Keys, virtual keys and applications

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based on joint works with
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cortipom25, Le Croisic, June 13

Plan

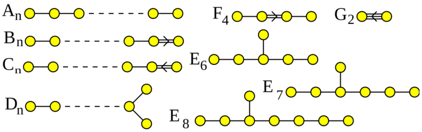
- I. Set up: Demazure and Demazure atom crystals/Schubert varieties
- II. Non-symmetric Cauchy identities/ LPP (last passage percolation model)
- III. Computation of key maps under virtualization
- IV. Illustration $B \hookrightarrow C$

I. Set up

- Let G be a simply-connected semisimple algebraic group over $\mathbb{C}$, and fix $T \subseteq B \subseteq G$, T a maximal torus and B a Borel subgroup of G. Let B^- be the corresponding opposite Borel subgroup, that is, it is the unique Borel subgroup of G with the property $B \cap B^- = T$.

- Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{b}$ ($\mathfrak{b}^-$) the Borel (opposite Borel) subalgebra, W the Weyl group (endowed with the strong Bruhat order) and I an index set for the vertices of the Dynkin diagram of $\mathfrak{g}$.
- Example: $G = SL_{n+1}(\mathbb{C}) = \{A \in GL_{n+1}(\mathbb{C}) : det(A) = 1\}$
  T the subgroup of diagonal matrices.
  B the subgroup of upper triangular matrices.
  B^- the subgroup of lower triangular matrices.
  $B^- = P_{w_0} B P_{w_0}$, w_0 the longest element of $W = \mathfrak{S}_n$.
  $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ special linear Lie algebra. Dynkin diagram A_n.

  wiki/Dynkin-diagram



Demazure and opposite Demazure modules

- Let the Lie algebra $\mathfrak{g}$ be endowed with the usual Cartan data given by the weight lattice P, $P^+ \subseteq P$ the set of dominant integral weights, $\{\alpha_i : i \in I\} \subseteq P$ the simple roots, $\{h_i = \alpha_i^\vee : i \in I\} \subseteq P^\vee = Hom(P, \mathbb{Z})$ the simple co-roots, fundamental weights $\{\omega_i, i \in I\}$, and canonical pairing $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Z}$ with Cartan matrix $C = (\langle \alpha_i^\vee, \alpha_j \rangle)_{i,j \in I}$ and Weyl group W.

- For $\lambda \in P^+$ let $V(\lambda)$ be the irreducible highest weight G-module over $\mathbb{C}$ with highest weight λ, and b_λ its highest weight vector.

- The W-**orbit of** λ under the action of W is the set of *extremal vectors* or *keys* in $V(\lambda)$

$$\mathcal{O}(\lambda) = \{wb_\lambda := \mathbf{b_{w\lambda}} : w \in W^\lambda\} \subseteq V(\lambda).$$

- The *Demazure module* and *opposite Demazure module*.

  For $w \in W$, $wb_\lambda \in \mathcal{O}(\lambda)$, we define the B-submodule $V_w(\lambda) \subseteq V(\lambda)$, resp. the B^--submodule $V^w(\lambda) \subseteq V(\lambda)$

$$V_w(\lambda) = \mathcal{U}(\mathfrak{b}).V(\lambda)_{w\lambda} \qquad V^w(\lambda) = \mathcal{U}(\mathfrak{b}^-).V(\lambda)_{w\lambda}.$$

  $\mathcal{U}(\mathfrak{b})$ is the enveloping algebra of the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$.
  $V(\lambda)_{w\lambda}$ is the one dimensional weight space of $V(\lambda)$ with extremal weight $w\lambda$.

Demazure modules and Schubert varieties

- Demazure modules $V_w(\lambda)$ were originally described as the space of global sections of a line bundle L_λ on a Schubert variety $X_w \subseteq G/B = \{gB : g \in G\}$ the full flag variety, $w \in W$,

$$G/B = \bigsqcup_{w \in W} BwB/B = \bigsqcup_{w \in W} B^- wB/B,$$

$$X_w = \bigsqcup_{v \leq w} \mathring{X}_v, \qquad X^w = \bigsqcup_{v \geq w} \mathring{X}^v = w_0 X_{w_0 w} \subseteq G/B.$$

  This description exhibits via the Borel-Weil theorem the natural correspondence between Schubert varieties and Demazure modules:

$$H^0(X_w, L_\lambda) \simeq V_w(\lambda)^*, \qquad H^0(X^w, L_\lambda) \simeq V^w(\lambda)^*, \quad w \in W.$$

- For any $w', w \in W$, $w' \leq w$ if and only if $X_{w'} \subseteq X_w \subseteq X_{w_0} = G/B$; the canonical restriction map:
  - $H^0(X_w, L_\lambda) \longrightarrow H^0(X_{w'}, L_\lambda)$ is surjective, and
  - induces an inclusion $\{b_\lambda\} = V_e(\lambda) \hookrightarrow V_{w'}(\lambda) \hookrightarrow V_w(\lambda) \hookrightarrow V_{w_0}(\lambda) = V(\lambda)$.

- Similarly for opposite Schubert varieties, $w' \leq w$ if and only if $G/B = X^e \supseteq X^{w'} \supseteq X^w$
  - $H^0(X^{w'}, L_\lambda) \longrightarrow H^0(X^w, L_\lambda)$ is surjective, and
  - induces an inclusion $V(\lambda) = V^e(\lambda) \hookleftarrow V^{w'}(\lambda) \hookleftarrow V^w(\lambda) \hookleftarrow V^{w_0}(\lambda) = \{b_{w_0\lambda}\}$.

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$$H^0(X_w, L_\lambda) \simeq V_w(\lambda)^*, \qquad H^0(X^w, L_\lambda) \simeq V^w(\lambda)^*, \quad w \in W.$$
- For any $w', w \in W$, $w' \leq w$ if and only if $X_{w'} \subseteq X_w \subseteq X_{w_0} = G/B$; the canonical restriction map:
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  - induces an inclusion $V(\lambda) = V^e(\lambda) \hookleftarrow V^{w'}(\lambda) \hookleftarrow V^w(\lambda) \hookleftarrow V^{w_0}(\lambda) = \{b_{w_0\lambda}\}$.
- **Richardson variety** $w' \leq w$ if and only if $X_w^{w'} := X_w \cap X^{w'} \neq \emptyset$.

Crystals and Demazure crystals

- The crystal graph $\mathcal{B}(\lambda)$ of $V(\lambda)$ is a finite directed graph with vertices given by the crystal basis of $V(\lambda)$ and edges corresponding to deformations of the Chevalley operators of the representation.

- It was shown by Littelmann and Kashiwara that any Demazure module $V_w(\lambda)$ has an associated **Demazure crystal** $\mathcal{B}_w(\lambda)$ that arises as an induced subgraph of $\mathcal{B}(\lambda)$.

  Similarly, an opposite Demazure module $V^w(\lambda)$ has an associated **opposite Demazure crystal** $\mathcal{B}^w(\lambda)$ in the sense that $V^w = w_0 V_{w_0 w}(\lambda)$ $(X^w = w_0 X_{w_0 w})$,

$$\mathcal{B}^w(\lambda) = \xi \mathcal{B}_{w_0 w}(\lambda), \quad \xi \text{ the Lusztig-Schützenberger involution.}$$

- For $w' \leq w \in W$,

$$\{b_\lambda\} = \mathcal{B}_e(\lambda) \subseteq \mathcal{B}_{w'}(\lambda) \subseteq \mathcal{B}_w(\lambda) \subseteq \mathcal{B}_{w_0}(\lambda) = \mathcal{B}(\lambda)$$

$$B(\lambda) = \mathcal{B}^e(\lambda) \supseteq \mathcal{B}^{w'}(\lambda) \supseteq \mathcal{B}^w \supseteq \mathcal{B}^{w_0}(\lambda) = \{b_{w_0\lambda}\}.$$

  - **Richardson crystal**

$$\mathcal{B}_w^{w'} := \mathcal{B}_w(\lambda) \cap \mathcal{B}^{w'}(\lambda) \neq \emptyset.$$

Key maps

- **Question:** For a given vertex $b \in \mathcal{B}(\lambda)$ it is natural to inquire what is
  (1) the smallest Demazure crystal $\mathcal{B}_w(\lambda)$ containing b.

  (2) the smallest opposite Demazure crystal $\mathcal{B}_{w'}(\lambda)$ containing b.

- For a fixed $b \in \mathcal{B}(\lambda)$, the answer to each of these questions is resp. given by the

  (1) **right key map**: $K^+ : b \in \mathcal{B}(\lambda) \mapsto b_{w\lambda} \in \mathcal{O}(\lambda) \subseteq \mathcal{B}(\lambda)$

  (2) **left key map**: $K^- : b \in \mathcal{B}(\lambda) \mapsto b_{w'\lambda} \in \mathcal{O}(\lambda) \subseteq \mathcal{B}(\lambda)$
  with $w \geq w'$ in W, called the **right key** respectively **left key** of b,

$$K^+_{|\mathcal{O}} = K^-_{|\mathcal{O}} = id.$$

- **Demazure atoms** respectively **opposite Demazure atoms** are the K^+ map fibers respectively K^- map fibers

$$\bar{\mathcal{B}}_w(\lambda) := \{x \in \mathcal{B}(\lambda) : K^+(x) = b_{w\lambda}\}$$

$$\mathcal{B}_w = \bigsqcup_{v \leq w} \bar{\mathcal{B}}_v(\lambda) = \{x \in \mathcal{B}(\lambda) : K^+(x) \leq b_{w\lambda}\}$$

$$\bar{\mathcal{B}}^w(\lambda) := \{x \in \mathcal{B}(\lambda) : K^-(x) = b_{w\lambda}\}$$

$$\mathcal{B}^w = \bigsqcup_{v \geq w} \bar{\mathcal{B}}^w(\lambda) = \{x \in \mathcal{B}(\lambda) : K^-(x) \geq b_{w\lambda}\}$$

$$\bar{\mathcal{B}}^w(\lambda) = \xi \bar{\mathcal{B}}_{w_0 w}(\lambda) \qquad (\mathring{X}^w = w_0 \mathring{X}_{w_0 w})$$

II. Non-symmetric Cauchy kernel identities and LPP (last passage percolation model)

- Cauchy kernel identity /Bicrystals and RSK correspondence

$$\psi : \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0}) \xrightarrow{1:1} \bigsqcup_{\lambda \in \mathcal{P}_{\min(m,n)}} \mathrm{B}(\lambda, m) \times \mathrm{B}(\lambda, n)$$

$$A \longmapsto (P(A), Q(A)).$$

- $$\prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \frac{1}{1 - x_i y_j} = \sum_{A \in \mathcal{M}_{m,n}} x^{\mathrm{wt}(P(A))} y^{\mathrm{wt}(Q(A))} = \sum_{\lambda \in \mathcal{P}_{\min(m,n)}} s_\lambda(x) s_\lambda(y).$$

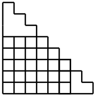
  $\mathrm{B}(\lambda, m)$ tableau crystal on the alphabet $[m]$ with highest weight element the key tableau $K(\lambda)$, $\lambda \in \mathcal{P}_{\min(m,n)}$.

  Bicrystal structure on $\mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ via ψ^{-1}, reverse column Schensted insertion.

  Danilov-Koshevoy 04, van Leeuwen 06, Choi-Kwon 18.

Restriction of RSK to Young shape matrices

- Stair RSK

$\varrho=$ 

The restriction of the RSK correspondence ψ to $\mathcal{M}^{\varrho}_{n,n}$, *$n \times n$ lower triangular matrices*, gives a one-to-one correspondence

$$\psi : \mathcal{M}^{\varrho}_{n,n} \xrightarrow{1:1} \bigsqcup_{\mu \in \mathbb{Z}^n_{\geq 0}} \overline{\mathrm{B}}^{\mu} \times \mathrm{B}_{\mu}$$ Lascoux 2000, Fu-Lascoux 09, A.-Emami 15, Choi-Kwon 18

$$A \mapsto (P, Q), \ {\scriptstyle K^+(Q) \leq K^-(P) = K(\mu)}$$

B_{μ} Demazure crystal consisting of all tableaux Q with right key $K^+(Q) \leq K(\mu)$.

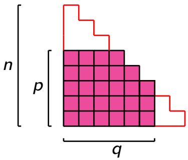
$\overline{\mathrm{B}}^{\mu}$ opposite Demazure atom crystal consisting of all tableaux P with left key $K^-(P) = K(\mu)$.

- $$\prod_{1 \leq j \leq i \leq n} \frac{1}{1 - x_i y_j} = \sum_{\mu \in \mathbb{Z}^n_{\geq 0}} \overline{\kappa}^{\mu}(x) \kappa_{\mu}(y)$$

LHS rewritten in the bases of Demazure and Demazure atom polynomials:
$\overline{\kappa}^{\mu}(x_1, \ldots, x_n) = \overline{\kappa}_{w_0 \mu}(x_n, \ldots, x_1)$ *opposite Demazure atom character* of $\overline{\mathrm{B}}^{\mu}$ and $\kappa_{\mu}(y)$ *Demazure character* of B_{μ}.

Truncated stair RSK

$n \geq q \geq p \geq 1, \quad \Lambda(p,q) =$



$$\psi : \mathcal{M}_{n,n}^{\Lambda(p,q)} \xrightarrow{1:1} \bigsqcup_{\mu \in \mathbb{Z}_{\geq 0}^p} \overline{\mathrm{B}}_p^{\mu} \times \mathrm{B}_{q,\widetilde{\mu}}$$

$$A \mapsto (P, Q), \qquad K^-(P) = K(\mu),\ K^+(Q) \leq K(\tilde{\mu}).$$

$$\prod_{(i,j) \in \Lambda(p,q)} \frac{1}{1 - x_i y_j} = \sum_{\mu \in \mathbb{Z}_{\geq 0}^p} \overline{\kappa}^{\mu}(x_{n-p+1}, \ldots, x_n) \kappa_{\widetilde{\mu}}(y_1, \ldots, y_q).$$

- $\widetilde{\mu} \in \mathbb{Z}_{\geq 0}^p$. A.-Emami 14, A.-Gobet-Lecouvey 22.

For $A = [a_{i,j}] \in \mathcal{M}_{n,n}$, the last passage percolation (time) associated to A:

$$\begin{aligned} perc(A) &= \max_{\pi \text{ in } A} \{\textstyle\sum \text{ entries along a path } \pi \text{ in } A \text{ with steps } \leftarrow, \downarrow \text{ starting in } (1,n) \text{ and ending in } (n,1)\} \\ &= \text{ maximal row lenght of } P \text{ (or } Q\text{) } (\psi(A) = (P,Q)). \end{aligned}$$

The random matrix $\mathcal{W} = \{W_{ij} : 1 \leq i,j \leq n\}$ of independent random variables w_{ij}, with values in $\mathbb{Z}_{\geq 0}$, called weights, where each follows a geometric distribution of parameter $u_i v_j$

$$\mathbb{P}(w_{i,j} = k) = (1 - u_i v_j)(u_i v_j)^k \text{ for any } k \in \mathbb{Z}_{\geq 0}.$$

Then

$$\mathbb{P}(\mathcal{W} = A) = \prod_{1 \leq i \leq n, 1 \leq j \leq n} (1 - u_i v_j) \prod_{1 \leq i \leq n, 1 \leq j \leq n} (u_i v_j)^{a_{i,j}}.$$

The law of the random variable $G = perc(\mathcal{W})$:

- **Schur measure**

$$\mathcal{M}_{n,n}, \qquad \mathbb{P}(G = k) = \prod_{1\le i\le n, 1\le j\le n} (1 - u_i v_j) \sum_{\lambda\in\mathcal{P}_n|\lambda_1=k} s_\lambda(u)s_\lambda(v).$$

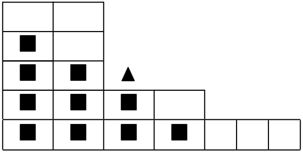
- **Demazure measures**
  - $$\mathcal{M}_{n,n}^{\Lambda(n,n)}, \quad \mathbb{P}(G = k) = \prod_{1\le j\le i\le n} (1 - u_i v_j) \sum_{\mu\in\mathbb{Z}^n_{\ge 0}|\max(\mu)=k} \overline{\kappa}^\mu(u)\kappa_\mu(v).$$
  - $\mathcal{W} \in \mathcal{M}_{n,n}^{\Lambda(p,q)}$,

$$\mathbb{P}(G = k) =$$

$$\prod_{(i,j)\in\Lambda(p,q)} (1 - u_i v_j) \sum_{(\mu_1,\dots,\mu_p)\in\mathbb{Z}^p_{\ge 0}|\max(\mu)=k} \overline{\kappa}_{(\mu_p,\dots,\mu_1)}(u_n,\dots,u_{n-p+1})\kappa_{\widetilde{\mu}}(v_1,\dots,v_q).$$

Cauchy identity for a general Young shape

- Augmented stair shape: Lascoux 2000, A.-Gobet-Lecouvey 2023

$\Lambda =$ 

The law of the random variable $G = perc(\mathcal{W})$:

$$\mathcal{W} \in \mathcal{M}_{n,n}^{\Lambda}, \quad \mathbb{P}(G = k) =$$
$$= \prod_{(i,j)\in\Lambda} (1 - u_i v_j).$$
$$\cdot \sum_{(\mu_1,\ldots,\mu_m)\in\mathbb{Z}^m | \max(\mu)=k} D_{\sigma(\Lambda,NW)}\overline{\kappa}_{(\mu_m,\ldots,\mu_1)}(u_n,\ldots,u_{n-m+1}) D_{\sigma(\Lambda,SE)}\kappa_{(\mu_1,\ldots,\mu_m)}(v_1,\ldots,v_m).$$

- Y a general Young shape: Feigen-Khoroshkin-Makedonskyi, 2024.

$$\prod_{(i,j)\in Y} \frac{1}{1 - x_i y_j} = \sum_{\bar{d} \text{ is } \bar{n}\text{-admissible}} \kappa_{hbs(\bar{d})}(x)\bar{\kappa}_{\bar{d}}(y).$$

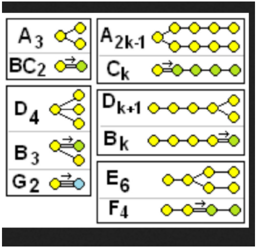
III. Target: Compute key maps under virtualization

- We introduce a Cartan type and crystal model-independent technique for computing both the key maps and the Schützenberger–Lusztig involution (evacuation) via *virtualization* of crystals.

- **Virtualization** is a method introduced by Kashiwara that embeds a highest weight crystal inside another of (potentially) different Lie type, provided the associated Dynkin diagrams are related via so-called diagram *folding*.

- The image of such an embedding equipped with an induced crystal structure is termed a **virtual crystal**.

  wiki/Dynkin-diagram



Crystals

- A (normal) $\mathfrak{g}$-crystal is a nonempty finite set $\mathcal{B}$ with a *weight* map $\mathrm{wt} : \mathcal{B} \to P$, *string operators* $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$, and *crystal operators* $e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ where $0 \notin \mathcal{B}$ is an auxiliary symbol, subject to the following conditions for all $i \in I$ and $b, b' \in \mathcal{B}$:
  - $\varphi_i(b) - \varepsilon_i(b) = \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$,
  - $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$ if $e_i(b) \in \mathcal{B}$,
  - $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$ if $f_i(b) \in \mathcal{B}$,
  - $b' = e_i(b)$ if and only if $b = f_i(b')$,
  - $\varepsilon_i(b) = \max\{k \geq 0 | e_i^k(b) \in \mathcal{B}\}$,
  - $\varphi_i(b) = \max\{k \geq 0 | f_i^k(b) \in \mathcal{B}\}$.
- Let $\mathcal{E}$ and $\mathcal{F}$ be the monoids generated by $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$, respectively. For $\lambda \in P^+$, $\mathcal{B}(\lambda)$ is the crystal graph associated to the highest weight $\mathfrak{g}$-module $V(\lambda)$ with highest weight vector b_λ, $\mathrm{wt}(b_\lambda) = \lambda$, and

$$\mathcal{F}\{b_\lambda\} = \mathcal{B}(\lambda) = \mathcal{E}\{b_{w_0\lambda}\}.$$

Virtualization

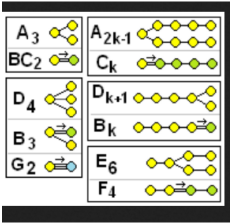
- For any Dynkin diagram D, denote by P_D the corresponding integral weight lattice and by ω_i^D the corresponding fundamental weights.
- Let X and Y be two Dynkin diagrams and let *aut* be an automorphism of Y such that distinct nodes of Y in the same *aut*-orbit are not connected by an edge.
- There is an *embedding* $\psi : X \hookrightarrow Y$ if there exists a bijection $\Psi : X \to Y/aut$, which preserves the edges, inducing a map $P_X \to P_Y$ given by the assignment

$$\omega_i^X \mapsto \gamma_i \sum_{j \in \Psi(i)} \omega_j^Y,$$

$$\alpha_i^X \mapsto \gamma_i \sum_{j \in \Psi(i)} \alpha_j^Y,$$

with γ_i given as in the Table:

X	**Y**	γ_i
C_n	A_{2n-1}	$\gamma_i = 1, 1 \leq i < n, \gamma_n = 2$
B_n	D_{n+1}	$\gamma_i = 2, 1 \leq i < n, \gamma_n = 1$
F_4	E_6	$\gamma_1 = \gamma_2 = 2, \gamma_3 = \gamma_4 = 1$
G_2	D_4	$\gamma_1 = 1, \gamma_2 = 3$
B_n	C_n	$\gamma_i = 2, 1 \leq i < n, \gamma_n = 1$
C_n	B_n	$\gamma_i = 1, 1 \leq i < n, \gamma_n = 2$



We have a natural embedding of the Weyl groups W^X into W^Y, identifying W^X with the set of elements $\widetilde{W}^X$ in W^Y that are fixed under the Dynkin symmetry:

$$W^X \cong \widetilde{W}^X := \langle \Pi_{j\in\psi(i)} \tilde{s}_j \mid i \in I^X \rangle \subset W^Y = \langle \tilde{s}_j \mid j \in I^Y \rangle,$$

via the group isomorphism $s_i \mapsto \Pi_{j\in\psi(i)} \tilde{s}_j$.

Virtualization

- Suppose X and Y are Dynkin diagrams with an embedding $\psi : X \hookrightarrow Y$. Let $(\tilde{\mathcal{B}}; \tilde{e}_j, \tilde{f}_j, \tilde{\varphi}_j, \tilde{\varepsilon}_j)_{j \in I^Y}$ be a normal $\mathfrak{g}_Y$-crystal. A **virtual** $\mathfrak{g}_X$**-crystal** is a subset $\mathcal{V} \subset \tilde{\mathcal{B}}$ such that $\mathcal{V}$ has a normal $\mathfrak{g}_X$-crystal structure where for any $i \in I^X$ the crystal operators are given by:

$$e_i^{\mathbf{v}} := \prod_{j \in \psi(i)} \tilde{e}_j^{\gamma_i}, \qquad f_i^{\mathbf{v}} := \prod_{j \in \psi(i)} \tilde{f}_j^{\gamma_i},$$

and for any choice of $j \in \psi(i)$, the string operators defined as:

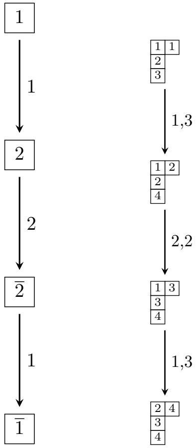
$$\varepsilon_i := \gamma_i^{-1} \tilde{\varepsilon}_j \quad \varphi_i := \gamma_i^{-1} \tilde{\varphi}_j.$$

- If a $\mathfrak{g}_X$-crystal $\mathcal{B}$ is isomorphic to a virtual $\mathfrak{g}_X$-crystal $\mathcal{V} \subset \tilde{\mathcal{B}}$, we call the associated isomorphism $\Upsilon_\psi : \mathcal{B} \to \mathcal{V}$ the **virtualization** map.

Properties:

- Virtualizations are closed under map composition and tensor products.
- Any virtualization maps the highest weight vector of $\mathcal{B}$ to the highest weight vector in $\mathcal{V}$, which coincides with that of $\tilde{\mathcal{B}}$ and therefore is unique (up to choice of embedding realization).

Example: virtualization $C_2 \hookrightarrow A_3$



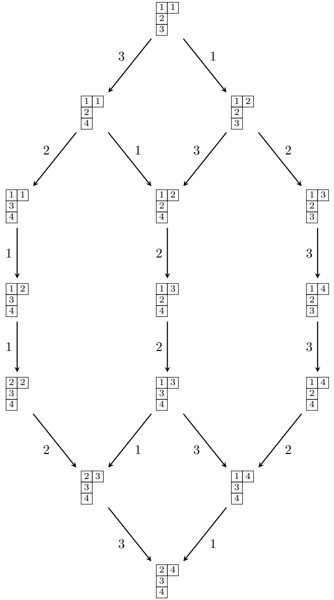


Figure: Left: C_2 standard crystal $\mathcal{B}(\omega_1^{C_2})$. Middle: virtual crystal $\mathcal{V}$, $f_1^{\mathsf{v}} = f_1^{C_3} \circ f_3^{C_3}$, $f_2^{\mathsf{v}} = f_2^{C_2} \circ f_2^{C_2}$. Right: $\mathcal{V} \subseteq \mathcal{B}(\omega_1^{A_3} + \omega_3^{A_3})$.

Dilation

- [Kashiwara 96] For any positive integer m, the m-**dilation** map $\mathbb{D}_m : \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(m\lambda)$ is the unique embedding such that
  - $\mathbb{D}_m(f_i b) = f_i^m \mathbb{D}_m(b)$, $\mathbb{D}_m(e_i b) = e_i^m \mathbb{D}_m(b)$,
  - $\varphi_i(\mathbb{D}_m(b)) = m\varphi_i(b)$, $\varepsilon_i(\mathbb{D}_m(b)) = m\varepsilon_i(b)$,
  - $\mathrm{wt}(\mathbb{D}_m(b)) = m\mathrm{wt}(b)$.

- The canonical realization of the m-dilation map $\mathbb{D}_m$ is the embedding $\Theta_m := G_m \circ \mathbb{D}_m$ where $G_m : \mathcal{B}(m\lambda) \longrightarrow \mathcal{F}\{b_\lambda^{\otimes m}\}$ is the unique crystal isomorphism mapping $b_{m\lambda} \to b_\lambda^{\otimes m}$,

$$\Theta_m := G_m \circ \mathbb{D}_m : \mathcal{B}(\lambda) \hookrightarrow \mathcal{F}\{b_\lambda^{\otimes m}\} \subset \mathcal{B}(\lambda)^{\otimes m}.$$

- [Kashiwara 96] Given any Dynkin diagram X and positive integer m, let $\Psi : X \to X$ be the automorphism determined by the assignment

$$\omega_i \mapsto m\omega_i \qquad \alpha_i \mapsto m\alpha_i$$

for each $\omega_i \in P$ and Υ_ψ the associated virtualization. Then, for any $\lambda \in P^+$, we have that $\Upsilon_\psi(\mathcal{B}(\lambda)) = \Theta_m(\mathcal{B}(\lambda))$. In particular, m-dilation is a virtualization

The left and right key pair and Lakshmibai-Seshadri paths

- Kashiwara has shown that when m is large enough we have a decomposition $\Theta_m(b) = b_{w\lambda} \otimes b' \otimes b_{w'\lambda}$ for any $b \in \mathcal{B}(\lambda)$ with $b_{w\lambda}, b_{w'\lambda}$, $w \geq w' \in W^\lambda$, extremals and $b' \in \mathcal{B}(\lambda)^{\otimes m-2}$.
- The extremal pair $(b_{w\lambda}, b_{w'\lambda})$ is independent of the choice of any such m, hence the following is well-defined.

Definition

For a given $b \in \mathcal{B}(\lambda)$ the **right key** (resp. **left key**) of b is the extremal vector $K^+(b) := b_{w\lambda}$ (resp. $K^-(b) := b_{w'\lambda}$).

The pair of keys $b_{w\lambda}$, $b_{w'\lambda}$ define the initial (resp. final) direction of the corresponding LS path in the isomorphic crystal of Lakshmibai-Seshadri paths.

$$K^+(\xi(b)) = \xi K^-(b)$$

How much does one have to iterate the tensor product?

The following provides a tight bound for the values of m and thus refines Kashiwara decomposition.

Theorem (A.-González-Huang-Torres 24)

Let $m \in \mathbb{N}$. For all $b \in \mathcal{B}(\lambda)$, there exist $b' \in \mathcal{B}(\lambda)^{\otimes(m-2)}$ and fixed $w \geq w' \in W^\lambda$ such that

$\Theta_m(b) = b_{w\lambda} \otimes b' \otimes b_{w'\lambda}$ *if and only if* $m \geq \ell = \max\{\mathit{length}(\rho) \mid \rho \textit{ is an i-string for } i \in I\}$.

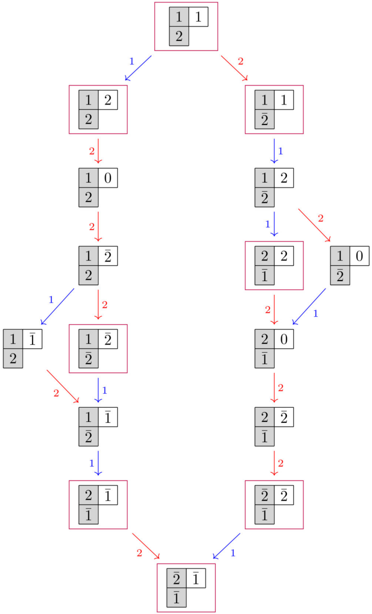
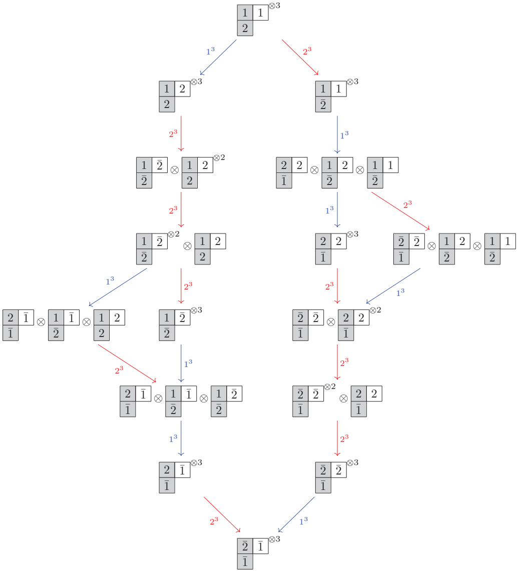


FIGURE 1. The B_2-crystal $\mathcal{B}(\lambda) = \mathrm{KN}_2^B(\lambda)$ for $\lambda = \omega_2^{B_2} + \omega_1^{B_2}$.

Left and right keys with $\ell = 3$ for the B_2-crystal $\mathcal{B}(\omega_1^{B_2} + \omega_2^{B_2})$



Demazure crystals and keys under virtualization

- For any virtualization map $\Upsilon_\psi : \mathcal{B}(\lambda) \to \mathcal{V} \subset \tilde{\mathcal{B}}(\psi(\lambda))$ and any positive integer m, $\Theta_m \Upsilon_\psi = \Upsilon_\psi^{\otimes m} \Theta_m$,

$$\begin{CD}
\mathcal{B}(\lambda) @>{\Upsilon_\psi}>> \mathcal{V} \subseteq \tilde{\mathcal{B}}(\psi(\lambda)) \\
@V{\Theta_m}VV @V{\Theta_m}VV \\
\mathcal{F}\{b_\lambda^{\otimes m}\} \subseteq \mathcal{B}(\lambda)^{\otimes m} @>>{\Upsilon_\psi^{\otimes m}}> \mathcal{F}\{b_{\psi(\lambda)}^{\otimes m}\} \subseteq \mathcal{B}(\psi(\lambda))
\end{CD}.$$

Theorem (A.-González-Huang-Torres 24)

Given a $\mathfrak{g}_X$-crystal $\mathcal{B}$ and Dynkin diagram embedding $\psi : X \to Y$ with virtualization map $\Upsilon : \mathcal{B} \to \mathcal{V} \subset \tilde{\mathcal{B}}$, with $\tilde{\mathcal{B}}$ a $\mathfrak{g}_Y$-crystal, the following holds:

1. $\Upsilon(\xi_{\mathcal{B}}(\mathcal{B})) = \xi_{\tilde{\mathcal{B}}}(\Upsilon(\mathcal{B}))$,
2. $\Upsilon(K^{\pm}(b)) = K^{\pm}(\Upsilon(b))$.

Thus, virtualization embeds Demazure crystals and atoms correspondingly, so that for any $w \in W^X$ we have
$\mathcal{B}_w(\lambda) \overset{\Upsilon}{\hookrightarrow} \tilde{\mathcal{B}}_{\psi(w)}(\psi(\lambda))$ *and* $\bar{\mathcal{B}}_w(\lambda) \overset{\Upsilon}{\hookrightarrow} \bar{\tilde{\mathcal{B}}}_{\psi(w)}(\psi(\lambda))$.

Virtualization $B \hookrightarrow C$ and jeu de taquin in type B

Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$ with fundamental weights $\omega_i^{B_n}$ given by

$$\omega_i^{B_n} = \begin{cases} (1^i, 0^{n-i}) & i \neq n \\ \frac{1}{2}(1^n) & i = n. \end{cases}$$

- The alphabets for Lie types B_n and C_n:

$$\mathsf{B}_n := \left\{1 \prec \cdots \prec n \prec 0 \prec \bar{n} \cdots \prec \bar{1}\right\}$$
$$\mathsf{C}_n := \left\{1 \prec \cdots \prec n \prec \bar{n} \prec \cdots \prec \bar{1}\right\}.$$

$\mathcal{B}(\omega_1^{B_n})$:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-2} \boxed{n-1} \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \boxed{\overline{n-1}} \xrightarrow{n-2} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

$\mathcal{B}(\omega_1^{C_n})$:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-2} \boxed{n-1} \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \boxed{\overline{n-1}} \xrightarrow{n-2} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

where $\mathrm{wt}\left(\boxed{i}\right) = \epsilon_i$, $\mathrm{wt}\left(\boxed{\bar{i}}\right) = -\epsilon_i$ for all $1 \leq i \leq n$, and $\mathrm{wt}\left(\boxed{0}\right) = 0$ (for type B_n only).

The splitting of a KN tableau in type B

The crystal $\mathrm{KN}_n^B(\lambda)$ of Kashiwara–Nakashima tableaux in type B_n with $\lambda = (\mu_0|\mu)$.

- **Example**

  Let $n = 3$ and $\lambda = \omega_1^{B_3} + \omega_2^{B_3} + 3\omega_3^{B_3} = (1^3|3,2,1)$ with spin part $\omega_3^{B_3} = (1^3)$.
  $T \in \mathrm{KN}_3^B(1^3|3,2,1)$
  $\mathrm{wt}(T) = \frac{1}{2}(1,-1,1) + (0,0,1) + (0,0,0) + (0,0,-1) = \frac{1}{2}(1,-1,1)$.
  $\mathrm{split}(T) \in \mathrm{KN}_3^C(\omega_3^{C_3} + 2(3,2,1))$ Kashiwara–Nakashima tableau in type C_3

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 0 & \bar{3} \\ \hline 3 & 3 & 0 \\ \cline{1-3} \bar{2} & \bar{2} \\ \cline{1-2}\end{array} \xrightarrow{\text{split}} \mathrm{split}(T) = \mathrm{split}(T_0)\cdots\mathrm{split}(T_3) = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & 3 & 3 & \bar{2} \\ \cline{1-5} \bar{2} & \bar{2} & \bar{1} \\ \cline{1-3}\end{array}.$$

$$\begin{aligned}\mathrm{wt}(\mathrm{split}(T)) &= 2(1,-1,1) + (-1,1,1) + (0,1,1) + (0,-1,-1) + 2(0,0,-1) \\ &= (1,-1,1).\end{aligned}$$

Theorem (A.-González-Huang-Torres 24)

Let $\mathfrak{C}|T \in \mathrm{KN}_n^B(\lambda)$ *be a Kashiwara–Nakashima tableau of type* B_n *with* $\lambda = (\mu_0|\mu)$. *Then*

1. $\Upsilon^{BC} : s\mathrm{KN}_n^B \hookrightarrow \mathrm{KN}_n^C(\omega_n^C), \mathfrak{C} \mapsto \Upsilon^{BC}(\mathfrak{C}) = \mathrm{split}(\mathfrak{C})$ *or un-shade of* $\mathfrak{C}$.
2. $\Upsilon^{BC}(\mathfrak{C}|T) = \mathrm{split}(\mathfrak{C}|T) = \mathrm{split}(\mathfrak{C})|\mathrm{split}(T) \in \mathrm{KN}_n^C(\omega_n^C|2\mu)$ *the splitting of* $\mathfrak{C}|T$.

The virtualization procedure on the non-spin part of an orthogonal tableau, is induced by the map defined by

$$i \mapsto \Upsilon^{BC}(i) = ii, \bar{i} \mapsto \Upsilon^{BC}(\bar{i}) = \bar{i}\,\bar{i}, 0 \mapsto \Upsilon^{BC}(0) = \bar{n}n \text{ for all } 1 \leq i \leq n$$

on words [**J. Pappe-S. Pfannerer-A. Schilling-M. C. Simone**, 2024] followed by *symplectic* insertion.

$$\begin{array}{ccc} w(T) = x_1 \otimes \cdots \otimes x_m \in \mathcal{B}(\omega_1^B)^{\otimes m} & \xrightarrow{\Upsilon^{\otimes m}} & \Upsilon^{BC}(x_1) \otimes \cdots \otimes \Upsilon^{BC}(x_m) \in \mathcal{B}(2\omega_1^C)^{\otimes m} \\ \Big\downarrow {\scriptstyle B_n-\text{insertion}} & & \Big\downarrow {\scriptstyle C_n-\text{insertion}} \\ T = [\emptyset \leftarrow x_1 \leftarrow \cdots \leftarrow x_m] \in \mathrm{KN}_n^B(\mu) & \xrightarrow[\Upsilon^{BC}]{} & \mathrm{split}(T) = [\emptyset \leftarrow \Upsilon^{BC}(x_1) \leftarrow \cdots \leftarrow \Upsilon^{BC}(x_m)] \in \mathrm{KN}_n^C(2\mu) \end{array}.$$

Corollary

Let T' be a orthogonal skew tableau potentially with a leftmost spin column. Then $\Upsilon^{BC} \circ \mathsf{rect}_B(T') = \mathsf{rect}_C \circ \Upsilon^{BC}(T')$ *which means*

$$\mathsf{rect}_B(T') = \mathsf{BJDT}(T') = \Upsilon^{-1} \circ \mathsf{rect}_C \circ \Upsilon^{BC}(T') \quad \textit{Lecouvey, 2002}$$

where $\Upsilon^{BC}(T') = \mathrm{split}(T')$*, and* $\mathsf{rect}_B(T')$ *might be obtained by the* B_n*-spin insertion scheme.*

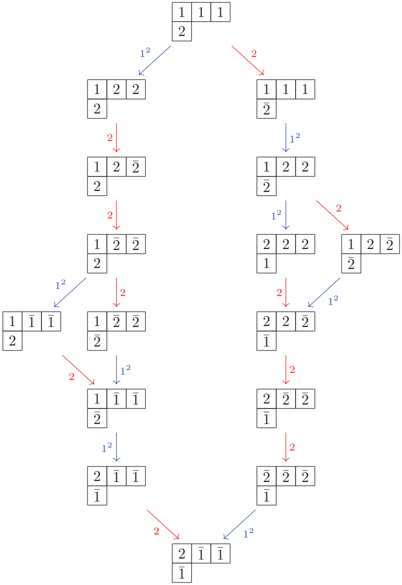


FIGURE 3. The virtual crystal $\Upsilon(\mathrm{KN}_2^B(\omega_2^{B_2}+\omega_1^{B_2}))$ embedded into $\mathrm{KN}_2^C(\omega_2^C+2\omega_1^C)$.