

SKEW RSK AND THE SWITCHING ON BALLOT TABLEAU PAIRS

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ABSTRACT. In arXiv:1808.06095 we have introduced the Knuth class of the word recording a sequence of locations for repeated internal insertion operations in the Sagan-Stanley skew RSK correspondence, with no prescribed external insertion of new cells, to be a preserver for the P -tableau. As a consequence the Benkart-Sottile-Stroomer switching involution on ballot tableau pairs allows a realization as a recursive internal insertion procedure. This amounts to explain the various presentations of Littlewood-Richardson (LR) commutators and their coincidence predicted by Pak and Vallejo with contributions by Danilov and Koshevoi. In particular, the aforesaid presentation provides internal insertion as an alternative to Schützenberger-Lusztig involution (or evacuation) to constructing the Gelfand-Tsetlin pair in the Henriques-Kamnitzer $\mathfrak{gl}_n$ -crystal commutator. In addition, the coincidence of LR commutators solves the Lecouvey-Lenart conjecture, recently further developed by Kumar-Torres, on bijections between the Kwon and Sundaram branching models.

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1. INTRODUCTION

We give a recursive presentation of the Benkart-Sottile-Stroomer (BSS) switching map [BSS96] on ballot tableau (also known as Littlewood-Richardson tableau) pairs based on the Sagan-Stanley internal row insertion procedure on skew-tableaux [SS90] (see also [RSSW01], and [IMS23] for recent developments). Our key tool is to observe that the Knuth class of an internal insertion order word of locations in a skew-tableau preserves the P -tableau in the Sagan-Stanley skew Robinson-Schensted-Knuth (RSK) correspondence with no prescribed external insertion of new cells. Our aim is to provide a clarification of, and, thereby, fulfill the original question raised by Pak and Vallejo in [PV10], with further contributions by Danilov and Koshevoi [DK05, DK08], on the coincidence and the involutive nature of all Littlewood-Richardson (LR) commutators. Danilov and Koshevoi [DK08] have proven the coincidence of the BSS switching commutator, referred to as ρ_1 in [PV10], with the Henriques-Kamnitzer crystal commutator in type A and hive commutator, here denoted Com_{HK} respectively Com_{HK}^h , [HK06a, HK06b].

LR commutators are combinatorial constructions that express in a bijective manner the symmetry of the $\mathfrak{gl}_n$ -tensor products. It means to specify combinatorial objects exhibiting the symmetry of the tensorands V_μ and V_ν in $V_\mu \otimes V_\nu \simeq V_\nu \otimes V_\mu$, by counting the multiplicity $c_{\mu,\nu}^\lambda = c_{\nu,\mu}^\lambda$ of the irreducible $\mathfrak{gl}_n$ -module V_λ in the decomposition into irreducibles of the tensor product $V_\mu \otimes V_\nu$. An alternative interpretation of these numbers is that they are the structure constants of the cohomology ring of a Grassmannian in the basis of Schubert classes. More generally, beyond Cartan type A , maps exhibiting the isomorphism of the tensor product of $\mathfrak{g}$ -crystals $A \otimes B$ and $B \otimes A$ are called *commutators*.

Here, we focus on the LR commutator or fundamental symmetry map referred to as ρ_3 in [PV10, Section 6.1] which corresponds to the commutativity bijections originally on ballot tableaux [Aze99, Aze08], and later detailed on ballot tableaux and hives in [AKT16, TKA18], and referred to as $\rho^{(n)}$ respectively $\sigma^{(n)}$. More precisely, our ρ_3 commutator, thanks to the BSS switching ρ_1 presentation on ballot tableau pairs as a recursive Sagan-Stanley internal row insertion, coincides with the Henriques-Kamnitzer $\mathfrak{gl}_n$ -crystal commutator [HK06a, HK06b] by providing the associated Gelfand-Tsetlin pair.

Henriques and Kamnitzer [HK06a], following an idea of A. Berenstein, show that the map $a \otimes b \mapsto \xi(\xi(b) \otimes \xi(a))$, with ξ the Lusztig-Schützenberger involution, does give a crystal isomorphism from $B(\mu) \otimes B(\nu)$ to $B(\nu) \otimes B(\mu)$ and thereby a crystal commutator. In other words, denoting by $LR(\lambda/\mu, \nu)$ the set of ballot tableaux of shape λ/μ and content μ , to conclude that a commutator ρ coincides with the Henriques-Kamnitzer $\mathfrak{gl}_n$ -crystal commutator [HK06b, HK06a], it is enough, from considerations on highest and lowest weights in a crystal, to show that the commutator $\rho : LR(\lambda/\mu, \nu) \rightarrow LR(\lambda/\nu, \mu)$ is such that for $T \in LR(\lambda/\mu, \nu)$ with left and right Gelfand-Tsetlin (GT) pair (G_μ, G_ν) , $\rho(T)$ has left and right Gelfand-Tsetlin (GT) pattern pair given by $(\xi(G_\nu), \xi(G_\mu))$. The commutator ρ_3 does so but replaces the Schützenberger involution with Sagan-Stanley internal row insertion or its reverse internal insertion. At this point it must be noted that any of those two Gelfand-Tsetlin patterns together with a triple of boundary partitions completely specify an LR tableau and the corresponding LR hive [Buc00, AKT16, KRV21].

The coincidence of Littlewood-Richardson (LR) commutators is instrumental on a Lecouvey-Lenart conjecture [LL20], recently further developed by Kumar-Torres [KT25a, KT25b], on bijections between the Kwon [Kwo18] and Sundaram [Sun86] branching models. As mentioned by Kumar-Torres in [KT25a], the only difference between their bijection and the bijection conjectured by Lecouvey-Lenart is the Littlewood-Richardson commutator used. While Lecouvey-Lenart use Henriques-Kamnitzer $\mathfrak{gl}_n$ -crystal commutator [HK06b, HK06a], Kumar-Torres use the one by Kushwaha-Raghavan-Viswanath [KRV21] on flagged hives studied in [KRV21, KRKV24]. The Lecouvey-Lenart conjecture is then positively answered thanks to the coincidence of LR commutators. We also note that a major fact in the settling of this conjecture is that Kumar-Torres bijection restricts to tableaux satisfying the Sundaram condition and those whose evacuation satisfy the Kwon condition by considering and recognizing that they can be embedded in the Kushwaha-Raghavan-Viswanath [KRV21] bijection on flagged hives. In other words, denoting by ${}^{-}LR_{\mu,\nu}^\lambda$ the set of left companions of $LR(\lambda/\mu, \nu)$, the Kumar-Torres bijection shows that the left companion of a Sundaram LR tableau is a tableau in ${}^{-}LR_{\mu,\nu}^\lambda$ satisfying the Kwon condition. See Section 5.4.

Other realisations for the Benkart-Sottile-Stroomer (BSS) switching commutator on ballot tableau pairs (that is, a tableau pair (Y, T) , written $Y \cup T$, with Y the Yamanouchi tableau of shape μ and T a ballot

tableau of shape λ/μ with $\mu \subseteq \lambda$), denoted ρ_1 in [PV10], are based on compositions of Schützenberger involutions [vL98, Section 3], [PV10], or on tableau sliding, as in the Thomas-Yong infusion involution [TY08, TY09, TY16]. The latter is realised in [TY08] via Fomin’s *jeu de taquin* growths [Sta98, Chapter 7, Appendix 1]. Beyond type A , Lenart [Len08] realises the Henriques-Kamnitzer $\mathfrak{g}$ -crystal commuter [HK06a, HK06b] via van Leeuwen’s *jeu taquin* [vL98] generalising the Fomin’s growth diagram presentation of *jeu de taquin* on Young tableaux for a broader set of root systems beyond the type A . For further details on coincidence of various LR commuters, $\rho_1, \rho_2, \rho'_2 = \rho_2^{-1}$ in [PV10], the Henriques-Kamnitzer $\mathfrak{gl}_n$ -crystal commuter [HK06a, HK06b] and hive commuter [HK06b], as well the commutativity bijection of Danilov and Koshevoi [DK05, DK08] on arrays, the Knutson-Tao-Woodward puzzles [KTW04] and the mosaic model [Pur08], we refer the reader to [AKT16, Section 12], [TKA18, Introduction] and [ACM25].

1.1. Sagan-Stanley internal insertion and our results. On skew tableaux there are two types of insertion [SS90]: external and internal both of which are based on the usual Schensted insertion operation. However the corresponding procedures on a skew tableau T are different. The former proceeds very similarly to the usual Schensted insertion. The later has two main steps, firstly one chooses an *inner corner* of T (see Section 3 for the definition) and bumps its entry, and, secondly, one inserts the bumped entry externally, in the row immediately below, in the usual manner. Eventually the bumping route lands at the end of some row of T where the last bumped entry settles and thus added at the end of that row of T . The *internal row insertion operation*, denoted ϕ_i if the row coordinate of the vacated inner corner is i (see Definition 3), is two-fold, adds one box, the vacant box, to the inner shape which in turn expands the outer shape in one box, the last bumped entry, but, contrary to the external insertion, without contributing with a new element to the multiset of entries of T .

The internal insertion procedure on a skew-tableau is an iteration of the internal row insertion operation and thus requires *a priori* in each iterative step an inner corner of that skew tableau. Such information is encoded by a second skew-tableau, sharing the inner border with the first, in the Sagan-Stanley skew-RSK correspondence [SS90]. On its turn the instructions that it provides can be translated into a *companion word*, Definition 1, listing the row coordinates of the entries, in the standard order, of the second skew-tableau. This word is the *internal row insertion order word* of the first tableau. The internal insertion procedure is not independent of the order of the chosen inner corners. However the Knuth class of the *companion word* of the second skew-tableau provides a set of internal insertion order words preserving the P -tableau in the Sagan-Stanley skew RSK correspondence, when the matrix prescribing external insertion of new cells is empty [SS90, RSSW01], as shown in Theorem 1.

Knuth relations on the companion word of the second skew tableau is a partial contribution to the question under what conditions is the P -tableau preserved in the skew RSK (see question (3) of [SS90, Section 9]). The internal insertion procedure in general is not independent of a particular sequence of chosen inner corners. Recently Imamura-Mucciconi-Sasamoto [IMS23] observed the same property for the invariance of P -tableau under Knuth relations on the companion word of the second skew tableau. Although Knuth relations do not capture completely the invariance of the P -tableau in the Sagan-Stanley skew RSK correspondence, (see Example 12), they are enough for the purpose of our paper. A nice observation [RSSW01] is that the rectification of a skew-tableau of inner shape, say μ , can either be calculated by using *jeu de taquin* or the internal insertion procedure by choosing, in the Sagan-Stanley internal skew procedure, an arbitrary second skew-tableau of inner shape μ , and *outer* shape an appropriate rectangle. That is, the rectification does not depend on the order of *jeu de taquin moves* nor on the internal insertion order words provided by the mentioned rectangular skew tableaux. In fact, it turns out that companion words of rectangle tableaux are anti-Yamanouchi words, therefore, Knuth equivalent when of the same content.

Two words π and π' are Knuth equivalent if and only if their P -tableaux under RSK correspondence are equal $P(\pi) = P(\pi')$ [Sta98]. Theorem 1 (Theorem 5) below is a natural generalization of this property for the skew RSK rephrased in Theorem 4 for the internal insertion location words in the Sagan-Stanley internal insertion correspondence.

The *companion word* $\mathcal{R}(U)$ of a skew tableau U defines an internal row insertion operator $\phi_{\mathcal{R}(U)}$ for any skew tableau T with the same inner shape as U . See Definition 1 respectively Definition 3 and its extensions (36), (37). If U is a skew tableau, with inner shape $\mu = (\mu_1, \dots, \mu_n)$, on the alphabet $[n]$, the companion

word of U factorizes into n maximal row words (possibly with some empty factors) $\mathcal{R}(U) = \mathcal{R}_n \cdots \mathcal{R}_1$ where $\mathcal{R}_i$ is the row word defined by the row coordinates of the i -cells in U for $i = 1, \dots, n$, and the internal insertion operator $\phi_{\mathcal{R}(U)}$ factorizes accordingly $\phi_{\mathcal{R}(U)} = \phi_{\mathcal{R}_n} \circ \cdots \circ \phi_{\mathcal{R}_1}$. We extend the action of $\phi_{\mathcal{R}(U)}$ on T to the pair $Y \cup T$ with $Y = Y_\mu$ the Yamanouchi tableau of shape μ , and denote it $\bar{\phi}_{\mathcal{R}(U)}$, by filling the vacated cell of T under the action of ϕ_i with i : $\bar{\phi}_i(Y \cup T) = Y_{(\mu_1, \dots, \mu_i+1, \mu_{i+1}, \dots, \mu_n)} \cup \phi_i(T)$. We refer the reader to Sections 2 and 3 for an explanation of undefined terms.

Let $YT(\lambda/\mu)$ be the set of all semistandard tableaux (SSYT) of shape λ/μ .

Theorem 1. [Theorem 5, Sagan-Stanley row internal insertion operators and Knuth relations]. Let $T \in YT(\alpha/\mu)$, $U, U' \in YT(\beta/\mu)$ and $P(T, U)$ respectively $P(T, U')$ the corresponding P -tableaux in the Sagan-Stanley internal insertion correspondence. Then

- (a) U and its standardization $\text{std } U$ have the same companion word, $\mathcal{R}(U) = \mathcal{R}(\text{std } U)$.
- (b) $P(T, U) = \phi_{\mathcal{R}(U)}(T)$.
- (c) $P(T, U) = P(T, \text{std } U)$ and internal row insertion commutes with standardization $\text{std}(P(T, U)) = P(\text{std } T, U) = P(\text{std } T, \text{std } U)$.
- (d) $P(T, U) = P(T, U') = \phi_{\mathcal{R}(U)}(T) = \phi_{\mathcal{R}(U')}(T)$ whenever $\mathcal{R}(U) \equiv \mathcal{R}(U')$ are Knuth equivalent.

Finally, Knuth equivalence of internal row insertion order words in Theorem 1, (d), means (see Proposition 3) that the composition of internal row insertion operators satisfy Knuth relations. The Knuth relations satisfied by the Sagan-Stanley internal row insertion operators are the key fact in Theorem 2 (Theorem 9) to show that Benkart-Sottile-Stroomer switching map [BSS96] on ballot tableau pairs, denoted ρ_1 , can be rephrased in the language of Sagan-Stanley internal row insertion operations or, equivalently, reverse internal insertion operations as next theorem states.

Let $n \geq 1$ and as usual put $[n] = \{1, \dots, n\}$. The set $\mathcal{LR}^{(n)}$ denotes the set of all ballot semistandard tableau pairs $Y_\mu \cup T$, say $\mu = (\mu_1, \dots, \mu_n)$, and T a ballot tableau of skew shape λ/μ , for some $\mu \subseteq \lambda = (\lambda_1, \dots, \lambda_n)$. The switching map on $\mathcal{LR}^{(n)}$ is denoted by $\rho_1^{(n)}$. For $1 \leq i \leq n$, let $(Y_\mu \cup T)^{[i]} := Y_{(\mu_1, \dots, \mu_i)} \cup T^{[i]} \in \mathcal{LR}^{(i)}$ be the restriction of $Y_\mu \cup T$ to the first i rows with $T^{[i]}$ of shape $(\lambda_1, \dots, \lambda_i)/(\mu_1, \dots, \mu_i)$. (See Section 2.4 and Definition 5 for precise definitions.)

Theorem 2. [Main Theorem 9] Let $n \geq 1$ and $Y_\mu \cup T \in \mathcal{LR}^{(n)}$ with T a ballot tableau of shape λ/μ and weight ν . For $1 \leq i \leq n$, let $(Y_\mu \cup T)^{[i]} \in \mathcal{LR}^{(i)}$ with $T^{[i]}$ of weight $\nu^{(i)}$. Consider the i th row word of $T^{[i]}$ where V_i is the row subword restricted to the entries in $[i-1]$, and $\hat{\nu}_i = \lambda_i - \mu_i - |V_i|$ is the number of entries equal to i . Put $(Y_\mu \cup T)^{[0]} = Y_{\nu^0} \cup H^{(0)} := \emptyset$, $\nu^0 := 0$, $\bar{\phi}_\emptyset = id$ and $\rho_1^{(0)}(\emptyset) := \emptyset$. Then, for $i = 1, \dots, n$, it holds

$$\rho_1^{(i)}[(Y_\mu \cup T)^{[i]}] = (\bar{\chi}_i^{\mu_i} \circ \bar{\phi}_{V_i} \circ \bar{\omega}_i^{\hat{\nu}_i}) \circ \rho^{(i-1)}[(Y_\mu \cup T)^{[i-1]}] \quad (1)$$

$$= \bar{\chi}_i^{\mu_i} \circ \bar{\omega}_i^{\hat{\nu}_i} \circ \bar{\phi}_{V_i}(Y_{\nu^{(i-1)}} \cup H^{(i-1)}) = Y_{\nu^{(i)}} \cup H^{(i)} \in \mathcal{LR}^{(i)}, \quad (2)$$

where $\bar{\omega}_i^{\hat{\nu}_i}$ adds the i th row word $i^{\hat{\nu}_i}$ to $Y_{\nu^{(i-1)}}$, $\bar{\chi}_i^{\mu_i}$ adds the row word i^{μ_i} at the end of the i th row of $\bar{\phi}_{V_i} \circ \bar{\omega}_i^{\hat{\nu}_i}(Y_{\nu^{(i-1)}} \cup H^{(i-1)})$ and $H^{(i)} \equiv Y_{(\mu_1, \dots, \mu_i)}$. In particular, all bumping routes of $\bar{\phi}_{V_i}$ are pairwise disjoint and terminate in the i th row.

This theorem is illustrated in Section 5.3. We observe that the recursive internal insertion presentation of switching $\rho_1^{(n)}$, supplied with *add operators* $\bar{\omega}_i$ and $\bar{\chi}_i$, on the ballot pair $Y_\mu \cup T$ in (1), (84),

$$\begin{aligned} \rho_1^{(n)}(Y_\mu \cup T) &= (\bar{\chi}_n^{\mu_n} \circ \bar{\phi}_{V_n} \circ \bar{\omega}_n^{\hat{\nu}_n}) \circ \cdots \circ (\bar{\chi}_2^{\mu_2} \circ \bar{\phi}_{V_2} \circ \bar{\omega}_2^{\hat{\nu}_2}) \circ (\bar{\chi}_1^{\mu_1} \circ \bar{\omega}_1^{\hat{\nu}_1})(\emptyset) \\ &= Y_\nu \cup H, \quad H \equiv Y_\mu, \end{aligned} \quad (3)$$

also produces the *companion tableau* $G_\nu(T)$ or the Gelfand-Tsetlin (GT) pattern of type ν and content $\lambda - \mu$ of T , defined by the nested sequence of partitions $\nu^{(1)} \subseteq \nu^{(2)} \subseteq \cdots \subseteq \nu^{(n)} = \nu$ such that $\nu^{(i)}$ is the content of the ballot tableau $T^{[i]} \in \mathcal{LR}^{(i)}$, for $i = 1, \dots, n$. Since $\rho_1^{(n)}$ is an involution, $\rho_1^{(n)}(Y_\nu \cup H) = Y_\mu \cup T$, this allows another presentation of the switching commutator $\rho_1^{(n)}$ via reverse internal row insertion that we call *deletion operator* $\rho^{(n)}$ in [AKT16]. Deletion operator $\rho^{(n)}$ (ρ_3 in [PV10]) just reverses the process as in (1),

(84) and gives

$$\rho^{(n)}(Y_\nu \cup H) = Y_\mu \cup T = \rho_1^{(n)}(Y_\nu \cup H) \quad (4)$$

by producing the GT pattern $G_\nu(T)$ of type ν and content $\lambda - \mu$ of T . This reverse process coincides with the deletion operations as explained for $\rho^{(n)}$ in [Aze99, AKT16] and translated for hives as $\sigma^{(n)}$ in [AKT16, TKA18].

For $i = 1, \dots, n$, let $\bar{\theta}_i := \bar{\chi}_i^{\mu_i} \bar{\phi}_{V_i} \bar{\omega}_i^{\hat{\nu}_i}$ in (1), (84). The operator defined by

$$\bar{\rho}^{(n)}(Y_\mu \cup T) := \bar{\theta}_n \cdots \bar{\theta}_2 \bar{\theta}_1(\emptyset) = (\bar{\chi}_n^{\mu_n} \circ \bar{\phi}_{V_n} \circ \bar{\omega}_n^{\hat{\nu}_n}) \cdots (\bar{\chi}_2^{\mu_2} \circ \bar{\phi}_{V_2} \circ \bar{\omega}_2^{\hat{\nu}_2}) \circ (\bar{\chi}_1^{\mu_1} \circ \bar{\omega}_1^{\hat{\nu}_1})(\emptyset). \quad (5)$$

is the translations of the operator $\bar{\sigma}^{(n)} = (\sigma^{(n)})^{-1}$ on hives [AKT16, TKA18] to ballot tableaux. That is, $\bar{\rho}^{(n)} = (\rho^{(n)})^{-1}$. From Theorem 2 and (3), one has

$$\rho_1^{(n)}(Y_\mu \cup T) = \bar{\rho}^{(n)}(Y_\mu \cup T) = Y_\nu \cup H, \quad H \equiv Y_\mu \text{ and shape } \lambda/\nu. \quad (6)$$

Since $\rho_1^{(n)}$ is an involution, it follows

$$\rho_1^{(n)} = \bar{\rho}^{(n)} = \rho^{(n)}. \quad (7)$$

Now, for $i = 1, \dots, n$, let

$$\delta_i := (\bar{\theta}_i)^{-1} = (\bar{\omega}_i^{\hat{\nu}_i})^{-1} (\bar{\phi}_{V_i})^{-1} (\bar{\chi}_i^{\mu_i})^{-1}$$

where the action of this operator is realised through deletion operations in the reverse process of (1). Note, from Theorem 2, all bumping routes of $\bar{\phi}_{V_i}$ are pairwise disjoint and terminate in the i th row, hence $(\bar{\phi}_{V_i})^{-1}$ is a reverse internal insertion operation and starts in row i . Since, $\bar{\rho}^{(n)}$ is reversible by reverse row internal insertion, it defines $\rho^{(n)}$ via the GT pattern G_ν that it produces.

We write

$$\rho^{(n)}(Y_\nu \cup H) = Y_\mu \cup T$$

in the sense that $\rho^{(n)}$ is defined by the production of the GT pattern of type ν of T given by the sequence of inner shapes in

$$Y_\nu \cup H, \delta_n(Y_\nu \cup H), \delta_{n-1}\delta_n(Y_\nu \cup H), \dots, \delta_2 \cdots \delta_{n-1}\delta_n(Y_\nu \cup H), \delta_1\delta_2 \cdots \delta_{n-1}\delta_n(Y_\nu \cup H) = \emptyset.$$

Hence

$$\rho^{(n)}(Y_\nu \cup H) = \rho_1^{(n)}(Y_\nu \cup H)$$

We also write

$$\rho^{(n)}(Y_\mu \cup T) = Y_\nu \cup H$$

in the sense that $\rho^{(n)}$ is defined by the production of the GT pattern of type μ of H given by the sequence of inner shapes in

$$Y_\mu \cup T, \delta_n(Y_\mu \cup T), \dots, \delta_2 \cdots \delta_{n-1}\delta_n(Y_\mu \cup T), \delta_1\delta_2 \cdots \delta_{n-1}\delta_n(Y_\mu \cup T) = \emptyset.$$

Then

$$\rho_1^{(n)}(Y_\mu \cup T) = \bar{\rho}^{(n)}(Y_\mu \cup T) = \rho^{(n)}(Y_\mu \cup T) = Y_\nu \cup H.$$

Thereby $\rho^{(n)}$ and $\bar{\rho}^{(n)}$ just provide another method to compute switching on ballot tableau pairs as well as the GT pattern pair in the Henriques–Kamnitzer crystal Com_{HK} and hive Com_{HK}^h commutators [HK06a, HK06b]. From the bijection, denoted φ , between hives and ballot tableaux [Buc00, AKT16, TKA18, KRV21, KT25a], one has the following corollary. (We warn the reader that a hive allows several representations, namely, vertex representation, as in [KT99], [Buc00] (and [KRV21] with a flag condition), edge representation as introduced by [RKT06], and gradient representation as in [AKT16].)

Corollary 1. The following commutators on ballot tableaux or hives coincide and are involutions

$$\rho_1 = \rho_2 = \rho_2' = \rho = \bar{\rho} = \rho_3 = Com_{HK} \quad (8)$$

$$\sigma = \bar{\sigma} = Com_{HK}^h \quad (9)$$

$$\varphi\sigma = Com_{HK}. \quad (10)$$

1.2. Lecouvey-Lenart and Kumar-Torres bijections between Sundaram and Kwon branching models coincide. Let $LR(\lambda/\mu, \nu)$ be the set of ballot (or LR) tableaux of shape λ/μ and content ν . Given $T \in LR(\lambda/\mu, \nu)$, we may associate a pair $(G_\mu(T), G_\nu(T))$ of semistandard tableaux (or GT), called the left and right companions of T , of shape μ and content the reverse of $\lambda - \nu$, respectively of shape ν and content $\lambda - \mu$. The right companion map c [KT25a] induces a bijection between the set $LR(\lambda/\mu, \nu)$ of ballot tableaux of skew shape λ/μ and content ν and the set $LR_{\mu, \nu}^\lambda$ of μ -dominant semi-standard Young tableaux G_ν of shape ν and content $\lambda - \mu$. The left companion map c^- map injects each $T \in LR(\lambda/\mu, \nu)$ to its left companion tableau $G_\mu(T)$ of shape μ and content reverse $\lambda - \nu$. The left companion tableau G_μ of $T \in LR(\lambda/\mu, \nu)$ can be characterized by the ν -dominance of its contre-tableau, that is, the contre-tableau of G_μ , with shape μ and weight $\lambda - \nu$, is ν -dominant. The set ${}^-LR_{\mu, \nu}^\lambda$ denotes the set of semi-standard Young tableaux of shape μ and content the reverse $\lambda - \nu$ whose contre-tableau is ν -dominant. The rectification of those contre-tableaux G_μ are exactly $\xi(G_\mu)$ that are still ν -dominant because rectification Knuth equivalence (rectification) preserves ν -dominance. Therefore $\xi({}^-LR_{\mu, \nu}^\lambda) = LR_{\nu, \mu}^\lambda$.

Corollary 2. The Henriques-Kamnitzer symmetry $LR_{\mu, \nu}^\lambda \xrightarrow{\sim} LR_{\nu, \mu}^\lambda$ can be defined by

$$G_\nu \xrightarrow{c^{-1}} T \xrightarrow{c^-} G_\mu(T) \rightarrow \xi(G_\mu(T)), \quad (11)$$

where ξ is the Lusztig-Schützenberger involution. That is, G is the left companion of $T \in LR(\lambda/\mu, \nu)$ if and only if $\xi(G)$ is the right companion of $\rho_1(T) = \rho^{(n)}(T)$. Moreover, $\xi(G_\mu)$ can be calculated by the reverse Sagan-Stanley internal insertion: it is the GT pattern of shape μ and content $\lambda - \nu$ produced by the sequence of inner shapes in

$$Y_\mu \cup T, \delta_n(Y_\mu \cup T), \dots, \delta_2 \cdots \delta_{n-1} \delta_n(Y_\mu \cup T), \delta_1 \delta_2 \cdots \delta_{n-1} \delta_n(Y_\mu \cup T) = \emptyset. \quad (12)$$

The commuter (11) concerning ballot tableaux can be translated for hives because $LR_{\mu, \nu}^\lambda$ and $Hive(\mu, \nu, \lambda)$ are in bijection thanks to [Buc00]. (We refrain from defining here hives and refer the reader to [Buc00, KRV21, KT25a].) It follows then that from a hive $h \in Hive(\nu, \mu, \lambda)$ we can injectively obtain simultaneously a ν -dominant tableau P_μ of shape μ and weight $\lambda - \nu$, that is, $P_\mu \in LR_{\nu, \mu}^\lambda$ and a μ -dominant contretableau P_ν of shape ν and weight $\lambda - \mu$, that is, $P_\nu \in {}^-LR_{\nu, \mu}^\lambda$ and $\xi(P_\nu) \in LR_{\mu, \nu}^\lambda$.

From the coincidence of LR commutators and the work of Kumar-Torres [KT25a], [KT25b] on flagged hives by Kushwaha-Raghavan-Viswanath [KRV21, KRKV24], the Lecouvey-Lenart conjecture [LL20] on bijections between the Kwon [Kwo18] and Sundaram [Sun86] branching models is settled. We refer the reader to Section 5.4 for notation and relevant definitions. The Lenart-Lecouvey conjecture says that the bijection defined by the Hendriques-Kamnitzer LR commutator between $LR_{\mu, \nu}^\lambda$ and $LR_{\nu, \mu}^\lambda$ restricts to a bijection between $LRS(\lambda/\mu, \nu)$, the set of LR tableaux in $LR(\lambda/\mu, \nu)$ satisfying the Sundaram property, and $LRK_{\nu, \mu}^\lambda$, the set of tableaux in $LR_{\nu, \mu}^\lambda$ such that their Schützenberger evacuation $evac_{2n}$ (or Lusztig-Schützenberger involution ξ) within the crystal $B(\mu, 2n)$, satisfy the Kwon property. This amounts to say that the left companions of $LRS(\lambda/\mu, \nu)$ are Kwon tableaux. Kumar-Torres [KT25a, KT25b] show then that the flagged hive by Kushwaha-Raghavan-Viswanath when restricted to a Sundaram LR tableau exhibits its GT pattern pair.

Let $LRS(\lambda, \mu) := \bigcup LRS(\lambda/\mu, \nu)$, and $LRK(\lambda, \mu) := \bigcup LRK_{\nu, \mu}^\lambda$ where in both cases the union is taken over all even partitions ν , that is, $\nu_{2i-1} = \nu_{2i}$, $i \geq 1$.

Theorem 3. [KT25a, KT25b] The bijection of Kushwaha-Raghavan-Viswanath [KRV21] between $LR_{\mu, \nu}^\lambda$ and $LR_{\nu, \mu}^\lambda$ restricts to a bijection between $LRS(\lambda/\mu, \nu)$ and $LRK_{\nu, \mu}^\lambda$.

Corollary 3 (Corollary 6). The Kumar-Torres bijection

$$LR(\lambda/\mu, \nu) \xrightarrow{\sim} LR_{\mu, \nu}^\lambda \xrightarrow{U} LR_{\nu, \mu}^\lambda$$

where U is the Kushwaha-Raghavan-Viswanath symmetry, and the Lecouvey-Lenart bijection

$$LR(\lambda/\mu, \nu) \xrightarrow{\sim} LR_{\mu, \nu}^\lambda \xrightarrow{U'} LR_{\nu, \mu}^\lambda$$

where U' is the LR commuter defined by Henriques–Kamnitzer, coincide. Thereby both restrict to a bijection between $LRS(\lambda/\mu, \nu)$ and $LRK_{\nu, \mu}^\lambda$ and to

$$LRS(\lambda, \mu) \xrightarrow{\sim} LRK(\lambda, \mu).$$

The Littlewood–Richardson commuter based on internal (or reversal) row insertion operations was first introduced in [Aze99] and called ρ_3 in [PV10]. The involutive nature of this commuter for tableaux and hives was completely detailed in [AKT16, TKA18] without making recourse of the BSS switching involution. Its coincidence with tableau switching was foreseen in [Aze08] which we fulfill here.

1.3. Organization of the paper. The paper is organized in seven sections. In Sections 2 and 3, we fix the basic notation to work with, introduce our main definitions and recall the skew RSK internal insertion correspondence. In particular, in Section 2.3 we recall the companion pair of a ballot tableau and its importance on characterizing LR commuters. In Section 4 we provide a preserver for the P -tableau. Theorem 1 is proved in Section 4.3 as a consequence of several lemmata. In Section 5 the recursive presentation of tableau switching on ballot tableau pairs in terms of the Sagan–Stanley internal row insertion is worked out; the Lecouvey–Lenart conjecture is settled as a consequence of the coincidence of LR commuters and the major contribution by Kumar–Torres bijection. In Section 6 the recursion on ballot tableau pairs is shown in Theorem 13. Theorem 2 (Main Theorem 9) is proved in Section 7.

This paper is an extension of the arXiv preprint [Aze18], also announced in [TKA18], with further results and applications.

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2. PRELIMINARY DEFINITIONS

2.1. Basics on Young tableaux. As usual $[n]$ denotes the set of positive integers $\{1, \dots, n\}$, $n \geq 1$, and, if $1 \leq d \leq n$, $[d, n]$ denotes the set $\{d, \dots, n\}$. A partition (or a normal shape) μ is a non negative integer vector in weakly decreasing order, $(\mu_1 \geq \dots \geq \mu_n \geq 0)$. It is identified with the Young diagram of shape μ , in the English convention, that is, n left justified rows of boxes with μ_i boxes in the i th row, for each i , numbering the rows and the columns in matrix style. The box or cell of the Young diagram in row i and column j will be denoted (i, j) with $1 \leq j \leq \mu_i$. Partitions are usually denoted by lowercase Greek letters. We write $|\mu| := \mu_1 + \dots + \mu_n$ for the number that μ partitions, and the number of positive parts in this summand is the length $\ell(\mu) \leq n$ of μ . The unique partition of length 0 is the null partition (0) , identified with $\emptyset$, the unique empty Young diagram. A *corner* of a Young diagram is a cell such that its removal still leaves a Young diagram.

For Young diagrams $\mu \subseteq \lambda$, the skew partition (or skew shape) λ/μ is the set-theoretic difference $\lambda \setminus \mu$. A semistandard Young tableau (SSYT) T of shape λ/μ is a filling of the boxes of λ/μ over a finite subset $[n]$ of the positive integers, such that the labels of each row weakly increase from left to right and the labels of each column strictly increase from top to bottom. The skew tableau T comprises an *inner border*, defined

by the unfilled *inner shape* μ , a filled skew shape λ/μ , and an *outer border*, defined by the *outer shape* λ . The labels of T are often referred to elements or entries of T . We denote by $\{T\}$ the *multiset of entries of* T counting the number of repetitions of each entry. If $\mu = \emptyset$, T is of partition (normal) shape. The unique empty skew tableau $\mu/\mu := \emptyset_\mu$, is the Young diagram of shape μ .

The (*row*) *reading word* of T is the word $w(T)$ on the alphabet $[n]$, read left to right across rows of T taken in turns from bottom to top. When needed, we also consider the *Kashiwara reading word* of T , also called north-western column reading word of T , that is, the word read from right to left across columns from top to bottom. The *content or weight* of T is the content of its reading word $w(T)$, that is, the vector $\gamma = (\gamma_1, \dots, \gamma_n)$ where γ_i records the number of i 's in T , for all i in the given alphabet. The length $|w(T)|$ of $w(T)$ is the number of letters which appear in $w(T)$. Equivalently, $|\lambda| - |\mu| = |w(T)| = |\gamma| := \gamma_1 + \dots + \gamma_s + \dots$. As usual, given the words u and v over an alphabet, uv denotes their concatenation. A word is said to be a *row word* if its letters weakly increase from right to left. The set of all SSYT's of shape λ/μ is denoted by $YT(\lambda/\mu)$. If we want to emphasize that the labels of the entries range on the set $\{1, \dots, n\}$ then we also write $YT(\lambda/\mu, n)$. For an illustration see (15).

Noting that $T \in YT(\lambda/\mu, n)$ is also realized by a sequence of nested partitions $\mu \subseteq \lambda^{(1)} \subseteq \dots \subseteq \lambda^{(n)} = \lambda$ where $\lambda^{(m)}/\mu$ defines the filling of the boxes of T on the alphabet $[m]$, for $1 \leq m \leq n$, the *restriction of* T *to the alphabet* $[m]$, $T|_{[m]} \in YT(\lambda^{(m)}/\mu, n)$, is the subtableau of T of content $(\gamma_1, \dots, \gamma_m)$, precisely realized by the subsequence $\mu \subseteq \lambda^{(1)} \subseteq \dots \subseteq \lambda^{(m)}$.

A tableau in $YT(\lambda/\mu)$ with $|\gamma|$ boxes is said to be *standard* if the entries are the numbers from 1 to $|\gamma|$, each occurring once. The *standard order* of the boxes on a semistandard Young tableau is given by the numerical ordering of the labels with priority, in the case of equality, given by rule *southwest=smaller, northeast=larger*.

Given $U \in YT(\lambda/\mu)$ of weight γ , the *standardisation* of U is the *standard tableau* $\text{std } U$ obtained by the standard order of the boxes of U . This means to renumber the entries of U in numerical order from 1 to $|\gamma|$, and, in case of equal entries, regard those to the left as smaller than those to the right. The tableau U is easily recovered from $\text{std } U$ and its weight γ . For an illustration see Example 5.

2.2. Basic calculus on Young tableaux. Recall the *Schensted row insertion* (here also called external Schensted row insertion) takes a SSYT T of partition shape, and an element m in the T -alphabet, and constructs a new tableau, denoted $T \cdot m$. For a word $w = w_1 \dots w_s$ on the T -alphabet, we recursively define the new tableau $T \cdot w = (\dots((T \cdot w_1) \cdot w_2) \dots) \cdot w_s$ [Ful97]. Recall the elementary *Knuth transformations* on words over a totally ordered alphabet and its compatibility with row Schensted insertion [Ful97, Sta98]. For letters x, y, z in a totally ordered alphabet, an elementary Knuth transformation is governed by the rules below. As usual we write $\equiv$ for Knuth equivalence between words,

$$yzx \equiv yxz, \text{ if } yz \cdot x = \begin{array}{|c|c|} \hline y & z \\ \hline \end{array} \cdot \begin{array}{|c|} \hline x \\ \hline \end{array} = \begin{array}{|c|c|} \hline x & z \\ \hline y & \\ \hline \end{array} = \begin{array}{|c|} \hline y \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline x & z \\ \hline \end{array} = y \cdot xz, \text{ equivalently, } x < y \leq z, \quad (13)$$

and

$$xzy \equiv zxy \text{ if } xz \cdot y = \begin{array}{|c|c|} \hline x & z \\ \hline \end{array} \cdot \begin{array}{|c|} \hline y \\ \hline \end{array} = \begin{array}{|c|c|} \hline x & y \\ \hline z & \\ \hline \end{array} = \begin{array}{|c|} \hline z \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline x & y \\ \hline \end{array} = z \cdot xy, \text{ equivalently, } x \leq y < z. \quad (14)$$

Two words w and w' are said to be *Knuth equivalent*, $w \equiv w'$, if they can be transformed into each other by a sequence of elementary Knuth transformations. Two skew-tableaux T and U are Knuth equivalent $T \equiv U$ if and only if $w(T) \equiv w(U)$. Equivalently, T and U have the same *rectification*, that is, the insertion tableaux $P(w(T)) = P(w(U))$ obtained by row Schensted insertion of the words $w(T)$ respectively $w(U)$ [Ful97, Sta98].

Remark 1. The row reading and column reading words of T are Knuth equivalent. The Kashiwara reading word and the reverse row reading word of T are Knuth equivalent. The P -tableau of the Schensted row insertion of a word $w = w_1 w_2 \dots w_s$ equals the P -tableau of the *Schensted column insertion* of the reverse word of w , $w_s \dots w_2 w_1$. That is, $P(w) = w_1 \cdot w_2 \cdot \dots \cdot w_s = w_s \leftarrow \dots \leftarrow w_2 \leftarrow w_1$ where $\leftarrow$ means *Schensted column insertion*. For instance, in (13), $yz \cdot x = x \leftarrow z \leftarrow y = y \cdot xz = z \leftarrow x \leftarrow y$. (Similarly for (13).)

Thereby, the P -tableau of the reading word of T , and the P -tableau of the Kashiwara reading word in the Schensted column insertion is the same.

If nothing in contrary is said we always consider Schensted row insertion.

Either for the Schensted row insertion or Schensted column insertion version of RSK, we have the following result: two words π and π' are Knuth equivalent if and only if their P -tableaux under RSK correspondence are equal $P(\pi) = P(\pi')$ [Ful97, Sta98]. In the usual RSK, Knuth relations completely characterize the words having the same P -tableau.

2.3. Ballot tableaux and companion pairs. We follow closely the references [AKT16, TKA18, ACM25] and we refer to them for additional details. A SSYT tableau is said to be *ballot* or *Littlewood-Richardson*(LR) tableau if the content of each suffix of its reading word, that is, the content of the subword read backwards from the end to any letter, is a partition. Such a word is also called Yamanouchi, ballot or reverse lattice word. The ballot tableau of shape μ is also called *Yamanouchi tableau*, Y_μ . In other words, it is the unique tableau of shape and content μ , that is, the entries of row i consist of i 's. Denote by $LR(\lambda/\mu, \nu)$, where $\mu, \nu \subseteq \lambda$ are partitions, the subset of $YT(\lambda/\mu)$ consisting of ballot tableaux of shape λ/μ and content ν . That is $T \in LR(\lambda/\mu, \nu)$ if and only if $T \equiv Y_\nu$.

Example 1. For instance, in (15), H and T are SSYT's of shape $(4, 3, 2)/(2, 1, 0)$, and $Y_{(3,2,1)}$ is the Yamanouchi tableau of shape $(3, 2, 1)$. The reading words of H , T and Y are 121312, 231211 and 322111 respectively. The two last words are Yamanouchi but not the first. Therefore T and Y are ballot but H is not,

$$H = \begin{array}{|c|c|c|c|} \hline & & 1 & 2 \\ \hline & 1 & 3 & \\ \hline 1 & 2 & & \\ \hline \end{array} \quad T = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & 1 & 2 & \\ \hline 2 & 3 & & \\ \hline \end{array} \quad Y_{(3,2,1)} = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline \end{array} \quad T \equiv Y_{(3,2,1)}. \quad (15)$$

A *key tableau* of shape π is the tableau Y_α of shape π whose content α is a permutation of its shape π or the columns are nested as sets. In particular, when α is the reverse of π , $rev \pi$, we say that $Y_{rev \pi}$ is the *anti-Yamanouchi* tableau of shape π . For instance, for $n = 4$ and $\mu = (2, 1, 0, 0)$ respectively $\nu = (3, 2, 1, 0)$, one has the anti-Yamanouchi tableaux of shapes μ respectively ν where the weight is the reverse of the shape

$$Y_{(0,0,1,2)} = \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 4 & \\ \hline \end{array} = \text{evac}_4 Y_{(2,1,0,0)}, \quad Y_{(0,1,2,3)} = \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline 3 & 4 & \\ \hline 4 & & \\ \hline \end{array} = \text{evac}_4 Y_{(3,2,1,0)}. \quad (16)$$

Let $B(\pi, n)$ be the crystal of tableaux of shape π on the alphabet $[n]$, and $\xi = \text{evac}_n$ the Schützenberger-Lusztig involution (or Schützenberger evacuation) on that crystal. The highest weight element of $B(\pi, n)$ is Y_π and the lowest element is $\xi(Y_\pi) = Y_{rev \pi}$.

A SSYT tableau T is said to be *anti-ballot* or an *opposite Littlewood-Richardson*(LR) tableau if its reading word $w(T)$ is anti-Yamanouchi, which means the content of each prefix, that is, the content of the subword read left to right from the beginning to any letter is a reverse partition. A tableau $T \in YT(\lambda/\mu, n)$ of content $rev \nu$ is said to be anti-ballot if $T \equiv \text{evac}_n(Y_\nu) = Y_{rev \nu}$. Denote by $opLR(\lambda/\mu, rev \nu)$, where $\mu, \nu \subseteq \lambda$ are partitions, the subset of $YT(\lambda/\mu, n)$ consisting of tableaux T of shape λ/μ and content $rev \nu$ such that $T \equiv \text{evac}_n(Y_\nu) = Y_{rev \nu}$.

Example 2. Let $n = 4$ and $T \in LR(\lambda/\mu, \nu)$ as in the previous example. Recalling that *reversal* [BSS96, ACM25] is the version of Schützenberger evacuation for skew-tableaux, one has that *reversal* of T is in $opLR(\lambda/\mu, rev \nu)$ with reading word 443324 equals to

$$\begin{array}{|c|c|c|c|} \hline & & 2 & 4 \\ \hline & 3 & 3 & \\ \hline 4 & 4 & & \\ \hline \end{array} \equiv Y_{(0,1,2,3)}. \quad (17)$$

Given $n \geq 1$ and a ballot tableau $T \in LR(\lambda/\mu, \nu)$ with $\ell(\lambda) \leq n$, one associates a *pair of companion tableaux* or *Gelfand-Tsetlin patterns* (GT) $(G_\mu(T), G_\nu(T))$, *left companion* respectively *right companion*

uniquely determined by T . (We identify a Gelfand-Tsetlin pattern of base (or type) κ with its natural tableau presentation of shape κ .)

Remark 2. In fact $T \in LR(\lambda/\mu, \nu)$ is completely specified either by its left or right companion tableau. In addition, they are linearly bijectively related, see [PV10, ACM25].

The left companion $G_\mu(T)$ of shape μ and content $rev(\lambda - \nu)$ is obtained from T by recording the sequence of partitions $\mu^{(n-r+1)}$ giving the shapes occupied by the entries including the empty entries identified with $0 < r$ in rows $r, r+1, \dots, n$ of T , for $r = 1, 2, \dots, n$. We get then the nested sequence of partitions $\mu^{(1)} \subseteq \mu^{(2)} \subseteq \dots \subseteq \mu^{(n)} = \mu$ defining $G_\mu(T)$.

The right companion $G_\nu(T)$ of shape ν and content $\lambda - \mu$ is obtained from T by recording the sequence of partitions ν^r giving the shapes occupied by the positive entries $\leq r$ in rows $1, 2, \dots, r$ of T , for $r = 1, 2, \dots, n$. We get then the nested sequence of partitions $\nu^{(1)} \subseteq \nu^{(2)} \subseteq \dots \subseteq \nu^{(n)} = \nu$ defining $G_\nu(T)$. Equivalently, the row r of $G_\nu(T)$ records the row coordinates of the r -cells of T for $r = 1, \dots, n$ [AKT16].

Example 3. Let $n = 6$, $\lambda = (6, 5, 5, 4, 3, 0)$, $\nu = (4, 4, 3, 2, 0, 0)$, $\mu = (4, 3, 2, 1, 0, 0)$ and $T \in LR(\lambda/\mu, \nu)$ as below. We illustrate T with its GT pattern pair $(G_\mu(T), G_\nu(T)) \in B(\mu, 6) \otimes B(\nu, 6)$

$$T = \begin{array}{|c|c|c|c|c|c|} \hline & & & & 1 & 1 \\ \hline & & & 1 & 2 & \\ \hline & & 1 & 2 & 3 & \\ \hline & 2 & 3 & 4 & & \\ \hline 2 & 3 & 4 & & & \\ \hline \end{array}, \quad G_\mu(T) = \begin{array}{|c|c|c|c|} \hline 2 & 2 & 2 & 4 \\ \hline 3 & 3 & 5 & \\ \hline 4 & 6 & & \\ \hline 6 & & & \\ \hline \end{array}, \quad \begin{array}{cccccc} & & & & 0 & \\ & & & & 3 & 0 \\ & & & 3 & 2 & 0 \\ & & 4 & 2 & 1 & 0 \\ & 4 & 3 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 & 0 \end{array}$$

GT pattern of type μ and weight $rev(\lambda - \nu) = (0, 3, 2, 2, 1, 2)$ defined by the nested sequence of partitions

$$\begin{aligned} \mu = \mu^{(6)} &= (4, 3, 2, 1, 0, 0) \supseteq \mu^{(5)} = (4, 3, 1, 0, 0) \supseteq \mu^{(4)} = (4, 2, 1, 0) \supseteq \\ &\supseteq \mu^{(3)} = (3, 2, 0) \supseteq \mu^{(2)} = (3, 0) \supseteq \mu^{(1)} = (0) \end{aligned} \quad (18)$$

and

$$G_\nu(T) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 & 5 \\ \hline 3 & 4 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array} \quad (19)$$

The Kashiwara reading word (right to left across columns top to bottom) of

$$G_\mu(T) \otimes Y_{rev\nu} = G_\mu(T) \otimes Y_{(0,0,2,3,4,4)} \equiv Y_{rev\lambda} \quad (20)$$

$$G_\mu(T) \otimes Y_{(002344)} \equiv 4252362346 \otimes 5645634563456 \quad (21)$$

$$\equiv G_\mu(T) \leftarrow 5 \leftarrow 6 \leftarrow 4 \leftarrow 5 \leftarrow 6 \leftarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow 6 \leftarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow 6 \quad (22)$$

$$= \begin{array}{|c|c|c|c|c|c|} \hline 2 & 2 & 2 & 3 & 4 & 6 \\ \hline 3 & 3 & 3 & 4 & 5 & \\ \hline 4 & 4 & 4 & 5 & 6 & \\ \hline 5 & 5 & 5 & 6 & & \\ \hline 6 & 6 & 6 & & & \\ \hline \end{array} = Y_{rev\lambda} \quad (23)$$

Remark 3. $G_\mu \otimes Y_{rev\nu} \equiv Y_{rev\lambda}$ is equivalent $Y_\nu \otimes C(G_\mu) \equiv Y_\lambda$ where $C(G_\mu)$ is the contre-tableau of G_μ . Note the word of the contre-tableau is the dual word of $w(G(\mu))$, that is, if $w = w_1 w_2 \dots w_s$ is a word in the alphabet $[m]$ then the dual is $(m - w_s) \dots (m - w_2)(m - w_1)$. Equivalently, $w(C(G_\mu))$ is ν -dominant,

and $w(Y_\nu)w(C(G_\mu))$ is a Yamanouchi word of weight λ (for details see [Ful97, Section 5.1, Appendix A.1] and [Buc00, Appendix]),

$$Y_\nu \otimes C(G_\mu) \equiv Y_\lambda. \quad (24)$$

Therefore, since $\xi(G_\mu)$ is the rectification of $C(G_\mu)$, $\xi(G_\mu) \equiv C(G_\mu)$ is ν -dominant,

$$Y_\nu \leftarrow C(G_\mu) = Y_{(443200)} \leftarrow \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 1 & 3 \\ \hline & 2 & 4 & 4 \\ \hline 3 & 5 & 5 & 5 \\ \hline \end{array} = Y_\lambda.$$

The Kashiwara reading word (right to left along columns top to bottom) of $Y_\mu \otimes G_\nu(T)$ is the Yamanouchi word of weight λ whose column insertion gives Y_λ :

$$Y_\mu \otimes G_\nu(T) \equiv 1121231234 \otimes 3524513451234 \equiv \quad (25)$$

$$Y_{(4,3,2,1,0,0)} \leftarrow 3 \leftarrow 5 \leftarrow 2 \leftarrow 4 \leftarrow 5 \leftarrow 1 \leftarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \quad (26)$$

$$= Y_\lambda \quad (27)$$

Remark 4. In the row reading word option the corresponding versions are $Y_{rev\nu} \bullet G_\mu(T) \equiv Y_{rev\lambda}$ respectively $G_\nu(T) \bullet Y_\mu \equiv Y_\lambda$ [Ful97, Section 5.2, Exercise 3].

We now collect a few facts that characterize the set of GT-patterns of shape μ and weight $rev(\lambda - \nu)$ respectively the set of GT-patterns of shape ν and weight $\lambda - \mu$ each of which specifying the set $LR(\lambda/\nu, \mu)$. We need some notation. We denote the set of those GT-patterns by ${}^{-}LR_{\mu,\nu}^\lambda$ respectively $LR_{\mu,\nu}^\lambda$. Recall that

$$B(\mu) \otimes B(\nu) \cong \bigoplus B(\lambda)^{c_{\mu,\nu}^\lambda}, \quad (28)$$

where the sum is taken for all partitions $\lambda \supseteq \mu, \nu$, and $c_{\mu,\nu}^\lambda = \#L(\lambda/\mu, \nu)$.

The set $LR_{\mu,\nu}^\lambda$ (${}^{-}LR_{\mu,\nu}^\lambda$) of semistandard tableaux G_ν (G_μ) of shape ν (μ) and content $\lambda - \mu$ ($rev(\lambda - \nu)$) are those in the crystal $B(\nu)$ ($B(\mu)$) such that $Y_\mu \otimes G_\nu$ ($G_\mu \otimes Y_{rev\nu}$) is the highest (lowest) weight element of weight λ ($rev\lambda$) in a connected component of $B(\mu) \otimes B(\nu)$ isomorphic to $B(\lambda)$.

Therefore, $G_\nu \in LR_{\mu,\nu}^\lambda$ if and only if it satisfies the left μ -dominance (the Kashiwara reading word of G_ν concatenated on the left with the canonical Yamanouchi word of weight μ (the Kashiwara reading word of Y_μ) is $\equiv Y_\lambda$).

On the other hand, $G_\mu \in {}^{-}LR_{\mu,\nu}^\lambda$ if and only if it satisfies the right $rev\nu$ -dominance, that is, the word of G_μ concatenated on the right with the canonical anti-Yamanouchi word of weight $rev\nu$ (the Kashiwara reading word of $Y_{rev\nu}$) is $\equiv Y_{rev\lambda}$. Equivalently, from (24), $Y_\nu \otimes C(G_\mu) \equiv Y_\lambda$, it amounts to say that $\xi(G_\mu) \in LR_{\mu,\nu}^\lambda$.

We know from [HK06b] that $LR_{\mu,\nu}^\lambda$, ${}^{-}LR_{\mu,\nu}^\lambda$ and $LR(\lambda/\nu, \mu)$ are in bijection. In fact, from [HK06b], $T \in LR(\lambda/\nu, \mu)$ has companion pair (G_μ, G_ν) if and only if $Y_\mu \otimes G_\nu \equiv Y_\lambda$ and $G_\mu \otimes Y_{rev\nu} \equiv Y_{rev\lambda}$ are the highest and lowest weight elements of a same connected component of $B(\mu) \otimes B(\nu)$ isomorphic to $B(\lambda)$.

We know from [PV10, HK06b] that $LR_{\mu,\nu}^\lambda$, ${}^{-}LR_{\mu,\nu}^\lambda$ and $LR(\lambda/\mu, \nu)$ are in bijection. In fact, from [HK06b], the pair $(G_\mu, G_\nu) \in LR_{\mu,\nu}^\lambda \times {}^{-}LR_{\mu,\nu}^\lambda$ is the companion pair of $T \in LR(\lambda/\mu, \nu)$ if and only if $Y_\mu \otimes G_\nu \equiv Y_\lambda$ and $G_\mu \otimes Y_{rev\nu} \equiv Y_{rev\lambda}$ are the highest respectively lowest weight elements of a same connected component of $B(\mu) \otimes B(\nu)$ isomorphic to $B(\lambda)$ where $T \in LR(\lambda/\mu, \nu)$ is the recording tableau in the column insertion [Nak93, Kwo18] of $G_\mu \leftarrow Y_{rev\nu}$ and $Y_\mu \leftarrow G_\nu$. In other words, each copy of $B(\lambda)$ is uniquely parameterized by a $T \in LR(\lambda/\mu, \nu)$. From [Kwo18] we then have an RSK version of (28)

$$B(\mu, n) \otimes B(\nu, n) \cong \bigoplus_{T \in LR(\lambda/\mu, \nu)} B(\lambda, n) \times \{T\}, \quad (29)$$

where λ is taken over all partitions of n such that $\mu, \nu \subseteq \lambda$.

Furthermore, from Henriques-Kamnitzer commuter [HK06b], (G_μ, G_ν) is the companion pair of some $T \in LR(\lambda/\mu, \nu)$ if and only if $(\xi(G_\nu), \xi(G_\mu))$ is the companion pair of some $H \in LR(\lambda/\nu, \mu)$. Note that the Henriques-Kamnitzer commuter subsumes Remark 3.

We then have the right companion bijection c

$$c : LR(\lambda/\mu, \nu) \longrightarrow LR_{\mu, \nu}^{\lambda}, T \mapsto G_{\nu}(T) \quad (30)$$

and the left companion bijection c^{-}

$$c^{-} : LR(\lambda/\mu, \nu) \longrightarrow {}^{-}LR_{\mu, \nu}^{\lambda} T \mapsto G_{\mu}(T). \quad (31)$$

The Henriques-Kamnitzer commuter [HK06b] Com_{HK} can be written as

$$Com_{HK} = \xi \circ c^{-} \circ c^{-1}$$

or in its left version

$$Com_{HK}^{-} = \xi \circ c \circ c^{-1}$$

$$Com_{HK} : LR_{\mu, \nu}^{\lambda} \xrightarrow{c^{-1}} LR(\lambda/\mu, \nu) \xrightarrow{c^{-}} {}^{-}LR_{\mu, \nu}^{\lambda} \xrightarrow{\xi} LR_{\nu, \mu}^{\lambda}, G_{\nu} \mapsto T \mapsto G_{\mu} \mapsto \xi(G_{\mu}), \quad (32)$$

or

$$Com_{HK}^{-} : {}^{-}LR_{\mu, \nu}^{\lambda} \xrightarrow{c^{-1}} LR(\lambda/\mu, \nu) \xrightarrow{c} LR_{\mu, \nu}^{\lambda} \xrightarrow{\xi} {}^{-}LR_{\nu, \mu}^{\lambda}, G_{\mu} \mapsto T \mapsto G_{\nu} \mapsto \xi(G_{\nu}). \quad (33)$$

Example 4. For $n = 6$ this is an illustration of the inverse of map c^{-} applied to $G_{\mu}(T)$ to get T :

$$\begin{aligned} G_{\mu}(T) = & \begin{array}{|c|c|c|c|} \hline 2 & 2 & 2 & 4 \\ \hline 3 & 3 & 5 & \\ \hline 4 & 6 & & \\ \hline 6 & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 2 & 2 & 2 & 4 \\ \hline 3 & 3 & 5 & \\ \hline 4 & & & \\ \hline & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & 1 \\ \hline & & 1 & \\ \hline & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 2 & 2 & 2 & 4 \\ \hline 3 & 3 & & \\ \hline 4 & & & \\ \hline & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & 1 \\ \hline & & 1 & 2 \\ \hline & & & \\ \hline 2 & & & \\ \hline \end{array}, \\ & \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & 1 \\ \hline & & 1 & 2 \\ \hline & & 2 & 3 \\ \hline 2 & 3 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 2 & 2 & 2 & \\ \hline 3 & 3 & & \\ \hline & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & 1 \\ \hline & & 1 & 2 \\ \hline & & 2 & 3 \\ \hline 2 & 3 & 4 & \\ \hline \end{array} = \tilde{T}, \quad \begin{array}{|c|c|c|c|c|c|} \hline & & & & 1 & 1 \\ \hline & & & 1 & 2 & \\ \hline & & 1 & 2 & 3 & \\ \hline & & 2 & 3 & 4 & \\ \hline 2 & 3 & 4 & & & \\ \hline \end{array} = T \end{aligned} \quad (34)$$

where $\widehat{\nu} = (2, 1, 1, 1, 0) = \lambda - \text{shape}(\tilde{T}) = \lambda - (4, 4, 4, 3, 3)$, and T is obtained from $\tilde{T}$ by adding $\widehat{\nu}_i$, i 's, to row i of $\tilde{T}$, for $i = 1, \dots, \ell(\lambda)$.

2.4. The companion word of a skew-tableau.

Definition 1. Let $U \in YT(\lambda/\mu, n)$ of weight γ . The *companion word* of U is defined to be the word $\mathcal{R}(U) = \mathcal{R}(\text{std } U) := u_{|\gamma|} \cdots u_2 u_1$ listing the row indices of the entries of $\text{std } U$, from the bigger to the smaller. Equivalently, to construct $u_{|\gamma|} \cdots u_2 u_1$, for $p = 1, \dots, |\gamma|$, put $u_p = i$, if the number p is in the i th row of $\text{std } U$. The companion word of T factorizes into n maximal row words (possibly with some empty factors) $\mathcal{R}(U) = \mathcal{R}_n \cdots \mathcal{R}_1$ where $\mathcal{R}_i$ is the row word defined by the row coordinates of the i -cells in T for $i = 1, \dots, n$.

Example 5. Let

$$\begin{aligned} U = & \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 3 \\ \hline & & 2 & 4 & \\ \hline 1 & 2 & 3 & & \\ \hline \end{array} \in YT(\lambda/(3, 2, 0), 4), \quad V = \begin{array}{|c|c|c|c|c|c|} \hline & & & 4 & 6 & \\ \hline & & 5 & 7 & & \\ \hline 4 & 5 & 6 & & & \\ \hline \end{array} \in YT(\lambda/(3, 2, 0), 6), \\ \text{std } U = \text{std } V = & \begin{array}{|c|c|c|c|c|} \hline & & & 2 & 6 \\ \hline & & 4 & 7 & \\ \hline 1 & 3 & 5 & & \\ \hline \end{array} \in YT(\lambda/(3, 2, 0), 7). \end{aligned}$$

The corresponding companion words are

$$\mathcal{R}(U) = \mathcal{R}(V) = \mathcal{R}(\text{std } U) = \mathcal{R}(\text{std } V) = 2132313$$

$$\mathcal{R}(U) = \mathcal{R}_4 \mathcal{R}_3 \mathcal{R}_2 \mathcal{R}_1 = 2132313$$

$$\mathcal{R}(V) = \mathcal{R}_7 \mathcal{R}_6 \mathcal{R}_5 \mathcal{R}_4 \mathcal{R}_3 \mathcal{R}_2 \mathcal{R}_1 = 2132313\emptyset\emptyset\emptyset$$

$$\mathcal{R}(\text{std}U) = \mathcal{R}(\text{std}V) = \mathcal{R}_7 \mathcal{R}_6 \mathcal{R}_5 \mathcal{R}_4 \mathcal{R}_3 \mathcal{R}_2 \mathcal{R}_1 = 2132313$$

Proposition 1. Let $U \in YT(\beta/\mu)$ and $\mathcal{R}(U)$ its companion word. If $\mu = \emptyset$ then $\mathcal{R}(U)$ is a Yamanouchi word $\equiv Y_\beta$. If $\mu \neq \emptyset$ and β has rectangle shape then $\mathcal{R}(U)$ is a anti-Yamanouchi word of content $(\beta_1^{\ell(\beta)}) - \mu = (\beta_1 - \mu_1, \beta_1 - \mu_2, \dots, \beta_1 - \mu_{\ell(\beta)})$ and thus $\equiv Y_{(\beta_1^{\ell(\beta)}) - \mu}$.

Proof. If $\mu = \emptyset$ it is a consequence of the definitions of companion word and Yamanouchi word [Ful97]. If $\mu \neq \emptyset$ and β has rectangle shape, it is enough to observe that the word $\mathcal{R}(U)$ is a reverse Yamanouchi word of content $(\beta_1 - \mu_1, \beta_1 - \mu_2, \dots, \beta_1 - \mu_{\ell(\beta)})$, that is, in each prefix of $\mathcal{R}(U)$ the number of $(i+1)$'s is at least equal to the number of i 's, for all i . $\square$

Example 6. The companion words of $Y_{(1,3,4)}$ respectively U

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & \\ \hline 3 & & & \\ \hline \end{array} = Y_{(1,3,4)}, \quad \mathcal{R}(Y_{134}) = 12231121, \quad U = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & \\ \hline 4 & & & \\ \hline \end{array} \quad \mathcal{R}(U) = 31221211$$

are Yamanouchi words of weight $(4, 3, 1)$.

Remark 5. If T is a ballot tableau of weight ν , $\mathcal{R}(T)$ is the word of the companion tableau or Gelfand-Tsetlin pattern G_ν of T . Let $U \in YT(4, 3, 2)/(2, 1)$ be a ballot tableau of weight $\nu = (3, 2, 1)$

$$U = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & 1 & 2 & \\ \hline 2 & 3 & & \\ \hline \end{array}$$

The companion word of U is $\mathcal{R}(U) = 323112 = \mathcal{R}_3 \mathcal{R}_2 \mathcal{R}_1 \equiv G_\nu$ which is precisely the reading word of the

companion tableau $G_\nu = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 3 & & \\ \hline \end{array}$ of the ballot tableau U .

Let T be a ballot tableau of weight $\nu = (4, 4, 3, 2)$

$$T = \begin{array}{|c|c|c|c|c|c|} \hline & & & & 1 & 1 \\ \hline & & & 1 & 2 & \\ \hline & & 1 & 2 & 3 & \\ \hline & 2 & 3 & 4 & & \\ \hline 2 & 3 & 4 & & & \\ \hline \end{array} \quad (35)$$

$$\mathcal{R}(T) = \emptyset 4534523451123 = \mathcal{R}_5 \mathcal{R}_4 \mathcal{R}_3 \mathcal{R}_2 \mathcal{R}_1 \equiv G_\nu = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 & 5 \\ \hline 3 & 4 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array}$$

Example 7. Companion words of rectangle shapes are anti-Yamanouchi words

$$U = \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & 1 & 3 & 4 \\ \hline 3 & 3 & 4 & 5 \\ \hline \end{array} \quad \text{std}U = \begin{array}{|c|c|c|c|} \hline & & & 2 \\ \hline & 1 & 5 & 7 \\ \hline 3 & 4 & 6 & 8 \\ \hline \end{array}$$

$$\mathcal{R}(U) = \mathcal{R}(\text{std}U) = 32323312 \equiv \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline \end{array} \equiv 3^4 2^3 1 \equiv \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & \\ \hline 3 & & & \\ \hline \end{array} = Y_{(4^3 - (4, 3, 1))}$$

If $\mu \subseteq \lambda \subseteq \gamma$ are partitions, we say that the shape of γ/λ extends the shape of λ/μ which in turn extends the shape μ . In general, when we say that the SSYT V extends the SSYT U it is meant that the shape of V extends the shape of U and we write $U \cup V$ for the object formed by gluing U and V together.

If S and T are SSYT's of shapes μ and λ/μ , $S \cup T$ is said to be of shape λ . If Z is another SSYT of shape γ/λ , $T \cup Z$ has shape λ/μ and we write $S \cup T \cup Z := (S \cup T) \cup Z = S \cup (T \cup Z)$. The tableaux S ,

T and Z are filled in any finite ordered alphabet consisting of positive integers. Sometimes it is convenient to look at $S \cup T$ as a SSYT of shape λ by adding a constant to all entries of T , for instance, the biggest entry of S . In view of these definitions if U is a tableau we say the subtableaux U_1 and U_2 decompose U if $U = U_1 \cup U_2$ and U_2 extends U_1 . When S and T are both ballot tableaux with S of normal shape, we say that we have a *ballot tableau pair* of normal shape. In this case S is the Yamanouchi tableau Y_μ .

Definition 2. Let $\mathcal{LR}^{(n)}$ denote the set of all ballot semistandard tableau pairs of partition shape where the length of the shape is $\leq n$.

3. SAGAN-STANLEY SKEW RSK AND INTERNAL ROW INSERTION OPERATORS

3.1. Internal insertion words of a skew tableau and their Knuth classes. Let $T \in YT(\lambda/\mu)$ with $\ell(\lambda) = n$. An *inner corner* of T is a cell (i, j) such that when added to the Young diagram of μ still results in a valid Young diagram. For instance, $(1, 4)$, $(2, 3)$, $(4, 2)$ and $(5, 1)$ are inner corners of T but $(6, 1)$ and

$(3, 3)$ are not, $T =$

		1
3		

 with $\ell(\lambda) = 5$.

Definition 3. [SS90] Let $T \in YT(\lambda/\mu)$ with $\ell(\lambda) = n$ and $1 \leq i \leq n + 1$. The Sagan-Stanley *internal (row) insertion operator* ϕ_i is an operation over T defined whenever $(i, \mu_i + 1)$ is an inner corner of T . We have to distinguish two cases.

- The cell $(i, \mu_i + 1)$ is an *empty cell* which means that $(i - 1, \mu_i + 1) \notin \lambda/\mu$. Then ϕ_i just adjoins the blank cell $(i, \mu_i + 1)$ to the Young diagram of μ , in row i of T .
- The cell $(i, \mu_i + 1) \in \lambda/\mu$. Then ϕ_i vacates the cell $(i, \mu_i + 1)$ of T , bumps its entry and inserts the bumped element, using the usual Schensted row insertion rules, into row $i + 1$ of T . The insertion then continues in a normal fashion, ending with an element settling at the end of some row $\leq n + 1$.

$$\phi_4\phi_1 \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline 3 & & \\ \hline \end{array} = \phi_4 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline 3 & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline 3 & & & \\ \hline \end{array}$$

With an internal row insertion operation ϕ_i on T , no new entry is added to the tableau T . Instead the skew-shape changes by adding one blank box at the end of row i of the inner shape μ , and, if it is the case, one filled box is added to the outer shape λ . The new tableau $\phi_i T$ has shape $(\lambda + e_t)/(\mu + e_i)$ with $i \leq t \leq n + 1$, and is Knuth equivalent to T . In particular, if T is a ballot tableau, then $\phi_i T$ is also ballot. Whenever the internal row insertion operator ϕ_i is defined on T , it can be easily extended to the tableau pair $Y \cup T$ with $Y = Y_\mu$, by putting

$$\bar{\phi}_i(Y \cup T) := \begin{cases} Y_{(\mu_1, \dots, \mu_i+1, \dots, \mu_n)} \cup \phi_i(T), & \text{if } 1 \leq i \leq n, \\ Y_{(\mu_1, \dots, \mu_n, 1)} \cup \phi_{n+1}(T), & \text{if } i = n + 1 \text{ and } \mu_n > 0. \end{cases} \quad (36)$$

$$\bar{\phi}_4\phi_1 \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 1 \\ \hline 3 & 3 & \\ \hline 3 & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & \\ \hline 3 & 3 & & \\ \hline 3 & & & \\ \hline 3 & & & \\ \hline \end{array}$$

If $Y \cup T \in \mathcal{LR}^{(n)}$, $\bar{\phi}_i(Y \cup T) \in \mathcal{LR}^{(n+1)}$.

Definition 4. Given $T \in YT(\alpha/\mu)$, an internal insertion order word for T is the companion word $\mathcal{R}(U)$ of any skew tableau U sharing the inner border with T , that is, $U \in YT(\beta/\mu)$ with $\mu \subseteq \beta$. We say $U \in YT(\beta/\mu)$ with $\mu \subseteq \beta$ is an internal insertion order tableau for T .

Given an internal insertion order word $u_{|\gamma|}u_{|\gamma|-1}\cdots u_2u_1$ of $T \in YT(\alpha/\mu)$ of content γ , one defines the corresponding *internal insertion operator*

$$\phi_{u_{|\gamma|}u_{|\gamma|-1}\cdots u_2u_1} := \phi_{u_{|\gamma|}} \circ \phi_{u_{|\gamma|-1}} \circ \cdots \circ \phi_{u_2} \circ \phi_{u_1}. \quad (37)$$

This is well defined because $\phi_{u_{|\gamma|}u_{|\gamma|-1}\cdots u_2u_1}T = \phi_{\mathcal{R}(U)}T$ for some $U \in YT(\beta/\mu)$. Then $P(T, U) := \phi_{\mathcal{R}(U)}T \in YT(\lambda/\beta)$.

The bijection below is a special case of the Sagan-Stanley skew-RSK correspondence, Theorem 6.11 in [SS90], when the matrix word $\pi = \emptyset$, and is denoted by **SS**. In this situation the skew-insertion procedure is reduced to the internal (Schensted) row insertion procedure. This correspondence, calculates a bijection between pairs of tableaux (T, U) sharing the inner border and pairs of tableaux (P, Q) sharing the outer border that preserves the Knuth equivalence class, $T \equiv P$ and $U \equiv Q$. Also the outer border of T equals the inner border of Q , and the outer border of U equals the inner border of P . (See also [RSSW01] for more details and properties.)

Theorem 4. [SS90, Theorem 6.1, Theorem 6.11], [RSSW01] (Sagan-Stanley internal row insertion correspondence.) Fix partitions $\mu \subseteq \alpha, \beta$. There is a bijection,

$$\begin{aligned} YT(\alpha/\mu) \times YT(\beta/\mu) &\longrightarrow \bigcup_{\substack{\lambda \\ |\lambda| = |\alpha| + |\beta| - |\mu|}} YT(\lambda/\beta) \times YT(\lambda/\alpha) \\ (T, U) &\xrightarrow{\text{SS}} (P, Q), \end{aligned} \quad (38)$$

where $P \equiv T$ and $Q \equiv U$. The P -tableau, $P(T, U) := P$ is given by $\phi_{\mathcal{R}(U)}T$. The Q -tableau $Q(T, U) := Q$ is the recording tableau.

Corollary 4. Fix partitions $\mu \subseteq \alpha, \beta$ and the above setting

- (1) If β is a rectangle shape with $\ell(\beta) \geq \ell(\mu)$. Then $P = \emptyset_\beta \cup \text{rect}(T)$ where $\text{rect}(T)$ is the rectification of T .
- (2) If $\alpha = \mu$ and $U \in YT(\beta/\mu)$, $P(\emptyset_\mu, U) = \emptyset_\beta$ equals to the Young diagram of shape β and $Q(\emptyset_\mu, U) = U$.
- (3) If $\mu = \emptyset$, $P(T, U) = T$ and $\mathcal{R}(U)$ is a Yamanouchi word of weight β .

Let U in the alphabet $[m]$. Set $T^{(0)} = T$ and $U^{(0)} = \emptyset_\lambda$. For $j = 1, \dots, m$, let $r_1^{(j)} \geq \cdots \geq r_{k_j}^{(j)}$ be the row coordinates of all j -cells in the standard order of U and define $T^{(j)} = \phi_{\mathcal{R}_{r_1^{(j)}}} \circ \cdots \circ \phi_{\mathcal{R}_{r_{k_j}^{(j)}}}(T^{(j-1)})$. Then define $U^{(j)}$ adding to $U^{(j-1)}$ j -cells so that the external shape of $U^{(j)}$ matches that of $T^{(j)}$. Finally set $P = T^{(m)}$ and $Q = U^{(m)}$. See Example 8 below.

The following exhibits a symmetry of the skew RSK map of tableaux proven in [SS90] not immediate from the definition, and the compatibility with standardization. Schensted insertion commutes with standardisation. In addition the effect of each operator ϕ_{u_i} on the shape of T will be the same as the effect on the shape of $\text{std}T$

Proposition 2. (a) [SS90, Theorem 3.3] If $\text{SS}(P, Q) = (P', Q')$, then $\text{SS}(Q, P) = (Q', P')$. That is, like P' and P , also the recording tableau Q' is obtained from Q following a number of internal insertions.

(b) $\text{std} \circ \text{SS}(P, Q) = \text{SS} \circ \text{std}(P, Q)$.

Unlike for the classical RSK correspondence, a detailed description of properties of Sagan and Stanley's algorithm has proven to be more challenging to obtain. A recent attempt was undertaken by Imamura-Mucciconi-Sasamoto [IMS23] by iterating the skew RSK correspondence, that is, the space of ordered pairs (P, Q) of tableaux of skew shape, which are built on repeated internal insertions in P whose locations are determined by Q

Given a tableau $T \in YT(\alpha/\mu)$, we would like to characterize the tableaux $U, U' \in YT(\beta/\mu)$ such that $P(T, U) = P(T, U')$ in (38). The theorem below shows that Knuth equivalence on the companion words $\mathcal{R}(U)$ and $\mathcal{R}(U')$ provides sufficient conditions for the equality. Given a skew tableau T , similarly to the ordinary RSK, Knuth equivalent internal insertion order words of T means a certain Knuth commutation of the corresponding internal insertion operators which gives rise to the same tableau P . Our theorem partially answers our aim and the second part of question (3) of Section 9, in [SS90].

Theorem 5. (Theorem 1) (Sagan-Stanley row internal insertion operators and Knuth relations.) Let $T \in YT(\alpha/\mu)$, $U, U' \in YT(\beta/\mu)$ and $P(T, U)$ respectively $P(T, U')$ the corresponding P -tableaux in the Sagan-Stanley internal insertion correspondence. Then

- (a) U and its standardization $\text{std } U$ have the same companion word, $\mathcal{R}(U) = \mathcal{R}(\text{std } U)$.
- (b) $P(T, U) = \phi_{\mathcal{R}(U)}(T)$.
- (c) $P(T, U) = P(T, \text{std } U)$ and internal row insertion commutes with standardization $\text{std}(P(T, U)) = P(\text{std } T, U) = P(\text{std } T, \text{std } U)$.
- (d) $P(T, U) = P(T, U') = \phi_{\mathcal{R}(U)}(T) = \phi_{\mathcal{R}(U')}(T)$ whenever $\mathcal{R}(U) \equiv \mathcal{R}(U')$ are Knuth equivalent.

The proof will be delayed until Section 4.3 and will follow from two lemmata, Lemma 6, Lemma 5 and Proposition 3.

Example 8. Let $\mu = (1)$, $\alpha = (3, 2, 0)$ and $\beta = (4, 2, 1)$. Below U , $\text{std } U$, and U' , $\text{std}(U')$ are internal insertion order tableaux of $T = \begin{array}{|c|c|c|} \hline & 1 & 3 \\ \hline 2 & 3 & \\ \hline \end{array}$. The corresponding internal insertion order words are Knuth equivalent, $\mathcal{R}(U) = 312211 = \mathcal{R}(\text{std } U) \equiv \mathcal{R}(U') = \mathcal{R}(\text{std } U') = 132211$,

$$U = \begin{array}{|c|c|c|c|} \hline & 1 & 1 & 2 \\ \hline 2 & 2 & & \\ \hline 3 & & & \\ \hline \end{array} \quad \text{std } U = \begin{array}{|c|c|c|c|} \hline & 1 & 2 & 5 \\ \hline 3 & 4 & & \\ \hline 6 & & & \\ \hline \end{array} \quad U' = \begin{array}{|c|c|c|c|} \hline & 1 & 1 & 3 \\ \hline 2 & 2 & & \\ \hline 3 & & & \\ \hline \end{array} \quad \text{std } U' = \begin{array}{|c|c|c|c|} \hline & 1 & 2 & 6 \\ \hline 3 & 4 & & \\ \hline 5 & & & \\ \hline \end{array}.$$

Knuth equivalence of internal insertion order words, $\mathcal{R}(U) \equiv \mathcal{R}(U')$, implies that the corresponding internal insertion operators $\phi_{\mathcal{R}(U)}$ and $\phi_{\mathcal{R}(U')}$ commute according to the Knuth relations, that is, $\phi_{\mathcal{R}(U)} = \phi_{\mathcal{R}(U')}$.

$$P(T, U) = \phi_3 \phi_{122} \phi_{11} T = \phi_{13} \phi_{22} \phi_{11} T = P(T, U') = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & 3 & \\ \hline & 3 & & \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline \end{array}.$$

$$Q(T, U) = \begin{array}{|c|c|c|c|} \hline & & & 2 \\ \hline & & 1 & \\ \hline 1 & 2 & & \\ \hline 2 & & & \\ \hline 3 & & & \\ \hline \end{array} \quad Q(T, U') = \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & & 1 & \\ \hline 1 & 2 & & \\ \hline 2 & & & \\ \hline 3 & & & \\ \hline \end{array}. \quad (39)$$

$$\mathcal{R}(U) \equiv \mathcal{R}(U') \text{ and } \mathcal{R}(Q(T, U)) = 513423 \equiv \mathcal{R}(Q(T, U')) = 153423.$$

Example 9. For the T and U above one has $n = 3$, $\alpha = (3, 2, 0)$, $\beta = (4, 2, 1)$, $\lambda = (4, 3, 2, 1, 1)$, $(P, Q) \in YT((4, 3, 2, 1, 1)/(4, 2, 1)) \times YT((4, 3, 2, 1, 1)/(3, 2, 0))$

$$\begin{array}{ccc} YT(\alpha/(1), 3) \times YT(\beta/(1), 3) & \longrightarrow & \bigcup_{\substack{\lambda \\ |\lambda|=11}} YT(\lambda/\beta, 3) \times YT(\lambda/\alpha, 3) \\ (T, U) & \xrightarrow{\text{ss}} & (P, Q). \end{array} \quad (40)$$

$$\begin{aligned}
T^{(0)} &= \begin{array}{|c|c|c|} \hline & 1 & 3 \\ \hline 2 & 3 & \\ \hline \end{array} & U^{(0)} = \emptyset_\alpha &= \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \\
\phi_{\mathcal{R}_1} = \phi_{11} \xrightarrow{\quad} T^{(1)} &= \begin{array}{|c|c|c|} \hline & & \\ \hline 1 & 3 & 3 \\ \hline 2 & & \\ \hline \end{array} & U^{(1)} &= \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline 1 & & \\ \hline \end{array} \\
\phi_{\mathcal{R}_2} = \phi_{122} \xrightarrow{\quad} T^{(2)} &= \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & 3 & \\ \hline 1 & 3 & & \\ \hline 2 & & & \\ \hline \end{array} & U^{(2)} &= \begin{array}{|c|c|c|c|} \hline & & & 2 \\ \hline & & 1 & \\ \hline 1 & 2 & & \\ \hline 2 & & & \\ \hline \end{array} \\
\phi_{\mathcal{R}_3} = \phi_3 \xrightarrow{\quad} P = T^{(3)} &= \begin{array}{|c|c|c|c|c|} \hline & & & 3 & \\ \hline & & 3 & & \\ \hline 1 & & & & \\ \hline 2 & & & & \\ \hline \end{array} & Q = U^{(3)} &= \begin{array}{|c|c|c|c|c|} \hline & & & 2 & \\ \hline & & 1 & & \\ \hline 1 & 2 & & & \\ \hline 2 & & & & \\ \hline 3 & & & & \\ \hline \end{array} \\
\mathbf{SS}(T, U) &= (P, Q)
\end{aligned}$$

The inverse Sagan-Stanley internal insertion correspondence $\mathbf{SS}^{-1}$, $\alpha = (3, 2, 0)$, $\beta = (4, 2, 1)$:

$$\begin{aligned}
YT(\lambda/\beta, 3) \times YT(\lambda/\alpha, 3) &\longrightarrow \bigcup_{\substack{\lambda \\ |\lambda|=11}} YT(\alpha/(1), 3) \times YT(\beta/(1), 3) \\
(P, Q) &\xrightarrow{\mathbf{SS}^{-1}} (T, U).
\end{aligned} \tag{41}$$

The inverse Sagan-Stanley *internal (row) insertion operator*, ϕ_i^{-1} , or Sagan-Stanley *deletion operator*, is denoted Δ_i . The *reverse companion word* of Q , $\text{rev}\mathcal{R}(Q) := \text{rev}\mathcal{R}_1 \text{rev}\mathcal{R}_2 \text{rev}\mathcal{R}_3 = 324315$ defines the *deletion operator*

$$\Delta_{\text{rev}\mathcal{R}(Q)} = \Delta_{\text{rev}\mathcal{R}_1 \text{rev}\mathcal{R}_2 \text{rev}\mathcal{R}_3} \tag{42}$$

$$= \Delta_{\text{rev}\mathcal{R}_1} \circ \Delta_{\text{rev}\mathcal{R}_2} \circ \Delta_{\text{rev}\mathcal{R}_3} \tag{43}$$

$$\begin{aligned}
P^{(0)} &= \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & 3 & \\ \hline & 3 & & \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline \end{array} & Q^{(0)} = \emptyset_\beta &= \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} & \Delta_{\mathcal{R}_3} = \Delta_5 \xrightarrow{\quad} P^{(1)} &= \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & 3 & \\ \hline 1 & 3 & & \\ \hline 2 & & & \\ \hline \end{array} & Q^{(1)} &= \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline 3 & & & \\ \hline \end{array} \\
\Delta_{\mathcal{R}_2} = \Delta_{431} \xrightarrow{\quad} P^{(2)} &= \begin{array}{|c|c|c|c|} \hline & & & \\ \hline 1 & 3 & 3 & \\ \hline 2 & & & \\ \hline \end{array} & Q^{(2)} &= \begin{array}{|c|c|c|c|} \hline & & & 2 \\ \hline 2 & 2 & & \\ \hline 3 & & & \\ \hline \end{array} & \Delta_{\mathcal{R}_1} = \Delta_{32} \xrightarrow{\quad} P^{(3)} &= \begin{array}{|c|c|c|c|} \hline & 1 & 3 & \\ \hline 2 & 3 & & \\ \hline \end{array} = T & Q^{(3)} &= \begin{array}{|c|c|c|c|c|} \hline & 1 & 1 & 2 & \\ \hline 2 & 2 & & & \\ \hline 3 & & & & \\ \hline \end{array} = U
\end{aligned}$$

$$\mathbf{SS}^{-1}(P, Q) = (T, U)$$

4. A PRESERVER FOR THE P -TABLEAU IN THE SKEW RSK CORRESPONDENCE

The Schensted row insertion takes a SSYT T of partition shape, and an element m in the T -alphabet, and constructs a new tableau, denoted $T \bullet m$ [Ful97]. For skew tableaux there are two types of row insertion: *external* and *internal* both of which based on Schensted insertion but with different procedures namely, in the former, the element in the T -alphabet to be inserted is added to the multiset $\{T\}$, and, in the latter, is picked in $\{T\}$ [SS90, Section 2].

4.1. External row insertion on skew tableaux. *Sagan-Stanley row external insertion* [SS90, Section 2] on skew-tableaux is similar to Schensted's original procedure. We start with a SSYT T of shape λ/μ and an element m in the T -alphabet to be added to T . To start, m replaces the smallest entry in the first row of T strictly larger than m ; in the case where m is bigger or equal than all entries in the first row, it is placed at that row's right end. If an entry was displaced from the first row then it is inserted into the second using the same rules as above. This process continues until some element comes to rest at the end of a row. The only need for caution in the skew case is when something is to be inserted into row i which is empty and this can only happen at the beginning or at the end of an insertion. It happens, when $\mu_i = \lambda_i$ and in this case we put the element m in cell $(i, \lambda_i + 1)$. We obtain a new skew-tableau denoted $T \cdot m$ with the same inner shape μ and $\{T \cdot m\} = \{T\} \cup \{m\}$.

If T is a skew-tableau and u is a word over the T -alphabet, $T \cdot u$ denotes the skew tableau with the same inner shape as T obtained by the *Sagan-Stanley row external insertion* of u in T . That is, if $u = u_1 \cdots u_n$, $T \cdot u = (((T \cdot u_1) \cdots) \cdot u_n)$. When T is of straight shape $T \cdot u$ coincides with the usual Schensted insertion [Ful97]. More generally, if U , V and W are tableaux of straight shape, $U \cdot V$ means $U \cdot w(V)$, and $(U \cdot V) \cdot W = U \cdot (V \cdot W)$.

We may look at Sagan-Stanley external insertion on a skew-tableau T of shape λ/μ as an insertion on a straight tableau $Y_\mu \cup T$ of shape λ , where $\ell(\mu)$ is added to each entry of T , that is, T is filled in the alphabet $\{\ell(\mu) + 1, \ell(\mu) + 2, \dots\}$. If u is a word over the alphabet $\{\ell(\mu) + 1, \ell(\mu) + 2, \dots\}$, then

$$Y_\mu \cup (T \cdot u) = (Y_\mu \cup T) \cdot u. \quad (44)$$

The following is a lemma on commutation and cancellation in the plactic monoid of Lascoux and Schützenberger

Lemma 1. Let u and u' be two words in a same ordered alphabet $A = [n]$.

- (a) [LS90, Proposition 2.3] Let $B \subseteq A$ be an interval and let $u|_B$ and let $u'|_B$ be the restrictions of u and u' to B . Then one has

$$u \equiv u' \Rightarrow u|_B \equiv u'|_B. \quad (45)$$

- (b) [RSSW01, Lemma 7.5] Let w, v be any words in the alphabet $[k]$ and $C = k \cdots 21$ be a column. Then

- (i) $C \cdot w \equiv w \cdot C$.
- (ii) $C \cdot w \equiv C \cdot v$ if and only if $w \equiv v$.
- (c) If $ux \equiv u'x$ with $x = a_1 \cdots a_k$ a row word in A such that the smallest letter a_1 in x is bigger or equal than all letters in the words u or u' then
 - (i) $ux \equiv u'x \equiv Tx$ is a tableau for some tableau T with entries in A less or equal than a_1 such that $u \equiv u' \equiv T$.

Lemma 2. (External insertion on skew-tableaux.) Let T be a skew-tableau and u and v two words over the T -alphabet. Then

- (a) $w(T \cdot u) \equiv w(T)u$, and
- (b) $T \cdot u = T \cdot v$ whenever $u \equiv v$.

Proof. (a) If T has straight shape then $w(T \cdot u) \equiv w(T)u$. In general, if T has shape λ/μ , using (44), one has $w[Y_\mu \cup (T \cdot u)] = w[(Y_\mu \cup T) \cdot u] \equiv w(Y_\mu \cup T)u$. From (45), it follows $w(T \cdot u) \equiv w(T)u$.

(b) Again using (44), as a consequence of the usual Schensted insertion on straight shapes, one has $Y_\mu \cup (T \cdot v) = (Y_\mu \cup T) \cdot v = (Y_\mu \cup T) \cdot u = Y_\mu \cup (T \cdot u)$ whenever $u \equiv v$ over the T -alphabet. Thus $Y_\mu \cup (T \cdot v) = Y_\mu \cup (T \cdot u)$ whenever $u \equiv v$ over the T -alphabet implies $T \cdot u = T \cdot v$ whenever $u \equiv v$ over the T -alphabet. $\square$

The reverse direction of Lemma 2, (b), that is, the cancelation law, is trivially true when T is just a Young diagram. Otherwise, unless additional conditions are satisfied as in Lemma 1, it is false in general. For instance, one has $yxz \equiv yzx$ for $x < y \leq z$ and $xz \not\equiv zx$ with $x < z$.

Indeed external insertion preserves Knuth equivalence: if T and Z are tableaux with the same inner shape, and $T \equiv Z$ and u and v are Knuth equivalent words, then from Lemma 2, $T \cdot u \equiv Z \cdot v$. In case T and Z are of the same straight shape then we have $T = Z$ and $T \cdot u = T \cdot v$.

4.2. Internal row insertion and internal bumping routes.

Definition 5. If $T \in YT(\lambda/\mu)$, with $\ell(\lambda) \leq n$, $T^{[i]} \in YT((\lambda_1, \dots, \lambda_i)/(\mu_1, \dots, \mu_i))$ denotes the tableau consisting of the first i rows of T , for $i = 0, 1, \dots, n$, that is, the restriction of the tableau T to the first i rows. We put $T^{[0]} = \emptyset$ and $T^{[n]} = T$.

Definition 6. Given the skew tableau $T \in YT(\lambda/\mu)$ and $0 \leq i \leq n$ where $\ell(\lambda) \leq n$, we say that T is factorized across the row i when we write $T = T' * T^{[i]}$, $0 \leq i \leq n$, with T' of shape $(\lambda_{i+1}, \dots, \lambda_n)/(\mu_{i+1}, \dots, \mu_n)$ is the restriction of T to the last $n - i$ rows.

In particular, let T be a SSYT of straight shape factored across the i th row, $T = T' * T^{[i]}$, with T' the tableau obtained by suppressing $T^{[i]}$ from T . Note that $T = T' * T^{[i]} = T' \cdot T^{[i]}$.

Let $T \in YT(\lambda/\mu)$ with $\ell(\lambda) = n$, assume $1 < 2 < \dots < n < 1 < 2 < \dots < n$ and fill in the inner shape $\emptyset_\mu$ with the coloured alphabet $1 < 2 < \dots < n$ so that we get the Yamanouchi tableau Y_μ . Let us factor $Y_\mu \cup T$ across the i th row. Then one has,

$$Y_\mu \cup T = (Y' \cup T') * (Y_\mu \cup T)^{[i]} = (Y' \cup T') * (Y_{(\mu_1, \dots, \mu_i)} \cup T^{[i]}) = (Y' \cup T') \cdot (Y_{(\mu_1, \dots, \mu_i)} \cup T^{[i]}) \quad (46)$$

where Y' and T' denote the restriction of Y_μ and T to the last $n - i$ rows of Y_μ and T respectively. Thereby, in the sense of (46), we may write

$$T = T' * T^{[i]} = T' \cdot T^{[i]}.$$

Assume that $(i, \mu_i + 1)$ is an inner corner of $T^{[i]}$ with entry x . It then follows that the action of the internal row insertion operator ϕ_i on T (3) may be read as an operation which bumps the entry x in the cell $(i, \mu_i + 1)$ of $T^{[i]}$ and then inserts externally the bumped element x in the subtableau T' . That is, ϕ_i on $T^{[i]}$ bumps the entry x and left justifies it in the $(i + 1)$ th row. Then

$$\phi_i(T^{[i]}) = x * T_-^{[i]} \quad (47)$$

$$\bar{\phi}_i(Y_{(\mu_1, \dots, \mu_i)} \cup T^{[i]}) = x * (Y_{(\mu_1, \dots, \mu_i+1)} \cup T_-^{[i]}), \quad (48)$$

where $T_-^{[i]}$ is $T^{[i]}$ with the entry x bumped out from the cell $(i, \mu_i + 1)$. It means that $T_-^{[i]} \in YT((\lambda_1, \dots, \lambda_i)/(\mu_1, \dots, \mu_i + 1))$, that is, $T^{[i]}$ with the left most entry of the i th row suppressed and the corresponding blank cell added to the i th row of the inner shape of $T^{[i]}$. Henceforth, from (46), (47) and (48),

$$\phi_i(T) = \phi_i[T' * T^{[i]}] = T' \cdot \phi_i(T^{[i]}) = T' \cdot [x * T_-^{[i]}] = (T' \cdot x) * T_-^{[i]}, \quad (49)$$

$$\begin{aligned} \bar{\phi}_i(Y_\mu \cup T) &= \bar{\phi}_i[(Y' \cup T') * (Y_{(\mu_1, \dots, \mu_i)} \cup T^{[i]})] = (Y' \cup T') \cdot \bar{\phi}_i(Y_{(\mu_1, \dots, \mu_i)} \cup T^{[i]}) \\ &= (Y' \cup T') \cdot [x * (Y_{(\mu_1, \dots, \mu_i+1)} \cup T_-^{[i]})] \\ &= [(Y' \cup T') \cdot x] * (Y_{(\mu_1, \dots, \mu_i+1)} \cup T_-^{[i]}) \\ &= (Y' \cup T') \cdot x * (Y_{(\mu_1, \dots, \mu_i+1)} \cup T_-^{[i]}). \end{aligned} \quad (50)$$

If $\mu_i = \lambda_i$ and $(i, \lambda_i + 1)$ is a blank inner corner then $x = \emptyset$ and $\phi_i T^{[i]} = \emptyset * T_-^{[i]} = T_-^{[i]}$, with $T_-^{[i]} \in YT((\lambda_1, \dots, \lambda_i + 1)/(\mu_1, \dots, \mu_i + 1))$ a. Also $\phi_i T = T' * T_-^{[i]}$.

As an aside, observe that $w(\phi_i T^{[i]}) = w(x * T_-^{[i]}) = x w(T_-^{[i]}) = w(T^{[i]})$, and obviously $\phi_i T^{[i]} \equiv T^{[i]}$. Therefore, using Lemma 2,

$$\begin{aligned} w(\phi_i T) &= w(T' \cdot x * T_-^{[i]}) = w(T' \cdot x) w(T_-^{[i]}) \equiv w(T') x w(T_-^{[i]}) = w(T') w(\phi_i T^{[i]}) \\ &= w(T') w(T^{[i]}) = w(T' * T^{[i]}) = w(T). \end{aligned} \quad (51)$$

Remark 6. Let $Y_{(\mu_1, \dots, \mu_{n+1})} \cup T \in \mathcal{LR}^{(n+1)}$ be a ballot tableau pair and $1 \leq m \leq n$. Consider the factorisation through row m ,

$$Y_{(\mu_1, \dots, \mu_{n+1})} \cup T = [Y_{(\mu_{m+1}, \dots, \mu_{n+1})} \cup \hat{T}] * [Y_{(\mu_1, \dots, \mu_m)} \cup T^{[m]}] \in \mathcal{LR}^{(n+1)}.$$

Then $T^{[m]}$ is a ballot tableau on the alphabet $[m]$, and $\widehat{T} \in YT((\lambda_{m+1}, \dots, \lambda_{n+1})/(\mu_{m+1}, \dots, \mu_{n+1}))$ consisting of the last $n+1-m$ rows of T is such that the word restricted to the alphabet $[m+1, n+1]$ satisfies the Yamanouchi condition.

Example 10. An illustration of (49) and (50) is given below:

$$\begin{aligned}
\bar{\phi}_3 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & 3 & & \\ \hline 4 & 3 & 4 & 4 & & \\ \hline 5 & 5 & & & & \\ \hline \end{array} &= \begin{array}{|c|c|c|c|} \hline 4 & 3 & 4 & 4 \\ \hline 5 & 5 & & \\ \hline \end{array} \cdot \bar{\phi}_3 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & 3 & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 4 & 3 & 4 & 4 \\ \hline 5 & 5 & & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 3 & 3 & 3 & & \\ \hline 2 & & & & & \\ \hline \end{array} \\
&= \begin{array}{|c|c|c|c|} \hline 4 & 3 & 4 & 4 \\ \hline 5 & 5 & & \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 2 \\ \hline \end{array} * \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 3 & 3 & 3 & & \\ \hline \end{array} \\
&= \begin{array}{|c|c|c|c|} \hline 4 & 2 & 4 & 4 \\ \hline 3 & 5 & & \\ \hline 5 & & & \\ \hline \end{array} * \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 3 & 3 & 3 & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 4 & 2 & 4 & 4 \\ \hline 3 & 5 & & \\ \hline 5 & & & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 3 & 3 & 3 & & \\ \hline \end{array} \\
&= \begin{array}{|c|c|c|c|} \hline 4 & 2 & 4 & 4 \\ \hline 3 & 5 & & \\ \hline 5 & & & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|} \hline 3 & 3 & 3 & 3 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline \end{array} \\
&= \begin{array}{|c|c|c|c|} \hline 3 & 3 & 3 & 3 \\ \hline 4 & 2 & 4 & 4 \\ \hline 3 & 5 & & \\ \hline 5 & & & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 3 & 3 & 3 & & \\ \hline 4 & 2 & 4 & 4 & & \\ \hline 3 & 5 & & & & \\ \hline 5 & & & & & \\ \hline \end{array} .
\end{aligned}$$

An *internal row insertion operator* $\bar{\phi}_i$ (ϕ_i) on $Y \cup T$ (T) determines a collection R_i of boxes, which are those where an element is bumped from a row, together with the box where the last bumped element lands and settles [Ful97]. Let us call to R_i the $\bar{\phi}_i$ -bumping route of $Y \cup T$. In particular, retain that whenever $\bar{\phi}_i$ acts on $Y \cup T$ and the $\bar{\phi}_i$ -bumping route terminates in some row $k > i$, this means that the k th row of $\bar{\phi}_i(Y \cup T)$ equals the k th row of $Y \cup T$ with the last bumped entry in $\{T\}$ added at the end. If $\ell(\lambda) \leq n$, the ϕ_i -bumping path (route) terminates in some row $\leq n+1$.

For instance, below $Y \cup T$, $Y \cup H \in \mathcal{LR}^{(5)}$, and $\bar{\phi}_3$ acting on $Y \cup T$ bumps 2 and fills the vacant cell with 3, then the bumped 2 is inserted in the 4th row and bumps 3, 3 bumps 5 which lands in the 6th row, and 5 is added to the 6th row. In (53), R_3 consists of the black boxes and the bumped numbers are 2, 3 and 5, highlighted with circles, the entries of the three last boxes of R_3 . The ϕ_1 -bumping routes R_1 in $Y \cup T$ and $Y \cup H$ are similarly displayed below in (53):

$$\begin{aligned}
Y \cup T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & 3 & & \\ \hline 1 & 3 & 4 & 4 & & \\ \hline 5 & 5 & & & & \\ \hline \end{array} & \bar{\phi}_3(Y \cup T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 3 & 3 & 3 & & \\ \hline 1 & 2 & 4 & 4 & & \\ \hline 3 & 5 & & & & \\ \hline 5 & & & & & \\ \hline \end{array} & \bar{\phi}_1(Y \cup T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 2 & 3 & & \\ \hline 1 & 3 & 3 & 4 & & \\ \hline 4 & 5 & & & & \\ \hline 5 & & & & & \\ \hline \end{array} \quad (52) \\
Y \cup H = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & 3 & & \\ \hline 1 & 3 & & & & \\ \hline 4 & 4 & & & & \\ \hline \end{array} & \bar{\phi}_1(Y \cup H) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 2 & 3 & & \\ \hline 1 & 3 & 3 & & & \\ \hline 4 & 4 & & & & \\ \hline \end{array} .
\end{aligned}$$

The $\bar{\phi}_3$ -bumping routes R_3 , $\bar{\phi}_1$ -bumping route R_1 of $Y \cup T$, and $\bar{\phi}_1$ -bumping route of $Y \cup H$

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & \blacksquare & 3 & 3 & & \\ \hline 1 & \blacksquare & 4 & 4 & & \\ \hline \blacksquare & 5 & & & & \\ \hline \blacksquare & & & & & \\ \hline \end{array}
 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & \blacksquare & 1 \\ \hline 2 & 2 & 1 & \blacksquare & 2 & \\ \hline 3 & 2 & \blacksquare & 3 & & \\ \hline 1 & 3 & \blacksquare & 4 & & \\ \hline \blacksquare & 5 & & & & \\ \hline \blacksquare & & & & & \\ \hline \end{array}
 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & \blacksquare & 1 \\ \hline 2 & 2 & 1 & \blacksquare & 2 & \\ \hline 3 & 2 & \blacksquare & 3 & & \\ \hline 1 & 3 & \blacksquare & & & \\ \hline 4 & 4 & & & & \\ \hline \end{array} . \quad (53)$$

One often says a is a $\bar{\phi}_i$ -bumped element of the skew-tableau T to mean an entry a of T that is bumped under the action of $\bar{\phi}_i$ on T . Thanks to (49), one has as a consequence of the Row External Bumping Lemma in Section 1.1 of [Ful97]. This Lemma is instrumental in the proof of the Main Theorem and clearly shows that the internal insertion operators do not obey a naive commutation but instead a Knuth relation commutation as we shall see in the next section.

Lemma 3. ([Ful97] Row internal bumping routes) Consider $1 \leq i \leq j \leq n$. Let R_j and R'_i be the pair of bumping routes of $\bar{\phi}_j$ on $Y \cup T$ and $\bar{\phi}_i$ on $\bar{\phi}_j(Y \cup T)$ respectively; and let R_i and R'_j be the pair of bumping routes of $\bar{\phi}_i$ on $Y \cup T$ and $\bar{\phi}_j$ on $\bar{\phi}_i(Y \cup T)$ respectively. Let B and B' be the corresponding pair of new boxes. Then it holds

(a) R_j is strictly left of R'_i and B is strictly left of and weakly below B' :

$$B \ B' \quad \text{or} \quad B \ B'.$$

(b) R'_j is weakly left of R_i and B' is weakly left of and strictly below B :

$$B' \quad \text{or} \quad B' B. \quad (54)$$

In particular, R'_j goes always strictly below the bottom box B of R_i by $\bar{\phi}_j$ -bumping the element in B , this necessarily happens in the case of the left hand side of (54), or by passing strictly to the left of B . Henceforth, if x is $\bar{\phi}_i$ -bumped and y is $\bar{\phi}_j$ -bumped from the same row then $y < x$. Moreover, if B was created in the $(n+1)$ th row then one has B and the last $\bar{\phi}_i$ -bumped element resting in B is $\bar{\phi}_j$ -bumped out to be settled in B' and is strictly bigger than the element $\bar{\phi}_j$ -inserted in B .

Example 11. An illustration of the internal insertion bumping routes lemma:

$$Y \cup T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & & \\ \hline 3 & 3 & 2 & 3 & & \\ \hline 4 & 1 & 3 & 4 & & \\ \hline 1 & 4 & 5 & & & \\ \hline 2 & & & & & \\ \hline \end{array}
 \quad \bar{\phi}_3 \bar{\phi}_1(Y \cup T) = \bar{\phi}_3 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \blacksquare & 1 & \\ \hline 2 & 2 & 2 & \textcircled{1} & & \\ \hline 3 & 3 & 2 & \textcircled{2} & & \\ \hline 4 & 1 & 3 & \textcircled{3} & & \\ \hline 1 & 4 & \textcircled{4} & & & \\ \hline 2 & \textcircled{5} & & & & \\ \hline \end{array}
 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \blacksquare & 1 & \\ \hline 2 & 2 & 2 & \textcircled{1} & & \\ \hline 3 & 3 & \blacksquare & \textcircled{2} & & \\ \hline 4 & 1 & \textcircled{2} & \textcircled{3} & & \\ \hline 1 & \textcircled{3} & \textcircled{4} & & & \\ \hline 2 & \textcircled{4} & & & & \\ \hline \textcircled{5} & & & & & \\ \hline \end{array} \quad (55)$$

$$\bar{\phi}_1 \bar{\phi}_3(Y \cup T) = \bar{\phi}_1 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & & \\ \hline 3 & 3 & \blacksquare & 3 & & \\ \hline 4 & 1 & \textcircled{2} & 4 & & \\ \hline 1 & \textcircled{3} & 5 & & & \\ \hline 2 & \textcircled{4} & & & & \\ \hline \end{array}
 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \blacksquare & 1 & \\ \hline 2 & 2 & 2 & \textcircled{1} & & \\ \hline 3 & 3 & \blacksquare & \textcircled{2} & & \\ \hline 4 & 1 & \textcircled{2} & \textcircled{3} & & \\ \hline 1 & \textcircled{3} & \textcircled{4} & & & \\ \hline 2 & \textcircled{4} & \textcircled{5} & & & \\ \hline \end{array}, \quad (56)$$

$$\bar{\phi}_1 \bar{\phi}_3(Y \cup T) \neq \bar{\phi}_3 \bar{\phi}_1(Y \cup T).$$

4.3. Proof of Theorem 1: Internal row insertion operators satisfy Knuth relations. In this section, for a fixed T , a sufficient condition on U for the coincidence of $P(T, U)$ in the Sagan-Stanley internal row insertion bijection in Theorem 4 is provided. The sufficient condition does not involve the tableau U directly but rather its companion word. The companion word of U , Definition 1, encodes the inner corners for the action of the sequence of internal insertion operators acting on T . Before giving the proof of Theorem 1 (Theorem 5) we start with some warmup results.

When the shape β of U is a rectangle of height $\geq \ell(\mu)$ and width $\geq \alpha_1$, then the part of P occupying the rows $> \ell(\beta)$ is a rectification of T . Then the internal insertion procedure is independent of a particular sequence of inner corners in T chosen [RSSW01]. We may therefore have $P(T, U) = P(T, U')$ with $U \neq U'$ and as we have seen this happens for example when the shape β of U and U' is a rectangle of height $\geq \ell(\mu)$ and width $\geq \mu_1$. We have the following characterization which only takes into account the shape of U and is independent of T .

Lemma 4. Let $T \in YT(\alpha/\mu)$ and $U \in YT(\beta/\mu)$ where β is of rectangle shape. Then $\mathcal{R}(U) \equiv \ell(\beta)^{\beta_1 - \mu_{\ell(\beta)}} \dots 2^{\beta_1 - \mu_2} 1^{\beta_1 - \mu_1}$, is a reverse Yamanouchi word of content $(\beta_1^{\ell(\beta)}) - \mu = (\beta_1 - \mu_1, \beta_1 - \mu_2, \dots, \beta_1 - \mu_{\ell(\beta)})$, and, therefore, $P(T, U) = P(T, U')$ is the skew tableau with inner shape the rectangle diagram $\emptyset_\beta$ where below it is the rectification of the subtableau of T consisting of the first β_1 columns, and to the right of it is the subtableau of T consisting of the last $\max\{\alpha_1 - \beta_1, 0\}$ columns of T .

Knuth equivalence on $\mathcal{R}(U)$ and $\mathcal{R}(U')$ provide only sufficient conditions for the equality of $P(T, U) = P(T, U')$. In fact, we may have $P(T, U) = P(T, U')$ and $\mathcal{R}(U) \neq \mathcal{R}(U')$. The example below illustrates this fact.

Example 12. (1) Companion words of U and U' :

$$U = \begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline 2 & 2 & \\ \hline \end{array} \quad stdU = \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 3 & 4 & \\ \hline \end{array} \quad U' = \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 1 & 2 & \\ \hline \end{array} \quad stdU' = \begin{array}{|c|c|c|} \hline & 2 & 4 \\ \hline 1 & 3 & \\ \hline \end{array}$$

$$\mathcal{R}(U) = 2211 = \mathcal{R}(stdU), \quad \mathcal{R}(U') = \mathcal{R}(stdU') = 1212.$$

(b) $\mathcal{R}(U)$ and $\mathcal{R}(U')$ are internal insertion order words of $T = \begin{array}{|c|c|c|} \hline & 1 & 3 \\ \hline 2 & 3 & \\ \hline \end{array}$, and

$$\phi_{2211}T = \begin{array}{|c|c|c|} \hline & & \\ \hline & & 3 \\ \hline 1 & 3 & \\ \hline 2 & & \\ \hline \end{array} = \phi_{1212}T$$

but $R(U) \neq R(U')$. However, this property does not hold for every given T having $\mathcal{R}(U)$ and $\mathcal{R}(U')$ as insertion words which shows that necessary and sufficient conditions for the equality $P(T, U) = P(T, U')$ also depend on T .

For $T = \begin{array}{|c|c|c|c|} \hline & 1 & 2 & 3 \\ \hline 3 & 3 & & \\ \hline \end{array}$, one has

$$\phi_{2211}T = \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & & & \\ \hline 1 & 2 & & \\ \hline 3 & 3 & & \\ \hline \end{array} \neq \phi_{1212}T = \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & & 2 & \\ \hline 1 & 3 & & \\ \hline 3 & & & \\ \hline \end{array}$$

(2) Companion words of V and V' below are internal insertion order words of

$$T' = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 & \\ \hline 3 & & \\ \hline \end{array}, \quad V = \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 1 & \\ \hline 3 & & \\ \hline \end{array} \equiv V' = \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 3 & \\ \hline 1 & & \\ \hline \end{array}, \quad R(V) = 312 \neq R(V') = 213,$$

$$\phi_{312}T' = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array} \neq \phi_{213}T' = \begin{array}{|c|c|c|} \hline & & \\ \hline & 1 & \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array}$$

$$(3) \quad W_1 = \begin{array}{|c|c|} \hline & 1 \\ \hline 1 & 2 \\ \hline \end{array}, W_2 = \begin{array}{|c|c|} \hline & 1 \\ \hline 2 & 2 \\ \hline \end{array}, R(W_1) = 212 \equiv R(W_2) = 221, 221 \equiv 212 \text{ but } 2211 \not\equiv 1212$$

$$\phi_{R(W_2)}T = \phi_{221}T = \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & & \\ \hline 1 & & 3 \\ \hline 2 & & \\ \hline \end{array} = \phi_{212}T = \phi_{R(W_1)}T$$

but as we have seen above $\phi_{2211}T = \phi_{1212}T$. On the other hand, if $T = \begin{array}{|c|c|c|} \hline & 1 & 3 \\ \hline 2 & 3 & \\ \hline \end{array}$, $U = \begin{array}{|c|c|} \hline & 1 \\ \hline 2 & \\ \hline \end{array}$

$$\text{and } V = \begin{array}{|c|c|} \hline & 3 \\ \hline 1 & \\ \hline \end{array}, \phi_2\phi_1T = \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & & \\ \hline 1 & & 3 \\ \hline 2 & & \\ \hline \end{array} \neq \phi_1\phi_2T = \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & 1 & \\ \hline 2 & 3 & \\ \hline \end{array}. \text{ However with } U' = \begin{array}{|c|c|} \hline & 2 \\ \hline 1 & \\ \hline 3 & \\ \hline \end{array} \text{ and}$$

$$V' = \begin{array}{|c|c|} \hline & 3 \\ \hline 1 & \\ \hline 2 & \\ \hline \end{array}, R(U') = 312 \equiv R(V') = 132, \text{ one has } \phi_{312}T = \phi_{132}T = \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & 1 & \\ \hline & 3 & \\ \hline 2 & & \\ \hline \end{array}.$$

The following lemmata and proposition show that Sagan-Stanley internal row insertion operators satisfy Knuth relations. Let T be a SSYT of shape λ/μ with $\ell(\lambda) = n$. When we write $\phi_u(T)$, for some word u , it is assumed that there exists $U \in YT(\beta/\mu)$ such that $\mathcal{R}(U) = u$, that is, u is an internal insertion order word for T .

Observe that if m is the largest entry of $U \in YT(\beta/\mu)$ with γ its content, and $1 \leq d \leq m$, we may decompose $U = U_{[d]} \cup U_{[d+1, m]}$ and $\text{std}U = (\text{std}U)_{[d]} \cup (\text{std}U)_{[d+1, m]}$ with $q = \gamma_1 + \dots + \gamma_d$, and $\mathcal{R}(U) = \mathcal{R}(U_{[d]})\mathcal{R}(U_{[d+1, m]})$. On the other hand, if for some $V \in YT(\gamma/\epsilon)$, $\mathcal{R}(\text{std}V) = u_3u_2u_1 = ijk$ and $1 \leq i \leq k < j$ (respectively $\mathcal{R}(\text{std}V) = u_3u_2u_1 = kji$ and $1 \leq i < k \leq j$) then γ is not a row nor a column and it is easily checked that there exists $V' \in YT(\gamma/\epsilon)$ such that $\mathcal{R}(\text{std}V') = jik$ (respectively $\mathcal{R}(\text{std}V') = kij$). Without loss of generality, we may consider $i < k = i+1 < j = i+2$ or $i = k < j = i+1$,

$$V = \begin{array}{|c|c|} \hline & 2 \\ \hline 1 & \\ \hline 3 & \\ \hline \end{array} \text{ or } \tilde{V} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \text{ and } V' = \begin{array}{|c|c|} \hline & 3 \\ \hline 1 & \\ \hline 2 & \\ \hline \end{array} \text{ or } \tilde{V}' = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, \mathcal{R}(V) = jik \equiv \mathcal{R}(V') = ijk, \mathcal{R}(\tilde{V}) = iji \equiv$$

$$\mathcal{R}(\tilde{V}') = jii; \text{ and } i < k = i+1 < j = i+2, V = \begin{array}{|c|c|} \hline & 1 \\ \hline & 3 \\ \hline 2 & \\ \hline \end{array}, V' = \begin{array}{|c|c|} \hline & 2 \\ \hline & 3 \\ \hline 1 & \\ \hline \end{array}, \mathcal{R}(V) = kji \equiv \mathcal{R}(V') = kij.$$

Reciprocally if $\mathcal{R}(\text{std}U) = wz$ then $U = Z \cup W$ with $\mathcal{R}(\text{std}W) = w$ and $\mathcal{R}(\text{std}Z) = z$. Let $\mathcal{R}(\text{std}U) = u_2jiku_1$ with $1 \leq i \leq k < j$ and u_1, u_2 words. Decompose $\text{std}U = U_1 \cup V \cup U_2$ such that $\mathcal{R}(U_1) = u_1$, $\mathcal{R}(V) = ijk$ and $\mathcal{R}(U_2) = u_2$. Then $U' := U_1 \cup V' \cup U_2$ has $\mathcal{R}(U') = u_2jiku_1$. Therefore if u is an internal insertion order word of T and $u' \equiv u$ then u' is also an internal insertion order word of T . Lemmata and proposition below show that the plactic class of an internal insertion order word of T gives rise to the same P -tableau of T in the Sagan-Stanley Theorem 4.

Lemma 5. Let $T \in YT(\lambda/\mu)$ with $\ell(\lambda) = n$. Let $n-1 \leq \ell(\mu) \leq n$. Then

$$\bar{\phi}_i \bar{\phi}_n \bar{\phi}_k T = \bar{\phi}_n \bar{\phi}_i \bar{\phi}_k T, \quad 1 \leq i \leq k < n. \quad (57)$$

In addition, if $\ell(\mu) = n$,

$$\bar{\phi}_i \bar{\phi}_{n+1} \bar{\phi}_k T = \bar{\phi}_{n+1} \bar{\phi}_i \bar{\phi}_k T, \quad 1 \leq i \leq k < n+1. \quad (58)$$

Proof. Recall that we are assuming that $(k, \mu_k + 1)$ is an inner corner of T .

We start with identity (57). Let $T = U * V$ be the factorization of T across the k th row, $1 \leq i \leq k < n$. Let $\phi_k V = \alpha * V'$ where $|\alpha| = 0, 1$ such that α is the empty word if and only if $\mu_k = \lambda_k$ and in this case V' is obtained from V by adding an empty box at the end of row k , otherwise the cell $(k, \mu_k + 1)$ is vacated from the entry α which migrates to the cell $(k+1, 1)$. Let $\phi_i V' = \beta * V''$. Since $i \leq k$, by Lemma 3, (a),

if $|\alpha| = 0$ then $|\beta| = 0$ and $\phi_i(V') = V''$, and if $|\alpha| = 1$, then either $|\beta| = 0$ or $\alpha \leq \beta$. Then, using the factorization (49), one has to analyse the cases

I. $|\alpha| = 0$. One has $\bar{\phi}_k V = V'$ and $\bar{\phi}_i V' = V''$, and it follows

$$\begin{aligned}\bar{\phi}_i \bar{\phi}_n \phi_k(U * V) &= \bar{\phi}_i \bar{\phi}_n(U * V') = \bar{\phi}_n U * \bar{\phi}_i V' = \bar{\phi}_n U * V'' = \bar{\phi}_n \bar{\phi}_i(U * V') \\ &= \bar{\phi}_n \bar{\phi}_i \bar{\phi}_k(U * V).\end{aligned}$$

II. $|\alpha| = 1$.

(a) If $|\beta| = 0$, one has $\phi_k V = \alpha * V'$ and $\phi_i V' = V''$, and

$$\begin{aligned}\bar{\phi}_i \phi_n \bar{\phi}_k(U * V) &= \bar{\phi}_i(\bar{\phi}_n(U \cdot \alpha) * V') = \bar{\phi}_n(U \cdot \alpha) * V'' = \phi_n \bar{\phi}_i(U \cdot \alpha) * V' \\ &= \phi_n \phi_i \phi_k(U * V).\end{aligned}$$

(b) If $|\beta| = 1$, one has $\bar{\phi}_k V = \alpha * V'$ and $\bar{\phi}_i V' = \beta * V''$, with $\alpha \leq \beta$, that is, $\phi_i \phi_k V = \alpha \beta * V''$, $\alpha \leq \beta$, and

$$\begin{aligned}\bar{\phi}_i \bar{\phi}_n \phi_k(U * V) &= \bar{\phi}_i \bar{\phi}_n(U \cdot \bar{\phi}_k V) = \bar{\phi}_i \bar{\phi}_n[(U \cdot \alpha) * V'] \\ &= \bar{\phi}_n(U \cdot \alpha) \cdot \bar{\phi}_i V' = [\bar{\phi}_n(U \cdot \alpha) \cdot \beta] * V'' \\ &= [\bar{\phi}_n(U \cdot \alpha \beta)] * V'', \quad \alpha \leq \beta; \text{ Lemma 3, (a), external bumping version,} \\ &= \bar{\phi}_n(U \cdot \alpha \beta * V'') = \bar{\phi}_n(U \cdot \bar{\phi}_i \bar{\phi}_k V) = \bar{\phi}_n \bar{\phi}_i(U \cdot \bar{\phi}_k V) = \bar{\phi}_n \bar{\phi}_i \bar{\phi}_k(U * V).\end{aligned}$$

It is similarly checked that (58) holds. $\square$

Lemma 6. Let $T \in YT(\lambda/\mu)$ with $\ell(\lambda) = n$. Let $n - 1 \leq \ell(\mu) \leq n$. Then

$$\bar{\phi}_k \bar{\phi}_i \bar{\phi}_n T = \bar{\phi}_k \bar{\phi}_n \bar{\phi}_i T, \quad 1 \leq i < k \leq n. \quad (59)$$

In addition if $\ell(\mu) = n$,

$$\bar{\phi}_k \phi_i \bar{\phi}_{n+1} T = \bar{\phi}_k \bar{\phi}_{n+1} \bar{\phi}_i T, \quad 1 \leq i < k \leq n + 1. \quad (60)$$

Proof. We first consider (59) with $1 \leq i < k < n$. Let $T = U * W * V$ be the factorization of T across the $(k - 1)$ th and $(n - 1)$ th rows. Since $\ell(\lambda) = n$ and $n - 1 \leq \ell(\mu) \leq n$, let $U = \emptyset_{\mu_n} \cdot u$ with $u = u_1 \dots u_r$, $r = \lambda_n - \mu_n$, be the n th row of T with $|u| \geq 0$ and $\mu = (\mu_1 \dots, \mu_n)$, $\mu_n \geq 0$, and $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_n > 0$. Using the factorization (49), one has

$$\begin{aligned}\bar{\phi}_k \bar{\phi}_i \bar{\phi}_n(U * W * V) &= \bar{\phi}_k \phi_i[(\bar{\phi}_n U) * W * V] \\ &= \bar{\phi}_k[\bar{\phi}_n U \cdot \bar{\phi}_i(W * V)], \quad i < n, \\ &= \bar{\phi}_k[(\phi_n U \cdot \beta) * W' * V'], \quad \phi_i(W * V) = \beta * W' * V', \quad |\beta| \geq 0, \\ &= (\phi_n U \cdot \beta) \cdot \phi_k W' * V'\end{aligned} \quad (61)$$

and

$$\begin{aligned}\bar{\phi}_k \bar{\phi}_n \bar{\phi}_i(U * W * V) &= \bar{\phi}_k \phi_n[U \cdot \phi_i(W * V)] = \phi_k \phi_n(U \cdot \beta * W' * V') \\ &= \bar{\phi}_n(U \cdot \beta) \cdot \bar{\phi}_k W' * V'.\end{aligned} \quad (62)$$

We want to show that (61) and (62) are equal, that is, $\bar{\phi}_n(U \cdot \beta) = \bar{\phi}_n U \cdot \beta$. We have two main cases, either the ϕ_i -bumping route reaches the n th row or not.

I. The ϕ_i -bumping route does not reach the n -th row, that is, $|\beta| = 0$. Then $\bar{\phi}_n(U \cdot \beta) = \bar{\phi}_n U = \phi_n U \cdot \beta$, and (61) = (62) = $\bar{\phi}_n U \cdot \bar{\phi}_k W' * V'$.

II. The ϕ_i -bumping route reaches the n -th row. That is $|\beta| = 1$, and β is the ϕ_i -bumped element from the $(n - 1)$ th row to the n th row. Since $i < k$, by Lemma 3, (b), $\phi_k W' = @ * W''$ with $\beta > @$. Then

$$(61) = (\bar{\phi}_n \cdot \beta) \cdot \bar{\phi}_k W' * V' = (\bar{\phi}_n U \cdot \beta \cdot @) * W'' * V',$$

$$(62) = \bar{\phi}_n(U \cdot \beta) \cdot \bar{\phi}_k W' * V' = \bar{\phi}_n(U \cdot \beta) \cdot @ * W'' * V'.$$

Next we consider the following cases according to the length of u in the n th row U of T .

(a) $|u| = 0$. One has $U = \emptyset_{\mu_n}$, $\bar{\phi}_n(U \cdot \beta) = \bar{\phi}_n(\emptyset_{\mu_n} \cdot \beta) = \beta * \emptyset_{\mu_n+1}$. Then, since $\beta > @$, it follows

$$\begin{aligned} \bar{\phi}_n U \cdot \beta \cdot @ &= \emptyset_{\mu_n+1} \cdot \beta \cdot @ = \beta * \emptyset_{\mu_n+1} @, \\ \phi_n(U \cdot \beta) \cdot @ &= \phi_n(\emptyset_{\mu_n} \beta) \cdot @ = \beta * \emptyset_{\mu_n+1} @, \text{ and} \\ (61) = (62) &= \beta * \emptyset_{\mu_n+1} @ * W'' * V'. \end{aligned} \tag{63}$$

(b) $|u| = 1$. One has $U = \emptyset_{\mu_n} u_1$, $\bar{\phi}_n U \cdot \beta = u_1 * \emptyset_{\mu_n+1} \beta$ and

$$\bar{\phi}_n(U \cdot \beta) = \bar{\phi}_n(\emptyset_{\mu_n} u_1 \cdot \beta) = \begin{cases} \bar{\phi}_n(\emptyset_{\mu_n} u_1 \beta), & u_1 \leq \beta, \\ \phi_n(\emptyset_{\mu_n} u_1 * \beta), & u_1 > \beta. \end{cases}$$

Henceforth

$$\begin{aligned} \phi_n U \cdot \beta \cdot @ &= (u_1 * \emptyset_{\mu_n+1} \beta) \cdot @ = \begin{cases} u_1 \beta * \emptyset_{\mu_n+1} @, & u_1 \leq \beta, \\ \beta_{u_1} * \emptyset_{\mu_n+1} @, & u_1 > \beta, \end{cases} \\ \bar{\phi}_n(U \cdot \beta) \cdot @ &= \begin{cases} \phi_n(\emptyset_{\mu_n} u_1 \beta) \cdot @ = u_1 * \emptyset_{\mu_n+1} \beta \cdot @ = u_1 \beta * \emptyset_{\mu_n+1} @, & u_1 \leq \beta, \\ \phi_n(u_1 * \emptyset_{\mu_n+1} \beta) @ = \beta_{u_1} * \emptyset_{\mu_n+1} @, & u_1 > \beta. \end{cases} \end{aligned}$$

Thus (61) = (62) are equal to $u_1 \beta * \emptyset_{\mu_n+1} @ * W'' * V'$, if $u_1 \leq \beta$, or $\beta_{u_1} * \emptyset_{\mu_n+1} @ * W'' * V'$, otherwise.

(c) $|u| \geq 2$. One has $U = \emptyset_{\mu_n} u_1 \cdots u_r$, with $r \geq 2$, and $\phi_n U = u_1 * \emptyset_{\mu_n+1} u_2 \cdots u_r$.

Either $u_r \leq \beta$ or $u_r > \beta$.

(i) $u_r \leq \beta$.

Since $\beta > @$, let $x := \min\{z \in \{u_2, \dots, u_r, \beta\} : z > @\}$ and $(u_2 \dots u_r \beta) \cdot @ =: x * u'$. Note $x \geq u_2 \geq u_1$. Henceforth

$$\bar{\phi}_n(U \cdot \beta) \cdot @ = \phi_n(\emptyset_{\mu_n} u_1 \cdots u_r \beta) \cdot @ = u_1 * (\emptyset_{\mu_n+1} u_2 \cdots u_r \beta) \cdot @ = u_1 x * \emptyset_{\mu_n+1} u',$$

and

$$\phi_n U \cdot \beta \cdot @ = u_1 * (\emptyset_{\mu_n+1} u_2 \cdots u_r) \cdot \beta \cdot @ = u_1 * (\emptyset_{\mu_n+1} u_2 \cdots u_r \beta) \cdot @ = u_1 x * \emptyset_{\mu_n+1} u'.$$

Therefore, (61) = (62) = $u_1 x * \emptyset_{\mu_n+1} u' * W'' * V'$.

(ii) It remains to study when $u_r > \beta$.

(ii.1) $u_1 > \beta$. One has $@ < \beta < u_1 \leq u_2 \leq \dots \leq u_r$,

$$\begin{aligned} \bar{\phi}_n(U \cdot \beta) \cdot @ &= \bar{\phi}_n(u_1 * \emptyset_{\mu_n} \beta u_2 \cdots u_r) \cdot @ = \beta_{u_1} * (\emptyset_{\mu_n+1} u_2 \cdots u_r) \cdot @ \\ &= \beta_{u_1} u_2 * \emptyset_{\mu_n+1} @ u_3 \cdots u_r, \end{aligned}$$

and

$$\begin{aligned} \bar{\phi}_n U \cdot \beta \cdot @ &= (u_1 * \emptyset_{\mu_n+1} u_2 \cdots u_r) \cdot \beta \cdot @ = u_1 u_2 * (\emptyset_{\mu_n+1} \beta u_3 \cdots u_r) \cdot @ \\ &= \beta_{u_1} u_2 * \emptyset_{\mu_n+1} @ u_3 \cdots u_r. \end{aligned}$$

Thus, (61) = (62) = $\beta_{u_1} u_2 * \emptyset_{\mu_n+1} @ u_3 \cdots u_r * W'' * V'$, where $u_1 u_2 \beta \equiv u_1 \beta u_2 \equiv \beta_{u_1} u_2$, with $\beta < u_1 \leq u_2$.

(ii.2) $u_r \geq \dots \geq u_i > \beta \geq u_{i-1} \geq \dots \geq u_1$, for some $i \in \{2, \dots, r\}$. One has

$$U \cdot \beta = \emptyset_{\mu_n} u_1 \cdots u_{i-1} u_i \cdots u_r \cdot \beta = u_i * \emptyset_{\mu_n} u_1 \cdots u_{i-1} \beta u_{i+1} \cdots u_r.$$

Let

$$x := \min\{z \in \{u_2, \dots, u_{i-1}, \beta\} : z > @\}$$

and $(u_2 \cdots u_{i-1} \beta u_{i+1} \cdots u_r) \cdot @ =: x * u'$. Note that $u_r \geq \dots \geq u_{i+1} \geq u_i > \beta \geq x \geq u_2 \geq u_1$. Henceforth

$$\begin{aligned} \bar{\phi}_n(U \cdot \beta) \cdot @ &= \bar{\phi}_n(u_i * \emptyset_{\mu_n} u_1 \cdots u_{i-1} \beta u_{i+1} \cdots u_r) \cdot @ = u_i * (\emptyset_{\mu_n+1} u_2 \cdots u_{i-1} \beta u_{i+1} \cdots u_r) \cdot @ \\ &= u_i x * \emptyset_{\mu_n+1} u', \end{aligned}$$

and

$$\begin{aligned}\bar{\phi}_n U \cdot \beta \cdot @ &= (u_1 * \emptyset_{\mu_n+1} u_2 \cdots u_r) \cdot \beta \cdot @ = u_1 u_i * (\emptyset_{\mu_n+1} u_2 \cdots u_{i-1} \beta u_{i+1} \cdots u_r) \cdot @ \\ &= u_1 u_i \cdot x * \emptyset_{\mu_n+1} u', \quad u_i > x \\ &= \overset{u_1}{u_i} x * \emptyset_{\mu_n+1} u'.\end{aligned}$$

Again, (61) = (62) = $\overset{u_1}{u_i} x * \emptyset_{\mu_n+1} u' * W'' * V'$ and $u_i u_1 x \equiv u_1 u_i x \equiv \overset{u_1}{u_i} x$, with $u_1 \leq x < u_i$.

We now study for $1 \leq i < k = n$, which is similarly checked.

$$\begin{aligned}\bar{\phi}_n \phi_i \phi_n (U * V) &= \phi_n (\bar{\phi}_n U \cdot \phi_i V), \quad \bar{\phi}_i V = \beta * V', \quad \ell(\beta) = 0, 1 \\ &= \phi_n (\phi_n U \cdot \beta) * V',\end{aligned}\tag{64}$$

$$\bar{\phi}_n \phi_n \phi_i (U * V) = \bar{\phi}_n \phi_n (U * \phi_i V) = \phi_n \phi_n [(U \cdot \beta) * V'].\tag{65}$$

I. $\ell(\beta) = 0$.

$$(64) = \phi_n (\phi_n U) * V' = \phi_n^2 U * V' = (65).$$

II. $\ell(\beta) = 1$.

(a) $\ell(u) = 0$. One has $U = \emptyset_{\mu_n} \Rightarrow \phi_n U = \emptyset_{\mu_n+1} \Rightarrow \phi_n (U \cdot \beta) = \beta * \emptyset_{\mu_n+1}$. Therefore

$$(64) = \bar{\phi}_n (\bar{\phi}_n U \cdot \beta) * V' = \bar{\phi}_n (\beta * \emptyset_{\mu_n+1}) * V' = \beta * \emptyset_{\mu_n+2} * V'.$$

and

$$(65) = \bar{\phi}_n \phi_n (U \cdot \beta) * V' = \bar{\phi}_n^2 (y\beta) * V' = \bar{\phi}_n (\beta * \emptyset_{\mu_n+1}) * V' = \beta * \emptyset_{\mu_n+2} * V'.$$

(b) $\ell(u) = 1$.

One has $U = \emptyset_{\mu_n} u_1$, $\phi_n U = u_1 * \emptyset_{\mu_n+1}$ and $\phi_n (U \cdot \beta) = \phi_n (\emptyset_{\mu_n} u_1 \cdot \beta)$. Then

$$\begin{aligned}(64) = \phi_n \phi_n (U \cdot \beta) * V' &= \phi_n^2 (\emptyset_{\mu_n} \beta) * V' \\ &= \phi_n (\beta * \emptyset_{\mu_n+1}) * V' = \begin{cases} u_1 \beta * \emptyset_{\mu_n+2} * V', & u_1 \leq \beta, \\ \overset{\beta}{u_1} * V', & u_1 > \beta, \end{cases}\end{aligned}$$

$$\begin{aligned}(65) = \bar{\phi}_n \bar{\phi}_n (U \cdot \beta) * V' &= \bar{\phi}_n^2 (y u_1 \cdot \beta) * V' = \begin{cases} \bar{\phi}_n^2 (\emptyset_{\mu_n} u_1 \beta) * V', & u_1 \leq \beta, \\ \bar{\phi}_n^2 (u_1 * y\beta) * V', & u_1 > \beta, \end{cases} \\ &= \begin{cases} \bar{\phi}_n (u_1 * \emptyset_{\mu_n+1} \beta) * V', & u_1 \leq \beta, \\ \bar{\phi}_n (\overset{\beta}{u_1} u_1 * \emptyset_{\mu_n+1}) * V', & u_1 > \beta, \end{cases} \\ &= \begin{cases} u_1 \beta * \emptyset_{\mu_n+2} * V', & u_1 \leq \beta, \\ \overset{\beta}{u_1} * \emptyset_{\mu_n+2} * V', & u_1 > \beta. \end{cases}\end{aligned}$$

(c) $\ell(u) \geq 2$.

Let $U = y u_1 \cdots u_r$, $r \geq 2$.

(i) $u_r \leq \beta$

$$\bar{\phi}_n (\phi_n U \cdot \beta) = \bar{\phi}_n (u_1 * \emptyset_{\mu_n+1} u_2 \cdots u_r \beta) = u_1 u_2 * \emptyset_{\mu_n+1} u_3 \cdots u_r \beta.$$

$$\bar{\phi}_n \bar{\phi}_n (U \cdot \beta) = \bar{\phi}_n (u_1 * \emptyset_{\mu_n+1} u_2 \cdots u_r \beta) = u_1 u_2 * \emptyset_{\mu_n+1} u_3 \cdots u_r \beta.$$

(ii) $u_1 > \beta$

$$\begin{aligned}\phi_n (\phi_n U \cdot \beta) &= \phi_n (u_1 * \emptyset_{\mu_n+1} u_2 \cdots u_r \cdot \beta) = \phi_n (u_1 u_2 * \emptyset_{\mu_n+1} \beta u_3 \cdots u_r) \\ &= \overset{\beta}{u_1} u_2 * \emptyset_{\mu_n+2} u_3 \cdots u_r * V'.\end{aligned}$$

$$\begin{aligned}\bar{\phi}_n \phi_n (U \cdot \beta) * V' &= \phi_n \phi_n (u_1 * y \beta u_2 \cdots u_r) * V' = \phi_n (\overset{\beta}{u_1} * \emptyset_{\mu_n+1} u_2 u_3 \cdots u_r) * V' \\ &= \overset{\beta}{u_1} u_2 * \emptyset_{\mu_n+2} u_3 \cdots u_r * V'.\end{aligned}$$

(iii) $u_r \geq \cdots \geq u_i > \beta \geq u_{i-1} \geq \cdots \geq u_1$, for some $i \in \{2, \dots, r\}$.

One has $U\beta = \emptyset_{\mu_n} u_1 \cdots u_{i-1} u_i \cdots u_r \bullet \beta = u_i * y u_1 \cdots u_{i-1} \beta u_{i+1} \cdots u_r$.

$$(64) = \bar{\phi}_n[(u_1 * y u_2 \cdots u_r) \bullet \beta] * V' = \bar{\phi}_n(u_1 u_i * \emptyset_{\mu_n+1} u_2 \cdots u_{i-1} \beta u_{i+1} \cdots u_r) * V' \\ = \overset{u_1}{u_i} \overset{u_2}{*} \emptyset_{\mu_n+2} u_3 \cdots u_{i-1} \beta u_{i+1} \cdots u_r * V'$$

$$(65) = \bar{\phi}_n \bar{\phi}_n(U\beta) * V' = \bar{\phi}_n^2[(u_i * y u_1 \cdots u_{i-1} \beta u_{i+1} \cdots u_r)] * V' \\ = \bar{\phi}_n(\overset{u_1}{u_i} * \emptyset_{\mu_n+1} u_2 \cdots u_{i-1} \beta u_{i+1} \cdots u_r) * V' \\ = \overset{u_1}{u_i} \overset{u_2}{*} \emptyset_{\mu_n+2} u_3 \cdots u_{i-1} \beta u_{i+1} \cdots u_r * V'.$$

It is similarly checked that (60) holds. $\square$

Remark 7. Let F be the tableau restricted to the rows strictly below the n th row of $\bar{\phi}_k \bar{\phi}_i \bar{\phi}_n T = \bar{\phi}_k \bar{\phi}_n \bar{\phi}_i T$, for $1 \leq i < k \leq n$. If w and w' are the words consisting of the elements of T successively bumped out from the n -th row under the action of $\bar{\phi}_k \bar{\phi}_i \bar{\phi}_n$ and $\bar{\phi}_k \bar{\phi}_n \bar{\phi}_i$, for $1 \leq i < k \leq n$, on T respectively, then $w \equiv w' \equiv F$. This easily follows from the fact that F is the external insertion of the elements of T successively bumped out of the n -th row. This also applies to the action of $\bar{\phi}_i \bar{\phi}_n \bar{\phi}_k$ and $\bar{\phi}_n \bar{\phi}_i \bar{\phi}_k$, for $1 \leq i \leq k < n$, on T .

Remark 8. Thanks to (3) and (36), lemmata 5 and 6 are generalized to $Y_\mu \cup T$.

Example 13. Illustration of Lemma 5, and of previous observations with $n = 3$,

$$\begin{aligned} \bar{\phi}_3 \bar{\phi}_1 \bar{\phi}_2 T &= \bar{\phi}_3 \bar{\phi}_1 \bar{\phi}_2 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & 3 & & \\ \hline \end{array} = \bar{\phi}_3 \bar{\phi}_1 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 3 & 1 & 3 & 3 & & \\ \hline 2 & & & & & \\ \hline \end{array} \\ &= \bar{\phi}_3 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 1 & 2 & \\ \hline 3 & 1 & 2 & 3 & & \\ \hline 2 & 3 & & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 1 & 2 & \\ \hline 3 & 3 & 2 & 3 & & \\ \hline 1 & 3 & & & & \\ \hline 2 & & & & & \\ \hline \end{array} \quad w = 231 \equiv \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \\ \bar{\phi}_1 \bar{\phi}_3 \bar{\phi}_2 T &= \bar{\phi}_1 \bar{\phi}_3 \bar{\phi}_2 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & 3 & & \\ \hline \end{array} = \bar{\phi}_1 \bar{\phi}_3 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & \\ \hline 3 & 1 & 3 & 3 & & \\ \hline 2 & & & & & \\ \hline \end{array} = \bar{\phi}_1 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 & & \\ \hline 1 & & & & & \\ \hline 2 & & & & & \\ \hline \end{array} \\ &= \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 1 & 2 & \\ \hline 3 & 3 & 2 & 3 & & \\ \hline 1 & 3 & & & & \\ \hline 2 & & & & & \\ \hline \end{array} = \bar{\phi}_3 \bar{\phi}_1 \bar{\phi}_2 T, \quad w = 231 \equiv w' = 213 \equiv \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}. \end{aligned}$$

Proposition 3. (Knuth relations of internal row insertion operators.) Let $Y \cup T$ be a tableau pair with $Y = Y_\mu$, T a SSYT of shape λ/μ and $\ell(\lambda) = n$. Suppose that kij with $1 \leq i < k \leq j \leq n+1$, or ijk with $1 \leq i \leq k < j \leq n+1$, are internal insertion order words of T . Then, it holds

$$\bar{\phi}_k \bar{\phi}_i \bar{\phi}_j(Y \cup T) = \bar{\phi}_k \bar{\phi}_j \bar{\phi}_i(Y \cup T), \quad 1 \leq i < k \leq j \leq n+1, \quad (66)$$

$$\bar{\phi}_i \bar{\phi}_j \bar{\phi}_k(Y \cup T) = \bar{\phi}_j \bar{\phi}_i \bar{\phi}_k(Y \cup T), \quad 1 \leq i \leq k < j \leq n+1. \quad (67)$$

More generally, if $u \equiv u'$ are internal insertion order words of T , then

$$\bar{\phi}_u(Y \cup T) = \bar{\phi}_{u'}(Y \cup T). \quad (68)$$

Proof. Let $1 < j \leq n$ and $Y_\mu \cup T = (Y' \cup T') * (Y_{(\mu_1, \dots, \mu_j)} \cup T^{[j]})$ with $Y' \cup T'$ the restriction of $Y \cup T$ to the last $n-j$ rows of $Y \cup T$. Lemmas 5 and 6 guarantee that

$$\bar{\phi}_k \bar{\phi}_i \bar{\phi}_j(Y_{(\mu_1, \dots, \mu_j)} \cup T^{[j]}) = \bar{\phi}_k \bar{\phi}_j \bar{\phi}_i(Y_{(\mu_1, \dots, \mu_j)} \cup T^{[j]}).$$

Let F be the restriction of $\bar{\phi}_k \bar{\phi}_i \bar{\phi}_j(Y_{(\mu_1, \dots, \mu_j)} \cup T^{[j]}) = \bar{\phi}_k \bar{\phi}_j \bar{\phi}_i(Y_{(\mu_1, \dots, \mu_j)} \cup T^{[j]})$ to the rows strictly below row j . Recalling Remark 7, the action of $\bar{\phi}_k \bar{\phi}_i \bar{\phi}_j$ and $\bar{\phi}_k \bar{\phi}_j \bar{\phi}_i$ on $Y \cup T$ inserts words w and w' respectively, which are Knuth equivalent to F , into $Y' \cup T'$,

$$\bar{\phi}_k \bar{\phi}_i \bar{\phi}_j(Y \cup T) = [(Y' \cup T').w] * \bar{\phi}_k \bar{\phi}_i \bar{\phi}_j(Y_{(\mu_1, \dots, \mu_j)} \cup T^{[j]}),$$

and

$$\bar{\phi}_i \bar{\phi}_j \bar{\phi}_k(Y \cup T) = [(Y' \cup T').w'] * \bar{\phi}_k \bar{\phi}_j \bar{\phi}_i(Y_{(\mu_1, \dots, \mu_j)} \cup T^{[j]}).$$

It follows from Lemma 2, (b), that $(Y' \cup T').w = (Y' \cup T').w' = (Y' \cup T').F$, and thus (66) holds,

$$\begin{aligned} \bar{\phi}_k \bar{\phi}_i \bar{\phi}_j(Y \cup T) &= [(Y' \cup T').F] * \bar{\phi}_k \bar{\phi}_i \bar{\phi}_j(Y_{(\mu_1, \dots, \mu_j)} \cup T^{[j]}) \\ &= [(Y' \cup T').F] * \bar{\phi}_k \bar{\phi}_i \bar{\phi}_j(Y_{(\mu_1, \dots, \mu_j)} \cup T^{[j]}) = \bar{\phi}_i \bar{\phi}_j \bar{\phi}_k(Y \cup T). \end{aligned} \quad (69)$$

Equality (67) follows similarly. Equality (68) follows from (66) and (67) and the definition of Knuth equivalent words. $\square$

Example 14. Illustration of this proposition using (55) in Example 11

(a) Consider $Y \cup T$ as in (55) and $213 \equiv 231$,

$$\bar{\phi}_2 \bar{\phi}_1 \bar{\phi}_3((55)) = \bar{\phi}_2 \bar{\phi}_3 \bar{\phi}_1((55)) =$$

1	1	1	■	1
2	2	2	■	
3	3	■	①	
4	1	②	②	
1	③	③		
2	④	④		
⑤				

(b) $312 \equiv 132$,

$$\begin{aligned} \bar{\phi}_3 \bar{\phi}_1 \bar{\phi}_2 &= \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & 3 & & \\ \hline 1 & 3 & 4 & 4 & & \\ \hline 5 & 5 & & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 4 \\ \hline 5 & 5 & & \\ \hline \end{array} \cdot \bar{\phi}_3 \bar{\phi}_1 \bar{\phi}_2 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & 3 & & \\ \hline 1 & 3 & 4 & 4 & & \\ \hline 5 & 5 & & & & \\ \hline \end{array} \\ &= \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 4 \\ \hline 5 & 5 & & \\ \hline \end{array} \cdot \bar{\phi}_3 \bar{\phi}_1 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & \\ \hline 3 & 1 & 3 & 3 & & \\ \hline 2 & & & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 4 \\ \hline 5 & 5 & & \\ \hline \end{array} \cdot \bar{\phi}_3 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 1 & 2 & \\ \hline 3 & 1 & 2 & 3 & & \\ \hline 2 & 3 & & & & \\ \hline \end{array} \\ &= \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 4 \\ \hline 5 & 5 & & \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array} \cdot \bar{\phi}_3 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 1 & 2 & \\ \hline 3 & 1 & 2 & 3 & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 4 \\ \hline 5 & 5 & & \\ \hline \end{array} \cdot \bar{\phi}_1 \bar{\phi}_3 \bar{\phi}_2 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & 3 & & \\ \hline \end{array} \\ &= \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 4 \\ \hline 5 & 5 & & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 1 & 2 & \\ \hline 3 & 3 & 2 & 3 & & \\ \hline 1 & 3 & & & & \\ \hline 2 & & & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 4 \\ \hline 5 & 5 & & \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 1 & 2 & \\ \hline 3 & 3 & 2 & 3 & & \\ \hline \end{array} \\ &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 4 \\ \hline 2 & 4 & & \\ \hline 3 & 5 & & \\ \hline 5 & & & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 1 & 2 & \\ \hline 3 & 3 & 2 & 3 & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 1 & 2 & \\ \hline 3 & 3 & 2 & 3 & & \\ \hline 1 & 1 & 3 & 4 & & \\ \hline 2 & 4 & & & & \\ \hline 3 & 5 & & & & \\ \hline 5 & & & & & \\ \hline \end{array}. \end{aligned}$$

5. SWITCHING ON BALLOT TABLEAU PAIRS AS A RECURSIVE SAGAN-STANLEY INTERNAL ROW INSERTION

5.1. Littlewood-Richardson rules and commutation symmetry. Let n be a fixed positive integer and let $x = (x_1, x_2, \dots, x_n)$ be a sequence of indeterminates. Then, for each partition λ of length $\leq n$, there exists a Schur function $s_\lambda(x)$ which is a homogeneous symmetric polynomial in the x_k of total degree $|\lambda|$. The product of two Schur functions is explicitly given by the Littlewood-Richardson rule which amounts to finding how many SSYT's satisfy certain conditions.

Theorem 6. [LR34, Tho78, Sch77] The Littlewood-Richardson (LR) rule. The coefficients appearing in the expansion of a product of Schur polynomials s_μ and s_ν

$$s_\mu(x) s_\nu(x) = \sum_{\lambda} c_{\mu\nu}^{\lambda} s_{\lambda}(x) \quad (70)$$

are given by $c_{\mu\nu}^{\lambda} = \#\{\text{ballot SSYT of shape } \lambda/\mu \text{ and content } \nu\}$. The coefficients $c_{\mu\nu}^{\lambda}$ are known as Littlewood-Richardson (LR) coefficients, and the ballot SSYT's are also known as Littlewood-Richardson tableaux.

The Schubert structure coefficients of the product in $H^*(G(d, n))$, the cohomology of the Grassmannian $G(d, n)$, (as a $\mathbb{Z}$ -module), are also given by the LR rule. The connection with $H^*(G(d, n))$, the cohomology of the Grassmannian $G(d, n)$ is due to L. Lesieur, [Les47].

$$\sigma_{\mu}\sigma_{\nu} = \sum_{\lambda \subseteq d \times (n-d)} c_{\mu\nu}^{\lambda} \sigma_{\lambda}. \quad (71)$$

The Schur structure coefficients (70) are not only Schubert structure coefficients (71). They are also multiplicities in tensor products of $GL_n(\mathbb{C})$ -representations and in induction products of $\mathfrak{S}_n$ -representations.

The *rectification* of a SSYT T is the unique SSYT of normal shape whose reading word is Knuth equivalent to that of T . Using the notion of rectification of a SSYT, the LR rule may also be formulated in the language of M.-P. Schützenberger's *jeu de taquin* [Sta98]. The SSYT's U and V are said to be *jeu de taquin* equivalent if one can be obtained from another by a sequence of *jeu de taquin* slides [Sta98, Ful97]. Recall that each stage of *jeu de taquin slide* converts the reading word of a semistandard Young tableau into a Knuth equivalent one, and *jeu de taquin* commutes with standardisation.

Theorem 7. Littlewood-Richardson rule's *jeu de taquin* version ([Sta98], Appendix 1.) Fix a standard tableau S of shape ν . Then

$$c_{\mu\nu}^{\lambda} = \#\{\text{SYT of shape } \lambda/\mu \text{ whose rectification is } S\}.$$

(The special choice of $S = \text{std}Y_{\nu}$ relates this version with the LR tableau version above.)

The rectification of a SSYT T does not depend on the order of *jeu de taquin slides*. We can then consider Y_{μ} to be the inner shape of a SSYT of shape λ/μ and content ν and then look at Y_{μ} as a set of instructions to tell where *jeu de taquin* contracting slides start to rectify T . (Standardise Y_{μ} and T . The *jeu de taquin* slides start with the biggest entry of $\text{std}Y_{\mu}$, seen as a hole. The "hole" will slide until it reaches the outer boundary [Sta98]. Then proceed similarly with *jeu de taquin* slides into the remaining entries of $\text{std}Y_{\mu}$, in decreasing numerical order. When $\text{std}T$ is rectified to some S the elements of $\text{std}Y_{\mu}$ are the entries of the skew shape λ/ν and encode the order in which the boxes were vacated in the *jeu de taquin* sliding process.) Such tableau sliding process correspond to a particular presentation of the tableau switching procedure [BSS96] on tableau pairs of partition shape. (When $S = \text{std}Y_{\nu}$ one has the tableau sliding presentation for ballot tableau pairs of normal shape.)

Tableau switching process is outlined in the next subsection. Let ρ_1 be the involution map that the tableau switching procedure calculates on tableau pairs. We call it the *switching involution*. The tableau sliding presentation of ρ_1 is called *infusion* by Thomas and Yong in [TY09, TY16], and it can be translated to the language of Fomin's *jeu de taquin growths* (see Fomin's Appendix 1 in [Sta98]) which shows that *infusion* is an involution. This approach is realised by Thomas and Yong in [TY08] to exhibit an involution for the commutation of LR coefficients.

5.2. The tableau switching map. Switching [BSS96] is an operation that takes a pair of tableaux $U \cup V$ and moves them through each other giving another such pair $V' \cup U'$ of the same shape, in a way that

preserves Knuth equivalence, $V \equiv V'$ and $U \equiv U'$, and the shape of their union. Loosely speaking, the switching algorithm may be realised as a mixture of Schützenberger's *jeu de taquin* and its reverse process in the sense that it calculates an involutive map on pairs of tableaux and if $U \cup V$ has normal shape then V' is the rectification of V and U is the rectification of U' . The *switching map* on $U \cup V$ is processed through local moves to interchanging two vertically or horizontally adjacent letters $\mathbf{u}$ and $\mathbf{v}$ from U and V respectively. The intermediate objects have to be defined when the switching procedure moves U and V through each other. A *perforated tableau* U of shape λ is a filling of some of the boxes in λ with integers satisfying some restrictions: whenever $\mathbf{x}$ and $\mathbf{x}'$ are entries of U where $\mathbf{x}'$ is to the north-west of $\mathbf{x}$, then

$$\text{if not in the same column, } \begin{smallmatrix} \mathbf{x}' \\ \mathbf{x} \end{smallmatrix}, \quad \mathbf{x}' \rightarrow_s \mathbf{x} \Rightarrow \mathbf{x} \geq \mathbf{x}', \text{ and} \quad (72)$$

$$\text{if in the same column } \begin{smallmatrix} \mathbf{x}' \\ \mathbf{x} \end{smallmatrix} \Rightarrow \mathbf{x} > \mathbf{x}'. \quad (73)$$

We may switch an integer with the neighbour empty box $\blacksquare$ to the south, east, north or west, in a perforated tableau, so that the result is still a perforated tableau:

$$\text{contracts } U \quad \begin{smallmatrix} \blacksquare \\ \mathbf{x} \end{smallmatrix} \xrightarrow{s} \begin{smallmatrix} \mathbf{x} \\ \blacksquare \end{smallmatrix} \quad \begin{smallmatrix} \blacksquare & \mathbf{x} \end{smallmatrix} \xrightarrow{s} \begin{smallmatrix} \mathbf{x} & \blacksquare \end{smallmatrix}$$

$$\text{expands } U \quad \begin{smallmatrix} \blacksquare & \mathbf{x} \end{smallmatrix} \xrightarrow{s} \begin{smallmatrix} \mathbf{x} \\ \blacksquare \end{smallmatrix} \quad \begin{smallmatrix} \blacksquare & \mathbf{x} \end{smallmatrix} \xrightarrow{s} \begin{smallmatrix} \mathbf{x} & \blacksquare \end{smallmatrix}$$

A *perforated tableau pair* $U \cup V$ of shape λ is the superimposing of two perforated tableaux U and V of shape λ , so that together they completely fill λ . If $\mathbf{u}$ and $\mathbf{v}$ are vertically or horizontally adjacent letters from U and V respectively, then an interchanging of $\mathbf{u}$ with $\mathbf{v}$ is a *switch*, written $\mathbf{u} \leftrightarrow_s \mathbf{v}$, provided it produces a new perforated tableau pair,

$$\begin{smallmatrix} \mathbf{u} \\ \mathbf{v} \end{smallmatrix} \leftrightarrow_s \begin{smallmatrix} \mathbf{v} \\ \mathbf{u} \end{smallmatrix} \quad \mathbf{u} \mathbf{v} \leftrightarrow_s \mathbf{v} \mathbf{u}.$$

We collect the following elementary perforated tableau pairs with the corresponding elementary moves. For the sake of clarity, the entries of U are drawn in red:

$$\begin{array}{ccc} \mathbf{u} & \mathbf{v} & \leftrightarrow_s \mathbf{v} \mathbf{u}, \\ \mathbf{u} & \mathbf{v} & \leftrightarrow_s \mathbf{v} \mathbf{u}, \quad a > v, \\ a & & a \end{array} \quad \begin{array}{ccc} \mathbf{u} & \leftrightarrow_s & \mathbf{v} \\ \mathbf{v} & & \mathbf{u} \\ \mathbf{u} & a & \leftrightarrow_s \mathbf{v} a, \quad a \geq v. \\ \mathbf{v} & & \mathbf{u} \end{array}$$

The above switches recover the *jeu de taquin* and *reverse jeu de taquin* switches when the $\mathbf{u}$ -entry is seen as an empty entry. An example of a sequence of switches

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & \\ \hline 3 & 1 & 2 & 3 & & \\ \hline \end{array} \xleftrightarrow{s} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & \\ \hline 1 & 2 & 3 & 3 & & \\ \hline \end{array} \xleftrightarrow{s} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 2 & 2 & 2 & \\ \hline 2 & 2 & 3 & 3 & & \\ \hline \end{array} \xleftrightarrow{s} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 2 & 2 & 2 & \\ \hline 2 & 2 & 3 & 3 & & \\ \hline \end{array}.$$

Theorem 8. [BSS96, Theorem 2.3] The Switching Procedure on tableau pairs, calculates *switching* the unique map on tableau pairs with the following properties

- (1) Start with a tableau pair $U \cup V$;
- (2) Switch integers from U with integers from V until it is no longer possible to do so. This produces a new tableau pair $S \cup H$.

$$U \cup V \xleftrightarrow{s} S \cup H,$$

where $U \equiv H$ and $V \equiv S$. (If $U \cup V$ has normal shape, U is the rectification of H and S the rectification of V). If subtableaux decompose U , V can switch with U in stages. Similarly if V decomposes, U can switch with V in stages. Switching is an *involution on tableau pairs* denoted by ρ_1 .

Definition 7. We also write $\rho_1^{(n)}$ when the switching ρ_1 is acting on tableau pairs with shape length $\leq n$.

Corollary 5. The map $\rho_1^{(n)}$ is an involution in $\mathcal{LR}^{(n)}$, for all $n \geq 1$. Moreover $c_{\mu,\nu}^\lambda = c_{\nu,\mu}^\lambda$.

For example, for $n = 3$, $\mu = (4, 4, 1)$ and $\nu = (3, 2, 1)$,

$$\begin{aligned}
Y_{(4,4,1)} \cup V &= \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & \\ \hline 3 & 1 & 2 & 3 & & \\ \hline \end{array} \xleftrightarrow{s} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & \\ \hline 1 & 2 & 3 & 3 & & \\ \hline \end{array} \xleftrightarrow{s} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 2 & 2 & 2 & \\ \hline 2 & 2 & 3 & 3 & & \\ \hline \end{array} \\
&\xleftrightarrow{s} Y_{(3,2,1)} \cup H = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 2 & 3 & & \\ \hline \end{array} = \rho_1^{(3)}(Y_{(4,4,1)} \cup V) \in \mathcal{LR}^{(3)}.
\end{aligned}$$

5.3. Henriques-Kamnitzer $\mathfrak{gl}_n$ -crystal commutator and Sagan-Stanley internal row insertion. For definitions, in this section, we refer the reader to [AKT16, Section 12], [ACM25, Section 3.4] and [KT25a]. One way to conclude that a ballot tableau pair commutator ρ coincides with the Henriques-Kamnitzer $\mathfrak{gl}_n$ -crystal commutator Com_{HK} [HK06b, HK06a] is to show that $\rho(Y_\mu \cup T) = Y_\nu \cup H$ with $H \equiv Y_\mu$ and $T \equiv Y_\nu$ satisfy the following: the corresponding left and right Gelfand-Tsetlin (GT) pattern pairs $(G_\mu(T), G_\nu(T))$ respectively $(G_\nu(H), G_\mu(H))$ are related through the Schützenberger involution ξ , $G_\mu(H) = \xi(G_\nu(T))$ and $G_\nu(H) = \xi(G_\mu(T))$. We shall show that our commutator (ρ_3 in [PV10]) based on the Sagan-Stanley internal insertion will produce such GT patterns in Theorem 2 thanks to the coincidence of the commutator ρ_1 with Com_{HK} . We will work on an illustration as a motivation for Theorem 2.

Given $Y \cup T \in \mathcal{LR}^{(n)}$ with $Y = Y_\mu$ and T a ballot tableau of shape λ/μ and weight ν , we define for $i = 1, \dots, n$, the partition $\nu^{(i)}$ to be the content of the ballot tableau $T^{[i]}$, and the partition $\hat{\nu} = (\hat{\nu}_1, \dots, \hat{\nu}_{n-1}, \hat{\nu}_n = \nu_n) \subseteq \nu$ where $\hat{\nu}_i$ is the number of i 's in row i of T , for $i = 1, \dots, n$.

The Sagan-Stanley internal row insertion correspondence (38) applied to the pair $(\emptyset_{\hat{\nu}}, U)$ and $U \in YT(\nu/\hat{\nu})$ with $\hat{\nu} \subseteq \nu$, produces P equals to $\emptyset_\nu$ the Young diagram of shape ν . Let G_ν be the *Gelfand-Tsetlin (GT) pattern* of type ν or the *companion tableau* of T . Note G_ν is defined by the nested sequence $\nu^{(1)} \subseteq \nu^{(2)} \subseteq \dots \subseteq \nu^{(n)} = \nu$. We use the skew GT pattern $G_{\nu/\hat{\nu}}$, as internal insertion order tableau, and the parts of μ to be added properly, to construct an LR commutator as we explain next. The skew GT pattern $G_{\nu/\hat{\nu}}$ is the skew tableau of shape $\nu/\hat{\nu}$ obtained by vacating the cells in the Young diagram of shape $\hat{\nu}$ in the GT pattern G_ν . See Example 15

The internal insertion order word $\mathcal{R}(G_{\nu/\hat{\nu}}) = V_n \cdots V_3 V_2 V_1$ is decomposed into row words V_i , $i = 1, \dots, n$. The row word V_i is precisely the i th row word of T restricted to the alphabet $[i-1]$, and $|V_i| = \lambda_i - \mu_i - \hat{\nu}_i$, for $i = 1, \dots, n$. Thus V_1 is the empty word, $\phi_{V_1} = 1$, $\phi_{\mathcal{R}(G_{\nu/\hat{\nu}})} = \phi_{V_n} \cdots \phi_{V_3} \phi_{V_2}$ and

$$\bar{\phi}_{\mathcal{R}(G_{\nu/\hat{\nu}})}(\emptyset_{\hat{\nu}}) = \bar{\phi}_{V_n} \cdots \bar{\phi}_{V_3} \bar{\phi}_{V_2}(\emptyset_{\hat{\nu}}) = \emptyset_\nu,$$

the Young diagram of shape ν . For $i = 1, \dots, n$, one needs the operator $\bar{\chi}_i$, to be iterated μ_i times, written $\bar{\chi}_i^{\mu_i} = \underbrace{\bar{\chi}_i \circ \dots \circ \bar{\chi}_i}_{\mu_i}$, over $\bar{\phi}_{V_i}(\bar{\chi}_{i-1}^{\mu_{i-1}} \bar{\phi}_{V_{i-1}}) \cdots (\bar{\chi}_2^{\mu_2} \bar{\phi}_{V_2}) \bar{\chi}_1^{\mu_1}(\emptyset_{\hat{\nu}})$, to recursively adding in each iteration one i

at the end of i th row. Given $Y_\mu \cup T \in \mathcal{LR}^{(n)}$ this procedure encodes $Y_\mu \cup T$ in the form $(\emptyset_{\hat{\nu}}, G_{\nu/\hat{\nu}}, \mu)$ and gives the μ -augmented internal insertion operator an involution in $\mathcal{LR}^{(n)}$

$$\bar{\phi}_{\mathcal{R}(G_{\nu/\hat{\nu}})}^\mu(\emptyset_{\hat{\nu}}) = (\bar{\chi}_n^{\mu_n} \bar{\phi}_{V_n}) \cdots (\bar{\chi}_3^{\mu_3} \bar{\phi}_{V_3}) (\bar{\chi}_2^{\mu_2} \bar{\phi}_{V_2}) \bar{\chi}_1^{\mu_1}(\emptyset_{\hat{\nu}}) = \emptyset_\nu \cup H \in \mathcal{LR}^{(n)}, \quad (74)$$

where H is a ballot tableau of shape λ/ν and content μ .

Example 15. Let $\lambda = (6, 5, 5, 4, 3)$, $\nu = (4, 4, 3, 2, 0)$, $\hat{\nu} = (2, 1, 1, 1, 0)$ and $\mu = (4, 3, 2, 1, 0)$

$$\begin{aligned}
 Y_\mu \cup T &= \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 1 & 2 & \\ \hline 3 & 3 & 1 & 2 & 3 & \\ \hline 4 & 2 & 3 & 4 & & \\ \hline 2 & 3 & 4 & & & \\ \hline \end{array} & G_\nu(T) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 & 5 \\ \hline 3 & 4 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array} & G_{\nu/\hat{\nu}}(T) = \begin{array}{|c|c|c|c|} \hline & & 2 & 3 \\ \hline & 3 & 4 & 5 \\ \hline & 4 & 5 & \\ \hline & 5 & & \\ \hline \end{array} & \text{std} G_{\nu/\hat{\nu}} = \begin{array}{|c|c|c|c|} \hline & & 1 & 3 \\ \hline & 2 & 5 & 8 \\ \hline & 4 & 7 & \\ \hline & 6 & & \\ \hline \end{array} \\
 Y_\mu &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & & \\ \hline 4 & & & \\ \hline \end{array} & \mathcal{R}(G_{\nu/\hat{\nu}}) = \mathcal{R}_5 \mathcal{R}_4 \mathcal{R}_3 \mathcal{R}_2 \emptyset = 234 23 12 1 = V_5 V_4 V_3 V_2.
 \end{aligned} \tag{75}$$

Then for $n = 5$,

$$\bar{\phi}_{\mathcal{R}(G_{\nu/\hat{\nu}})}(\emptyset_{\hat{\nu}}) = \bar{\phi}_{\mathcal{R}_5 \mathcal{R}_4 \mathcal{R}_3 \mathcal{R}_2 \emptyset}(\emptyset_{\hat{\nu}}) = \bar{\phi}_{234} \bar{\phi}_{23} \bar{\phi}_{12} \bar{\phi}_1(\emptyset_{\hat{\nu}}) = \emptyset_\nu.$$

The μ -augmentation of $\bar{\phi}_{\mathcal{R}(G_{\nu/\hat{\nu}})}$:

$$\begin{aligned}
 \bar{\phi}_{\mathcal{R}(G_{\nu/\hat{\nu}})}^\mu(\emptyset_{\hat{\nu}}) &= \bar{\chi}_5^{\mu_5} \bar{\phi}_{\mathcal{R}_5} \bar{\chi}_4^{\mu_4} \bar{\phi}_{\mathcal{R}_4} \bar{\chi}_3^{\mu_3} \bar{\phi}_{\mathcal{R}_3} \bar{\chi}_2^{\mu_2} \bar{\phi}_{\mathcal{R}_2} \bar{\chi}_1^{\mu_1} \bar{\phi}_\emptyset(\emptyset_{\hat{\nu}}) \\
 &= \bar{\phi}_{234} \bar{\chi}_4 \bar{\phi}_{23} \bar{\chi}_3^2 \bar{\phi}_{12} \bar{\chi}_2^3 \bar{\phi}_1 \bar{\chi}_1^4(\emptyset_{\hat{\nu}}) \\
 &= \emptyset_\nu \cup H
 \end{aligned}$$

gives

$$\begin{aligned}
 \emptyset_{\hat{\nu}} &= \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \xrightarrow{\bar{\chi}_1^4} \begin{array}{|c|c|c|c|c|c|} \hline & & 1 & 1 & 1 & 1 \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array} \xrightarrow{\bar{\chi}_2^3 \bar{\phi}_1} \begin{array}{|c|c|c|c|c|} \hline & & 1 & 1 & 1 \\ \hline & 1 & 2 & 2 & 2 \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} \xrightarrow{\bar{\chi}_3^2 \bar{\phi}_1 \bar{\phi}_2} \begin{array}{|c|c|c|c|c|c|} \hline & & & & 1 & 1 \\ \hline & & & 1 & 2 & 2 \\ \hline & & & 1 & 2 & 3 \\ \hline & & & & & \\ \hline \end{array} \\
 &\xrightarrow{\bar{\chi}_4 \bar{\phi}_2 \bar{\phi}_3} \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 1 \\ \hline & & & 2 & 2 \\ \hline & & 1 & 3 & 3 \\ \hline & 1 & 2 & 4 & \\ \hline \end{array} \xrightarrow{\bar{\phi}_2 \bar{\phi}_3 \bar{\phi}_4} \begin{array}{|c|c|c|c|c|c|} \hline & & & & 1 & 1 \\ \hline & & & & 2 & \\ \hline & & & 2 & 3 & \\ \hline & & 1 & 3 & & \\ \hline 1 & 2 & 4 & & & \\ \hline \end{array} = \emptyset_\nu \cup H.
 \end{aligned} \tag{76}$$

where H is a ballot tableau of content μ and shape λ/ν . This augmented operator creates the skew GT pattern $G_{\nu/\hat{\nu}}$ defined by the sequence of inner shapes read in (76) onwards and displayed in (77).

The inverse of the μ -augmented operator is the μ -deletion operator

$$\Delta^\mu(\emptyset_\nu \cup H) = (\bar{\chi}_1^{\mu_1})^{-1} \Delta_2^{\lambda_2 - \nu_2 - \hat{\mu}_2} (\bar{\chi}_2^{\mu_2})^{-1} \Delta_3^{\lambda_3 - \nu_3 - \hat{\mu}_3} (\bar{\chi}_3^{\mu_3})^{-1} \Delta_4^{\lambda_4 - \nu_4 - \hat{\mu}_4} (\bar{\chi}_4^{\mu_4})^{-1} \Delta_5^{\lambda_5 - \nu_5 - \hat{\mu}_5}(\emptyset_\nu \cup H)$$

which reads (76) backwards and thereby creates $G_{\nu/\hat{\nu}}$ the nested sequence of inner shapes

$$\nu = (4, 4, 3, 2, 0) = \hat{\nu}^{(5)} \supseteq (4, 3, 2, 1, 0) = \hat{\nu}^{(4)} \supseteq (4, 2, 1, 1, 0) = \hat{\nu}^{(3)} \supseteq (3, 1, 1, 1, 0) = \hat{\nu}^{(2)} \supseteq (2, 1, 1, 1, 0) = \hat{\nu} \tag{77}$$

Example 16. Let $\lambda = (6, 5, 5, 4, 3)$, $\nu = (4, 4, 3, 2, 0)$, $\hat{\mu} = (2, 1, 1, 0, 0)$ and $\mu = (4, 3, 2, 1)$

$$\begin{aligned}
 Y_\nu \cup H &= \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & \\ \hline 3 & 3 & 3 & 2 & 3 & \\ \hline 4 & 4 & 1 & 3 & & \\ \hline 1 & 2 & 4 & & & \\ \hline \end{array} & G_\mu(H) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 4 & 5 \\ \hline 2 & 3 & 5 & \\ \hline 3 & 4 & & \\ \hline 5 & & & \\ \hline \end{array} & G_{\mu/\hat{\mu}}(H) = \begin{array}{|c|c|c|c|} \hline & & 4 & 5 \\ \hline & 3 & 5 & \\ \hline & 4 & & \\ \hline 5 & & & \\ \hline \end{array} \\
 Y_\nu &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & \\ \hline 4 & 4 & & \\ \hline \end{array} & \mathcal{R}(G_{\mu/\hat{\mu}}) = \mathcal{R}_5 \mathcal{R}_4 \mathcal{R}_3 \emptyset \emptyset = 124 13 2 \emptyset \emptyset.
 \end{aligned} \tag{78}$$

Then $\bar{\phi}_{\mathcal{R}(G_{\mu/\hat{\mu}})}(\emptyset_{\hat{\mu}}) = \bar{\phi}_{124}\bar{\phi}_{13}\bar{\phi}_2(\emptyset_{\hat{\mu}}) = \emptyset_{\mu}$, and $\bar{\phi}_{\mathcal{R}(G_{\mu/\hat{\mu}})}^{\nu}(\emptyset_{\hat{\mu}}) = \bar{\phi}_{124}\bar{\chi}_4^2\bar{\phi}_{13}\bar{\chi}_3^3\bar{\phi}_2\bar{\chi}_2^4\bar{\chi}_1^4(\emptyset_{\hat{\mu}}) = \emptyset_{\mu} \cup T$ are displayed below

$$\begin{aligned}
 \emptyset_{\hat{\mu}} &= \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \xrightarrow{\bar{\chi}_1^4} \begin{array}{|c|c|c|c|c|} \hline & & 1 & 1 & 1 & 1 \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array} \xrightarrow{\bar{\chi}_2^4} \begin{array}{|c|c|c|c|c|} \hline & & 1 & 1 & 1 & 1 \\ \hline & & 2 & 2 & 2 & 2 \\ \hline & & & & & \\ \hline \end{array} \xrightarrow{\bar{\chi}_3^3\bar{\phi}_2} \begin{array}{|c|c|c|c|c|} \hline & & 1 & 1 & 1 & 1 \\ \hline & & 2 & 2 & 2 & \\ \hline & & 2 & 3 & 3 & 3 \\ \hline \end{array} \\
 &\xrightarrow{\bar{\chi}_4^2\bar{\phi}_1\bar{\phi}_3} \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 1 & 1 \\ \hline & & & 1 & 2 & 2 \\ \hline & & & 2 & 3 & 3 \\ \hline & & 2 & 3 & 4 & 4 \\ \hline \end{array} \xrightarrow{\bar{\phi}_1\bar{\phi}_2\bar{\phi}_4} \begin{array}{|c|c|c|c|c|} \hline & & & & 1 & 1 \\ \hline & & & & 1 & 2 \\ \hline & & & 1 & 2 & 3 \\ \hline & & 2 & 3 & 4 & \\ \hline & 2 & 3 & 4 & & \\ \hline \end{array} = \emptyset_{\mu} \cup T. \tag{79}
 \end{aligned}$$

The deletion operator is obtained from (79) backwards while recording the inner shapes to define $G_{\mu/\hat{\mu}}$ the skew companion tableau of H

$$\mu = (4, 3, 2, 1, 0) \supseteq \hat{\mu}^{(4)} = (3, 2, 2, 0, 0) \supseteq \hat{\mu}^{(3)} = (2, 2, 1, 0, 0) \supseteq \hat{\mu}^{(2)} = (2, 1, 1, 0, 0) \supseteq \hat{\mu} = (2, 1, 1, 0, 0).$$

$$\Delta^{\nu}(\emptyset_{\mu} \cup T) = (\bar{\chi}_1^{\nu_1})^{-1} \Delta_2^{\lambda_2 - \mu_2 - \hat{\nu}_2} (\bar{\chi}_2^{\nu_2})^{-1} \Delta_3^{\lambda_3 - \mu_3 - \hat{\nu}_3} (\bar{\chi}_3^{\nu_3})^{-1} \Delta_4^{\lambda_4 - \mu_4 - \hat{\nu}_4} (\bar{\chi}_4^{\nu_4})^{-1} \Delta_5^{\lambda_5 - \mu_5 - \hat{\nu}_5} (\emptyset_{\mu} \cup T) = \emptyset_{\nu} \cup H$$

$$H \equiv Y_{\mu}$$

Remark 9. [Buc00, PV10, HK06b, AKT16, TKA18, ACM25] Recall $\text{evac}G_{\mu}(T) = G_{\mu}(H)$ where $\text{evac} = \xi$ denotes the Schützenberger evacuation

$$\text{evac}G_{\mu}(T) = \text{evac} \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 5 & & \\ \hline 5 & & & \\ \hline \end{array} = G_{\mu}(H) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 4 & 5 \\ \hline 2 & 3 & 5 & \\ \hline 3 & 4 & & \\ \hline 5 & & & \\ \hline \end{array}$$

To avoid the skew GT patterns $G_{\nu/\hat{\nu}}$ and $G_{\mu/\hat{\mu}}$ in the previous examples, and since $G_{\nu} = Y_{\hat{\nu}} \cup G_{\nu/\hat{\nu}}$ and similarly $G_{\mu} = Y_{\hat{\mu}} \cup G_{\mu/\hat{\mu}}$, we get supplied with another adding operator $\bar{\omega}_i$ regarding the partitions $\hat{\nu} = (\hat{\nu}_1, \dots, \hat{\nu}_n)$ or $\hat{\mu} = (\hat{\mu}_1, \dots, \hat{\mu}_n)$ as follows.

Put $(Y \cup T)^{[0]} := \emptyset$. For $i = 0, 1, \dots, n$, let $(Y_{\mu} \cup T)^{[i]} = Y_{(\mu_1, \dots, \mu_i)} \cup T^{[i]} \in \mathcal{LR}^{(i)}$ where $T^{[i]}$ has content $\nu^{(i)}$, and row i consists of V_i , the row subword restricted to the entries in $[i-1]$, followed with $\hat{\nu}_i$ i's. For each $i = 1, \dots, n$, we now consider the operator $\bar{\omega}_i$, to be iterated $\hat{\nu}_i$ times, to contributing, in each iteration over

$$\begin{aligned}
 Y_{\nu^{(i-1)}} \cup H^{(i-1)} &= (\bar{\chi}_{i-1}^{\mu_{i-1}} \bar{\phi}_{V_{i-1}} \bar{\omega}_{i-1}^{\hat{\nu}_{i-1}}) \cdots (\bar{\chi}_2^{\mu_2} \bar{\phi}_{V_2} \bar{\omega}_2^{\hat{\nu}_2}) (\bar{\chi}_1^{\mu_1} \bar{\omega}_1^{\hat{\nu}_1})(\emptyset), \\
 H^{(i-1)} &\equiv Y_{(\mu_1, \dots, \mu_{i-1})}, \text{ skew shape } (\lambda_1, \dots, \lambda_{i-1})/\nu^{(i-1)}, \quad H^{(0)} = \emptyset,
 \end{aligned}$$

with one i to the i th row of the inner shape $Y_{\nu^{(i-1)}}$. This allows to give the following recursive presentation of the switching map $\rho_1^{(n)}$ on ballot tableau pairs in $\mathcal{LR}^{(n)}$, for $n \geq 1$,

$$\rho_1^{(n)}(Y_{\mu} \cup T) = (\bar{\chi}_n^{\mu_n} \bar{\phi}_{V_n} \bar{\omega}_n^{\hat{\nu}_n}) \cdots (\bar{\chi}_3^{\mu_3} \bar{\phi}_{V_3} \bar{\omega}_3^{\hat{\nu}_3}) (\bar{\chi}_2^{\mu_2} \bar{\phi}_{V_2} \bar{\omega}_2^{\hat{\nu}_2}) (\bar{\chi}_1^{\mu_1} \bar{\omega}_1^{\hat{\nu}_1})(\emptyset) \tag{80}$$

$$= (\bar{\chi}_n^{\mu_n} \bar{\phi}_{V_n} \bar{\omega}_n^{\hat{\nu}_n}) \rho^{(n-1)}[(Y_{\mu} \cup T)^{[n-1]}] \tag{81}$$

$$= Y_{\nu} \cup H, \text{ with } H \equiv Y_{\mu}. \tag{82}$$

The Yamanouchi tableau $Y_{\hat{\nu}}$ is constructed recursively by means of $\hat{\nu}_i$ iterations of the operator $\bar{\omega}_i$, for $i = 1, \dots, n$. In particular, if V is the empty word and μ is the zero partition, then $\hat{\nu} = \nu$ and $Y_{\nu} = \bar{\omega}_n^{\nu_n} \cdots \bar{\omega}_2^{\nu_2} \bar{\omega}_1^{\nu_1}(\emptyset)$. Observe that $\bar{\omega}_i$ commutes with $\bar{\phi}_{V_i}$.

Example 17. Let us resume to Example 15 where $\nu = (4, 4, 3, 2, 0)$, $\hat{\nu} = (2, 1, 1, 1, 0)$ and $\mu = (4, 3, 2, 1)$. We illustrate

$$\rho_1^{(5)}(Y_{\mu} \cup T) = \bar{\phi}_2 \bar{\phi}_3 \bar{\phi}_4 \bar{\chi}_4^{\mu_4} \bar{\phi}_2 \bar{\phi}_3 \bar{\omega}_4^{\mu_3} \bar{\phi}_1 \bar{\phi}_2 \bar{\omega}_3^{\mu_2} \bar{\phi}_1 \bar{\omega}_2 \bar{\chi}_1^{\mu_1} \bar{\omega}_1^2(\emptyset).$$

Recall $\mathbf{s}$ denotes the switching operator.

$$\begin{aligned}
(Y \cup T)^{(0)} &= \emptyset \\
(Y \cup T)^{[1]} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 2 \end{bmatrix} \\
(Y \cup T)^{[2]} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 2 \\ 3 & 3 & 1 & 2 & 3 \end{bmatrix} \\
(Y \cup T)^{[3]} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 2 \\ 3 & 3 & 1 & 2 & 3 \\ 4 & 2 & 3 & 4 \end{bmatrix} \\
(Y \cup T)^{[4]} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 2 \\ 3 & 3 & 1 & 2 & 3 \\ 4 & 2 & 3 & 4 \end{bmatrix} \\
(Y \cup T)^{[5]} &= Y \cup T.
\end{aligned}$$

We check $\rho_1^{(i)}(Y_\mu \cup T)^{[i]} = (\bar{\chi}_i^{\mu_i} \bar{\phi}_{V_i} \bar{\omega}_i^{\hat{\nu}_i}) \rho^{(i-1)}[(Y_\mu \cup T)^{[i-1]}]$, for $i = 1, \dots, 5$.

$$\begin{aligned}
(Y \cup T)^{[1]} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{s} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \end{bmatrix} \\
&= \rho_1^{(1)}[(Y \cup T)^{[1]}] \\
&= Y_{\hat{\nu}_1} \cup H^{(1)}, \quad H^{(1)} \equiv Y_{(\mu_1)}
\end{aligned}$$

$$\begin{aligned}
(Y \cup T)^{[0]} &= Y_0 \cup H^{(0)} = \emptyset \xrightarrow{\omega_1^2} \begin{bmatrix} 1 & 1 \end{bmatrix} \xrightarrow{\chi_1^{\mu_1}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 \end{bmatrix} \\
&= \chi_1^{\mu_1} \omega_1^{\hat{\nu}_1} (Y \cup T)^{[0]} \\
&= \rho_1^{(1)}[(Y \cup T)^{[1]}] \\
&= Y_{\hat{\nu}_1} \cup H^{(1)}, \quad H^{(1)} \equiv Y_{(\mu_1)}
\end{aligned}$$

$$\begin{aligned}
(Y \cup T)^{[2]} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{s} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \end{bmatrix} \xrightarrow{s} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \end{bmatrix} \xrightarrow{s} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 \end{bmatrix} \\
&= \rho_1^{(2)}[(Y \cup T)^{[2]}] \\
&= Y_{\nu^{(2)}} \cup H^{(2)}, \quad H^{(2)} \equiv Y_{(\mu_1, \mu_2)}
\end{aligned}$$

$$\begin{aligned}
\rho_1^{(1)}[(Y \cup T)^{[1]}] &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 \end{bmatrix} \xrightarrow{\omega_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 \end{bmatrix} \xrightarrow{\bar{\phi}_1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 \end{bmatrix} \xrightarrow{\chi_2^{\mu_2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 \end{bmatrix} \\
&= \bar{\chi}_2^{\mu_2} \bar{\phi}_1 \bar{\omega}_2 \rho_1^{(1)}[(Y \cup T)^{[1]}] \\
&= \rho_1^{(2)}[(Y \cup T)^{[2]}]
\end{aligned}$$

$$\begin{aligned}
(Y \cup T)^{[3]} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 2 \\ 3 & 3 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{s} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \end{bmatrix} \xrightarrow{s} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 \\ 2 & 3 & 2 & 3 & 3 \end{bmatrix} \xrightarrow{s} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 3 & 1 & 2 & 3 & 3 \end{bmatrix} \\
&= \rho_1^{(3)}[(Y \cup T)^{[3]}] \\
&= Y_{\nu^{(3)}} \cup H^{(3)} \quad H^{(3)} \equiv Y_{(\mu_1, \mu_2, \mu_3)}
\end{aligned}$$

$$\begin{aligned}
\rho_1^{(4)}[(Y \cup T)^{[4]}] &\xrightarrow{\bar{\phi}_4} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & \color{red}{1} & \color{red}{1} \\ \hline 2 & 2 & 2 & \color{red}{2} & \color{red}{2} & \\ \hline 3 & 3 & \color{red}{1} & \color{red}{3} & \color{red}{3} & \\ \hline 4 & 4 & \color{red}{2} & \color{red}{4} & & \\ \hline \color{red}{1} & & & & & \\ \hline \end{array} \xrightarrow{\bar{\phi}_3} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & \color{red}{1} & \color{red}{1} \\ \hline 2 & 2 & 2 & \color{red}{2} & \color{red}{2} & \\ \hline 3 & 3 & 3 & \color{red}{3} & \color{red}{3} & \\ \hline 4 & 4 & \color{red}{1} & \color{red}{4} & & \\ \hline \color{red}{1} & \color{red}{2} & & & & \\ \hline \end{array} \xrightarrow{\bar{\phi}_2} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & \color{red}{1} & \color{red}{1} \\ \hline 2 & 2 & 2 & 2 & \color{red}{2} & \\ \hline 3 & 3 & 3 & \color{red}{2} & \color{red}{3} & \\ \hline 4 & 4 & \color{red}{1} & \color{red}{3} & & \\ \hline \color{red}{1} & \color{red}{2} & \color{red}{4} & & & \\ \hline \end{array} \\
&= \bar{\phi}_2 \bar{\phi}_3 \bar{\phi}_4 \rho_1^{(4)}[(Y \cup T)^{[4]}] \\
&= \rho_1^{(5)}[Y \cup T] \\
&= Y_\nu \cup H \quad H \equiv Y_\mu.
\end{aligned}$$

Observe that $\widehat{\nu}_1 = \nu^{(1)} = (2) \subseteq \nu^{(2)} = (3, 1) \subseteq \nu^{(3)} = (4, 2, 1) \subseteq \nu^{(4)} = (4, 3, 2, 1) \subseteq \nu^{(5)} = (4, 4, 3, 2, 0)$ defines $G_\nu(T)$.

The switching map on ballot tableau pairs of normal shape can now be translated to the language of internal row insertion operations.

Theorem 9. (Theorem 2) (Main Theorem) Let $n \geq 1$ and $Y \cup T \in \mathcal{LR}^{(n)}$ with $Y = Y_\mu$ and T a ballot tableau of shape λ/μ and weight ν . For $1 \leq i \leq n$, let $(Y_\mu \cup T)^{[i]} = Y_{(\mu_1, \dots, \mu_i)} \cup T^{[i]} \in \mathcal{LR}^{(i)}$ with $T^{[i]}$ of weight $\nu^{(i)}$. Consider the i th row word of $T^{[i]}$ where V_i is the row subword restricted to the entries in $[i-1]$, and $\widehat{\nu}_i = \lambda_i - \mu_i - |V_i|$ is the number of entries equal to i . Put $(Y_\mu \cup T)^{[0]} = Y_{\nu^0} \cup H^{(0)} := \emptyset$, $\nu^0 := 0$, $\bar{\phi}_\emptyset = id$ and $\rho_1^{(0)}(\emptyset) := \emptyset$. Then, for $i = 1, \dots, n$, it holds

$$\rho_1^{(i)}[(Y_\mu \cup T)^{[i]}] = (\bar{\chi}_i^{\mu_i} \circ \bar{\phi}_{V_i} \circ \bar{\omega}_i^{\widehat{\nu}_i}) \circ \rho^{(i-1)}[(Y_\mu \cup T)^{[i-1]}] \quad (83)$$

$$= \bar{\chi}_i^{\mu_i} \circ \bar{\omega}_i^{\widehat{\nu}_i} \circ \bar{\phi}_{V_i}(Y_{\nu^{(i-1)}} \cup H^{(i-1)}) = Y_{\nu^{(i)}} \cup H^{(i)} \in \mathcal{LR}^{(i)}, \quad (84)$$

where $\bar{\omega}_i^{\widehat{\nu}_i}$ adds the i th row word $i^{\widehat{\nu}_i}$ to $Y_{\nu^{(i-1)}}$, $\bar{\chi}_i^{\mu_i}$ adds the row word i^{μ_i} at the end of the i th row of $\bar{\phi}_{V_i} \circ \bar{\omega}_i^{\widehat{\nu}_i}(Y_{\nu^{(i-1)}} \cup H^{(i-1)})$ and $H^{(i)} \equiv Y_{(\mu_1, \dots, \mu_i)}$. In particular, all bumping routes of $\bar{\phi}_{V_i}$ are pairwise disjoint and terminate in the i th row.

5.4. Lecouvey-Lenart and Kumar-Torres bijections between Kwon and Sundaram branching models. For detailed definitions pertaining this section we refer to [LL20, KRV21, Kwo18, KT25a]. A fundamental fact of Kumar-Torres bijection is that it restricts to tableaux satisfying the Sundaram condition and those whose evacuation satisfy the Kwon condition by considering and recognizing that they can be embedded in the Kushwaha-Raghavan-Viswanath [KRV21] bijection on flagged hives. To settle the Lecouvey-Lenart conjecture [LL20], it remains to know the coincidence of LR commutators. Note that in our notation the role of λ and ν are swapped in [KT25a].

We fix a positive integer n , and assume, unless otherwise stated, that $\ell(\lambda) \leq 2n-1$ and $\ell(\mu) \leq n$. Let $m = 2n$. A Littlewood-Richardson tableau of shape λ/μ and weight ν satisfies the *Sundaram property* if for each $i = 0, \dots, \ell(\nu)/2$, the entry $2i+1$ appears in row $n+i$ or above in the Young diagram of λ . The set of $T \in LR(\lambda/\mu, \nu)$ satisfying the Sundaram property is denoted by $LRS(\lambda/\mu, \nu)$ and called the set of *Sundaram LR tableaux*.

Denote by $LRS_{\mu, \nu}^\lambda$ the subset of $LR_{\mu, \nu}^\lambda$ consisting of the right companions of $LRS(\lambda/\mu, \nu)$. Then those tableaux consist of the tableaux in $LR_{\mu, \nu}^\lambda$ satisfying the flag property that the entries in the k th row are bounded above by $n + \lfloor k/2 \rfloor$, for $k = 1, \dots, 2n$, [KT25a, Proposition 4.7]. We call them *Sundaram right companions*. Denote by ${}^-LRS_{\mu, \nu}^\lambda$ the subset of ${}^-LR_{\mu, \nu}^\lambda$ consisting of the left companions of $LRS(\lambda/\mu, \nu)$.

A semistandard tableau of shape μ and $\ell(\mu) \leq n$ is said to satisfy the *Kwon property* if the entries in row i are at least $2i-1$, for $i = 1, \dots, n$. Denote by $LRK_{\nu, \mu}^\lambda$ the subset of $LR_{\nu, \mu}^\lambda$ consisting of tableaux G such that their Schützenberger evacuation $\xi = \text{evac}_{2n}$ within the crystal $B(\mu, 2n)$, $\xi(G) = \text{evac}_{2n}(G)$, satisfy the Kwon property.

The Sundaram branching rule (see [KT25a] and [Sun86] for the definition) states the following.

Theorem 10. [Sun86] The branching coefficient c_μ^λ equals the cardinality of the set

$$LRS(\lambda, \mu) := \bigcup LRS(\lambda/\mu, \nu),$$

where the union is taken over all even partitions ν , that is, $\nu_{2i-1} = \nu_{2i}$, $i \geq 1$.

The Kwon's branching rule as reformulated by Lecouvey-Lenart [LL20, Lemma 6.11] says the following.

Theorem 11. [Kwo18, LL20] The branching coefficient c_μ^λ equals the cardinality of the set

$$LRK(\lambda, \mu) := \bigcup LRK_{\nu, \mu}^\lambda$$

where the union is taken over all even partitions ν , that is, $\nu_{2i-1} = \nu_{2i}$, $i \geq 1$.

Theorem 12 (Kumar-Torres Theorem 3.6 [KT25a]). The composition

$$LR(\lambda/\mu, \nu) \xrightarrow{\sim} LR_{\mu, \nu}^\lambda \xrightarrow{U} LR_{\nu, \mu}^\lambda$$

restricts to a bijection

$$LRS(\lambda/\mu, \nu) \xrightarrow{\sim} LRK_{\nu, \mu}^{\lambda} \quad (85)$$

where $U : LR_{\mu, \nu}^{\lambda} \xrightarrow{U} LR_{\nu, \mu}^{\lambda}$ is the LR commuter by Kushwaha–Raghavan–Viswanath [KRV21]. (See [KT25a] for definitions). Therefore, the above composition induces a bijection between $LRS(\lambda, \mu)$ and $LRK(\lambda, \mu)$.

From [AKT16, Section 12] we know the LR commuter U coincides with the Henriques-Kamnitzer $\mathfrak{gl}_n$ -crystal commuter because it produces the same GT pattern pair. The Kumar-Torres bijection together with the coincidence of LR commuters gives the corollary:

Corollary 6. The Kumar-Torres bijection

$$LR(\lambda/\mu, \nu) \xrightarrow{\sim} LR_{\mu, \nu}^{\lambda} \xrightarrow{U} LR_{\nu, \mu}^{\lambda}$$

and the Lecouvey-Lenart bijection

$$LR(\lambda/\mu, \nu) \xrightarrow{\sim} LR_{\mu, \nu}^{\lambda} \xrightarrow{U'} LR_{\nu, \mu}^{\lambda}$$

where U' is the LR commuter defined by Henriques–Kamnitzer, coincide. Thereby both restrict to a bijection

$$LRS(\lambda, \mu) \xrightarrow{\sim} LRK(\lambda, \mu).$$

Corollary 7. The Henriques-Kamnitzer commuter (32) or (33) restrict to LRS tableaux and gives

$$LRS_{\mu, \nu}^{\lambda} \xrightarrow{\sim} LRS(\lambda/\mu, \nu) \xrightarrow{\sim} {}^{-}LRS_{\mu, \nu}^{\lambda} \xrightarrow{\xi} LRK_{\nu, \mu}^{\lambda}, \quad G_{\nu} \mapsto T \mapsto G_{\mu} \mapsto \xi(G_{\mu}), \quad (86)$$

Therefore, ${}^{-}LRS_{\mu, \nu}^{\lambda} = \xi(LRK_{\nu, \mu}^{\lambda})$ which are precisely those tableaux satisfying the Kwon condition in ${}^{-}LR_{\mu, \nu}^{\lambda}$.

Example 18. Motivated by the question raised by the authors in [KT25a, Remark 3.7], we now illustrate the Lecouvey-Lenart and Kumar-Torres bijections with our LR commuter based on the Sagan-Stanley internal insertion. We consider [KT25a, Example 4.11] where $n = 3$, $m = 6$, $\ell(\lambda) = 4$, $\ell(\nu) = 4$, $\ell(\mu) = 3$, and $\mu = (2, 1, 1, 0, 0, 0)$, $\nu = (4, 4, 2, 1, 0, 0) \subseteq \lambda = (5, 4, 3, 3, 0, 0)$:

$$Y_{\mu} \cup T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & & \\ \hline 2 & 3 & 4 & & \\ \hline \end{array}, \quad T \in LRS(\lambda/\mu, \nu) \subseteq LR(\lambda/\mu, \nu). \quad (87)$$

The right and left companions of the Sundaram LR tableau T are respectively

$$G_{\nu}(T) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 & & \\ \hline 4 & & & \\ \hline \end{array} \in LRS_{\mu, \nu}^{\lambda}, \quad G_{\mu}(T) = \begin{array}{|c|c|} \hline 3 & 3 \\ \hline 4 & \\ \hline 6 & \\ \hline \end{array} \in {}^{-}LRS_{\mu, \nu}^{\lambda} \subseteq {}^{-}LR_{\mu, \nu}^{\lambda} \quad (88)$$

where $G_{\mu}(T)$ of weight $rev(\lambda - \nu) = (0, 0, 2, 1, 0, 1)$ is defined by the nested sequence

$$(2, 1, 1, 0, 0, 0) \supseteq (21000) \supseteq (2100) \supseteq (200) \supseteq (00) \supseteq (00).$$

We may check that $G_{\mu}(T)$ satisfy the Kwon condition: the entries in row i are at least $2i - 1$, for $i = 1, 2, 3$: entries in row 1 are $3 \geq 1$, entries in row 2 are $4 \geq 3$, entries in row 3 are $6 \geq 6 - 1$.

$$evac_6 G_{\mu}(T) = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \in LRK_{\nu, \mu}^{\lambda} \subseteq LR_{\nu, \mu}^{\lambda} \subseteq B(\mu, 6), \quad \text{weight } \lambda - \nu = (1, 0, 1, 2, 0, 0) \quad (89)$$

The Henriques-Kamnitzer LR commuter, the LR commuter by Kushwaha–Raghavan–Viswanath and our commuter all of them send $(G_{\mu}(T), G_{\nu}(T))$ to $(evac_6 G_{\nu}(T), evac_6 G_{\mu}(T))$:

$$Y_\mu \cup T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & & \\ \hline 2 & 3 & 4 & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 2 & 2 & \\ \hline 3 & 2 & 3 & & \\ \hline 2 & 3 & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & \\ \hline 3 & 3 & 3 & & \\ \hline 2 & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & \\ \hline 2 & 3 & 3 & & \\ \hline \end{array} \quad (90)$$

$$\rightarrow \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & \\ \hline 2 & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline 1 \\ \hline \end{array} \rightarrow \emptyset \quad (91)$$

One then has the GT pattern of type μ defined by the red nested sequence of partitions

$$(1) \subseteq (1) \subseteq (1, 1) \subseteq (2, 1, 1),$$

that gives the tableau $evac_6 G_\mu(T)$ as in the bijection by Kumar-Torres based on the LR commuter by Kushwaha–Raghavan–Viswanath,

$$evac_6 G_\mu(T) = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \xrightarrow{c^{-1}} Y_\nu \cup H = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & \\ \hline 3 & 3 & 2 & & \\ \hline 4 & 1 & 3 & & \\ \hline \end{array}$$

$$G_\nu(H) = \begin{array}{|c|c|c|c|} \hline 3 & 3 & 3 & 5 \\ \hline 4 & 4 & 6 & 6 \\ \hline 5 & 5 & & \\ \hline 6 & & & \\ \hline \end{array} = evac_6(G_\nu(T)) \quad G_\mu(H) = evac_6 G_\mu(T) \in LRK_{\mu,\nu}^\lambda$$

6. RECURSION OF SWITCHING ON BALLOT TABLEAU PAIRS

Switching can be performed in stages whenever a decomposition of Y or T in the tableau pair $Y \cup T$ is considered. This property allows to exhibit a recursion of the switching map on ballot tableau pairs $Y_\mu \cup T$ by reducing the size of the partition μ . Due to the switching condition (73), switches in a tableau pair $Y \cup T$ where Y or T is a ballot tableau are such that the entries i in the i th row of a ballot tableau can not be switched upwards. Thereby switching T with the rows of Y_μ in stages incurs that the length of the word in the n th row of T restricted to the alphabet $[n-1]$ eventually reduces. Because an entry i in row i of Y_μ either switches horizontally with an entry of T or vertically with an entry of T below row i and further moves of that i will never occur with entries of T above row i . Recall Remark 6.

Lemma 7. Let $2 \leq l \leq n$ and $T \in YT((\lambda_l, \dots, \lambda_{n+1})/(\mu_l))$, $\mu_l > 0$, of weight $(\alpha_1, \dots, \alpha_{l-1}, \nu_l, \dots, \nu_{n+1})$, with $(\nu_l, \dots, \nu_{n+1})$ a partition. For some $\beta \subseteq (\lambda_l, \dots, \lambda_{n+1})$, assume the decomposition $Y_{(\mu_l)} \cup T = Y_{(\mu_l)} \cup A \cup B$ where $A = T_{[l-1]} \in YT(\beta/(\mu_l))$ has weight $(\alpha_1, \dots, \alpha_{l-1})$, and $B = T_{[l, n+1]}$ is a ballot tableau of shape $(\lambda_l, \dots, \lambda_{n+1})/\beta$ and weight $(\nu_l, \dots, \nu_{n+1})$. Then if $F(n+1)^{\nu_{n+1}}$ is the $(n-l+2)$ th row of T with F a word in $[n]$, one has

$$\rho_1^{(n-l+2)}(Y_{(\mu_l)} \cup T) = S \cup Q,$$

such that $S = S_1 \cup S_2 \equiv T$, with $S_1 = S_{[l-1]} \equiv A$ and $S_2 = S_{[l, n+1]} \equiv B \equiv Y_{(\nu_l, \dots, \nu_{n+1})}$, and $Q \equiv Y_{(\mu_l)}$. Moreover, if $\hat{F}(n+1)^{\nu_{n+1}}$ is the $(n-l+2)$ th row of S and D is the $(n-l+2)$ th row of Q then $\hat{F}$ is a subword of F and $|\hat{F}| + |D| = |F|$.

Proof. Let S be the rectification of T . The rectification can proceed in stages by switching $Y_{(\mu_l)}$ with A and B in stages. Observe that since T restricted to the alphabet $[l, n+1]$ is the ballot tableau B of skew shape $(\lambda_l, \dots, \lambda_{n+1})/\beta$ and weight $(\nu_l, \dots, \nu_{n+1})$, the ν_{n+1} entries $n+1$ in the last row of T stay there until the end of the rectification of T . Therefore $\rho_1^{(n-l+2)}(Y_{(\mu_l)} \cup T) = S \cup Q$, such that $S = S_1 \cup S_2 \equiv T$, with $S_1 = S_{[l-1]} \equiv A$ and $S_2 = S_{[l, n+1]} \equiv B \equiv Y_{(\nu_l, \dots, \nu_{n+1})}$, and $Q \equiv Y_{(\mu_l)}$. In particular, the last row of S has ν_{n+1} entries $n+1$.

If $F = \emptyset$ then $Y_{(\mu_l)} \cup T = Y_{(\mu_l)} \cup ((n+1)^{\nu_{n+1}} * T^{[n-l+1]}) = Y_{(\mu_l)} \cup [A \cup ((n+1)^{\nu_{n+1}} * B^{[n-l+1]})]$. Hence $\rho_1^{(n-l+2)}(Y_{(\mu_l)} \cup T) = [S_1 \cup ((n+1)^{\nu_{n+1}} * S_2^{[n-l+1]})] \cup (\emptyset * Q^{[n-l+1]})$ with $S_1 \equiv A$, $S_2 = (n+1)^{\nu_{n+1}} * S_2^{[n-l+1]} \equiv$

$Y_{(\nu_1, \dots, \nu_{n+1})}$ and $S = S_1 \cup S_2 \equiv T$. Thus

$$\begin{aligned} \rho_1^{(n-l+2)}(Y_{(\mu_l)} \cup T) &= (n+1)^{\nu_{n+1}} * \rho_1^{(n-l+1)}(Y_{(\mu_l)} \cup T^{[n-l+1]}) \\ &= ((n+1)^{\nu_{n+1}} * S^{[n-l+1]}) \cup (\emptyset * Q^{[n-l+1]}), \end{aligned}$$

where $S = ((n+1)^{\nu_{n+1}} * S^{[n-l+1]}) \equiv T$ and $Q = \emptyset * Q^{[n-l+1]} \equiv Y_{(\mu_l)}$. The last rows, $\hat{F}$ of S , and D of Q are both the empty word.

Let $F \neq \emptyset$ and $Y_{(\mu_l)} = l^{\mu_l}$. Either an entry l of $Y_{(\mu_l)}$ reaches in the switching process the row next to the last or not. In the later case $D = \emptyset$ and $\hat{F} = F$. In the former case, there exists a perforated tableau pair where the row next to the last has one entry of $Y_{(\mu_l)}$. Then the two last rows are either of the form:

(a)

$$\begin{array}{ccccccc} \cdots & \textcolor{red}{l} & a & \cdots & n^{\hat{\nu}_n} & & \\ \cdots & b & \cdots & (n+1)^{\nu_{n+1}} & & & \end{array}, \text{ with } 1 \leq a, b \leq n \text{ and } \hat{\nu}_n \leq \nu_n,$$

and the next switches either are

$$\begin{array}{ccccccc} \cdots & \textcolor{red}{l} & a & \cdots & n^{\hat{\nu}_n} & & \\ \cdots & b & \cdots & (n+1)^{\nu_{n+1}} & & & \end{array} \xrightarrow{s} \begin{array}{ccccccc} \cdots & b & a & \cdots & n^{\hat{\nu}_n} & & \\ \cdots & \textcolor{red}{l} & \cdots & (n+1)^{\nu_{n+1}} & & & \end{array} \xrightarrow{s} \begin{array}{ccccccc} \cdots & b & a & \cdots & n^{\hat{\nu}_n} & & \\ \cdots & \textcolor{red}{l} & \cdots & (n+1)^{\nu_{n+1}} & & & \end{array}, \text{ if } b \leq a \leq n,$$

or

$$\begin{array}{ccccccc} \cdots & \textcolor{red}{l} & a & \cdots & n^{\hat{\nu}_n} & & \\ \cdots & b & \cdots & (n+1)^{\nu_{n+1}} & & & \end{array} \xrightarrow{s} \begin{array}{ccccccc} \cdots & a & \textcolor{red}{l} & \cdots & n^{\hat{\nu}_n} & & \\ \cdots & b & \cdots & (n+1)^{\nu_{n+1}} & & & \end{array}, \text{ if } a < b \leq n;$$

or (b)

$$\begin{array}{ccccccc} \cdots & & \textcolor{red}{l} & \cdots & n & n & \cdots n \\ \cdots & n+1 & \cdots & n+1 & \cdots & n+1 & \end{array} \xrightarrow{\text{horizontal switch } s} \begin{array}{ccccccc} \cdots & & & \cdots & n^{\hat{\nu}_n} & \textcolor{red}{l} & \\ \cdots & & & (n+1)^{\nu_{n+1}} & & & \end{array}.$$

In any case, $S = [\hat{F}(n+1)^{\nu_{n+1}}] * S^{[n-l+1]}$ with $\hat{F}$ a subword of F , and $Q = D * Q^{[n-l+1]} \equiv Y_{(\mu_l)}$ such that $D = l^{|D|}$ and $|\hat{F}| + |D| = |F|$. $\square$

Example 19. Below one illustrates the previous lemma.

- (1) $l = 3 < n = 4$, $\mu_3 = 3$, $T \in YT((7, 6, 4)/(3))$ of content $(2, 1; 5, 3, 3)$ such that $T_{|\{1, 2\}} \in YT((4, 2)/(3))$ has content $(\alpha_1 = 2, \alpha_2 = 1)$ and $T_{|\{3, 4, 5\}}$ a ballot tableau of skew-shape $(7, 6, 4)/(4, 2)$ content $\nu = (\nu_3 = 5, \nu_4 = 3, \nu_5 = 3)$ and $F = 3$.

$$\begin{aligned} Y_{(3)} \cup T &= \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{red}{3} & \textcolor{red}{3} & \textcolor{red}{3} & 1 & 3 & 3 & 3 \\ \hline 1 & 2 & 3 & 4 & 4 & 4 & \\ \hline 3 & 5 & 5 & 5 & & & \\ \hline \end{array} \xrightarrow{s} \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{red}{3} & \textcolor{red}{3} & 1 & 3 & 3 & 3 & \textcolor{red}{3} \\ \hline 1 & 2 & 3 & 4 & 4 & 4 & \\ \hline 3 & 5 & 5 & 5 & & & \\ \hline \end{array} \xrightarrow{s} \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{red}{3} & 1 & \textcolor{red}{3} & 3 & 3 & 3 & \textcolor{red}{3} \\ \hline 1 & 2 & 3 & 4 & 4 & 4 & \\ \hline 3 & 5 & 5 & 5 & & & \\ \hline \end{array} \\ &\xrightarrow{s} \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{red}{3} & 1 & 3 & 3 & 3 & 3 & \textcolor{red}{3} \\ \hline 1 & 2 & \textcolor{red}{3} & 4 & 4 & 4 & \\ \hline 3 & 5 & 5 & 5 & & & \\ \hline \end{array} \xrightarrow{s} \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{red}{3} & 1 & 3 & 3 & 3 & 3 & \textcolor{red}{3} \\ \hline 1 & 2 & 4 & 4 & 4 & \textcolor{red}{3} & \\ \hline 3 & 5 & 5 & 5 & & & \\ \hline \end{array} \xrightarrow{s} \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 3 & 3 & \textcolor{red}{3} \\ \hline \textcolor{red}{3} & 2 & 4 & 4 & 4 & \textcolor{red}{3} & \\ \hline 3 & 5 & 5 & 5 & & & \\ \hline \end{array} \\ &\xrightarrow{s} \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 3 & 3 & \textcolor{red}{3} \\ \hline 2 & 4 & 4 & 4 & \textcolor{red}{3} & \textcolor{red}{3} & \\ \hline 3 & 5 & 5 & 5 & & & \\ \hline \end{array} = S \cup Q, \quad S \equiv T, \quad Q = \emptyset * Q^{[2]} \equiv Y_{(3)}, \quad D = \emptyset, \quad F = \hat{F}, \quad \hat{F}5^{\nu_5} = F5^{\nu_5} = 35^3. \end{aligned}$$

- (2) $l = 3 < n = 4$, $\mu_3 = 3$, $H \in YT((7, 6, 4)/(3))$ of content $(2, 1; 5, 4, 2)$ such that $H_{|\{1, 2\}} \in YT((4, 2)/(3))$ has content $(\alpha_1 = 2, \alpha_2 = 1)$ and $H_{|\{3, 4, 5\}}$ a ballot tableau of skew-shape $(7, 6, 4)/(4, 2)$ content $\nu = (\nu_3 = 5, \nu_4 = 4, \nu_5 = 2)$ and $F = 34$.

$$Y_{(3)} \cup H = \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{red}{3} & \textcolor{red}{3} & \textcolor{red}{3} & 1 & 3 & 3 & 3 \\ \hline 1 & 2 & 3 & 4 & 4 & 4 & \\ \hline 3 & 4 & 5 & 5 & & & \\ \hline \end{array} \xrightarrow{s} \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{red}{3} & \textcolor{red}{3} & 1 & 3 & 3 & 3 & \textcolor{red}{3} \\ \hline 1 & 2 & 3 & 4 & 4 & 4 & \\ \hline 3 & 4 & 5 & 5 & & & \\ \hline \end{array} \xrightarrow{s} \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{red}{3} & 1 & \textcolor{red}{3} & 3 & 3 & 3 & \textcolor{red}{3} \\ \hline 1 & 2 & 3 & 4 & 4 & 4 & \\ \hline 3 & 4 & 5 & 5 & & & \\ \hline \end{array}$$

$$\begin{array}{ccccccc}
\begin{array}{|c|c|c|c|c|c|c|} \hline \color{red}{3} & 1 & 3 & 3 & 3 & 3 & \color{red}{3} \\ \hline 1 & 2 & \color{red}{3} & 4 & 4 & 4 & \\ \hline 3 & 4 & 5 & 5 & & & \\ \hline \end{array} & \xrightarrow{s} & \begin{array}{|c|c|c|c|c|c|c|} \hline \color{red}{3} & 1 & 3 & 3 & 3 & 3 & \color{red}{3} \\ \hline 1 & 2 & 4 & 4 & 4 & 4 & \color{red}{3} \\ \hline 3 & 4 & 5 & 5 & & & \\ \hline \end{array} & \xrightarrow{s} & \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 3 & 3 & \color{red}{3} \\ \hline \color{red}{3} & 2 & 4 & 4 & 4 & 4 & \color{red}{3} \\ \hline 3 & 4 & 5 & 5 & & & \\ \hline \end{array} & \xrightarrow{s} & \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 3 & 3 & \color{red}{3} \\ \hline 2 & \color{red}{3} & 4 & 4 & 4 & 4 & \color{red}{3} \\ \hline 3 & 4 & 5 & 5 & & & \\ \hline \end{array} \\
\downarrow s & & \downarrow s & & & & \\
\begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 3 & 3 & \color{red}{3} \\ \hline 2 & 4 & 4 & 4 & 4 & 4 & \color{red}{3} \\ \hline 3 & \color{red}{3} & 5 & 5 & & & \\ \hline \end{array} & \xrightarrow{s} & \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 3 & 3 & \color{red}{3} \\ \hline 2 & 4 & 4 & 4 & 4 & 4 & \color{red}{3} \\ \hline 3 & 5 & 5 & \color{red}{3} & & & \\ \hline \end{array} & = S \cup Q, \quad S \equiv H, \quad Q = 3 * Q^{[2]}, \quad D = 3, \quad F = 34, \quad \widehat{F} = 3
\end{array}$$

with $\widehat{F}5^{\nu_5} = 35^2$ a subword of $F5^{\nu_5} = 345^2$.

Lemma 8. Let $n \geq 1$ and $Y_\mu \cup T \in \mathcal{LR}^{(n+1)}$, with $\mu = (\mu_1, \dots, \mu_n, 0)$ a non zero partition and $T \equiv Y_\nu$. Suppose $\lambda_{n+1} = \nu_{n+1} \geq 0$, that is, the $(n+1)$ th row of T is the word $(n+1)^{\nu_{n+1}}$. Then

$$\rho_1^{(n+1)}(Y_\mu \cup T) = (n+1)^{\nu_{n+1}} * \rho_1^{(n)}(Y_\mu \cup T^{[n]}) = Y_\nu \cup Q,$$

where $Q = \emptyset * Q^{[n]} \equiv Y_\mu$.

Proof. One has $Y \cup T = (n+1)^{\nu_{n+1}} * (Y \cup T^{[n]})$ a ballot tableau pair. Hence the switching procedure on $Y \cup T$ only comprises the entries of $T^{[n]}$, all $\leq n$, and the entries of Y_μ . Thus $\rho_1^{(n+1)}(Y_\mu \cup T) = (n+1)^{\nu_{n+1}} * \rho_1^{(n)}(Y_\mu \cup T^{[n]}) = Y_\nu \cup Q$ and the $(n+1)$ th row of Q is empty. $\square$

Let $n \geq 1$ and $Y_\mu \cup T \in \mathcal{LR}^{(n+1)}$, with $\mu = (\mu_1, \dots, \mu_n, 0)$ a non zero partition, and $T \equiv Y_\nu$. Next theorem uses switching into stages. For some $1 \leq d \leq n$ with $\mu_d > 0$, we decompose $Y_\mu = Y_{(\mu_1, \dots, \mu_{d-1})} \cup Y_{(\mu_d, \dots, \mu_n)}$, and thereby decomposing $Y_\mu \cup T = Y_{(\mu_1, \dots, \mu_{d-1})} \cup Y_{(\mu_d, \dots, \mu_n)} \cup T$. Then switch T with $Y_{(\mu_d, \dots, \mu_n)}$ to get $Y_{(\mu_1, \dots, \mu_{d-1})} \cup S \cup Q$, with $S \equiv T$ and $Q \equiv Y_{(\mu_d, \dots, \mu_n)}$ consisting of the entries of Y_μ moved out to the skew shape of T . The choice of d is made with the purpose to reduce the length of the $(n+1)$ th row word F of T , restricted to the alphabet $[n]$. We have the guarantee that this happens with $d = 1$ but at this point, when T is rectified, F is empty. (Note that when $d = 1$ we have the full switch of Y with Y_μ which gives $S \cup Q$, with $S = Y_\nu \equiv T$, therefore the $(n+1)$ th row of S is empty, and $Q \equiv Y_{(\mu_1, \dots, \mu_n)}$.) The choice of d is made when for the first time an entry d of Y_μ reaches the $(n+1)$ th row. When this happens and T is full switched with $Y_{(\mu_d, \dots, \mu_n)}$, we stop the switching. At this stage the $(n+1)$ th row D of Q comprises only entries equal to d and the word F is reduced to a subword $\widehat{F}$ of length $|F| - |D|$ with $|D| > 0$.

Theorem 13. Let $n \geq 1$ and $Y_\mu \cup T \in \mathcal{LR}^{(n+1)}$, with $\mu = (\mu_1, \dots, \mu_n, 0)$ a non zero partition, and $T \equiv Y_\nu$. Suppose $\lambda_{n+1} - \nu_{n+1} \geq 1$, that is, the $(n+1)$ th-row of T is the word $F(n+1)^{\nu_{n+1}}$ with F a non empty word in the alphabet $[n]$. Then, there exists $1 \leq d \leq n$, with $\mu_d > 0$, such that

$$Y_\mu \cup T \xrightarrow{s} Y_{(\mu_1, \dots, \mu_{d-1})} \cup S \cup Q = [\widehat{F}(n+1)^{\nu_{n+1}} * (Y_{(\mu_1, \dots, \mu_{d-1})} \cup S^{[n]})] \cup [D * Q^{[n]}] \quad (92)$$

where $S \equiv Y_\nu$ and $Q = D * Q^{[n]} \equiv Y_{(\mu_d, \dots, \mu_n)}$ is over the alphabet $[d, n]$, with $Q^{[d-1]} = \emptyset$, and $D = d^{|D|}$. In addition, the $(n+1)$ th row $\widehat{F}(n+1)^{\nu_{n+1}}$ of S is such that $\widehat{F}$ is a strict subword of F whose length satisfies $|D| + |\widehat{F}| = |F| > |\widehat{F}| \geq 0$. Also,

$$\rho_1^{(n+1)}(Y_\mu \cup T) = \begin{cases} Y_\nu \cup Q, & \text{with } |D| = \lambda_{n+1} - \nu_{n+1}, \text{ if } d = 1, \\ \rho_1^{(n+1)}[Y_{(\mu_1, \dots, \mu_{d-1})} \cup S] \cup Q, & \text{if } d > 1. \end{cases} \quad (93)$$

Proof. We handle the proof by induction on the length $l := \ell(\mu) \geq 1$. Let $n \geq l$ and $v := \lambda_{n+1} - \nu_{n+1} = |F| > 0$ the number of entries $\leq n$ in the $(n+1)$ th row of T .

For $l = 1$, $Y_{(\mu_1)} \cup T \in \mathcal{LR}^{(n+1)}$ with $\mu_1 > 0$, and, therefore, the $(n+1)$ th row of T is of the form $n^v(n+1)^{\nu_{n+1}}$ with $1 \leq v := \lambda_{n+1} - \nu_{n+1} \leq \mu_1$. Rectify T with *jeu de taquin* slides looking at the entries of $Y_{(\mu_1)}$ as holes. Then

$$Y_{\mu_1} \cup T \xrightarrow{s} Y_\nu \cup Q,$$

where $Q = D * Q^{[n]} \equiv Y_{\mu_1}$. Since the shape of $Y_{(\mu_1)} \cup T$ is preserved in the switching procedure, $|D| = |F| = v$ and $D = 1^v$. In the case $l = 1$, (92) and (93) hold with $d = 1$, and $\widehat{F} = \emptyset$.

Let $l > 1$, and assume the statement true for $1, \dots, l-1$. Then $n+1 \geq l+1 \geq 3$, and consider the factorisation

$$Y_{(\mu_1, \dots, \mu_l)} \cup T = [Y_{(\mu_l)} \cup \widehat{T}] * [Y_{(\mu_1, \dots, \mu_{l-1})} \cup T^{[l-1]}] \in \mathcal{LR}^{(n+1)},$$

where $T^{[l-1]}$ is a ballot tableau on the alphabet $[l-1]$, and $\widehat{T} \in YT((\lambda_l, \dots, \lambda_{n+1})/(\mu_l))$ consists of the last $n-l+2$ rows of T whose word restricted to the alphabet $[l, n+1]$ satisfies the Yamanouchi condition. Then, by Lemma 7, the switching procedure gives

$$Y_{(\mu_l)} \cup \widehat{T} \xrightarrow{s} S \cup Q, \quad (94)$$

with $S \equiv \widehat{T}$ (the rectification of $\widehat{T}$) in the alphabet $\{1, \dots, n+1\}$, and $Q = D * Q^{[n-l+1]} \equiv Y_{(\mu_l)}$ in the alphabet $\{l\}$. If $|D| > 0$ the last row of S is the word $\widehat{F}(n+1)^{\nu_{n+1}}$ with $\widehat{F}$ a subword of F . Observing that $T = \widehat{T} * T^{[l-1]} \equiv S * T^{[l-1]}$ is a ballot tableau, we have in addition

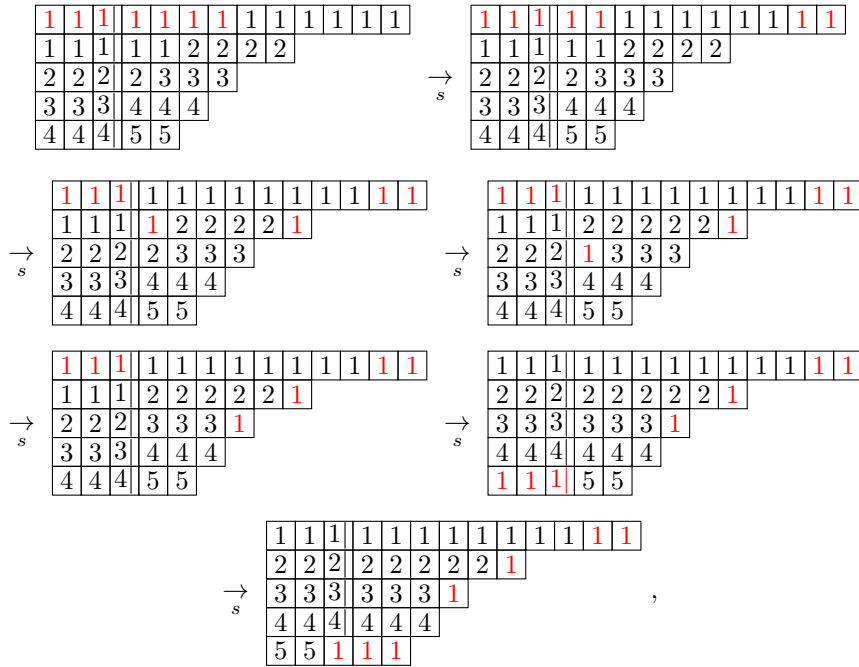
$$Y_{(\mu_1, \dots, \mu_l)} \cup T \xrightarrow{s} [Y_{(\mu_1, \dots, \mu_{l-1})} \cup (S * T^{[l-1]})] \cup (Q * \emptyset^{l-1}),$$

where $Y_{(\mu_1, \dots, \mu_{l-1})} \cup (S * T^{[l-1]})$ is a ballot tableau pair. If $|D| > 0$, (92) and (93) holds with $d = l$. Otherwise, $|D| = 0$ and henceforth the $(n+1)$ th row of S (94) is the $(n+1)$ th row of T with length λ_{n+1} . Thereby

$$Y_{(\mu_1, \dots, \mu_l)} \cup T \xrightarrow{s} [Y_{(\mu_1, \dots, \mu_{l-1})} \cup (S * T^{[l-1]})] \cup (\emptyset * Q^{[n-l+1]} * \emptyset^{l-1}),$$

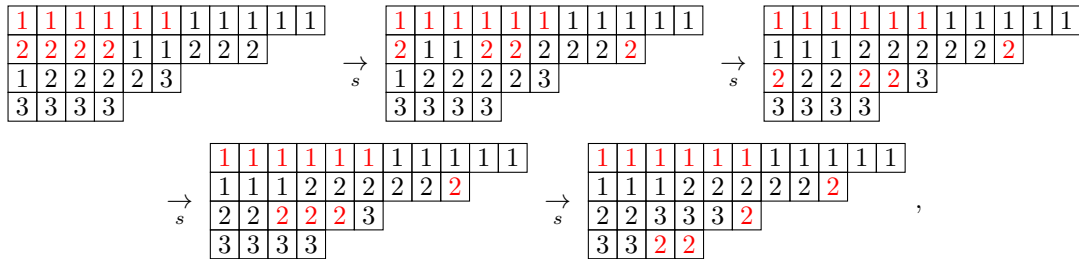
where $Y_{(\mu_1, \dots, \mu_{l-1})} \cup (S * T^{[l-1]}) \in \mathcal{LR}^{(n+1)}$ is in the case $l-1$, and, by inductive hypothesis, we get (92) and (93) with $1 \leq d \leq l-1$. $\square$

Example 20. (I). $\ell(\mu) = 1$. $n = 4$, $\mu_1 = 7$, $v = 3$, $d = 1$,



$F = 4^3$, $D = 1^3$ and $Q = 1^3 * \emptyset * 1 * 1 * 1^2$ of skew shape $\lambda/(11, 8, 6, 6, 2)$.

(II). $\ell(\mu) = 2$, $n = 3$, $v = 4$, $d = 2$



$F = 3^4$, $\nu_4 = 0$, $\widehat{F} = 3^2$, $D = 2^2$ and $Q = 2^2 * 1 * 1 * \emptyset$ of skew shape $\lambda/(11, 8, 5, 2)$.

We first recall the following property. The major result of the next statement is that the reading word $\widehat{F}(n+1)^{\nu_{n+1}}GCn^{\nu_n}$ (96) of the two last rows of $Y_{(\mu_1, \dots, \mu_{d-1})} \cup S$, and the concatenation of the $(n+1)$ th row $F(n+1)^{\nu_{n+1}}$ (97) of $Y_\mu \cup T$ with the n th row $\widehat{G}Cn^{\nu_n}$ (97) of $Y_{(\mu_1, \dots, \mu_{d-1})} \cup R$, that is, $F(n+1)^{\nu_{n+1}}GCn^{\nu_n}$, are related through Knuth transformations. Since GC is a row word on the alphabet $[n-1]$, $\widehat{G}$ a subword of G and F is a word in the alphabet $[n]$ and $\widehat{F}$ a subword of F , from Lemma 1, (a), (45), it means that $\widehat{F}GCn^{\nu_n} \equiv F\widehat{G}Cn^{\nu_n}$ and from Lemma 1, (c), $\widehat{F}GC \equiv F\widehat{G}C$ or $\widehat{F}G \equiv F\widehat{G}$.

Corollary 8. Let $n \geq 1$, $Y_\mu \cup T \in \mathcal{LR}^{(n+1)}$ and assume the assumptions of previous theorem on $Y_\mu \cup T$. Consider the equality (92), for some $1 \leq d \leq n$ and $\mu_d > 0$, where we detach the two last rows of S and Q ,

$$Y_\mu \cup T = F(n+1)^{\nu_{n+1}} * (Y \cup T)^{[n]} \xrightarrow{s} [Y_{(\mu_1, \dots, \mu_{d-1})} \cup S] \cup Q \quad (95)$$

$$= [\widehat{F}(n+1)^{\nu_{n+1}} * GCn^{\nu_n} * (Y_{(\mu_1, \dots, \mu_{d-1})} \cup S^{[n-1]})] \cup [D * X * Q^{[n-1]}], \quad (96)$$

with $Y_{(\mu_1, \dots, \mu_{d-1})} \cup S^{[n]} = GCn^{\nu_n} * (Y_{(\mu_1, \dots, \mu_{d-1})} \cup S^{[n-1]})$, $Q^{[n]} = X * Q^{[n-1]}$ such that X is a row word on the alphabet $[d, n]$, and GC , a row word on the alphabet $[n-1]$, with the factor G satisfying $|G| = |F| = |\widehat{F}| + |D|$. Then

$$(Y_\mu \cup T)^{[n]} \xrightarrow{s} [Y_{(\mu_1, \dots, \mu_{d-1})} \cup R] \cup P, \quad (97)$$

where $P = DX * Q^{[n-1]} \equiv Q$, and $R \equiv T^{[n]}$ is such that $Y_{(\mu_1, \dots, \mu_{d-1})} \cup R = \widehat{G}Cn^{\nu_n} * (Y_{(\mu_1, \dots, \mu_{d-1})} \cup S^{[n-1]})$, with $\widehat{G}$ a row subword of G so that $|\widehat{G}| = |\widehat{F}|$, $(|\widehat{G}| + |D| = |F| = |G|)$ and $\widehat{F}G \equiv F\widehat{G}$ are Knuth equivalent. Also

$$\begin{aligned} \rho_1^{(n)}[(Y_\mu \cup T)^{[n]}] &= \rho_1^{(n)}(Y_{(\mu_1, \dots, \mu_{d-1})} \cup R) \cup (DX * Q^{[n-1]}) \\ &= \rho_1^{(n)}[\widehat{G}Cn^{\nu_n} * (Y_{(\mu_1, \dots, \mu_{d-1})} \cup S^{[n-1]})] \cup (DX * Q^{[n-1]}). \end{aligned} \quad (98)$$

Proof. Switching back in the two last rows of $Y_{(\mu_1, \dots, \mu_{d-1})} \cup S \cup Q$, (96), and factoring $G = AB$ into two row words A and B , with $|A| = |\widehat{F}|$ and $|B| = |D|$, one has

$$[\widehat{F}(n+1)^{\nu_{n+1}} \cup D] * [GCn^{\nu_n} \cup X] = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline A & B & C & n^e & \cdots & n^h & \textcolor{red}{X} & & & & \\ \hline \widehat{F} & n+1^{|D|} & \cdots & n+1^e & \textcolor{red}{D} & & & & & & \\ \hline \end{array} \xleftrightarrow{s} \quad (99)$$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline G_1 & F_2 & G_3 & F_4 & \cdots & G_{k-1} & F_k & C & n^e & \cdots & n^h & \textcolor{red}{X} \\ \hline F_1 & \textcolor{red}{D}_2 & F_3 & \textcolor{red}{D}_4 & \cdots & F_{k-1} & \textcolor{red}{D}_k & n+1^{|C|} & \cdots & n+1^{|D|} & & \\ \hline \end{array} \xleftrightarrow{s} \quad (100)$$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline G_1 & \textcolor{red}{D}_2 & G_3 & \textcolor{red}{D}_4 & \cdots & G_{k-1} & \textcolor{red}{D}_k & C & n^e & \cdots & n^h & \textcolor{red}{X} \\ \hline F_1 & F_2 & F_3 & F_4 & \cdots & F_{k-1} & F_k & n+1^{|C|} & \cdots & n+1^{|D|} & & \\ \hline \end{array} \quad (101)$$

where $\textcolor{red}{D} = \textcolor{red}{D}_2\textcolor{red}{D}_4 \cdots \textcolor{red}{D}_k$, $F = F_1F_2 \cdots F_k$, $\widehat{F} = F_1F_3 \cdots F_{k-1}$ and $\widehat{G} := G_1G_3 \cdots G_{k-1}$ a row subword of $G = AB = G_1F_2G_3 \cdots G_{k-1}F_k$, with $|\widehat{G}| + |D| = |G| = |F| = |\widehat{F}| + |D|$. The subword of F with $\widehat{F}$ suppressed, $F \setminus \widehat{F}$, and $\widehat{G}$ are complementary row subwords of G .

In every step of the *jeu de taquin* sliding or *reverse jeu de taquin* sliding process the reading word is transformed into a Knuth equivalent one [Ful97, Sta98]. Looking at the red letters as holes, the row reading

words $\widehat{F}ABC = \widehat{F}GC$ of (99) (or (100)), and $F\widehat{G}C$ of (101), restricted to the alphabet $[n]$ (in black), are Knuth equivalent.

$$\widehat{F}GC \equiv F\widehat{G}C \Leftrightarrow \widehat{F}G \equiv F\widehat{G}.$$

The last row of (101) is $F(n+1)^{\nu_{n+1}}$. Let $Y' := Y_{(\mu_1, \dots, \mu_{d-1})}$. It then follows from (95),

$$\begin{aligned} (Y_\mu \cup T)^{[n]} &\xrightarrow{s} \begin{bmatrix} G_1 & D_2 & G_3 & D_4 & \cdots & D_k & C & n & \cdots & n & X \end{bmatrix} * [Y' \cup S^{[n-1]} \cup Q^{[n-1]}] \\ &\xrightarrow{s} \begin{bmatrix} \widehat{G} & C & n & \cdots & n & D & X \end{bmatrix} * [Y' \cup S^{[n-1]} \cup Q^{[n-1]}] \\ &= Y' \cup R \cup P = [\widehat{G}Cn^{\nu_n} \cup DX] * [Y' \cup S^{[n-1]} \cup Q^{[n-1]}], \end{aligned} \quad (102)$$

where $Y' \cup R = \widehat{G}Cn^{\nu_n} * (Y' \cup S^{[n-1]})$, and P is $Q^{[n]}$ with D attached to the left of its n th row X . $\square$

7. PROOF OF THE MAIN THEOREM

Let $u = u_v \cdots u_1$ be an internal inserting order word of a tableau T . By $\bar{\phi}_u$ -bumping routes on $Y \cup T$, we mean the collection of $\bar{\phi}_{u_i}$ -bumping routes on $\bar{\phi}_{u_{i-1} \dots u_1}(Y \cup T)$ for $i = 1, \dots, v$.

Lemma 9. Let $u = u_v \cdots u_2 u_1$ be a row word to be an internal insertion order word of T . Then

- (a) the plactic class of u is reduced to the sole u .
- (b) the $\bar{\phi}_u$ -bumping routes on T are pairwise disjoint.
- (c) if the $\bar{\phi}_{u_i}$ -bumping route lands in row $1 \leq k \leq n+1$, the $\bar{\phi}_{u_{i+1}}$ -bumping route lands strictly to the right in a row $\leq k$, for $i = 1, \dots, n-1$.

Proof. (a) The plactic class of a row tableau has a sole element. (b) and (c) follow from Lemma 3 (a). $\square$

If $T \in YT(\lambda/\mu)$ is a ballot tableau and $\ell(\lambda) \leq n$, and $1 \leq k \leq \lambda_n - \mu_n$, then $T_{\leftarrow k}$ denotes the ballot tableau in $YT((\lambda_1, \dots, \lambda_{n-1}, \lambda_n - k)/\mu)$ obtained from T first by suppressing, in the n th row, the first k filled boxes and then pushing the remaining $\lambda_n - \mu_n - k$ boxes k steps to the left.

$$n = 3, \mu = (6, 4, 1), \lambda = (11, 9, 6), \lambda_3 - \mu_3 = 6, k = 3 < 6,$$

$$Y_\mu \cup T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 2 & 2 & 3 \end{bmatrix} \quad Y_\mu \cup T_{\leftarrow 3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} \quad (103)$$

We are now ready to prove the main result.

Proof of Theorem 2.

We may reduce the statement to the case $\mu = (\mu_1, \dots, \mu_{n-1}, \mu_n = 0)$. Let $\hat{\mu} := \mu - (\mu_n^n)$. Performing horizontal switches in $Y \cup T = Y_{(\mu_n^n)} \cup Y_{\hat{\mu}} \cup T \xrightarrow{s} Y_{\hat{\mu}} \cup T \cup Z$, where Z is the unique ballot tableau of shape $\lambda/(\lambda - (\mu_n^n))$ and content (μ_n^n) . Thus

$$\rho_1^{(n)}(Y \cup T) = \rho_1^{(n)}(Y_{\hat{\mu}} \cup T) \cup Z = \bar{\chi}_n^{\mu_n} [\rho_1^{(n)}(Y_{\hat{\mu}} \cup T) \cup (\emptyset * Z^{[n-1]})]. \quad (104)$$

Similarly, $(Y \cup T)^{[n-1]} \xrightarrow{s} (Y_{\hat{\mu}} \cup T)^{[n-1]} \cup Z^{[n-1]}$. Hence, $\rho_1^{(n-1)}[(Y \cup T)^{[n-1]}] = \rho_1^{(n-1)}[(Y_{\hat{\mu}} \cup T)^{[n-1]}] \cup Z^{[n-1]}$.

Assuming that (84) has been proved in the case of $\mu_n = 0$, and using (104), we then may write

$$\begin{aligned} \rho_1^{(n)}(Y \cup T) &= \rho_1^{(n)}(Y_{\hat{\mu}} \cup T) \cup Z = \bar{\chi}_n^{\mu_n} [\rho_1^{(n)}(Y_{\hat{\mu}} \cup T) \cup (\emptyset * Z^{[n-1]})] \\ &= \bar{\chi}_n^{\mu_n} \{ \bar{\phi}_{V_n} \bar{\omega}_n^{\nu_n} \rho_1^{(n-1)}[(Y_{\hat{\mu}} \cup T)^{[n-1]}] \cup Z^{[n-1]} \} \\ &= \bar{\chi}_n^{\mu_n} \bar{\phi}_{V_n} \bar{\omega}_n^{\nu_n} \left(\rho_1^{(n-1)}[(Y_{\hat{\mu}} \cup T)^{[n-1]}] \cup Z^{[n-1]} \right) \\ &= \bar{\chi}_n^{\mu_n} \bar{\phi}_{V_n} \bar{\omega}_n^{\nu_n} \rho_1^{(n-1)}[(Y \cup T)^{[n-1]}]. \end{aligned} \quad (105)$$

Observe (105) just says that the bumping routes of $\bar{\phi}_{V_n}$, all of them landing in the n th row, will not change when $\rho_1^{(n-1)}[(Y_{\hat{\mu}} \cup T)^{[n-1]}] \in \mathcal{LR}^{(n-1)}$ is extended with the tableau $Z^{[n-1]}$. Each route of ϕ_V follows the available path in $\rho_1^{(n)}[(\bar{Y} \cup T)^{[n-1]}]$ and remains there until landing in the n th row.

Let $\mu_n = 0$. We now show, by induction on $n \geq 1$ and $|V_n| \geq 0$, that

$$\rho_1^{(n)}(Y \cup T) = \bar{\phi}_{V_n} \bar{\omega}^{\hat{V}_n} \rho_1^{(n-1)}[(Y \cup T)^{[n-1]}] = \bar{\omega}^{\hat{V}_n} \bar{\phi}_{V_n} \rho_1^{(n-1)}[(Y \cup T)^{[n-1]}], \quad (106)$$

where $Y = Y(\mu_1, \dots, \mu_{n-1}, 0)$ and V_n is the word of the n th row of T restricted to the alphabet $[n-1]$. In addition, all bumping routes of $\bar{\phi}_{V_n}$ terminate in the n th row.

If $n = 1$ then $|V_1| = 0$, $\mu_1 = 0$, $T = Y_{(\nu_1)}$, $\hat{V}_1 = \nu_1$, and $\rho_1^{(1)}(\emptyset \cup Y_{(\nu_1)}) = Y_{(\nu_1)} \cup \emptyset$. Then $\rho_1^{(1)}(\emptyset \cup Y_{(\nu_1)}) = Y_{(\nu_1)} \cup \emptyset$ is obtained from $\emptyset$ by adding the row 1^{ν_1} . Thus $\rho_1^{(1)}(\emptyset \cup Y_{(\nu_1)}) = \bar{\omega}_1^{\nu_1} \emptyset = \bar{\omega}_1^{\nu_1} \rho_1^{(0)}[(\emptyset \cup Y_{(\nu_1)})^{[0]}]$, with $\bar{\phi}_{V_1} = id$.

Suppose that (106) holds for $n \geq 1$, and let us prove for $n+1$. Assume $Y \cup T$ with $n+1$ rows where $Y = Y(\mu_1, \dots, \mu_n, 0)$.

Let $v := |V_{n+1}| \geq 0$ and let $F := V_{n+1}$ be the $(n+1)$ th row word of T restricted to the alphabet $[n]$. Since $\mu_{n+1} = 0$, then $F(n+1)^{\nu_{n+1}}$ is the $(n+1)$ th row of $Y \cup T$ and detaching the $(n+1)$ th row, $Y \cup T = F(n+1)^{\nu_{n+1}} * (Y \cup T)^{[n]}$. We want to show that

$$\rho_1^{(n+1)}(Y \cup T) = \bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_F \rho_1^{(n)}[(Y \cup T)^{[n]}]. \quad (107)$$

If $v = 0$, $Y \cup T = (n+1)^{\nu_{n+1}} * (Y \cup T)^{[n]}$. Therefore, by Lemma 8

$$\rho_1^{(n+1)}(Y \cup T) = (n+1)^{\nu_{n+1}} * \rho_1^{(n)}[(Y \cup T)^{[n]}] = \bar{\omega}_{n+1}^{\nu_{n+1}} \rho_1^{(n)}[(Y \cup T)^{[n]}],$$

with $\bar{\phi}_F$ the identity.

If $v \geq 1$, the $(n+1)$ th row of $Y \cup T$ is the word $F(n+1)^{\nu_{n+1}}$ with F a no empty word on the alphabet $[n]$. We shall now use induction on v .

Step 1. We pass from the ballot pair $Y \cup T$ to a ballot pair $Y' \cup S$ with $(n+1)$ th row word $\hat{F}(n+1)^{\nu_{n+1}}$ so that $\hat{F}$ is a strict subword of F .

From Theorem 13, there exists $Y' := Y(\mu_1, \dots, \mu_{d-1})$, for some $1 \leq d \leq n$, with $\mu_d > 0$, such that

$$\begin{aligned} Y \cup T &= F(n+1)^{\nu_{n+1}} * (Y \cup T)^{[n]} \xrightarrow{s} Y' \cup S \cup Q \\ &= [\hat{F}(n+1)^{\nu_{n+1}} \cup D] * [(Y' \cup S)^{[n]} \cup Q^{[n]}], \end{aligned} \quad (108)$$

where $S = \hat{F}(n+1)^{\nu_{n+1}} * S^{(n)} \equiv T$, $\hat{F}$ is a strict subword of F , and $Q = D * Q^{[n]} \equiv Y(\mu_d, \dots, \mu_n)$ is over the alphabet $\{d, \dots, n\}$ and has the $(n+1)$ th row $D = d^{|D|}$ such that $|D| = |F| - |\hat{F}| > 0$. Therefore

$$\rho_1^{(n+1)}(Y \cup T) = \rho_1^{(n+1)}(Y' \cup S) \cup Q. \quad (109)$$

Since $\hat{F}$ is the $(n+1)$ th row of $Y' \cup S$, restricted to the entries $\leq n$, and $0 \leq |\hat{F}| < v$, by induction on v , we may write

$$\rho_1^{(n+1)}(Y' \cup S) = \bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_{\hat{F}} \rho_1^{(n)}[(Y' \cup S)^{[n]}], \quad (110)$$

where all $\bar{\phi}_{\hat{F}}$ -bumping routes terminate in the $(n+1)$ th row.

To reach (107), one has, so far, from (109) and (110),

$$\rho_1^{(n+1)}(Y \cup T) = \bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_{\hat{F}} \rho_1^{(n)}[(Y' \cup S)^{[n]}] \cup (D * Q^{[n]}), \quad (111)$$

with $\hat{F}$ a strict subword of F such that $|\hat{F}| + |D| = |F|$.

Step 2. One has to relate the n th row of $(Y' \cup S)^{[n]}$ with the n th row of $(Y' \cup T)^{[n]}$.

This requires Corollary 8 and, in particular, the analysis of the reading words in (99), (100) and (101).

Step 2.1. We first analyse $\rho_1^{(n)}[(Y' \cup S)^{[n]}]$ of (111).

From Corollary 8 one has $Y' \cup S^{[n]} = GCn^{\nu_n} * (Y' \cup S^{[n-1]})$ with G and C row words in the alphabet $[n-1]$, such that $|G| + |\hat{F}| + |D| = v = |F|$. In addition $Y' \cup (S^{[n]})_{\leftarrow v}^n = Cn^{\nu_n} * (Y' \cup S^{[n-1]}) \in \mathcal{LR}^{(n)}$ with

C a row word in the alphabet $[n-1]$. By induction on n , one has

$$\rho_1^{(n)}(Y' \cup (S^{[n]})_{\leftarrow v}) = \bar{\phi}_C \omega_n^{\hat{\nu}_n} \rho_1^{(n-1)}(Y' \cup S^{[n-1]}),$$

and

$$\begin{aligned} \rho_1^{(n)}(Y' \cup S^{[n]}) &= \rho_1^{(n)}[GCn^{\hat{\nu}_n} * (Y' \cup S^{[n-1]})] \\ &= \bar{\phi}_G \bar{\phi}_C \bar{\omega}_n^{\hat{\nu}_n} \rho_1^{(n-1)}(Y' \cup S^{[n-1]}) \\ &= \bar{\phi}_G \rho_1^{(n)}(Y' \cup (S^{[n]})_{\leftarrow v}), \end{aligned} \quad (112)$$

where all $\bar{\phi}_G$ -bumping routes (also $\bar{\phi}_C$ -bumping routes) will end up in the n th row.

Step 2.2. We now analyse $\rho_1^{(n)}(Y \cup T^{[n]})$.

From Corollary 8, one has

$$Y \cup T^{[n]} \xrightarrow{s} Y' \cup R \cup P = [\widehat{G}Cn^{\hat{\nu}_n} \cup \textcolor{red}{DX}] * [Y' \cup S^{[n-1]} \cup Q^{[n-1]}],$$

where $Y' \cup R = \widehat{G}Cn^{\hat{\nu}_n} * (Y' \cup S^{[n-1]})$ with $\widehat{G}$ a subword of G , and $P = DX * Q^{[n-1]}$ such that $|\widehat{G}| = |F| - |D| = v - |D|$ and $\widehat{F}G \equiv F\widehat{G}$ Knuth equivalent.

Observe that

$$Y' \cup (S^{[n]})_{\leftarrow v} = Y' \cup R \xleftarrow[v-|D|]{} = Cn^{\hat{\nu}_n} * (Y' \cup S^{[n-1]}). \quad (113)$$

Again by induction on n , and using the identity (113), one has

$$\begin{aligned} \rho_1^{(n)}(Y' \cup R \xleftarrow[v-|D|]{}) &= \bar{\phi}_C \bar{\omega}_n^{\hat{\nu}_n} \rho_1^{(n-1)}(Y' \cup S^{[n-1]}) \\ \rho_1^{(n)}(Y' \cup R) &= \rho_1^{(n-1)}[\widehat{G}Cn^{\hat{\nu}_n} * (Y' \cup S^{[n-1]})] \\ &= \bar{\phi}_{\widehat{G}} \bar{\phi}_C \bar{\omega}_n^{\hat{\nu}_n} \rho_1^{(n-1)}(Y' \cup S^{[n-1]}) \\ &= \bar{\phi}_{\widehat{G}} \rho_1^{(n)}(Y' \cup R \xleftarrow[v-|D|]{}), \end{aligned} \quad (114)$$

where all $\bar{\phi}_{\widehat{G}}$ -bumping routes will end up in the n th row. Therefore from Corollary 8, (98), and (114),

$$\begin{aligned} \rho_1^{(n)}(Y \cup T^{[n]}) &= \rho_1^{(n)}(Y' \cup R) \cup P = \bar{\phi}_{\widehat{G}} \rho_1^{(n)}(Y' \cup R \xleftarrow[v-|D|]{}) \cup P \\ &= \bar{\phi}_{\widehat{G}} \rho_1^{(n)}(Y' \cup R \xleftarrow[v-|D|]{}) \cup (DX * Q^{[n-1]}). \end{aligned} \quad (115)$$

We are now in conditions to go back to (111).

Step 4. Going back to (111).

$$\begin{aligned} \rho_1^{(n+1)}(Y \cup T) &= \bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_{\widehat{F}} \rho_1^{(n)}(Y' \cup S^{[n]}) \cup (\textcolor{red}{D} * \textcolor{red}{Q}^{[n]}), \text{ using (111),} \\ &= \bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_{\widehat{F}} [\bar{\phi}_G \rho_1^{(n)}(Y' \cup (S^{[n]})_{\leftarrow v})] \cup (\textcolor{red}{D} * \textcolor{red}{Q}^{[n]}), \text{ using (112),} \\ &= \left(\bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_{\widehat{F}} \bar{\phi}_G \rho_1^{(n)}(Y' \cup R \xleftarrow[v-|D|]{}) \right) \cup (\textcolor{red}{D} * \textcolor{red}{Q}^{[n]}), \text{ using (113),} \end{aligned} \quad (116)$$

with all $\bar{\phi}_G$ -bumping routes ending up in the n th row, and the $\bar{\phi}_{\widehat{F}}$ -bumping routes in the $(n+1)$ th row. The cardinality of the $\bar{\phi}_G$ -bumping routes is $v = v - |D| + |D| = |\widehat{G}| + |D| = |\widehat{F}| + |D|$.

Key Step 5. Knuth relations of internal insertion operators.

From Proposition 3, since $\widehat{F}G \equiv F\widehat{G}$, one has

$$\bar{\phi}_{\widehat{F}G} \rho_1^{(n)}(Y' \cup R \xleftarrow[v-|D|]{}) = \bar{\phi}_{F\widehat{G}} \rho_1^{(n)}(Y' \cup R \xleftarrow[v-|D|]{}). \quad (117)$$

Another key fact and a simple observation is that the number v of bumping routes landing in the n th row and the number $v - |D|$ of bumping routes landing in the $(n + 1)$ th row is the same for $\bar{\phi}_F \bar{\phi}_{\widehat{G}}$ and $\bar{\phi}_{\widehat{F}} \bar{\phi}_G$ when acting on $\rho_1^{(n)}(Y' \cup R \xleftarrow{v-|D|})$. Thereby,

$$(116) = \bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_F \bar{\phi}_{\widehat{G}} \rho_1^{(n)}(Y' \cup R \xleftarrow{v-\ell(D)}) \cup (D * Q^{[n]}), \text{ using (117),} \quad (118)$$

$$= \bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_F \rho_1^{(n)}(Y' \cup R) \cup (D * X * Q^{[n-1]}), \text{ using (114).} \quad (119)$$

From (118) to (119), $|\widehat{G}| = v - |D|$ bumping routes are executed by $\bar{\phi}_{\widehat{G}}$ and land in row n . This implies that in the action of $\bar{\phi}_F$ over $\rho_1^{(n)}(Y' \cup R)$, $|D|$ of the $v = |F|$ pairwise disjoint bumping routes will still land and settle in the n th row, which means when settling to adding $|D|$ new entries at the end of the n th row of $\rho_1^{(n)}(Y' \cup R)$, while $v - |D|$ of them will land in the $(n + 1)$ th row. Recall that since F is a row word, from Lemma 3, (a), the v bumping routes are pairwise disjoint and, more importantly, the $|D|$ bumping routes settling in the n th row are necessarily the last to be executed. This means that if we attach D at the end of the n th row $\rho_1^{(n)}(Y' \cup R)$, the rightmost $|D|$ bumping routes of $\bar{\phi}_F$ when landing to the n th row will meet the entire row D and bumps it out to the $(n + 1)$ th row. Thus recalling that $P = DX * Q^{[n-1]}$ and identity (115),

$$(119) = \bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_F [\rho_1^{(n)}(Y' \cup R) \cup (DX * Q^{[n-1]})] \quad (120)$$

$$= \bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_F [\rho_1^{(n)}(Y' \cup R) \cup P], \quad (121)$$

$$= \bar{\phi}_F \bar{\omega}_{n+1}^{\nu_{n+1}} \rho_1^{(n)}(Y \cup T^{[n]}), \text{ using (115).} \quad (122)$$

We have shown $\rho_1^{(n+1)}(Y \cup T) = (122)$, that is, identity (107),

$$\rho_1^{(n+1)}(Y \cup T) = \bar{\omega}_{n+1}^{\nu_{n+1}} \bar{\phi}_F \rho_1^{(n)}[(Y \cup T)^{[n]}]. \quad \square$$

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