know under which conditions an LR sequence (or an

LR tableau) of type (a, b, c) exists. (A good account for this, can be found in [20].)

Following the terminology of [20], given the partitions $a \subseteq c$, let $LR(a, c)$ be the set of partitions b for which there is at least one LR sequence of type (a, b, c). Our matrix problem has therefore the following combinatorial translation: given the partitions $a \subseteq c$, determine the partitions b in $LR(a, c)$. I. Zaballa [20] has given a necessary condition for this problem by showing that $LR(a, c) \subseteq [\mathbf{w}, \mathbf{n}]$, where $\mathbf{w}$ and $\mathbf{n}^*$ ($*$ means conjugate) are, respectively, the difference partitions associated with the skew-diagrams c/a and its conjugate, and $\mathbf{w}$ and $\mathbf{n}$ are the minimal and maximal elements of $LR(a, c)$ with respect to the majorization order of integer partitions. Hence, given partitions $a \subseteq c$ and b, there are matrices A, B and C, with invariant partitions a, b and c, respectively, such that $AB = C$ only if $\mathbf{w} \preceq b \preceq \mathbf{n}$ ($\preceq$ stands for majorization order). The converse is not in general true. We call $[\mathbf{w}, \mathbf{n}]$ the admissible interval for the invariant factors of a product of two matrices A and B, where a and c are the invariant partitions of A and AB respectively.

Our aim, in this work, is to give a method for determining $LR(a, c) \cap [\mathbf{w}, \mathbf{n}]$. Also explicit bijections between Littlewood–Richardson sequences of conjugate shape and weight and between Littlewood–Richardson sequences of dual shape and equal weight are presented by means of certain permutations of LR sequences. Purely combinatorial techniques are used.

The structure of the paper is the following. In Section 2 we present the basic combinatorial tools: dual and conjugate of a Young diagram and a skew-diagram; dual and conjugate of a tableau (see [2] for duality); and the role played by the majorization order in this context. To a skew-diagram c/a and its conjugate we can associate the difference partitions $\mathbf{w} = c - a$ and $\mathbf{n}^* = c^* - a^*$ [9], respectively, that is, the partitions defined, respectively, by the length of each row and each column of c/a by decreasing order. These two partitions are related by the order of majorization, *i.e.*, $\mathbf{w} \preceq \mathbf{n}$. I. Zaballa [20] proofs this relation using inequalities in the context of LR sequences. We deduce this property by using a purely combinatorial argument on the skew-diagram c/a where the relation between the partitions $\mathbf{w}$ and $\mathbf{n}$ becomes apparent.

In Section 3 we exhibit explicit bijections between LR tableaux of conjugate shape and weight and between LR tableaux of dual shape and equal weight. Our construction is based on transpositions of consecutive integers of an LR tableau [3]. Although it is known from the representation theory of the symmetric group $\mathcal{S}_n$ that there exists a bijection between $LR(a,b,c)$ and $LR(a^*,b^*,c^*)$ (see [10, p. 110], [12]) our aim is to present explicitly such a bijection. There are several equivalent ways of defining an LR sequence (see, for example, [2, 3] and [19]). The terminology in [2] is slightly different from that in [19]. Namely, in [2] opposite LR sequences are called in [19] increasing LR sequences. I. Zaballa [19, Theorem 3.1] has shown a connection between decreasing (the usual LR sequences), increasing LR sequences (or opposite LR sequences) and increasing LR sequences of conjugate shape. More precisely, he constructed a bijection between decreasing LR sequences of type (a,b,c) and increasing LR sequences of type (a^*,b^*,c^*). Following his ideas [19], to exhibit an explicit bijective mapping between LR sequences of conjugate shape and weight we need only to transform LR sequences of type (a,b,c) into opposite LR sequences (or, in the terminology of [19], increasing LR sequences) of the same type and show that this transformation is a bijection.

Finally, in Section 4, given $a \subseteq c$ and following the terminology of [20], we identify the partitions b of the set $LR(a,c)$. As mentioned above, I. Zaballa has shown, in the context of LR sequences [20], that $LR(a,c) \subseteq [\mathbf{w},\mathbf{n}]$, where $\mathbf{w}$ is the minimal element and $\mathbf{n}$ the maximal element with respect to the majorization order. Furthermore, he has exhibited the corresponding minimal and maximal LR sequences using inequalities involving the partitions a and c.

On the basis of Sections 2 and 3, it is worth to mention a different approach to Zaballa's result. In [9] a complete characterization of Young *id*-tableaux in terms of the difference partition is given. (*id* denotes the identity permutation. For the definition of ε-tableau, where ε is a permutation of apropriate order, see [2] and Section 2.C.) That is, there exists an *id*-tableau of type (a,b,c) if and only if $b \in [\mathbf{w},\hat{\mathbf{1}}]$, where $\hat{\mathbf{1}}$ is the top element of the lattice of integer partitions of $|\mathbf{w}|$ ordered by majorization ($|\ |$ means weight). In fact, there is only one *id*-tableau of type $(a,\mathbf{w},c)$ which is an LR tableau. Since the majorization order is self dual under the map which sends each partition to its conjugate, we have the equivalent reasoning: there exists an *id*-tableau of type (a^*,b^*,c^*) if and only if $b \in [\hat{\mathbf{0}},\mathbf{n}]$ (recall that $\mathbf{n}^* = c^* - a^*$), where $\hat{\mathbf{0}}$ means the bottom element of the lattice considered above. By symmetry, there is only one *id*-tableau of type $(a^*, \mathbf{n}^*, c^*)$ which is an *LR* tableau. Thus, since $\mathbf{w} \preceq \mathbf{n}$, given an *id*-tableau of type (a, b, c) there is an *id*-tableau of type (a^*, b^*, c^*) if and only if $b \in [\mathbf{w}, \mathbf{n}]$. According to Section 3, since there is only one *id*-tableau of type $(a^*, \mathbf{n}^*, c^*)$, which is precisely an *LR* tableau, there is also only one *LR* tableau of type $(a, \mathbf{n}, c)$. Therefore, since for each *LR* tableau of shape c/a there is always an *LR* tableau of conjugate shape and weight, it follows that $LR(a, c) \subseteq [\mathbf{w}, \mathbf{n}]$. (We remark that taking into account [2, Algorithm 2.17 and Theorem 2.23] the characterization given in [9] of Young *id*-tableaux can be extended to ε-tableaux).

2. COMBINATORICS OF DIAGRAMS, SKEW-DIAGRAMS AND TABLEAUX

A. Partitions and Diagrams

A partition is a (finite or infinite) sequence of non negative integers $a = (a_1, a_2, \ldots, a_n, \ldots)$ by decreasing order, almost all zero. The number $|a| = a_1 + a_2 + \cdots + a_n + \cdots$ is called the *weight* of a; the maximum value of i for which $a_i > 0$ is called the *length* of a and is denoted by $l(a)$. If $a_i = 0$, for $i > n$, we shall write $a = (a_1, a_2, \ldots, a_n)$. For example, we regard $(3, 2, 2, 1)$, $(3, 2, 2, 1, 0, 0)$ and $(3, 2, 2, 1, 0, \ldots)$ as the same partition.

Sometimes we find it useful to use the notation

$$a = (x_1^{m_1}, \ldots, x_k^{m_k}),$$

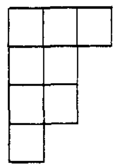
where $x_1 > x_2 > \cdots > x_k > 0$ and $x_i^{m_i}$, with $m_i \geq 0$, means that the integer x_i appears m_i times as a part of a.

Let $a = (a_1, a_2, \ldots, a_n)$ be a partition of length r. The *Young diagram* of a may be defined as the set of points $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that $1 \leq j \leq a_i$ and $1 \leq i \leq r$. We draw these diagrams with the first coordinate i (row index) strictly increasing from top to bottom and the second coordinate j (the column index) strictly increasing from left to right. For example, the Young diagram of the partition

$(3, 2, 2, 1)$ is

$$\begin{array}{lll} \bullet(1,1) & \bullet(1,2) & \bullet(1,3) \\ \bullet(2,1) & \bullet(2,2) & \\ \bullet(3,1) & \bullet(3,2) & \\ \bullet(4,1) & & \end{array}$$

We often replace the dots by boxes, in which case the diagram is

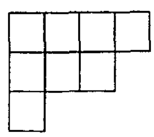


(1)

Thus the boxes of the Young diagram may be referred by their coordinates. We shall identify a partition with its diagram.

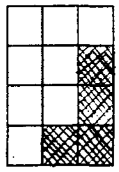
Given two partitions a and b, we write $a \leq b$ or $a \subseteq b$ to mean $a_i \leq b_i$, for all i. Geometrically, this means that the Young diagram of a is contained in the Young diagram of b.

The *conjugate* of a partition a is the partition $a^* := (a_1^*, a_2^*, \ldots, a_s^*)$ given by $a_k^* := \#\{i : a_i \geq k\}$ ($\#$ means cardinality), for $k = 1, \ldots, s$, where $s = a_1$. Geometrically, a^* is the partition whose diagram is the transpose of the a diagram. For example, the conjugate of $(3, 2, 2, 1)$ is $(4, 3, 1)$, whose diagram (2) is the transpose of (1)



(2)

The *M-dual* of a partition $a = (a_1, a_2, \ldots, a_n)$, with length r and $M \geq a_1$, is the partition $a_M^- := (M - a_r, M - a_{r-1}, \ldots, M - a_2, M - a_1)$. Geometrically, a_M^- is the partition whose diagram is the complementary of the diagram of a in (M^r), reading the rows from bottom to top and the columns from right to left (turn the sheet upside down). When $M = a_1$ we just write a^- and call it the *dual partition* of a. (A discussion of the properties of a_M^- can be found in [2].) For example, the dual of $(3, 2, 2, 1)$ is $(2, 1, 1)$, whose diagram is the shaded region in the picture below



(3)

turning the sheet upside down.

Observe that the two operations, conjugate and dual, are permutable, *i.e.*, $(a^-)^* = (a^*)^-$. Geometrically, $(a^-)^*$ is defined by the boxes of the complementary of a in (a_1^r) reading the rows from right to left and the columns from bottom to top (rotate the sheet $\pi/2$ anti-clockwise and reflect it vertically). Or, equivalently, $(a^*)^-$ is defined by the boxes of the complementary of a^* in $(a_1^{*a_1})$ reading the rows from bottom to top and the columns from right to left. For example, $((3,2,2,1)^*)^-$ is $(3,1)$ whose diagram is the shaded region in (3) rotating the sheet $\pi/2$ anti-clockwise and reflecting it vertically.

Given partitions a and b, we define $a \cup b$ to be the partition whose parts are those of a and b, arranged in decreasing order; and $a + b$ to be the partition which is the sum of the sequences a and b.

B. The Lattice of Partitions

Let $\mathcal{P}_k$ be the set of all partitions with weight k. Let a and b be partitions. We say that *a majorizes b*, written $a \succeq b$, if $a_1 + \cdots + a_r \geq b_1 + \cdots + b_r$, for $r = 1, \ldots, k$ [5, 9, 14].

$(\mathcal{P}_k, \succeq)$ is a lattice with maximal element (k) and minimal element (1^k) and is self dual under the map which sends each partition a to its conjugate a^*. Henceforth we shall denote the top element by $\hat{\mathbf{1}}$ and the bottom element by $\hat{\mathbf{0}}$. Note that $\succeq$ is linear if and only iff $k \leq 5$. For a discussion of these properties see, for instance [5] and [14].

Geometrically, $a \succeq b$ if and only if the diagram of b is obtained from a by "lowering" at least one box in the Young diagram of a. This means that *a dominates b* by *rows*. This is equivalent to say that b^* *dominates* a^* by *columns* or that b^* is obtained from a^* by "lifting" at least one box in the Young diagram of a^*. For example, $(3,2,2,1) \succeq (3,2,1,1,1)$ and $(3,2,1,1,1)^* \succeq (3,2,2,1)^*$.

Let $a \succ b$. Then *a covers b* if and only if a is obtained from b by "lifting" exactly one box in the diagram of b to the next available position such that the transfer must either be from some b_{i+1} to b_i or from b^*_{i-1} to b^*_i (see [5] and [14]).

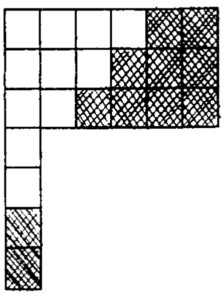
Given $a \preceq b$, we denote by $[a, b]$ the interval defined by all partitions $a \preceq x \preceq b$. We say that $a = a^0 \prec a^1 \prec a^2 \prec \cdots \prec a^r = b$ is a *connected chain* from a to b if a^i covers a^{i-1}, for $i = 1, \ldots, r$.

C. Skew-diagrams, Tableaux and Dual Tableaux

Let a and c be partitions such that $a \subseteq c$. We define

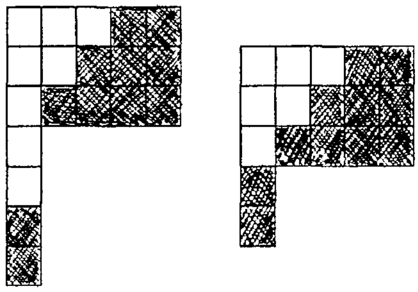
$$c/a := \{(i, j) \in c \,:\, (i, j) \notin a\},$$

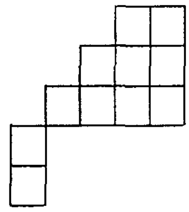
called a *skew-diagram modulo a*. The number $|c/a| := |c| - |a|$ is called the *weight* of c/a, and $l(c/a) := \#\{i : c_i - a_i \neq 0\}$ the number of rows of c/a. For example, if $a = (4, 3, 2, 1, 1)$ and $c = (6^3, 1^4)$, the skew-diagram c/a is the shaded region in the picture below



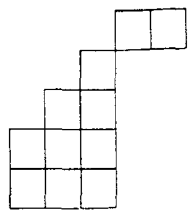
(4)

Let $\{j_1, \ldots, j_k\} = \{j : a^*_j = c^*_j\}$ and $\{i_1, \ldots, i_r\} = \{i : c_i = a_i\}$ Let c' and a' be the diagrams obtained from c and a deleting respectively the rows $i_1, \ldots, i_r$ and the columns $j_1, \ldots, j_k$. We do not distinguish the skew-diagrams c/a and c'/a'. For example, the shaded region of (4) and the following shaded regions are regarded as the same skew-diagram

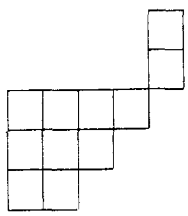


$c/a =$  (5)

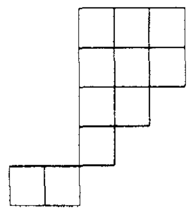
The *conjugate* of a skew-diagram c/a is the skew-diagram $(c/a)^* := c^*/a^*$. $(c/a)^*$ is obtained from c/a by transposition. For example, the conjugate of the skew-diagram (5) is

$(c/a)^* =$  (6)

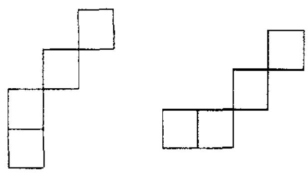
The *dual* of a skew-diagram c/a is the skew-diagram $(c/a)^- := ((M^k) \cup a^-_M)/c^-_M = ((c_1^k) \cup a^-_{c_1})/c^-$, where $M \geq c_1$ and $k = l(c) - l(a)$. $(c/a)^-$ is obtained from c/a by a vertical and a horizontal reflection. Or just read the rows of c/a from bottom to top and the columns from right to left (turn the sheet upside down). For example, the dual of (5) is

$(c/a)^- =$  (7)

As before $((c/a)^-)^* = ((c/a)^*)^-$. $((c/a)^*)^-$ is obtained from c/a reading the rows along the columns of c/a from right to left, and the columns along the rows of c/a from bottom to top (rotate the sheet $\pi/2$ anti-clockwise and reflect it vertically); or, from $(c/a)^*$ turning the sheet upside down. For example, the dual of the conjugate of (5) is the dual of (6) and the conjugate of (7).

$((c/a)^*)^- =$  (8)

A skew-diagram is called a vertical [horizontal] *m-strip*, where $m > 0$, if it has m boxes and at most one box in each row (column). For example,

 (9)

are vertical and horizontal 4-*strips*, respectively.

Clearly, the conjugate of a vertical [horizontal] strip is a horizontal [vertical] strip; and the dual of a vertical [horizontal] is also a vertical [horizontal] strip. In (9), one is the dual of the conjugate of the other.

Two strips of a skew-diagram are disjoint if they have no boxes in common.

Given c/a, we define a sequence of vertical strips and a sequence of horizontal strips, called the **V**-*sequence* and the **H**-*sequence* of c/a, respectively. We say that $\mathbf{V} = (V_1, \ldots, V_f)$ is the **V**-sequence of c/a if f is the length of the longest row of c/a and V_i is the vertical strip defined by the i-th box of each row of c/a, counting from left to right, for $i = 1, \ldots, f$; and $\mathbf{H} = (H_1, \ldots, H_l)$ is the **H**-sequence of c/a if l is the length of the longest column of c/a and H_i is the horizontal strip defined by the i-th box of each column of c/a, counting from bottom to top.

Let $(\mathbf{H}^*)^- := ((H_1^*)^-, \ldots, (H_l^*)^-)$. We remark that $(\mathbf{H}^*)^-$ is the **V**-sequence of $((c/a)^*)^-$.

Let $\mathcal{S}_t$ be the symmetric group of degree t, and *id* the identity permutation. As usual $(u_1\, u_2, \ldots, u_t)$ denotes a cycle in $\mathcal{S}_t$.

A *Young tableau* $\mathcal{T}$ [16] of type (a, b, c) with $b^* = (m_1, \ldots, m_t)$ is a skew-diagram c/a with a *labelling* $\tau = \{\tau_{ij} : (i,j) \in c \text{ and } (i,j) \notin a\}$ of the boxes of c/a with positive integers $1, \ldots, t$, where the labels τ_{ij} of the boxes (i,j) are as follows:

(a) For all (i, j), $(i+1, j)$ and $(i, j+1)$ in c/a

$$\tau_{ij} < \tau_{ij+1} \text{ and } \tau_{ij} \leq \tau_{i+1j}. \tag{10}$$

Labels are strictly increasing along rows from left to right and increasing along columns from top to bottom.

(b) For each $k \in \{1, \ldots, t\}$, $\#\{(i,j) : \tau_{ij} = k\} = m_{\varepsilon(k)}$, for some permutation $\varepsilon \in \mathcal{S}_t$.

The skew-diagram c/a is called the *shape* of the tableau $\mathcal{T}$ and b the *weight* of $\mathcal{T}$. Henceforth we regard a tableau as a skew-diagram with a labelling τ satisfying conditions (a) and (b).

If $\mathcal{T}$ is a tableau of type (a, b, c) and we want to stress that the multiplicity of the labels k is $m_{\varepsilon(k)}$, with $\varepsilon \in \mathcal{S}_t$, *i.e.*, the labels multiplicity order is given by $\varepsilon \in \mathcal{S}_t$, we say that $\mathcal{T}$ is an ε-*tableau*. The number of ε-tableaux and *id*-tableaux of type (a, b, c) is the same, see [2, Theorem 2.23].

A Young tableau $\mathcal{T}$ of type (a, b, c) may also be regarded as the sequence of partitions

$$(a^0, a^1, \ldots, a^t)$$

such that $a = a^0 \subseteq a^1 \subseteq \cdots \subseteq a^t = c$ and each skew-diagram a^k/a^{k-1} is a vertical strip labelled by $k, 1 \leq k \leq t = b_1$, where b^* is $(|a^1/a^0|, \ldots, |a^t/a^{t-1}|)$ by decreasing order. In this work, as mentioned, we shall think of a tableau as a labelled skew-diagram.

The *indexing sets* $J_1, \ldots, J_t$ of $\mathcal{T}$ [1, 2] are the subsets of $\{1, \ldots, n\}$ such that, for $k = 1, \ldots, t, J_k$ is defined by the row indices of the boxes of c/a labelled by k. Clearly, b^* is $(\#J_1, \#J_2, \ldots, \#J_t)$ by decreasing order. We identify the indexing sets $J_1, \ldots, J_t$ of $\mathcal{T}$ with the vertical strips a^k/a^{k-1} of c/a, $1 \le k \le t$.

Let us denote by *op* the reverse permutation of $\mathcal{S}_t$. The *dual* of $\mathcal{T} = (a^0, a^1, \ldots, a^t)$ [2] is the tableau $\mathcal{T}^-$ of shape $(c/a)^-$ with indexing sets defined by the vertical strips $(a^{op(i)}/a^{op(i+1)})^-$ labelled by i, for $i = 1, \ldots, t$ (we convention $op(t+1) = 0$). Equivalently, if $J_1, \ldots, J_t$ are the indexing sets of $\mathcal{T}$ then its dual is the tableau of shape $(c/a)^-$ with indexing sets $J_t^-, \ldots, J_1^-$. So $\mathcal{T}^-$ is a tableau of type $(c_M^-, b, (M^k) \cup a_M^-)$ with $M \ge c_1$ and $k = l(c) - l(a)$ (we may consider $M = c_1$). Clearly, there is a bijection between the tableaux of shape c/a and weight b and their duals which send $\mathcal{T}$ to $\mathcal{T}^-$. We call this bijection the canonical one.

Example 1

$$\begin{array}{cccc} & & 2 & 4 \\ & 1 & 3 & 5 \\ 2 & 3 & & \\ 2 & & & \end{array} \qquad \begin{array}{cccc} & & 1 & 4 \\ & 1 & 2 & 5 \\ 2 & 3 & & \\ 2 & & & \end{array} \qquad \begin{array}{cccc} & & 1 & 4 \\ & 1 & 3 & 5 \\ 1 & 2 & & \\ 2 & & & \end{array}$$

are ε-tableaux of type $((2,1,0,0), (3,2,1^3)^*; (4,4,2,1))$, with $\varepsilon = (23)(12)$, (12) and $id \in \mathcal{S}_5$ respectively, and their duals are

$$\begin{array}{cccc} & & 4 & 2 \\ & 5 & 3 & 1 \\ 4 & 3 & & \\ 4 & & & \end{array} \qquad \begin{array}{cccc} & & 5 & 2 \\ & 5 & 4 & 1 \\ 4 & 3 & & \\ 4 & & & \end{array} \qquad \begin{array}{cccc} & & 5 & 2 \\ & 5 & 3 & 1 \\ 5 & 4 & & \\ 4 & & & \end{array}$$

turning the sheet upside-down.

D. Difference Partitions

Given a skew-diagram c/a we define the partition $c - a$ as $(c_1 - a_1, \ldots, c_n - a_n)$ by decreasing order, called the *difference* partition of c/a. (In [9], T. Y. Lam calls difference partition the conjugate of $c - a$). Equivalently, $c - a$ is defined by the length of each row of c/a by decreasing order. We shall write $(c-a)_i$ for the i-th component of $c - a$.

Geometrically, $(c-a)^* = (v_1, \ldots, v_f)$, with f the length of the longest row of c/a, is defined inductively as follows: for $i = 1, \ldots, f, v_i$ is the number of rows of c/a after deleting the first $i-1$ boxes of each row of c/a. Or, equivalently, $(c-a)^* = (|V_1|, \ldots, |V_f|)$.

The difference partition of $(c/a)^*$ is $c^* - a^*$, being defined by the length of each column of c/a by decreasing order. Geometrically, $(c^* - a^*)^* = (h_1, \ldots, h_l)$, with l the length of the longest column of c/a, is defined inductively as follows: for $= 1, \ldots, l$, h_i is the number of columns of c/a after deleting the last $i-1$ boxes of each column of c/a. Or, equivalently, $(c^* - a^*)^* = (|H_1|, \ldots, |H_l|)$.

Note that the difference partition of c/a and its dual are the same, so $(c^* - a^*)^*$ is the conjugate of the difference partition of $((c/a)^*)^-$.

For example, the difference partitions of (5) and (6) and their conjugate are respectively $c-a = (4,3,2,1,1)$, $(c-a)^* = (5,3,2,1)$ and $c^* - a^* = (3,3,2,2,1)$, $(c^* - a^*)^* = (5,4,2)$.

THEOREM 1 *There is one and only one id-tableau of type* $(a, c-a, c)$ *and of type* $(c_M^{*-}, c^* - a^*, (M^k) \cup a_M^{*-})$, *respectively. There is at least one id-tableau of type* $(a, (c^* - a^*)^*, c)$.

Proof Consider the **V**-sequence of c/a and label each box of the vertical strip V_i by i. Clearly, $c - a = (|V_1|, \ldots, |V_f|)^*$ and c/a with this labelling is an *id*-tableau and it is the only one of shape c/a and weight $c-a$. Consider the **H**-sequence of c/a and label the boxes of the horizontal strip H_i, from bottom to top and from left to right by $1, \ldots, |H_i|$. Clearly, $(c^* - a^*)^* = (|H_1|, \ldots, |H_l|)$ and c/a with this labelling is an *id*-tableau of type $(a, (c^* - a^*)^*, c)$. To prove that there is only one *id*-tableau of type $(c_M^{*-}, c^* - a^*, (M^k) \cup a_M^{*-})$ use duality and the fact that there is only one *id*-tableau of type $(a^*, c^* - a^*, c^*)$ or recall that $(\mathbf{H}^*)^-$ is the **V**-sequence of $((c/a)^*)^-$ whose difference partition is $c^* - a^* = (|H_1|, \ldots, |H_l|)^*$ (note that $|(H_i^*)^-| = |H_i|$). ■

Remark 1 According to this theorem, there is only one *id*-tableau of type $(a^*, c^* - a^*, c^*)$ but, in general, there are more than one *id*-tableau of type $(a, (c^* - a^*)^*, c)$. For example, if $a = (1,1,0)$ and $c = (3,3,2)$, the sequences $((1,1,0), (2,2,1), (3,2,2), (3,3,2))$ and $((1,1,0), (2,2,1), (3,3,1), (3,3,2))$ are both *id*-tableaux of type $((1,1,0), (3,2,1), (3,3,2))$ where $(c^* - a^*)^* = (3,2,1)$.

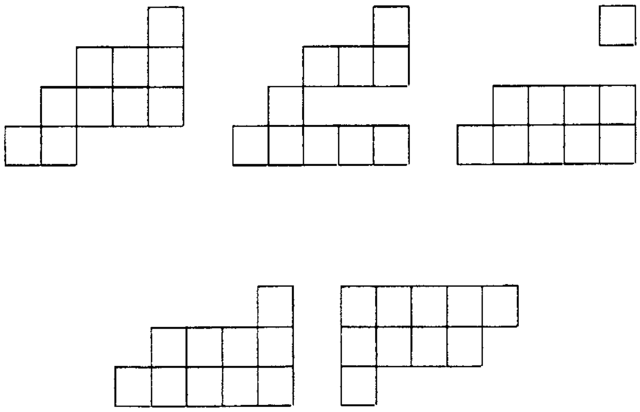
The Young diagram of $(c^*-a^*)^* = (h_1,\dots,h_l)$ may also be obtained from c/a by the following procedure:

Let $c^0 := c$ and $a^0 := a$.

For $i=1,\dots,l-1$, push up [down] the first [last] box of each column of c^{i-1}/a^{i-1}. After this operation we obtain a row with h_i boxes and a skew-diagram c^i/a^i, such that $(c^{i*}-a^{i*})^* = (h_{i+1},\dots,h_l)$.

Pushing to the left all the rows [pushing to the right and turning the sheet upside down] we obtain finally the Young diagram of $(c^*-a^*)^* = (h_1,\dots,h_l)$.

For example,



From this geometrical procedure we get

THEOREM 2 *Let $a \subseteq c$. Then $c-a \preceq (c^*-a^*)^*$.*

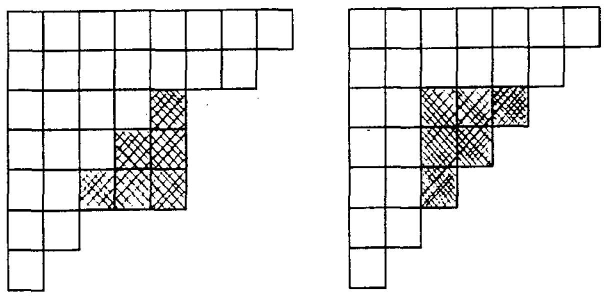
Proof Recall that $c-a$ is the length of each row of c/a by decreasing order. On the other hand, for $i=1,\dots,l, h_i$ is the length of the horizontal strip defined by the last boxes of each column of c^{i-1}/a^{i-1}. Fix $i\in\{1,\dots,l\}$, and let $w^i := (h_1,\dots,h_i)\cup c^i-a^i$. According to the procedure above, we have transformed the sequence of integers $(c_1^{i-1}-a_1^{i-1},\dots,c_n^{i-1}-a_n^{i-1})$ into $x=(c_1^{i-1}-a_1^{i-1}-\varepsilon_1,\dots,c_r^{i-1}-a_r^{i-1}+\Sigma_{j\neq r}\varepsilon_j,\dots,c_n^{i-1}-a_n^{i-1}-\varepsilon_n)$ where $\varepsilon_j\geq 0$ and $c_r^{i-1}-a_r^{i-1}+\Sigma_{j\neq r}\varepsilon_j=h_i$, for some r. Clearly, $h_i=(c^{i-1}-a^{i-1})_1+\delta$, for some $\delta\geq 0$. So x by decreasing order is $((c^{i-1}-a^{i-1})_1+\delta,(c^{i-1}-a^{i-1})_2-\delta_2,\dots,(c^{i-1}-a^{i-1})_n-\delta_n)\succeq c^{i-1}-a^{i-1}$, where $\Sigma\delta_j=\delta$. On the other hand, x by decreasing order is $(h_i)\cup c^i-a^i$. Therefore, $c^{i-1}-a^{i-1}\preceq$

$(h_i) \cup c^i - a^i$ and $(h_1, \ldots, h_{i-1}) \cup c^{i-1} - a^{i-1} \preceq (h_1, \ldots, h_{i-1}, h_i) \cup c^i - a^i$. That is, $w^{i-1} \preceq w^i$. Hence, $c - a = w^0 \preceq w^1 \preceq w^2 \preceq \cdots \preceq w^l = (c^* - a^*)^*$. ■

THEOREM 3 *$c - a = (c^* - a^*)^*$ if and only if c/a or $(c/a)^-$ is a Young diagram.*

Proof Straightforward. ■

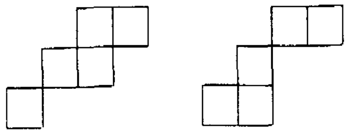
The geometric conditions given for the shape of c/a by this theorem are equivalent to those given in [20, Corollary 3.9]. Graphically, they mean that a tableau of shape c/a whose partitions $c - a$ and $(c^* - a^*)^*$ coincide should have one of the following forms, where the shaded region represents the shape



Remark 2 Given u and $v \in \mathcal{P}_n$ such that $u \preceq v$, there are not always partitions $a \subseteq c$ satisfying $u = c - a$ and $v = (c^* - a^*)^*$. Let A_u be the set of partitions $v \in \mathcal{P}_n$ for which a skew-diagram c/a with $c - a = u$ and $(c^* - a^*)^* = v$ exists. Clearly, $A_u \subseteq [u, \hat{\mathbf{1}}]$. The following examples show that A_u is not always the whole interval $[u, \hat{\mathbf{1}}]$.

1. If $u = \hat{\mathbf{1}}$, $A_u = \{\hat{\mathbf{1}}\}$ and if $u = \hat{\mathbf{0}}$, $A_u = [\hat{\mathbf{0}}, \hat{\mathbf{1}}]$.
2. If $u = (2^3)$, $A_u = \{(2,2,2), (3,2,1), (4,2), (5,1), (6)\}$. The partitions $(4,1,1)$, $(3^2) \notin A_u$.
   In general, if $u = (m^k)$, with $k > 2$ and $m \geq 2$, $((k-1)(m-1) + m, 1^{k-1}) \notin A_u$.
3. If $u \in \mathcal{P}_n$ is such that $l(u) = 2$, then $A_u = [u, \hat{\mathbf{1}}]$.

Remark 3 Given the skew-diagram c/a, the difference partitions $c - a$ and $c^* - a^*$ do not, in general, characterize completely c/a. For example,



have the same difference partitions $c - a = (2, 2, 1)$ and $c^* - a^* = (2, 1, 1, 1)$.

3. *LR* TABLEAUX AND *LR* TABLEAUX OF DUAL AND CONJUGATE SHAPE

In [9, Theorem 3.1] a complete characterization of Young *id*-tableaux with prescribed shape, in terms of the difference partition is given. That is, there is an *id*-tableau of type (a, b, c) if and only if $b \in [c - a, \hat{\mathbf{1}}]$, where $\hat{\mathbf{1}}$ is the top element of the lattice of partitions of $|c - a|$ ordered by majorization. (In fact, this characterization can be extended to an ε-tableau, see [2, Algorithm 2.17 and Theorem 2.23].) Hence, from Theorem 2, given a skew-diagram c/a, the interval $[c - a, (c^* - a^*)^*]$ provides a complete characterization of the *id*-tableaux of type (a, b, c) for which at least one *id*-tableau of type (a^*, b^*, c^*) exists. In general, given $b \in [c - a, (c^* - a^*)^*]$, the number of *id*-tableaux of type (a, b, c) is not equal to the number of *id*-tableaux of type (a^*, b^*, c^*) (see Theorem 1 and Remark 1). On the other hand, we shall see that the number of *LR* tableaux of type (a, b, c) is the number of *LR* tableaux of type (a^*, b^*, c^*). Also the number of *LR* tableaux of type (a, b, c) is the number of *LR* tableaux of dual shape and weight b.

In what follows we shall be concerned mainly with certain special *id*-tableaux and *op*-tableaux, although in Subsection B also some special ε-tableaux will take place. These cases will be clear from the context.

A. *LR* and LR_{op} Tableaux

In this subsection we begin by introducing the concepts of *LR*, dual *LR* and opposite *LR* tableaux. The dual *LR* tableau is the geometric translation of the opposite *LR* tableau: the dual of an *LR* tableau is an opposite LR tableau of dual shape. In fact, there is a bijection between the LR tableaux with prescribed type and their duals [2, Theorem 2.15]. Using these notions, in Corollary 3 we restate Theorem 3.1 of I. Zaballa [19] which says that there is a bijection between LR tableaux of type (a, b, c) and the opposite LR tableaux (increasing LR tableaux in the terminology of [19]) of type (a^*, b^*, c^*).

DEFINITION 1 Let $J = \{x_1, \ldots, x_s\}$ and $K = \{y_1, \ldots, y_m\}$ be finite sets of integers, where we are assuming that $x_1 > \cdots > x_s$ and $y_l > \cdots > y_m$. Then we write $J \geq K$ (or $K \leq J$) whenever $s \geq m$ and $x_i \geq y_i$, for $i = 1, \ldots, m$.

Observe that $J \geq K$ if and only if there is $A \subseteq J$ such that $\#A = \#K$ and $A \geq K$.

DEFINITION 2 Let J and K be the finite sets of integers defined above, where we are assuming that $x_1 < \cdots < x_s$ and $y_1 < \cdots < y_m$. We write $J \geq_{op} K$ (or $K \leq_{op} J$) whenever $s \leq m$ and $x_i \geq y_i$, for $i = 1, \ldots, s$.

This is equivalent to saying that there is $A \subseteq K$, such that $\#A = \#J$ and $J \geq A$.

DEFINITION 3 [1] Let $\mathcal{T}$ be a tableau of type (a, b, c) with indexing sets $J_1, \ldots, J_t$. We say that $\mathcal{T}$ is a Littlewood–Richardson (LR for short) tableau or a Littlewood–Richardson sequence if

$$J_1 \geq \cdots \geq J_t. \tag{11}$$

This definition is an equivalent formulation of what is usually called the Littlewood–Richardson sequence [8, 10, 11].

DEFINITION 4 [2] Let $\mathcal{T}$ be a tableau of type (a, b, c) with indexing sets $J_1, \ldots, J_t$. We say that $\mathcal{T}$ is an opposite Littlewood–Richardson (LR_{op} for short) tableau or an opposite Littlewood–Richardson sequence if

$$J_1 \geq_{op} \cdots \geq_{op} J_t. \tag{12}$$

THEOREM 4 *A tableau $\mathcal{T}$ is an LR_{op} tableau if and only if $\mathcal{T}^-$ is an LR tableau.*

Proof See [2, Theorem 2.15]. ■

That is, there is a bijection between LR tableaux of type (a, b, c) and LR_{op} tableaux of dual shape and weight b.

DEFINITION 5 Let $\mathcal{T}$ be a tableau of type (a, b, c), with labelling τ. Let $k \in \{1, \ldots, l(b)\}$ and $S = \{(x_1, j_1), \ldots, (x_{b_k}, j_{b_k})\}$ be an horizontal b_k-strip of c/a, where we are assuming $x_1 \geq \cdots \geq x_{b_k}$. We say that S is a b_k-string of $\mathcal{T}$ if $\tau_{x_r, j_r} = r$, for $r = 1, \ldots, b_k$.

If $J_1, \ldots, J_t$ are the indexing sets of $\mathcal{T}$ then $S = \{(x_1, j_1), \ldots, (x_{b_k}, j_{b_k})\}$ is an horizontal b_k-string of $\mathcal{T}$ if and only if $x_i \in J_i$, for $i = 1, \ldots, b_k$.

Two strings of $\mathcal{T}$ are said to be disjoint if their strips are disjoint. Under the conditions (10) of a tableau, it is clear that two horizontal strings $S = \{(x_1, j_1), \ldots, (x_{b_k}, j_{b_k})\}$ and $S' = \{(y_1, g_1), \ldots, (y_{b_s}, g_{b_s})\}$ are disjoint if and only if $x_i \neq y_i$, for $i = 1, \ldots, \min\{b_k, b_s\}$. So, without ambiguity, we may identify the strings S and S' with $\{x_1, \ldots, x_{b_k}\}$ and $\{y_1, \ldots, y_{b_s}\}$, *i.e.*, we identify the string boxes with the row indices.

We say that $S \geq S'$ if $b_k \geq b_s$ and $x_i \geq y_i$, for $i = 1, \ldots, b_k$. If S' and S are disjoint then $x_i > y_i$, for all i, and, in this case, $S > S'$.

Example 2

					2
			1	3	5
	1	2	3	6	
1	2	4			

There are no 6-horizontal strings and there are 2 and 3-horizontal strings. $S_2 = \{3, 3\}$ and $S_3 = \{4, 4, 3\}$ are disjoint strings but $S'_2 = \{4, 3\}$ and S_3 are not.

DEFINITION 6 Let $\mathcal{T}$ be a tableau of type (a, b, c) with $b = (b_1, \ldots, b_m), m = l(b)$. We say that $(S_{b_1}, \ldots, S_{b_m})$ is a complete sequence of horizontal strings (for short complete sequence of strings) of $\mathcal{T}$ if the strings are pairwise disjoint.

Clearly, $(\# S_{b_1}, \ldots, \# S_{b_m}) = (\# J_1, \ldots, \# J_t)^*$.

Remark 4 By Definition 5, a tableau with a complete sequence of strings is necessarily an *id*-tableau.

Example 3

1.

$$\mathcal{T} = \begin{array}{ccccc} \cdot & \cdot & \cdot & 1 & 4 \\ \cdot & \cdot & 2 & 3 & \\ 1 & 2 & 3 & & \\ 1 & 2 & & & \\ 1 & & & & \end{array}$$

The sequences of strings ($S_4 = \{5,4,3,1\}$, $S_3 = \{4,3,2\}$, $S_2 = \{3,2\}$, $S_1 = \{1\}$), ($S'_4 = \{3,3,3,1\}$, $S'_3 = \{4,4,2\}$, $S'_2 = \{5,2\}$, $S'_1 = \{1\}$) and ($S''_4 = \{4,3,2,1\}$, $S''_3 = \{5,4,3\}$, $S''_2 = \{3,2\}$, $S''_1 = \{1\}$) are complete.

2.

$$\mathcal{T} = \begin{array}{cccc} \cdot & \cdot & 1 & 2 \\ \cdot & \cdot & 2 & 3 \\ 1 & 2 & 4 & \\ 3 & & & \end{array}$$

has no complete sequence of strings.

We have characterized an LR tableau by means of vertical strips *i.e.*, the indexing sets $J_1, \ldots, J_t$ (see Definition 3). Next theorem describes an LR tableau in terms of horizontal strings.

THEOREM 5 *Let $\mathcal{T}$ be a tableau of type (a,b,c) where $b = (b_1, b_2, \ldots, b_m)$. $\mathcal{T}$ is an LR tableau if and only if $\mathcal{T}$ has a complete sequence of strings; and, in this case, $\mathcal{T}$ has a complete sequence of strings $(S_{b_1}, \ldots, S_{b_m})$ satisfying $S_{b_1} > S_{b_2} > \cdots > S_{b_m}$.*

Proof For the "*if*" part, suppose that $(S_{b_1}, \ldots, S_{b_m})$ is a complete sequence of strings of $\mathcal{T}$. Let $r \in \{1, \ldots, b_1\}$ and (x,y) be a box of c/a labelled by r. Then (x,y) belongs exactly to one string S_{b_k}, for some $k \in \{1, \ldots, m\}$ with $b_k \geq r$. By definition of a string (Definition 5), if $r > 1$, there exists always a box $(x', y') \in S_{b_k}$, marked with $r-1$, such that $x' \geq x$. This means that $J_{r-1} \geq J_r$. So, $\mathcal{T}$ is an LR tableau.

The "*only if*" part will be handle by induction on the number n of rows of c/a. We shall show that $\mathcal{T}$ has a complete sequence of strings $(S_{b_1}, \ldots, S_{b_m})$ satisfying $S_{b_1} > S_{b_2} > \cdots > S_{b_m}$. If $n = 1$, c/a has exactly one row and there is only the string H_1.

If $n = 2$, then $b = (b_1)$ or $b = (b_1, b_2)$. In case $b = (b_1)$, the two rows of c/a are necessarily disconnected, meaning that $c^* - a^* = (1^{b_1})$, and there is only one string H_1 formed by all the boxes of c/a. In case $b = (b_1, b_2)$, let S_{b_1} be the string defined by the lowest b_1 boxes in the horizontal strip H_1 of c/a labelled by $1, 2, \ldots, b_1$ and S_{b_2} be the string defined by the remaining boxes in the first row of c/a. Clearly, $S_{b_1} > S_{b_2}$.

Let $n > 2$. Consider the string S_{b_1} defined by the lowest b_1 boxes in the horizontal strip H_1 of c/a labelled by $1, 2, \ldots, b_1$. Since $\mathcal{T}$ is an LR tableau, we have the following property:

$$(x, j) \in S_{b_1} \text{ and } (x, j+1) \in H_1 \Rightarrow (x, j+1) \in S_{b_1}. \tag{13}$$

So we may consider the tableau $\mathcal{T}'$ obtained from $\mathcal{T}$ deleting the string S_{b_1}. $\mathcal{T}'$ is an LR tableau with weight $(b_2, \ldots, b_m)$ and $n-1$ rows. (Note that the last row of c/a is necessarily labelled by consecutive integers starting with 1.) By induction, $\mathcal{T}'$ has a sequence of strings $S_{b_2} > \cdots > S_{b_m}$. Since $S_{b_1} \subseteq H_1$ and is defined by the lowest boxes labelled by $1, \ldots, b_1$, it is clear that $S_{b_1} > S_{b_2} > \cdots > S_{b_m}$. ■

COROLLARY 1 *The id-tableau of type $(a, c-a, c)$ is an LR tableau. There is an LR tableau of type $(a, (c^* - a^*)^*, c)$.*

Proof The indexing sets of the *id*-tableau of type $(a, c-a, c)$ may be identified with $\mathbf{V} = (V_1, \ldots, V_f)$ which satisfy $V_1 \geq \cdots \geq V_f$. On the other hand, $\mathbf{H} = (H_1, \ldots, H_l)$ is a complete sequence of strings of the *id*-tableau of type $(a, (c^* - a^*)^*, c)$ constructed in Theorem 1. ■

We remark that $c - a = (|V_1|, \ldots, |V_f|)^* \preceq (|S_{b_1}|, \ldots, |S_{b_m}|) \preceq (|H_1|, \ldots, |H_l|) = (c^* - a^*)^*$.

The foregoing theorem says that a tableau $\mathcal{T}$ with weight $(b_1, \ldots, b_m)$ is an LR tableau if and only if it has a sequence of strings $S_{b_1} > S_{b_2} > \cdots > S_{b_m}$. This sequence is the *maximal sequence of strings of $\mathcal{T}$* with respect to the lexicographic order in the set of all complete sequences of strings of $\mathcal{T}$.

In Example 3, (S_4, S_3, S_2, S_1) is the maximal sequence of strings of $\mathcal{T}$. $\mathbf{H} = (H_1, \ldots, H_l)$ is the maximal sequence of strings of the tableau of type $(a, (c^* - a^*)^*, c)$ constructed in Theorem 1.

PROPERTIES 1 *The maximal sequence of strings* $(S_{b_1}, S_{b_2}, \ldots, S_{b_m})$ *of an LR tableau* $\mathcal{T}$ *satisfies the following properties*:

1. *If* $(x, j) \in S_{b_r}$ *and* $(x', j) \in S_{b_{r'}}$, *then*

$$x' > x \Leftrightarrow r' < r.$$

2. *If* $(x, j) \in S_{b_r}$ *and* $(x, j') \in S_{b_{r'}}$, *then*

$$j' > j \Leftrightarrow r' \leq r.$$

3. *If* $S_{b_r} = \{(x_r^1, j_r^1), \ldots, (x_r{}^{b_r}, j_r^{b_r})\}$ *and* $S_{b_{r+1}} = \{(x_{r+1}^1, j_{r+1}^1), \ldots, (x_{r+1}^{b_{r+1}}, j_{r+1}^{b_{r+1}})\}$, *then, for all* $k = 1, \ldots, b_{r+1}$,

$$x_r^k > x_{r+1}^k \quad \text{and} \quad j_{r+1}^k \geq j_r^k. \qquad \blacksquare$$

DEFINITION 7 Let $\mathcal{T}$ be an LR tableau with shape c/a and maximal sequence of strings $S_{b_1} > S_{b_2} > \cdots > S_{b_m}$. We define the labelling ρ in $((c/a)^*)^-$ such that $\rho_{x'y'} = r$ if $(x', y') \in (S_{b_r}^*)^-$, for all r, called the labelling induced by the maximal sequence of strings of $\mathcal{T}$.

Let $J_1, \ldots, J_t$ be the indexing sets of $\mathcal{T}$ and identify each J_i with the vertical strip of c/a labelled by i. Define the labelling ρ' in $((c/a)^*)^-$ such that, for $i = 1, \ldots, t$,

$$J_i = \{x_1^i > \cdots > x_{m_i}^i\} \Rightarrow \rho'_{x'y'} = r,$$

whenever $(x', y') = ((x_r^i)^*)^-$, $1 \leq r \leq m_i$. $(((x_r^i)^*)^-$ means the coordinates of the image of the box $x_r^i \in J_i$ by the map that sends the vertical strip J_i to $(J_i^*)^-)$.

Clearly, $\rho' = \rho$ (recall that $x_r^i \in J_i \Leftrightarrow x_r^i \in S_{b_r}$.) So the labellings induced by the indexing sets and the maximal sequence of strings of $\mathcal{T}$ are the same. For short, we call ρ the *labelling* of $((c/a)^*)^-$ *induced* by $\mathcal{T}$.

THEOREM 6 *Let $\mathcal{T}$ be an LR tableau of type (a, b, c), with maximal sequence of strings $(S_{b_1}, \ldots, S_{b_m})$ and indexing sets $J_1, \ldots, J_t$. Then $((c/a)^*)^-$ with the labelling induced by $\mathcal{T}$ is an LR-tableau of type $((c_M^*)^-, b^*, (M^k) \cup a_M^{*-})$ with indexing sets $(S_{b_1}^*)^-, \ldots, (S_{b_m}^*)^-$ and maximal sequence of strings $((J_1^*)^-, \ldots, (J_t^*)^-)$.*

Proof Let ρ be the induced labeling in $((c/a)^*)^-$. ■

Claim $((c/a)^*)^-$ with the labelling ρ is an LR tableau of type $((c_M^*)^-, b^*, (M^k) \cup a_M^{*-})$.

Using 2 of Properties 1 the labels of $((c/a)^*)^-$ are increasing along columns (read c/a along rows from right to left); and using 1 of Properties 1, the labels of $((c/a)^*)^-$ are strictly increasing along rows (read c/a along columns from bottom to top). Finally, from 3 of Properties 1, we conclude that $(S_{b_1}^*)^-, \ldots, (S_{b_t}^*)^-$ are the indexing sets of $((c/a)^*)^-$ and $((J_1^*)^-, \ldots, (J_t^*)^-)$ its maximal sequence of strings. ■

From this theorem it follows

COROLLARY 2 *There exist a bijection ν between LR tableaux of type (a, b, c) and LR tableaux of type $((c_M^*)^-, b^*, (M^k) \cup a_M^{*-})$ defined by $\nu(\mathcal{T})$ equals the tableau of shape $((c/a)^*)^-$ with the labelling induced by $\mathcal{T}$.* ■

Since there is a bijection between LR tableaux of type (a, b, c) and their duals which are LR_{op} tableaux, (see Theorem 4), it follows

COROLLARY 3 [19, Theorem 3.1] *There exist a bijection ϕ between LR tableaux of type (a, b, c) and LR_{op} tableaux of type (a^*, b^*, c^*) defined by $\phi(\mathcal{T}) = (\nu(\mathcal{T}))^-$.* ■

Next example illustrates Corollary 3.

Example 4 Let $a = (3, 1, 0, 0)$, $b = (4, 2, 2, 1)^*$ and $c = (5, 4, 3, 1)$. The LR tableaux of type (a, b, c) and the LR_{op} tableaux of type (a^*, b^*, c^*) are given by first and last labelled skew-diagrams, respectively:

1.

$$c/a = \begin{array}{ccccc} & & & 1 & 4 \\ & 1 & 2 & 3 & \\ 1 & 2 & 3 & & \\ 1 & & & & \end{array} \qquad ((c/a)^*)^- = \begin{array}{cccc} & & & 1 \\ & & 2 & 4 \\ & 1 & 2 & \\ & 1 & 3 & \\ 1 & 2 & & \end{array} \qquad (c/a)^* = \begin{array}{cccc} & & 3 & 4 \\ & 2 & 4 & \\ & 3 & 4 & \\ 1 & 3 & & \\ 4 & & & \end{array}$$

2.

$$
c/a = \begin{array}{ccccc} & & & 1 & 3 \\ & 1 & 2 & 4 & \\ 1 & 2 & 3 & & \\ 1 & & & & \end{array}
\qquad
((c/a)^*)^- = \begin{array}{cccc} & & & 2 \\ & & 1 & 4 \\ & 1 & 2 & \\ & 1 & 3 & \\ 1 & 2 & & \end{array}
\qquad
(c/a)^* = \begin{array}{cccc} & & 3 & 4 \\ & 2 & 4 & \\ & 3 & 4 & \\ 1 & 4 & & \\ 3 & & & \end{array}
$$

B. An Algorithm to Transform *LR* Tableaux into LR_{op} Tableaux and *vice-versa*

For establishing an explicit bijection between the LR tableaux of type (a, b, c) and the LR tableaux of type (a^*, b^*, c^*) or the LR tableaux of type $(c_M^-, b, (M^k) \cup a_M^-)$, it remains to exhibit a process of transforming an LR tableau of type (a, b, c) into an LR_{op} tableau of type (a, b, c) and show that the process is reversible.

Algorithm 1 *Let T be an LR tableau with shape c/a and indexing sets $J_1 \geq J_2$.*

1. *Define*

$$J'_1 := \min\{A \subseteq J_1 : \#A = \#J_2,\ A \geq J_2\}$$
$$J'_2 := J_2 \cup (J_1 \setminus J'_1).$$

2. *Define*

$$a'^1/a := J'_1,$$
$$a'^2/a'^1 := J'_2.$$

■

We observe that the minimum in step 1 is with respect to the order relation given in Definition 1. Also observe that $J_1 \cap J_2 \subseteq J'_1$.

Clearly, $(a, a'^1, a'^2 = c)$ is an LR_{op} tableau of shape c/a with indexing sets J'_1, J'_2.

Let **T** be the set of all tableaux of type (a, b, c) with $l(b^*) = t$, such that the vertical strips labelled by k and $k+1$, respectively, satisfy the LR rule, and **L** the set of all tableaux of type (a, b, c) with $l(b^*) = t$, such that the vertical strips labelled by k and $k+1$, respectively, satisfy the LR_{op} rule.

Let $\mathcal{T} = (a^0, \ldots, a^k, a^{k+1}, \ldots, a^t)$ be a tableau in $\mathbf{T}$ with indexing sets $J_1, \ldots, J_t$. Let $(a^{k-1}, a'^k, a'^{k+1})$ be the LR_{op} tableau generated by Algorithm 1 when applied to (a^{k-1}, a^k, a^{k+1}). Then $\mathcal{T}' = (a^0, \ldots, a^{k-1}, a'^k, a'^{k+1}, \ldots, a^t)$ is a tableau in $\mathbf{L}$ with indexing sets $J_1, \ldots, J'_k, J'_{k+1}, \ldots J_t$.

We denote by $\psi^{(k\ k+1)}$ the map that sends $\mathcal{T}$ to $\mathcal{T}'$. Clearly, $\psi^{(k\ k+1)}$ is a bijection between $\mathbf{T}$ and $\mathbf{L}$. For this, let $\mathbf{L}^-$ and $\mathbf{T}^-$ be the sets of the duals of the elements of $\mathbf{L}$ and $\mathbf{T}$ respectively. Denote by $\phi^{(t-k, t-k+1)}$ the bijection between $\mathbf{L}^-$ and $\mathbf{T}^-$ induced by Algorithm 1 when applied to the vertical strips labelled by $t-k$ and $t-k+1$, respectively. Let λ and μ be the canonical bijections, respectively, between $\mathbf{L}$ and $\mathbf{L}^-$, and $\mathbf{T}$ and $\mathbf{T}^-$. Now, consider the diagram

$$\begin{tikzcd}
\mathbf{L} \arrow[r] \arrow[d, "\lambda"] & \mathbf{T} \\
\mathbf{L}^- \arrow[r, "\phi^{(t-k\ t-k+1)}"] & \mathbf{T}^- \arrow[u, "\mu^{-1}"']
\end{tikzcd}$$

and define the map $\psi_{op}^{(k\,k+1)}$ between the sets $\mathbf{L}$ and $\mathbf{T}$ such that $\psi_{op}^{(k\,k+1)} := \mu^{-1}\,\phi^{(t-k\,t-k+1)}\,\lambda$.

Then

$$\psi_{op}^{(k\,k+1)}\,\psi^{(k\,k+1)} = \psi^{(k\,k+1)}\,\psi_{op}^{(k\,k+1)} = id.$$

In particular, when $t = 2$, $\psi^{(1\ 2)}$ is a bijection between the set of LR tableaux of type (a, b, c) and the set of LR_{op} tableaux of type (a, b, c), defined by Algorithm 1.

The following algorithm defines the map $\psi_{op}^{(1\,2)}$ explicitly.

Algorithm 2 *Let $\mathcal{T}$ be an LR_{op} tableau with shape c/a and indexing sets $J_1 \geq_{op} J_2$.*

1. *Define*

$$\begin{aligned}
\tilde{J}_2 &:= \max\{A \subseteq J_2 : \#A = \#J_1,\ J_1 \geq A\} \\
\tilde{J}_1 &:= J_1 \cup (J_2 \setminus \tilde{J}_2).
\end{aligned}$$

2. *Define*

$$\tilde{a}^1/a := \tilde{J}_1,$$
$$\tilde{a}^2/\tilde{a}^1 := \tilde{J}_2.$$

■

We observe that the maximum in step 1 is with respect to the order relation given in Definition 1. Also observe that $J_1 \cap J_2 \subseteq \tilde{J}_2$.

Next algorithm gives a bijection between LR tableaux of type (a, b, c) and LR_{op} tableaux of type (a, b, c).

Algorithm 3 *Let $\mathcal{T}$ be an LR tableau with indexing sets $J_1, \ldots, J_t$.*

1. *Let $\varepsilon_0 := id$ and $\mathcal{T}^{(0,\varepsilon_0)} := \mathcal{T}$. Do $k := 0$ and go to 2.*
2. *For $j = 1, \ldots, t-k-1$, define inductively*

$$\mathcal{T}^{(j,\varepsilon_k,\ldots,\varepsilon_1,\varepsilon_0)} := \psi^{(t-j\,t-j+1)}(\mathcal{T}^{(j-1,\varepsilon_k,\ldots,\varepsilon_1,\varepsilon_0)}).$$

   Go to 4.
3. *Do $k := k+1$ and go to 2.*
4. *Define the cycle $\varepsilon_{k+1} := (k+1 \ldots t-1\, t)$ of $\mathcal{S}_t$.*
   Define $\mathcal{T}^{(0,\varepsilon_{k+1},\ldots,\varepsilon_1,\varepsilon_0)} := \mathcal{T}^{(t-k-1,\varepsilon_k,\ldots,\varepsilon_1,\varepsilon_0)}$.
   If $k = t-2$, define

$$\mathcal{T}^{(0,op)} := \mathcal{T}^{(0,\varepsilon_{t-1},\ldots,\varepsilon_1,\varepsilon_0)}$$

   and stop. Otherwise, go to 3. ■

This algorithm is based on the decomposition of the reverse permutation of $\mathcal{S}_t$,

$$op = (t-1\,t)\,(t-2\,t-1\,t)\ldots(k\ldots t-1\,t)\ldots(2\ldots t-1\,t)\,(1\ldots t-1\,t)$$

where $(k \ldots t-1\,t) = (k\,k+1)\ldots(t-1\,t)$, for $k = 1, \ldots, t-1$.

LEMMA 1 *Let $F, G \subseteq \{1, \ldots, n\}$ such that $F \geq G$. Let $B \subseteq G$ and $F' = \min\{A \subseteq F : \# A = \# B, A \geq B\}$. Then*

$$F \setminus F' \geq G \setminus B.$$

Proof It is an easy exercise. ■

THEOREM 7 *For* $j = 1, \ldots, t-1$, $\mathcal{T}^{(j,\varepsilon_0)}$ *defined in step* 2 *of Algorithm* 3 *is a tableau with indexing sets* $L_1, L_2, \ldots, L_t$ *satisfying*

1. $L_1 \geq \cdots \geq L_{t-j-1} \geq L_{t-j} \geq_{op} L_{t-j+1} \geq \cdots \geq L_t$.
2. $L_i = J_i$, *for* $i = 1, \ldots, t-j-1$.
3. $L_{t-j-1} \setminus L'_{t-j-1} \geq L_{t-j+1} \setminus \tilde{L}_{t-j+1} \geq \cdots \geq L_t \setminus \tilde{L}_t$, *where*

$$L'_{t-j-1} = \min\{A \subseteq L_{t-j-1} \,:\, \# A = \# L_{t-j},\ A \geq L_{t-j}\}$$

and, for $i = t-j+1, \ldots, t$, *we define inductively*

$$\begin{aligned}\tilde{L}_{t-j} &:= L_{t-j}\\ \tilde{L}_i &:= \max\{A \subseteq L_i \,:\, \# A = \# \tilde{L}_{i-1},\ \tilde{L}_{i-1} \geq A\}.\end{aligned}$$

Proof The proof will be handle by induction on j. Let $j = 1$. Clearly, attending to the definition of the map $\psi^{(t-1\,t)}$,

$$\mathcal{T}^{(1,\varepsilon_0)} := \psi^{(t-1\,t)}(\mathcal{T})$$

is a tableau with indexing sets $L_1, \ldots, L_t$, where $L_i = J_i$ for $i = 1, \ldots, t-2$, and

$$\begin{aligned}L_{t-1} &:= \min\{A \subseteq J_{t-1} \,:\, \# A = \# J_t,\ A \geq J_t\},\\ L_t &:= J_t \cup (J_{t-1} \setminus L_{t-1}).\end{aligned}$$

So, $L_1 \geq \cdots \geq L_{t-2} \geq L_{t-1} \geq_{op} L_t$.

Since $J_{t-1} = L_{t-1} \cup (J_{t-1} \setminus L_{t-1})$ and $L_{t-2} \geq J_{t-1}$, it follows, from Lemma 1, that

$$L_{t-2} \setminus L'_{t-2} \geq J_{t-1} \setminus L_{t-1}. \tag{14}$$

Claim $L_{t-2} \setminus L'_{t-2} \geq L_t \setminus \tilde{L}_t$.

Since $\psi_{op}^{(t-1\,t)}(\mathcal{T}^{(1,\varepsilon_0)}) = \mathcal{T}$, it is clear that $\tilde{L}_t = J_t$. On the other hand, $J_{t-1} \setminus L_{t-1} = L_t \setminus J_t$. Hence, $J_{t-1} \setminus L_{t-1} = L_t \setminus J_t = L_t \setminus \tilde{L}_t$ and, from (14), it follows $L_{t-2} \setminus L'_{t-2} \geq L_t \setminus \tilde{L}_t$.

Let $j \geq 2$. By induction,

$$\mathcal{T}^{(j,\varepsilon_0)} = \psi^{(t-j\,t-j+1)}(\mathcal{T}^{(j-1,\varepsilon_0)}),$$

where $\mathcal{T}^{(j-1,\varepsilon_0)}$ is a tableau with indexing sets $F_1, \ldots, F_t$ satisfying

$$F_1 \geq \cdots \geq F_{t-j-1} \geq F_{t-j} \geq F_{t-j+1} \geq_{op} F_{t-j+2} \geq \cdots \geq F_t,$$
$$F_i = J_i, \text{ for } i = 1, \ldots, t-j,$$

$$F_{t-j} \setminus F'_{t-j} \geq F_{t-j+2} \setminus \tilde{F}_{t-j+2} \geq \cdots \geq F_t \setminus \tilde{F}_t. \tag{15}$$

with

$$F'_{t-j} = \min\{A \subseteq F_{t-j} \,:\, \#A = \#F_{t-j+1},\, A \geq F_{t-j+1}\} \tag{16}$$

and, for $i = t-j+2, \ldots, t$,

$$\begin{aligned} \tilde{F}_{t-j+1} \,:&= F_{t-j+1} \\ \tilde{F}_i &= \max\{A \subseteq F_i \,:\, \#A = \#\tilde{F}_{i-1},\, \tilde{F}_{i-1} \geq A\}. \end{aligned} \tag{17}$$

Using the definition of the map $\psi^{(t-j\,t-j+1)}$ and since the vertical strips $F_1, \ldots, F_{t-j+1}$ of $\mathcal{T}^{(j-1,\varepsilon_0)}$ form an LR tableau it follows from the case $j = 1$ that $\mathcal{T}^{(j,\varepsilon_0)}$ is a tableau with indexing sets $L_1, \ldots, L_t$ satisfying

$$L_1 \geq \cdots \geq L_{t-j-1} \geq L_{t-j} \geq_{op} L_{t-j+1}\,,$$
$$L_{t-j+2} \geq \cdots \geq L_t\,,$$

with $L_i = F_i = J_i$, for $i = 1, \ldots, t-j-1$, $L_i = F_i$, for $t-j+2, \ldots, t$, and $L_{t-j} = F'_{t-j}$,

$$L_{t-j+1} = F_{t-j+1} \cup (F_{t-j} \setminus F'_{t-j}), \tag{18}$$

and

$$L_{t-j-1} \setminus L'_{t-j-1} \geq L_{t-j+1} \setminus \tilde{L}_{t-j+1}$$

where

$$L'_{t-j-1} = \min\{A \subseteq L_{t-j-1} \,:\, \#A = \#L_{t-j},\, A \geq L_{t-j}\}$$

and $\tilde{L}_{t-j+1} = \tilde{F}_{t-j+1}$.

So, from (15), (16), (17), (18), we have $L_{t-j+1} \geq \tilde{F}_{t-j+2} \cup (F_{t-j+2} \setminus \tilde{F}_{t-j+2}) = F_{t-j+2} = L_{t-j+2}$. On the other hand,

$$\begin{aligned} L_{t-j+1} \setminus \tilde{L}_{t-j+1} &= L_{t-j+1} \setminus F_{t-j+1} = F_{t-j} \setminus F'_{t-j} \geq F_{t-j+2} \setminus \tilde{F}_{t-j+2} \\ &= L_{t-j+2} \setminus \tilde{L}_{t-j+2} \end{aligned}$$

Finally, observe that $\tilde{L}_i = \tilde{F}_i$, for $i = t-j+2, \ldots, t-1, t$. ■

DEFINITION 8 Let $j \in \{1, \ldots, t-1\}$. A tableau $\mathcal{L}$ is called a $((t-j\ t-j+1), \ldots, (t-2\ t-1), (t-1\ t))$ LR tableau if there exists an LR tableau $\mathcal{T}$ such that $\mathcal{L} = \mathcal{T}^{(j,\varepsilon_0)}$, where $\mathcal{T}^{(j,\varepsilon_0)}$ is the tableau generated by step 2 of Algorithm 3 when applied to $\mathcal{T}$. ■

THEOREM 8 *Let $j \in \{1, \ldots, t-1\}$. A tableau $\mathcal{L}$ with indexing sets $L_1, L_2, \ldots, L_t$ is a $((t-j\ t-j+1), \ldots, (t-2\ t-1), (t-1, t))$ LR tableau if and only if it satisfies the conditions* (1), (2) *and* (3) *of Theorem* 7 *above.*

Proof The "*only if*" part is Theorem 7. To prove the "*if*" part, suppose conditions (1), (2) and (3) of Theorem 7 are satisfied. Let $\mathcal{F} = \psi_{op}^{(t-j\,t-j+1)} \mathcal{L}$. Then, by induction on j, $\mathcal{F} = \mathcal{T}^{(j-1,\varepsilon_0)}$, for some LR tableau $\mathcal{T}$. ■

COROLLARY 4 *For each $j = 1, \ldots, t-1$, $\psi^{(t-j\,t-j+1)}$ is a bijection between the sets of $((t-j\,t-j+1), \ldots, (t-2\,t-1)\,(t-1\ t))$ LR tableaux of type (a, b, c) and the $((t-j+1\ t-j+2), \ldots, (t-2\,t-1)\,(t-1\ t))$ LR tableaux of type (a, b, c).*

Proof Straightforward. ■

THEOREM 9 *For each $k \in \{0, 1, \ldots, t-2\}$ and $j \in \{1, \ldots, t-k-1\}$, $\mathcal{T}^{(j,\varepsilon_k,\ldots,\varepsilon_1,\varepsilon_0)}$ defined in step 2 of Algorithm 3, is a tableau with indexing sets $L_1, L_2, \ldots, L_t$ such that:*

1. $L_1 \geq_{op} \cdots \geq_{op} L_k \geq_{op} \tilde{L}_{k+1} \geq \cdots \geq \tilde{L}_{t-1} \geq \tilde{L}_t$, *where for $i = k+1, \ldots, t$, we define inductively*

$$\begin{aligned} \tilde{L}_k &:= L_k \\ \tilde{L}_i &:= \max\{A \subseteq L_i : \# A = \# \tilde{L}_{i-1},\ \tilde{L}_{i-1} \geq A\}. \end{aligned}$$

(*When $k = 0$ we agree $\tilde{L}_0 := \emptyset$.*)

2. $L_{k+1},\ldots,L_t$ *satisfy the conditions of a* $((t-j\ t-j+1),\ldots,(t-2\ t-1),(t-1\ t))$ *LR tableau.*

Proof The proof will be handle by induction on k. When $k=0$ the condition 1 is vacuous and condition 2 gives Theorem 7. In particular, if $j=t-1$ we have $L_1 \geq_{op} \tilde{L}_2 \geq \cdots \geq \tilde{L}_t$ and $L_2\setminus\tilde{L}_2 \geq \cdots \geq L_t\setminus\tilde{L}_t$. So, a tableau with indexing sets $L_2,\ldots,L_t$ is an LR tableau. Let $k\geq 1$ and $j\in\{1,\ldots,t-k-1\}$. Then $\mathcal{T}^{(j,\varepsilon_k,\ldots,\varepsilon_1,\varepsilon_0)}=\psi^{(t-jt-j+1)}(\mathcal{T}^{(j-1,\varepsilon_k,\ldots,\varepsilon_1,\varepsilon_0)})$. By induction on j, $\mathcal{T}^{(j-1,\varepsilon_k,\ldots,\varepsilon_1,\varepsilon_0)}$ is a tableau with indexing sets $F_1,\ldots,F_t$ such that:

(a) $F_1 \geq_{op} \cdots \geq_{op} F_k \geq_{op} \tilde{F}_{k+1} \geq \cdots \geq \tilde{F}_{t-1} \geq \tilde{F}_t$, where for $i=k+1,\ldots,t$, we define inductively

$$\tilde{F}_k := F_k$$
$$\tilde{F}_i := \max\{A\subseteq F_i : \#A=\#\tilde{F}_{i-1},\ \tilde{F}_{i-1}\geq A\}.$$

(b) $F_{k+1},\ldots,F_t$ satisfy the conditions of a $((t-j+1\ t-j+2),\ldots,(t-2\ t-1),(t-1\ t))$ LR tableau.

Since a tableau with indexing sets $F_{k+1},\ldots,F_t$ satisfies the conditions of Theorem 7, the condition 2 follows.

To prove the condition 1 observe that $L_i=F_i$, for $i\neq t-j,t-j+1$, $L_{t-j}=F'_{t-j}, L_{t-j+1}=F_{t-j+1}\cup(F_{t-j}\setminus F'_{t-j})$. On the other hand, $\tilde{F}_{t-j-1}\geq\tilde{F}_{t-j}\geq\tilde{F}_{t-j+1}$ and $F_{t-j-1}\geq F_{t-j}\geq F_{t-j+1}$. So, there exist $Z\subseteq F_{t-j-1}$ and $Y\subseteq F_{t-j}$ such that $\tilde{F}_{t-j-1}\cup Z\geq\tilde{F}_{t-j}\cup Y\geq F_{t-j+1}$ and $\#(\tilde{F}_{t-j-1}\cup Z)=\#(\tilde{F}_{t-j}\cup Y)=\#F_{t-j+1}$. Therefore, $\tilde{F}_{t-j-1}\cup Z\geq\ \tilde{F}_{t-j}\cup Y\geq F'_{t-j}\geq F_{t-j+1}$. Hence, $\tilde{L}_{t-j-1}\geq\tilde{L}_{t-j}\geq\tilde{L}_{t-j+1}$ and condition 1 follows. ■

COROLLARY 5 $\mathcal{T}^{(0,op)}$ *is an* LR_{op} *tableau.*

Proof When $k=t-2$ and $j=1$ in Algorithm 3 we get a tableau with indexing sets $L_1\geq_{op}\cdots\geq_{op}L_{t-2}\geq_{op}L_{t-1}\geq\tilde{L}_t$, where $\tilde{L}_t=\max\{A\subseteq L_t : \#A=\#L_{t-1},\ L_{t-1}\geq A\}$. So, $L_1\geq_{op}\cdots\geq_{op}L_{t-2}\geq_{op}L_{t-1}\geq_{op}L_t$. ■

THEOREM 10 *Algorithm 3 defines a bijection between the set of LR tableaux of type (a,b,c) and the set of LR_{op} tableaux of type (a,b,c).*

Proof Let $\psi^{(t-k \ldots t-1\, t)} := \Pi_{j=1}^{k} \psi^{(t-j\; t-j+1)}$, for $k = 1, \ldots, t-1$. Then $\psi = \psi^{(t-1\;\; t)} \psi^{(t-2\; t-1\; t)} \ldots \psi^{(2 \ldots t-1\;\; t)} \psi^{(1\;\; 2\;\; \ldots t-1 t)}$ is a composition of bijective maps. ■

THEOREM 11 *Let ϕ be the bijection defined in Corollary 3. Then*

(a) *$\psi^{-1}\, \phi$ is a bijection between the LR tableaux of type (a, b, c) and the LR tableaux of type (a^*, b^*, c^*).*
(b) *the map $\mathcal{T} \to \psi^{-1}(\mathcal{T}^-)$ defines a bijection between the LR tableaux of type (a, b, c) and the LR tableaux of type $(c_M^-, b, (M^k) \cup a_M^-)$.*

Proof (a) It follows from Corollary 3 and Theorem 10. (b) It follows from Theorem 4 and Theorem 10. ■

COROLLARY 6 *There is only one LR tableau of type $(a, (c^* - a^*)^*, c)$.*

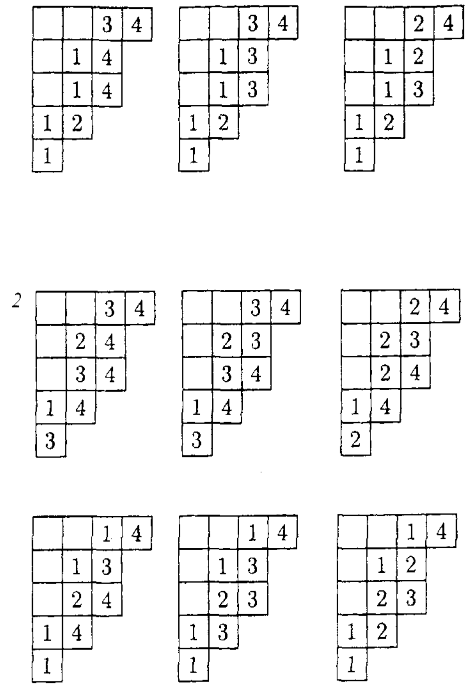
Proof There is only one LR tableau of type $(a^*, c^* - a^*, c^*)$. So, from the previous theorem, there is only one LR-tableau of type $(a, (c^* - a^*)^*, c)$. ■

The process described in the Algorithm 3 is quite easy to carry out. We will give two explicit constructions to impart the flavor of the algorithm.

Example 5 Let $a = (3, 1, 0, 0)$, $b = (4, 2, 2, 1)^*$ and $c = (5, 4, 3, 1)$ as in Example 4. We may check the algorithm to determine all the LR tableaux of type (a, b, c) and (a^*, b^*, c^*), respectively.

We start with the LR_{op} tableaux of type (a^*, b^*, c^*), determined in Example 4, to transform them into LR tableaux of type (a^*, b^*, c^*).

$$I \quad \begin{array}{cccc} \square & \square & 3 & 4 \\ \square & 2 & 4 & \\ \square & 3 & 4 & \\ 1 & 3 & & \\ 4 & & & \end{array} \qquad \begin{array}{cccc} \square & \square & 3 & 4 \\ \square & 2 & 4 & \\ \square & 3 & 4 & \\ 1 & 3 & & \\ 3 & & & \end{array} \qquad \begin{array}{cccc} \square & \square & 3 & 4 \\ \square & 2 & 4 & \\ \square & 2 & 4 & \\ 1 & 2 & & \\ 2 & & & \end{array}$$



4. THE ADMISSIBLE INTERVAL [w, n]

Given partitions $a \subseteq c$, we present an algorithm to describe all the partitions of $LR(a, c)$. This algorithm stresses the significance played by the majorization order among the partitions of $LR(a, c)$.

In this section the LR tableaux are taken under the point of view of Theorem 5. The symbols $\mathbf{w}$ and $\mathbf{n}$ will stand for the difference partitions $c - a$ and $(c^* - a^*)^*$, respectively.

For convenience, we shall assume that an LR tableau $\mathcal{T}$ of type (a, b, c) with $l(b) = m$, has an infinite number of infinitely long columns where the boxes in column r, outside the c-diagram, are labelled by ∞_r, with the convention $\infty_i < \infty_r$ iff $i < r$, and $\infty_r > i$, for all i and r positive integers. So, if $(S_{b_1}, \ldots, S_{b_m})$ is the maximal sequence of strings of $\mathcal{T}$, we define: for all $j > m$, $b_j := 0$ and $S_{b_j} := \emptyset$; and, for all

$j < 1$, $b_j := \infty$, with the convention $\infty > i$, for all i positive integer, and S_{b_j} by the boxes labelled by $\infty_1, \ldots, \infty_r, \ldots$ such that ∞_r is the label of the j-th box in column r, outside the c-diagram (counting from top to bottom). Under these conventions, we shall assume $(S_{b_1}, \ldots, S_{b_m}) = (\ldots, S_\infty, S_{b_1}, \ldots, S_{b_m}, \emptyset, \ldots)$ and $(b_1, \ldots, b_m) = (\ldots, \infty, b_1, \ldots, b_m, 0, \ldots)$.

Let $\mathcal{T}$ be an LR tableau of type (a, b, c), with maximal sequence of strings $(S_{b_1}, \ldots, S_{b_m})$. For $i \in \{1, \ldots, m\}$, we denote by $\mathcal{T}^{(i)}$ the LR tableau of shape c^i/a and weight $(\ldots, \infty, b_{i+1}, \ldots, b_m, 0, \ldots)$ obtained from $\mathcal{T}^{(i-1)}$ deleting the string S_{b_i}, where $\mathcal{T}^{(0)} := \mathcal{T}$ and $c^0 := c$. Clearly, $\mathcal{T}^{(m)} = a = c^m$.

Example 6

			2	∞_5	$\cdots$
		1	3	∞_5	$\cdots$
1	2	∞_3	∞_4	∞_5	$\cdots$
∞_1	∞_2	∞_3	∞_4	∞_5	$\cdots$
∞_1	∞_2	∞_3	∞_4	∞_5	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

LEMMA 2 *Let $\mathcal{T}$ be an LR tableau with maximal sequence of strings $(S_{b_1}, \ldots, S_{b_m})$. Suppose the boxes of the string S_{b_i}, with $i \in \{1, \ldots, m\}$, belonging to the first row of c/a are labelled by $x + 1, \ldots, x + u = b_i, u > 0, x > 0$, and the previous box belonging to the string S_{b_j} $(j > i)$ is such that $0 \leq b_j < x$ (if $b_j = 0$ we agree $j = m + 1$). Then if we change the labels $x + 1, \ldots, x + u$ to $b_j + 1, x + 1, \ldots, x + u - 1$ respectively, we obtain an LR tableau of type (a, b', c), where $b \succ b'$ and b' is obtained from b lowering exactly one box. Moreover, b covers b' in $\mathcal{P}_{|b|}$, if and only if either $b_{i+1} = b_j$ or $b_i = b_j + 2$ and $b_{i+1} = b_{i+2} = \cdots = b_{j-1} = b_{i-1}$.*

Proof Under the assumptions of the string S_{b_i} we may write $S_{b_i} = K_i \cup R_i$, where $|K_i| = x$ and R_i is defined by the last u boxes of S_{b_i} contained in the first row of c/a which are labelled by $x + 1, \ldots, x + u$. Note that, by Properties 1, since S_{b_i} has boxes in the first row of c/a, it follows that $b_i \geq x + 1 > b_{i+1}$ which implies $b_i > x \geq b_{i+1}$. On the other hand, $b_i > x > b_j$, therefore, $b_i \geq b_j + 2$.

Now, let $\mathcal{T}'$ be the tableau obtained from $\mathcal{T}$ changing the labels $x + 1, \ldots, x + u = b_i$ of R_i to $b_{j+1}, x + 1, \ldots, x + u - 1 = b_i - 1$, respectively.

Claim $\mathcal{T}'$ is an LR tableau of type (a, b', c) where b' is obtained from b lowering exactly one box from the i-th to the k-th row such that $b_k = b_j$ where $i < k \leq j$ and $b_{k-1} > b_j$.

Now let $\mathcal{F}$ be the tableau obtained from $\mathcal{T}^{(i-1)}$ changing the labels $x + 1, \ldots, x + u$ of R_i to $b_j + 1, x + 1, \ldots, x + u - 1 = b_i - 1$. Clearly, $\mathcal{F}$ is an LR tableau with maximal sequence of strings $(N_i, \ldots, N_m, N_{m+1})$ (if $b_j > 0$, $N_{m+1} = \emptyset$) such that $N_i = K_i \cup R'_i$, where R'_i is formed by the boxes of the the first row of the skew-diagram of $\mathcal{F}$ labelled by $x + 1, \ldots, x + u - 1 = b_{i-1}, N_{i+1} = S_{b_{i+1}}, \ldots, N_h = S_{b_h}$, for some $i \leq h < j$, with $b_i \geq b_h > b_{h+1} = \cdots = b_{h+g} = b_j$, and $g = j - h$, N_{h+1} is equal to $S_{b_{h+1}}$ plus the box of the first row of the skew-diagram of $\mathcal{F}$ labelled by $b_j + 1$, and $N_s = S_{b_s}$, for $s > h + 1$. So, $|N_i| = b_i - 1$ and $|N_{h+1}| = b_j + 1$. Let $\tilde{b}$ be the weight of $\mathcal{F}$. Then

$$(b_i, \ldots, b_m) \succ \tilde{b},$$

where $\tilde{b} = (b_i - 1, b_{i+1}, \ldots, b_h, b_{h+1} + 1, b_{h+2}, \ldots, b_{h+g}, b_{j+1}, \ldots, b_m)$, if $b_j > 0$, and $\tilde{b} = (b_i - 1, b_{i+1}, \ldots, b_m, 1)$, otherwise. Note that the k-th row claimed above is precisely $h + 1$.

So, $(b_1, \ldots, b_{i-1}, b_i, \ldots, b_m) \succ (b_1, \ldots, b_{i-1}) \cup \tilde{b}$.

Clearly, $\mathcal{T}'$ is the tableau obtained from $\mathcal{F}$ adjoining the strings $S_{b_{i-1}}, \ldots, S_{b_1}$. Therefore, the weight of $\mathcal{T}'$ is $b' = (b_1, \ldots, b_{i-1}) \cup \tilde{b}$.

■

Next example illustrates the lemma.

Example 7 1. With $i = 1$, $j = 5$ and $k = 4$,

$$\mathcal{T} = \begin{array}{cccccccccc} \square&\square&\square&\square&\square&\square&1&2&4&5\\ \square&\square&\square&\square&\square&1&2&&&\\ \square&\square&\square&1&2&3&&&&\\ \square&\square&1&2&3&&&&&\\ 1&2&3&&&&&&& \end{array} \qquad \mathcal{T}' = \begin{array}{cccccccccc} \square&\square&\square&\square&\square&\square&1&2&3&4\\ \square&\square&\square&\square&\square&1&2&&&\\ \square&\square&\square&1&2&3&&&&\\ \square&\square&1&2&3&&&&&\\ 1&2&3&&&&&&& \end{array}$$

$b = (5, 3, 3, 2, 2)$ $\qquad$ $b' = (4, 3, 3, 3, 2)$.

b does not cover b' in $\mathcal{P}_{14}$.

2. With $i = 2$, $j = 5$ and $k = i + 1 = 3$,

$$\mathcal{T} = \begin{ytableau} {} & {} & {} & {} & 1 & 2 & 3 & 6 & 7 \\ {} & {} & {} & 1 & 2 & 3 & 6 \\ {} & {} & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{ytableau} \qquad \mathcal{T}' = \begin{ytableau} {} & {} & {} & {} & 1 & 2 & 3 & 4 & 7 \\ {} & {} & {} & 1 & 2 & 3 & 6 \\ {} & {} & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{ytableau}$$

$$b = (7, 6, 3, 3, 3) \qquad b' = (7, 5, 4, 3, 3).$$

b covers b' in $\mathcal{P}_{22}$.

3. With $i = 1$ and $j = k = 4$,

$$\mathcal{T} = \begin{ytableau} {} & {} & 1 & 2 & 3 & 5 \\ {} & 1 & 2 & 3 & 4 \\ {} & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{ytableau} \qquad \mathcal{T}' = \begin{ytableau} {} & {} & 1 & 2 & 3 & 4 \\ {} & 1 & 2 & 3 & 4 \\ {} & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{ytableau}$$

$$b = (5, 4, 4, 3) \qquad b' = (4, 4, 4, 4).$$

b covers b' in $\mathcal{P}_{16}$.

4. With $i = 1$ and $j = i + 1 = k = 2$,

$$\mathcal{T} = \begin{ytableau} {} & {} & {} & {} & 1 & 2 & 3 & 5 & 6 & 7 & 8 & 10 & 11 \\ {} & {} & {} & 1 & 2 & 3 & 4 & 5 & 8 & 9 \\ {} & {} & 1 & 2 & 3 & 4 & 5 & 7 \\ {} & 1 & 2 & 3 & 4 & 6 \\ 1 & 2 & 3 & 4 & 5 \end{ytableau} \quad , b = (11, 8, 5, 5, 3),$$

$$\mathcal{T}' = \begin{ytableau} {} & {} & {} & {} & 1 & 2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 \\ {} & {} & {} & 1 & 2 & 3 & 4 & 5 & 8 & 9 \\ {} & {} & 1 & 2 & 3 & 4 & 5 & 7 \\ {} & 1 & 2 & 3 & 4 & 6 \\ 1 & 2 & 3 & 4 & 5 \end{ytableau} \quad , b' = (10, 9, 5, 5, 3).$$

b covers b' in $\mathcal{P}_{32}$.

THEOREM 12 *Let $\mathcal{T}$ be an LR tableau as in Lemma 2. Let $v = x - b_j \geq 1$ and $0 < r \leq \min\{u, v\}$. Then if we change the labels $x + 1, \ldots, x + u$ of the first row of c/a to $b_j + 1, \ldots, b_j + r, x + 1, \ldots, x + u - r = b_i - r$ respectively, then we obtain an LR tableau $\mathcal{T}'$*

of type (a, b', c) where $b \succ b'$ and b' is obtained from b lowering exactly r boxes. Moreover, there is a sequence of LR tableaux $(\mathcal{T}^l)_{l=0}^r$ of type (a, b^l, c), for, $l = 0, \ldots, r$, with $\mathcal{T}^0 = \mathcal{T}$, $\mathcal{T}^r = \mathcal{T}'$ such that $b^0 = b \succ b^1 \succ \cdots \succ b^r = b'$, where b^l is obtained from b^{l-1} lowering exactly one box, for $l = 1, \ldots, r$.

Proof For $l = 1, \ldots, r$, let $\mathcal{T}^l$ be the tableau obtained from $\mathcal{T}^{l-1}$ changing the labels $x + 1, \ldots, x + u - l + 1$ to $b_j + l$, $x + 1, \ldots, x + u - l$. From, the lemma above, $(\mathcal{T}^l)_{l=0}^r$ is a sequence of LR tableaux of type (a, b^l, c), for $l = 0, \ldots, r$, where b^l is obtained from b^{l-1} lowering exactly one box, for $l = 1, \ldots, r$. ■

COROLLARY 7 *Let $\mathcal{T}$ be an LR tableau with the first $k + u$ boxes of the first row of c/a labelled by $1, \ldots, k, x + 1, \ldots, x + u$, where $x > k \geq 0$. Let $v = x - k \geq 1$ and $r = \min\{u, v\}$. Let $\mathcal{T}'$ be the tableau obtained from $\mathcal{T}$ changing the labels $x + 1, \ldots, x + u$ of the first row of c/a to $k + 1, k + 2, \ldots, k + u$, respectively, then there exist a sequence of LR tableaux $(\mathcal{T}^l)_{l=0}^r$ of type (a, b^l, c), $l = 0, \ldots, r$, with $\mathcal{T}^0 = \mathcal{T}$, $\mathcal{T}^r = \mathcal{T}'$, such that $b^0 = b \succ b^1 \succ \cdots \succ b^r = b'$, where b^l is obtained from b^{l-1} lowering exactly one box, for $l = 1, \ldots, r$.*

Proof It is a particular case of the previous theorem with $r = \min\{u, v\}$. Therefore, if $r = u = v$, then $x + u - r = k + r = k + u = k + v = x$; if $r = v < u$, then $x + u - r = k + u > k + r = k + v = x$; if $r = u < v$, then $x + u - r = x = k + v > k + u = k + r$. ■

COROLLARY 8 *Let $\mathcal{T}$ be an LR tableau of type (a, b, c). Let k be the length of the first row of c/a. If $\mathcal{T}'$ is the tableau obtained from $\mathcal{T}$ changing the labels of the first row of c/a to $1, 2, \ldots, k$, respectively (reading from left to right) then $\mathcal{T}'$ is an LR tableau of type (a, b', c) where $b \succeq b'$.*

Proof By successive applications (from left to right) of the previous corollary we attain the result. ■

Example 8 In Example 7, 2, if we apply Lemma 2 to $\mathcal{T}'$, with $i = 1$ and $j = 3$, we obtain $\mathcal{T}''$ of type $(a,\ b = (6, 5, 5, 3, 3), c)$, with the first row labelled by $1, 2, 3, 4, 5$.

We remark that if we apply repeatedly the operation described in Lemma 2, that is, the procedure of Corollary 8 to each row of the skew-diagram c/a of $\mathcal{T}$, from top to bottom, we will reach the minimal LR tableau of type $(a, \mathbf{w}, c)$. Reversing these operations we obtain $\mathcal{T}$ from the minimal LR tableau of type $(a, \mathbf{w}, c)$.

In what follows we shall present an algorithm to construct systematically all the elements of $LR(a, c)$. Along the process all the LR tableaux of shape c/a are also exhibited. This algorithm consists in changing successively the labels of each row of the minimal LR tableau from the last to the first row of c/a. These changing of labels in each row of a tableau of shape c/a are described in the two following lemmas.

For this we need an additional definition.

DEFINITION 9 Given c/a, we define the length of the i-th step of c/a, written $s_i(c/a)$, (counting from top to bottom) as being the number of boxes in row i of c/a having no common sides with any of the boxes of the $i+1$-th row.

LEMMA 3 *Let $\mathcal{T}$ be an LR tableau with maximal sequence of strings $(S_{b_1}, \dots, S_{b_m})$, $m \geq 2$. Let $m \geq j > i \geq 1$. Suppose the string S_{b_j} has at least one box in the first row of c/a and the strings $S_{b_{j-1}}, \dots, S_{b_{j-g+1}}, S_{b_i}$, with $g = j - i \geq 1$, which have no boxes in the first row of c/a are such that $b_{j-1} = \cdots = b_{j-g+1} = b_i < b_{i-1}$. Moreover, the box immediately to the right of the box labelled by $b_j \in S_{b_j}$ is labelled by z. Then*

1. *If $z > b_i + 1$, it follows:*
   (a) *$s_1(c^{i-1}/a) = 0$. (The labels $b_j \in S_{b_j}$ and $b_{j-r} \in S_{b_{j-r}}$ are in the same column, for some $1 \leq r \leq g$). There is no LR tableau of shape c/a obtained from $\mathcal{T}$ changing the label b_j in the first row of c/a to b_{i+1}.*
   (b) *$s_1(c^{i-1}/a) > 0$. (The labels $b_j \in S_{b_j}$ and $b_{j-r} \in S_{b_{j-r}}$ are not in the same column, for all $1 \leq r \leq g$). There is an LR tableau of type $(a, \tilde{b}, c)$, with $\tilde{b} = (b_1, \dots, \underbrace{b_i + 1, \dots,}_{g}\, b_j - 1, b_{j+1}, \dots, b_m) \succ b$, obtained from $\mathcal{T}$ changing the label b_j in the first row of c/a to $b_i + 1$. Moreover, $\tilde{b}$ covers b iff either $g = 1$ or $b_i = b_j$.*
2. *If $z = b_i + 1$, there is no LR tableau of shape c/a obtained from $\mathcal{T}$ changing the label b_j in the first row of c/a to $b_i + 1$.*

Proof Attending to Properties 1 and the definition of an LR tableau, it is an easy exercise to check these conditions. It is clear that we may change the label b_j to $b_i + 1$ iff we are in case $1\cdot(b)$. (Recall the conventions made at the begining of this section and note that z is the label of a box belonging to a string $S_{b_{q-1}}$ with $q \leq i$).

Finally, observe that if S_{b_j} has one box in the first row of c/a then $b_{j+1} < b_j$. ■

The following is a generalization of the previous lemma.

PROPOSITION 1 *Let $\mathcal{T}$ be an LR tableau with maximal sequence of strings $(S_{b_1}, \ldots, S_{b_m})$, $m \geq 2$. Let $m \geq j > i \geq 1$. Suppose the string S_{b_j} has at least one box in the first row of c/a and the box to the right of the label $b_j \in S_{b_j}$ belonging to $S_{b_{q-1}}$ is labelled with z. Moreover, suppose the strings $S_{b_{j-1}}, \ldots, S_{b_{j-h+1}}, S_{b_q}$, with $h = j - q \geq 1$, have no boxes in the first row of c/a. Then*

1. *If, for some $g \in \{1, \ldots, h\}$, $b_{j-g} < b_{j-g-1}$, with $g \neq h$, or $b_q + 1 < z$, it follows:*
   (a) *$s_1(c^{j-g-1}/a) = 0$. (The labels $b_j \in S_{b_j}$ and $b_{j-r} \in S_{b_{j-r}}$ are in the same column, for some $1 \leq r \leq g$). There is no LR tableau of shape c/a obtained from $\mathcal{T}$ changing the label b_j in the first row of c/a to $b_{j-g} + 1$.*
   (b) *$s_1(c^{j-g-1}/a) > 0$. (The labels $b_j \in S_{b_j}$ and $b_{j-r} \in S_{b_{j-r}}$ are not in the same column, for all $1 \leq r \leq g$). There is an LR tableau of type $(a, \tilde{b}, c)$, with $\tilde{b} = (b_1, \ldots, \underbrace{b_{j-g} + 1, \ldots,}_{g} b_j - 1, b_{j+1}, \ldots, b_m) \succ b$, obtained from $\mathcal{T}$ changing the label b_j in the first row of c/a to $b_{j-g} + 1$. Moreover, $\tilde{b}$ covers b iff either $g = 1$ or $b_{j-g} = b_j$.*
2. *If $z = b_{j-g} + 1$, for all $g \in \{1, \ldots, h\}$, there is no LR tableau of shape c/a obtained from $\mathcal{T}$ changing the label b_j in the first row of c/a to $b_{j-g} + 1$, for all $g \in \{1, \ldots, h\}$.*

Proof We obtain the previous lemma with i equals the minimum $g \in \{1, \ldots, h\}$ for which $b_{j-g} < b_{j-g-1}$.

Attending to Properties 1 and the definition of an *LR* tableau, it is an easy exercise to check these conditions. If S_{b_q} has no boxes in the first row of c/a, then $b_q < z$. So, if $b_{j-g-1} > b_{j-g}$, for some $1 \leq g < h$, then $z > b_q \geq b_{j-g-1} > b_{j-g}$, and, therefore, $z > b_{j-g} + 1$, for some $1 \leq g < h$. As a consequence, in case 1 we have $b_{j-g} + 1 < z$, for some $g \in \{1, \ldots, h\}$. It is now clear that we may change the label b_j to $b_{j-g} + 1$, with $g \in \{1, \ldots, h\}$, iff we are in case 1·(*b*).

Finally, observe that if S_{b_j} has one box in the first row of c/a then $b_{j+1} < b_j$. ■

In the previous proposition, let $g_1 < g_2 \in \{1, \dots, h\}$ such that $b_{j-g_1} < b_{j-g_1-1} \le b_{j-g_2} \le b_{j-g_2-1}$. Suppose g_1 and g_2 satisfy 1·(b). Let $\mathcal{T}_1$ be the LR tableau obtained from $\mathcal{T}$ changing the label $b_j \in S_{b_j}$ to $b_{j-g_1} + 1$, and $\mathcal{T}_2$ the LR tableau obtained from $\mathcal{T}$ changing the label $b_j \in S_{b_j}$ to $b_{j-g_2} + 1$. Then $\mathcal{T}_2$ may be obtained from $\mathcal{T}_1$ changing the label $b_{j-g_1} + 1$ to $b_{j-g_2} + 1$.

The next example is an illustration of Lemma 3 and Proposition 1.

Example 9

1. With $i = 2, j = 4, z = 4, g = 2$ and $q = 2$,

$$\mathcal{T} = \begin{array}{cccccccc} \square & \square & \square & \square & 1 & 2 & 4 & 5 \\ \square & \square & 1 & 2 & 3 & & & \\ \square & 1 & 2 & 3 & & & & \\ 1 & 2 & 3 & & & & & \end{array}, b = (5, 3, 3, 2).$$

$z = b_3 + 1 = b_2 + 1 = 4$. There is no LR tableau obtained from $\mathcal{T}$ changing the label $2 \in S_{b_4}$ to 4.

2. With $i = 3, j = 4, z = 4, g = 1, 2$ and $q = 2$,

$$\mathcal{T} = \begin{array}{ccccccc} \square & \square & \square & 1 & 2 & 4 & 5 \\ \square & \square & 1 & 2 & & & \\ \square & 1 & 2 & 3 & & & \\ 1 & 2 & 3 & & & & \end{array}, b = (5, 3, 2, 2).$$

$z = b_2 + 1 > b_3 + 1$. There is no LR tableau obtained from $\mathcal{T}$ changing the label $2 \in S_{b_4}$ to 4 but we may change $2 \in S_{b_4}$ to 3.

3. With $i = 3, j = 4, z = 6, g = 1, 2$ and $q = 2$,

$$\mathcal{T} = \begin{array}{cccccccc} \square & \square & \square & \square & 1 & 2 & 6 & 7 \\ \square & \square & \square & 1 & 2 & 4 & & \\ \square & 1 & 2 & 3 & 5 & & & \\ 1 & 2 & 3 & 4 & & & & \end{array}, b = (7, 4, 2, 2).$$

$z = 6 > b_2 + 1$, $b_2 > b_3$ and $s_1(c^1/a) = 0$, $s_1(c^2/a) > 0$. There is no LR tableau obtained from $\mathcal{T}$ changing the label $2 \in S_{b_4}$ to 5, but we may change the label $2 \in S_{b_4}$ to 3.

4. With $i = 3, j = 5, z = \infty_7, b_4 = b_3 < b_2, g = 2, 3$ and $q = 2$,

$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|}
\hline
 & & & 1 & 2 & 3 \\
\hline
 & & 1 & 2 & 3 & 6 \\
\hline
 & 1 & 2 & 3 & 5 & \multicolumn{1}{c}{} \\
\cline{1-5}
1 & 2 & 3 & 4 & \multicolumn{2}{c}{} \\
\cline{1-4}
1 & 2 & 3 & 4 & \multicolumn{2}{c}{} \\
\cline{1-4}
\end{array}\ , b = (6, 4, 3, 3, 3).$$

We may change the label $3 \in S_{b_4}$ to 4 or 5. We obtain LR tableaux $\mathcal{T}_1$ and $\mathcal{T}_2$ of type $(a, b' = (6, 4, 4, 3, 2),\ c)$ and $(a,\ b' = (6, 5, 3, 3, 2),\ c)$ respectively. $\mathcal{T}_2$ may be obtained from $\mathcal{T}_1$ changing 4 to 5.

LEMMA 4 *Let $\mathcal{T}$ be an LR tableau with maximal sequence of strings $(S_{b_1}, \ldots, S_{b_m})$, $m \geq 2$. Suppose the string S_{b_i}, $m > i \geq 1$, has u boxes in the first row of c/a labelled by $x + 1, \ldots, x + u = b_i$, with $x > 0$, respectively, where $b_i = b_{i-1} = \cdots = b_{i-g+1} < b_{i-g}$, with $g \geq 1$. Moreover, the box immediately to the left of the box labelled by $x + 1 \in S_{b_i}$ belongs to the string S_{b_j}, and the box immediately to the right of the box labelled by $b_i \in S_{b_i}$ is labelled by z. Then,*

1. *If $z > b_i + 1$, it follows:*
   (a) *$s_1(c^{i-1}/a) = u$. (The labels $b_j \in S_{b_j}$ and $b_r \in S_{b_r}$ are in the same column, for some $i \leq r < j$.) There is no LR tableau obtained from $\mathcal{T}$ changing the labels $b_j, x + 1, \ldots, b_i$ in the first row of c/a to $x + 1, \ldots, b_i + 1$.*
   (b) *$s_1(c^{i-1}/a) > u$. (The labels $b_j \in S_{b_j}$ and $b_r \in S_{b_r}$ are not in the same column, for all $i \leq r < j$.) There is an LR tableau of type $(a, \tilde{b}, c)$, with $\tilde{b} = (b_1, \ldots, b_{i-g}, \underbrace{b_i + 1, \ldots, b_i}_{g}, \ldots, b_j - 1, \ldots,\ b_m)$, obtained from $\mathcal{T}$ changing the labels b_j, $x + 1, \ldots, b_i$ in the first row of c/a to $x + 1, \ldots, b_i + 1$. Moreover, $\tilde{b}$ covers b iff $j = i + 1$ and $g = 1$.*

2. *If $z = b_i + 1$, there is no LR tableau obtained from $\mathcal{T}$ changing the labels $b_j, x + 1, \ldots, b_i$ in the first row of c/a to $x + 1, \ldots, b_i + 1$.*

Proof It is an easy exercise to check these conditions. By Properties 1, clearly $b_j \leq x$ and if $s_1(c^{i-1}/a) > u$ the label of the box immediately below the box b_j (recall the conventions made at the begining of this section) is $> x \geq b_j$, therefore, we may change the labels b_j, $x + 1, \ldots, b_i$ in the first row of c/a to $x + 1, \ldots, b_i + 1$. Finally, note that, if $g > 1$ the strings $S_{b_{i-1}}, \ldots, S_{b_{i-g+1}}$ have no boxes in the first row of c/a. ■

The next example is an illustration of Lemma 4.

Example 10

1. With $i = 3$, $j = 5$, $b_1 > b_2 = b_3$, $s_1(c^2/a) = 2 = u$ and $z = \infty_{11}$,

$$\begin{array}{cccccccccc} \, & \, & \, & \, & 1 & 2 & 3 & 4 & 5 & 6 \\ \, & \, & \, & \, & 1 & 2 & 3 & 4 & 7 & \\ \, & \, & 1 & 2 & 3 & 4 & 5 & 6 & & \\ \, & 1 & 2 & 3 & 4 & 6 & & & & \\ 1 & 2 & 3 & 4 & 5 & & & & & \end{array} \quad , b = (7, 6, 6, 4, 4).$$

We cannot change the label $4 \in S_{b_5}$ to 5.

2. With $i = 2$, $j = 3$, $\infty = b_0 > b_1 = b_2$, $u = 1$, $s_1(c^1/a) = 2 > 1$ and $z = \infty_{11}$,

$$\begin{array}{cccccccccc} \, & \, & \, & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \, & \, & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \, & 1 & 2 & 3 & 4 & 5 & & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & & & & \\ 1 & 2 & 3 & 4 & & & & & & \end{array} \quad , b = (7, 7, 6, 5, 4).$$

We may change the label 6 of S_{b_3} and the label 7 of S_{b_2} to 7 and 8 respectively.

The next example is an illustration of Lemmas 3 and 4.

Example 11

I

$$\begin{array}{ccccccc} \, & \, & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & & & & \\ 1 & 2 & & & & & \end{array} \qquad \begin{array}{ccccccc} \, & \, & 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 3 & & & & \\ 1 & 2 & & & & & \end{array} \qquad \begin{array}{ccccccc} \, & \, & 1 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & & & & \\ 1 & 2 & & & & & \end{array}$$

$b = (5, 3, 2)$ $\qquad b = (6, 2, 2)$ $\qquad b = (6, 3, 1)$

$$\begin{array}{ccccccc} \, & \, & 1 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & & & & \\ 1 & 2 & & & & & \end{array} \qquad \begin{array}{ccccccc} \, & \, & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & & & & \\ 1 & 2 & & & & & \end{array}$$

$b = (7, 2, 1)$ $\qquad b = (7, 3)$.

$LR\,(a, c) \subseteq [(5, 3, 2), (7, 3)]$. $(5, 4, 1)$, $(5, 5)$, $(6, 4) \notin LR(a, c)$.

2.

$$\begin{array}{cccccc} \square & \square & \square & 1 & 2 & 3 \\ \square & \square & 1 & 2 & & \\ 1 & 2 & 3 & 4 & & \end{array} \qquad \begin{array}{cccccc} \square & \square & \square & 1 & 2 & 5 \\ \square & \square & 1 & 2 & & \\ 1 & 2 & 3 & 4 & & \end{array} \qquad \begin{array}{cccccc} \square & \square & \square & 1 & 3 & 5 \\ \square & \square & 1 & 2 & & \\ 1 & 2 & 3 & 4 & & \end{array}$$

$b = (4,3,2)$ $\qquad$ $b = (5,2,2)$ $\qquad$ $b = (5,3,1)$

$$\begin{array}{cccccc} \square & \square & \square & 1 & 3 & 4 \\ \square & \square & 1 & 2 & & \\ 1 & 2 & 3 & 4 & & \end{array} \qquad \begin{array}{cccccc} \square & \square & \square & 1 & 5 & 6 \\ \square & \square & 1 & 2 & & \\ 1 & 2 & 3 & 4 & & \end{array}$$

$b = (4,4,1)$ $\qquad$ $b = (6,2,1)$.

$(4,3,2) \prec (4,4,1) \prec (5,3,1) \prec (6,2,1)$
$(4,3,2) \prec (5,2,2) \prec (5,3,1)$
$LR(a,c) = [(4,3,2),(6,2,1)]$.

3.

$$\begin{array}{ccccccc} \square & \square & 1 & 2 & 3 & 4 & 5 \\ \square & \square & 1 & 2 & 3 & & \\ 1 & 2 & 3 & 4 & 5 & & \end{array} \qquad \begin{array}{ccccccc} \square & \square & 1 & 2 & 3 & 4 & 6 \\ \square & \square & 1 & 2 & 3 & & \\ 1 & 2 & 3 & 4 & 5 & & \end{array} \qquad \begin{array}{ccccccc} \square & \square & 1 & 2 & 3 & 6 & 7 \\ \square & \square & 1 & 2 & 3 & & \\ 1 & 2 & 3 & 4 & 5 & & \end{array}$$

$b = (5,5,3)$ $\qquad$ $b' = (6,4,3)$ $\qquad$ $b'' = (7,3,3)$.

$LR(a,c) = [(5,5,3),(7,3,3)]$.

4.

$$\begin{array}{cccccc} \square & \square & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & & & \\ 1 & 2 & 3 & & & \end{array} \qquad \begin{array}{cccccc} \square & \square & 1 & 2 & 4 & 5 \\ 1 & 2 & 3 & & & \\ 1 & 2 & 3 & & & \end{array} \qquad \begin{array}{cccccc} \square & \square & 1 & 4 & 5 & 6 \\ 1 & 2 & 3 & & & \\ 1 & 2 & 3 & & & \end{array}$$

$b = (4,3,3)$ $\qquad$ $b' = (5,3,2)$ $\qquad$ $b'' = (6,3,1)$.

b' does not cover b and b'' does not cover b'. $LR(a,c) \subseteq [(4,3,3),(6,3,1)]$. $(4,4,2)$, $(5,4,1)$, $(6,2,2) \notin LR(a,c)$.

We say that an LR tableau $\mathcal{T}$ of shape c/a contains an LR tableau $\hat{\mathcal{T}}$ of shape $\hat{c}/\hat{a}$ if $\hat{\mathcal{T}}$ is precisely the LR tableau defined by the last $l(\hat{c}/\hat{a})$ rows of $\mathcal{T}$.

Next theorem shows that the reverse of the operation described in Lemma 2 is given by Lemma 3 or Lemma 4.

THEOREM 13 *Let $\mathcal{F}$ be an LR tableau of shape c/a in the conditions of Lemma 2 and $\mathcal{F}'$ the LR tableau of type (a, b', c) obtained from $\mathcal{F}$ by changing the labels $x+1, \dots, x+u \in S_{b_i}$ to b_j+1, $x+1, \dots, x+u-1$. Then $\mathcal{F}$ is obtained from $\mathcal{F}'$ by applying the operation described in Lemma 3, 1·(b), if $u = 1$, and the operation described in Lemma 4, 1·(b), if $u > 1$.*

Proof Since the string S_{b_i} has u boxes in the first row of c/a labeled by $x+1, \dots, x+u$, the label of the box immediately under the label $x+1 \in S_{b_i}$ is $\geq x+1$ and, on the other hand, by Properties 1, $b_{i+1} \leq x$. Therefore, the label immediately under the label $x+1 \in S_{b_i}$ belongs to a string S_{b_h} with $h < i$.

Let $(S_{b'_1}, \dots, S_{b'_m}, S_{b'_{m+1}})$ be the maximal sequence of strings of $\mathcal{F}'$ (possibly $S_{b'_{m-1}} = \emptyset$).

1*st Case*. $u = 1$.

$S_{b'_i}$, with $b'_i = x$, has no boxes in the first row of c/a and the tableau $\mathcal{F}'$ is in the conditions of Lemma 3.1, (*b*), with $s_1(c^{i-1}/a) > 0$. Then $\mathcal{F}$ is obtained from $\mathcal{F}'$ by applying the operation described in Lemma 3.1, (*b*), that is, changing the label b_j+1 in the first row of c/a to $x+1$.

2*nd Case*. $u > 1$.

$S_{b'_i}$, with $b'_i = x+u-1 \geq x+1$, has at least one box in the first row of c/a. Naming $u-1$ by "*new u*", the tableau $\mathcal{F}'$ is in the conditions of Lemma 4.1, (*b*) with $s_1(c^{i-1}/a) >$ "*new u*". Then $\mathcal{F}$ is obtained from $\mathcal{F}'$ by applying the operation described in Lemma 4.1 (*b*), that is, changing the labels b_j+1, $x+1, \dots, x+u-1$ to $x+1, \dots, x+u$ in the first row of c/a.

THEOREM 14 *Let $\hat{\mathcal{T}}$ be an LR tableau of shape $\hat{c}/\hat{a}$ and $\mathcal{T}$ the LR tableau of shape c/a obtained from $\hat{\mathcal{T}}$ by adjoining to the top of $\hat{c}/\hat{a}$ one row labeled by consecutive integers $1, 2, \dots, k$. Then $\tilde{\mathcal{T}}$ is an LR tableau of type (a, b, c) containing $\hat{\mathcal{T}}$ iff $\tilde{\mathcal{T}}$ is obtained from $\mathcal{T}$ by applying successively the operations described in Lemma 3 and 4. In this case, $b \succ \hat{b} \cup (k)$.*

Proof The "*if*" part is a consequence of Lemma 3 and 4. It remains to prove the "*only if*" part. If $\tilde{\mathcal{T}}$ is an LR tableau of shape c/a containing $\hat{\mathcal{T}}$ then, from Corollary 8, $\mathcal{T}$ is obtained from $\tilde{\mathcal{T}}$ by successive applications of Lemma 2. Reversing these operations we obtain $\tilde{\mathcal{T}}$ from $\mathcal{T}$. From the previous theorem, these reversing operations are described in Lemma 3 and 4. ■

Taking into account Theorem 14 we may now present an algorithm to construct all the elements of $LR(a, c)$. This algorithm starts with the minimal LR tableau. Since the operations in which this algorithm is based on are reversible it is clear that we may also construct an algorithm starting with the maximal LR tableau.

Algorithm 4 *Let $a \subseteq c$ and suppose c/a has n rows. For $i = 0, 1, \ldots, n-1$, let $(c/a)^{[n-i]}$ be the skew-diagram defined by the $n, \ldots, n-i$-th rows of c/a.*

1. *Let $\mathcal{T}^{\{0\}}$ be the minimal LR tableau of type $(a, \mathbf{w}, c)$.*
   Let $\mathcal{T}^{[n]}$ be the LR tableau of shape $(c/a)^{[n]}$.
2. *Do $i := 0$ and go to 3.*
3. *To each each LR tableau $\hat{\mathcal{T}} \in \mathcal{T}^{[n-i]}$, adjoin to the top of the skew-diagram of $\hat{\mathcal{T}}$ the $(n-i-1)$-th row of the skew-diagram of $\mathcal{T}^{\{0\}}$ such that the LR tableau obtained is of shape $(c/a)^{[n-i-1]}$. Apply the operations described in Lemma 3 and 4 to construct all LR tableaux of shape $(c/a)^{[n-i-1]}$ containing $\hat{\mathcal{T}} \in \mathcal{T}^{[n-i]}$. Denote by $\mathcal{T}^{[n-i-1]}$ the set of all LR tableaux of shape $(c/a)^{[n-i-1]}$.*
   Go to 4.
4. *Add the remaining rows of $\mathcal{T}^{\{0\}}$ to each LR tableau $\in \mathcal{T}^{[n-i-1]}$. We obtain LR tableaux of shape c/a. Denote this set by $\mathcal{T}^{\{i+1\}}$.*
   If $i = n-2$, stop. Otherwise, do $i := i+1$ and go to 3. ■

(We write $\mathcal{T}^{\{0\}}$ both for the minimal tableau of type $(a, \mathbf{w}; c)$ and for the set defined by this tableau. A similar convention is made with $\mathcal{T}^{[n]}$).

This algorithm produces a sequence of sets of LR tableaux of shape c/a:

$$\mathcal{T}^{\{0\}} \subseteq \mathcal{T}^{\{1\}} \subseteq \ldots \subseteq \mathcal{T}^{\{n-1\}}$$

such that, for $i = 0, \ldots, n-2$, the first $n-i-2$ rows of $\mathcal{T}^{\{i\}}$ and $\mathcal{T}^{\{i+1\}}$ are the first $n-i-2$ rows of $\mathcal{T}^{\{0\}}$. The set $\mathcal{T}^{\{i+1\}}$ is obtained from $\mathcal{T}^{\{i\}}$ by applying to the $n-i-1$-th row of each tableau of $\mathcal{T}^{\{i\}}$ the admissible operations described in Lemmas 3 and 4.

If $\mathcal{T}' \in \mathcal{T}^{\{i+1\}} \setminus \mathcal{T}^{\{i\}}$ is of type (a, b', c) then there exist always $\mathcal{T} \in \mathcal{T}^{\{i\}}$ of type (a, b, c) with $b' \succ b$, such that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by applying to the $n-i+1$-th row the operations defined in Lemmas 3 and 4. However, if $\mathcal{T}' \in \mathcal{T}^{\{i+1\}} \setminus \mathcal{T}^{\{i\}}$ is of type (a, b', c) and $\mathcal{T} \in \mathcal{T}^{\{i\}}$ is of type (a, b, c) with $b' \succ b$, we cannot say, in general, that $\mathcal{T}'$ is obtained from $\mathcal{T}$ using the operations defined in Lemmas 3 and 4.

We remark that the maximal LR tableau of type $(a, \mathbf{n}, c)$ is contained in $\mathcal{T}^{\{n-1\}} = \mathcal{T}^{\{1\}}$.

DEFINITION 10 Given partitions $v = (v_1, \ldots, v_n)$ and $y = (y_1, \ldots, y_n)$, we write $c/a = v_{op} + y$ if $a = v^-$ and $c = (v_1^n) + y$. In this case, $\mathbf{w} = (v_i + y_{n-i+1})_{1 \le i \le n}$ by decreasing order and $\mathbf{n} = (v_i + y_i)_{1 \le i \le n}$.

LEMMA 5

(a) *If c/a is equal to $(v_{op} + y)^*$ and c'/a' is equal to $([v + (x^n)]_{op} + y)^*$. Then, $b \in LR(a, c)$ iff $b \cup (n^x) \in LR(a', c')$.*
(b) *If c/a is equal to $v_{op} + y$ and c'/a' is equal to $[v + (x^n)]_{op} + y$. Then, $b \in LR(a, c)$ iff $b + (x^n) \in LR(a', c')$.*

Proof Since $b \in LR(a, c)$ iff $b^* \in LR(a^*, c^*)$ it is sufficient to prove (a).

First observe that c'/a' is obtained from c/a by adding x boxes under the last box of each column of c/a. So every LR tableau of shape c'/a' is obtained from an LR tableau of shape c/a by adding x boxes labelled by j under the last box of the j-th column of c/a, for $j = 1, \ldots, n$.

Therefore, $(S_{b_1}, \ldots, S_{b_m})$ is the maximal sequence of strings of an LR tableau of shape c/a iff $(H_1, \ldots, H_x, S_{b_1}, \ldots, S_{b_m})$ is the maximal sequence of strings of an LR tableau of shape c'/a', where $H_1, \ldots, H_x$ are the first x components of the $\mathbf{H}$-sequence of c'/a'. Hence, $b \in LR(a, c)$ iff $b \cup (n^x) \in LR(a', c')$. Equivalently, $b^* \in LR(a^*, c^*)$ iff $b^* + (x^n) \in LR(a'^*, c'^*)$. ■

LEMMA 6 *Let $\hat{T}$ be a tableau of shape $\hat{c}/\hat{a}$ with only one string $S_{\hat{b}_1}$. Then all LR tableaux of type (a, b, c) containing $\hat{T}$ by adjoining one row of length k to the top of $\hat{c}/\hat{a}$ are such that the b' s form a connected chain in $\mathcal{P}_{|b|}$. Moreover, if*

1. *$k \le \hat{b}_1$ and $s_1(c/a) = r$, we have the chain $b^0 = (\hat{b}_1, k) \prec (\hat{b}_1 + 1, k - 1) \prec \cdots \prec (\hat{b}_1 + r, k - r) = b^r$.*
2. *$k > \hat{b}_1$ and $s_1(c/a) = r_1 + r_2$ with $\hat{b}_1 + r_2 = k$, we have the chain $b^0 = (k, \hat{b}_1) \prec (k + 1, \hat{b}_1 - 1) \prec \cdots \prec (k + r_1, \hat{b}_1 - r_1) = b^{r_1}$.*

Proof It follows from Algorithm 4 and Theorems 13 and 14. ■

THEOREM 15 *If c/a has exactly two rows then $LR(a, c) = [\mathbf{w}, \mathbf{n}]$ and is a connected chain.*

Proof We are in the conditions of the previous lemma with k and $\hat{b}_1$ equal to the length of the first and second row of c/a respectively. So, $\mathbf{w} = b^0$ and $\mathbf{n} = b^r$ or b^{r_1}. $[\mathbf{w}, \mathbf{n}]$ is precisely the connected chain $b^0 = (\hat{b}_1, k) \prec (\hat{b}_1 + 1, k-1) \prec \cdots \prec (\hat{b}_1 + r, k-r) = b^r$ or $b^0 = (k, \hat{b}_1) \prec (k+1, \hat{b}_1 - 1) \prec \cdots \prec (k + r_1, \hat{b}_1 - r_1) = b^{r_1}$. ■

COROLLARY 9 *Let c/a or $(c/a)^-$ be equal to $v_{op} + y$, where either $v = (v_1, x^r)$ and $y = (k, 0^{n-1})$ or $v = (v_1^r, x)$ and $y = (k^r, 0)$, with $r + 1 = n$. Then $LR(a, c) = [\mathbf{w}, \mathbf{n}]$.*

Proof It is a consequence of the previous theorem and Lemma 5. Observe that, in the first case, $\mathbf{w} = (v_1, x + k) \cup (x^{r-1})$ by decreasing order and $\mathbf{n} = (v_1 + k, x^r)$ and, in the second case, $\mathbf{w} = ((v_1 + k)^{r-1}) \cup (x + k, v_1)$ by decreasing order and $\mathbf{n} = ((v_1 + k)^r, x)$. ■

THEOREM 16

(a) *$LR(a, c) = \{\mathbf{w} = \mathbf{n}\}$ iff c/a or $(c/a)^-$ is a partition.*
(b) *$LR(a, c) = \{\mathbf{w}, \mathbf{n}\}$, $\mathbf{w} \neq \mathbf{n}$ iff c/a or $(c/a)^-$ is equal to $v_{op} + y$, where either $v = (v_1^r, x^s)$, $y = (1, 0^{n-1})$ or $v = (v_1^r, (v_1 - 1)^s)$, $y = (k, 0^{n-1})$ or $v = (v_1^{n-1}, v_1 - 1)$, $y = (k^r, 0^{n-r})$ or $v = (v_1^{n-1}, x)$, $y = (1^r, 0^{n-r})$, with $v_1 > x \geq 0$, $k > 0$, $r, s > 0$, $r + s = n$.*

Proof

(a) Without using Theorem 3, this result also follows from Algorithm 4.
(b) Observe that $LR(a, c) = LR(c_M^-, (M^k) \cup a_M^-)$. On the other hand, from (a) and Algorithm 4, we conclude that for $i = 0, 1, \ldots, n-2$, $\# \mathcal{T}^{[n]} = \cdots = \# \mathcal{T}^{[n-i]}$ and $\# \mathcal{T}^{[n-i-1]} = \# \mathcal{T}^{[n-i]} + 1$ iff $(c/a)^{[n-i-1]}$ has the form indicated substituting n by $i + 2$. ■

In the following we give some examples of the set $LR(a, c)$.

Example 12

$$I \quad \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 1 & 2 & 3 \\ \hline 1 & 2 & 3 & 4 \\ \cline{1-4} 1 & 2 & 3 & 4 \\ \cline{1-4} 1 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 \\ \cline{1-4} 1 & 2 & 3 & 4 \\ \cline{1-4} 1 \\ \cline{1-1} \end{array}$$

$(4, 4, 3, 2, 1)$ $\qquad$ $(4, 4, 4, 2)$

			1	2
		1	2	5
1	2	3	4	
1	2	3	4	
1				

$(5,4,2,2,1)$

			1	2
		2	3	5
1	2	3	4	
1	2	3	4	
1				

$(5,4,3,2)$

2.

			2	3
		1	2	3
1	2	3	4	
1	2	3	4	
1				

$(4,4,3,3)$

3.

			1	2
		1	2	5
1	2	3	4	
1	2	3	4	
1				

$(5,4,2,2,1)$

			1	5
		1	2	5
1	2	3	4	
1	2	3	4	
1				

$(5,5,2,1,1)$

			2	5
		1	2	5
1	2	3	4	
1	2	3	4	
1				

$(5,5,2,2)$

			1	3
		1	2	5
1	2	3	4	
1	2	3	4	
1				

$(5,4,3,1,1)$

			2	3
		1	2	5
1	2	3	4	
1	2	3	4	
1				

$(5,4,3,2)$

4.

			1	2
		2	3	5
1	2	3	4	
1	2	3	4	
1				

$(5,4,3,2)$

			1	4
		2	3	5
1	2	3	4	
1	2	3	4	
1				

$(5,4,4,1)$

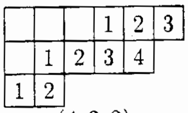
			1	5
		2	3	5
1	2	3	4	
1	2	3	4	
1				

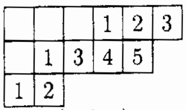
$(5,5,3,1)$

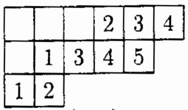
$$\begin{array}{cccc}
 & \underline{(5,5,3,1)} & & \\
\underline{(5,4,4,1)} & & \underline{(5,5,2,2)} & \\
 & \underline{(5,4,3,2)} & & \underline{(5,5,2,1,1)} \\
\underline{(4,4,4,2)} & & \underline{(5,4,3,1,1)} & \\
 & (5,3,3,3) & & \underline{(5,4,2,2,1)} \\
\underline{(4,4,3,3)} & (4,4,4,1,1) & (5,3,3,2,1) & \\
 & \underline{(4,4,3,2,1)} & &
\end{array}$$

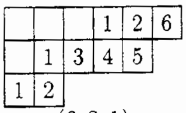
The underlined partitions are the elements of $LR(a, c)$.

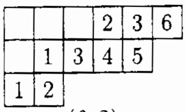
Example 13

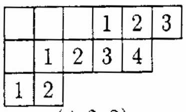

(4, 3, 2)

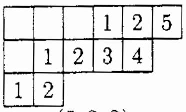

(5, 3, 1)

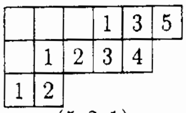

(5, 4)

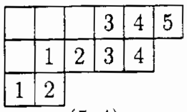

(6, 2, 1)

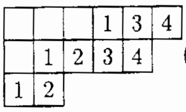

(6, 3)


(4, 3, 2)


(5, 2, 2)


(5, 3, 1)


(5, 4)


(4, 4, 1)

$$\begin{array}{ccc}
 & \underline{(6,3)} & \\
\underline{(6,2,1)} & & \underline{(5,4)} \\
 & \underline{(5,3,1)} & \\
\underline{(5,2,2)} & & \underline{(4,4,1)} \\
 & \underline{(4,3,2)} &
\end{array}$$

The underlined partitions are the elements of $LR(a, c)$.

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APPENDIX

After this paper was written the author was informed by A. Kovačec that in [6] and references therein other explicit bijections betwen LR sequences of conjugate shape and weight can be found based on properties of Schensted insertion or using a *jeu de taquin* approach. Thanks are given to A. Kovačec for his indication.