Partitions with full equivalence Schur support are monotone ribbons with full Schur support

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Specht and skew Specht modules, and general diagrams

- As λ ranges over all partitions of n, the Specht modules (over $\mathbb{C}$) of λ, S^λ, are the irreducible representations of $\mathfrak{S}_n$. The *Frobenius characteristic map*
$$ch(S^\lambda) = s_\lambda.$$
- Given the skew shape A, one can define the skew Spetch module S^A which has a decomposition into irreducibles given by the LR rule

$$S^A \cong \bigoplus_{\nu \in [r(A), c(A)']} (S^\nu)^{\oplus c_A^\nu} = S^{r(A)} \oplus \cdots \oplus (S^\nu)^{\oplus c_A^\nu} \oplus \cdots \oplus S^{c(A)'}$$

$$ch(S^A) = s_A = 1s_{r(A)} + \cdots + c_A^\nu s_\nu + \cdots + 1s_{c(A)'}$$

General diagrams

- A diagram D is an arbitrary finite collection of unit lattice boxes in the plane

$$A = \begin{array}{cccc} \square & \square & \cdot & \cdot \\ \square & \cdot & \square & \cdot \\ \square & \cdot & \cdot & \square \end{array} \qquad \begin{array}{ccc} \square & \square & \cdot \\ \cdot & \square & \square \\ \square & \cdot & \square \end{array}$$

- The construction of S^λ, λ a partition, allows a construction of the Specht module S^D for any diagram D. S^D may be decomposed into irreducible representations

$$S^D \cong \bigoplus_{\nu \in [\ ,\ ]} (S^\nu)^{\oplus c_D^\nu}$$

where c_D^ν is the number of times S^ν appears in the decomposition of S^D.

- This allows to define the Schur function associated to D

$$s_D := ch(S^D) = \sum_\nu c_D^\nu s_\nu.$$

It is not known a combinatorial description of the coefficients c_D^ν for D in general.

Is the D-support contained in an interval w.r.t. dominance order?

Computational evidence for diagrams D with at most 8 boxes in Ricky Liu PhD thesis (2010)

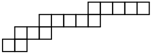
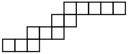
$$A = \begin{array}{cccc} \square & \square & \cdot & \cdot \\ \square & \cdot & \square & \cdot \\ \square & \cdot & \cdot & \square \end{array} \qquad s_D = s_{411} + 2s_{321} + s_{222}$$

$$\begin{array}{ccc} \square & \square & \cdot \\ \cdot & \square & \square \\ \square & \cdot & \square \end{array} \qquad s_D = s_{33} + 2s_{321} + s_{222}$$

$$\begin{array}{cccc} \square & \square & \square & \cdot \\ \square & \cdot & \cdot & \cdot \\ \cdot & \square & \cdot & \cdot \\ \cdot & \cdot & \square & \square \end{array} \qquad s_{43} + 2s_{421} + s_{4111} + 2s_{331} + s_{322} + s_{3211}$$

Our problem: overview

- *Connected ribbon shapes with rows of length at least two.*

$R_{(\mathbf{5532})} =$  $R_{(5253)} =$ 

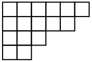
- **Problem** Given the *partition* α,
  - when do we have

$$supp[S^{R_\alpha}] = supp[S^{R_\beta}] \Leftrightarrow supp[s_{R_\alpha}] = supp[s_{R_\beta}]$$

  for any rearrangement β of α?
  - when does R_α have full Schur support?
- **Partial answer**:
  - *Gaetz-Hardt-Sridhar necessary condition* (2017). R_α has full equivalence class $\Rightarrow$ some inequalities on α are satisfied.
  - (A., Mamede) These inequalities characterize partitions α so that R_α has full Schur support.
    R_α has full equivalence class $\Rightarrow$ R_α has full Schur support.
- **Conjecture**: (A., Mamede, 2018)
  R_α has full equivalence class $\Leftrightarrow$ R_α has full Schur support.

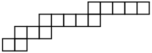
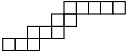
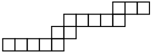
Ribbon shapes

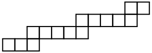
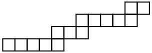
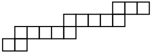
- A partition or straight shape ν

$\nu = (6532) =$  $\quad |\nu| = 6 + 5 + 3 + 2 = 16, \quad \ell(\nu) = 4$

- A skew-shape ν/μ, $\mu \subseteq \nu$

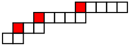
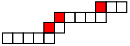
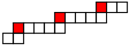
- **Connected ribbons** with **row lengths at least 2** are encoded by compositions α with parts at least 2

$R_{(\mathbf{5532})} =$  $R_{(5253)} =$  $R_{(3525)} =$ 

$R_{(2553)} =$  $R_{(2535)} =$  $R_{(3552)} =$ 

12 compositions in the orbit of the $\alpha^+ = (\mathbf{5532})$

- R_α: $\alpha^+ = (5532)$, $|\alpha| = 5+5+3+2 = 15$, $\ell(\alpha) = 4$

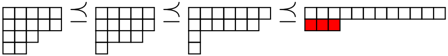
$(\mathbf{5532}) =$  $(3525) =$  $(3552) =$ 

- The **Schur interval** of the connected ribbons in the orbit of the partition α^+ is the interval, in the dominance lattice of partitions of $|\alpha|$,

$$[\alpha^+;\, \hat{\alpha} = (|\alpha| - \ell(\alpha) + 1, \ell(\alpha) - 1)] = \{\nu : \alpha^+ \preceq \nu \preceq \hat{\alpha}\}.$$

- $\alpha^+ = (5532)$, $\ell(\alpha) - 1 = 3$

$$(7611) \in [(5532), (15-3, 3)] = \{\nu : (5532) \preceq \nu \preceq (12, 3)\}$$



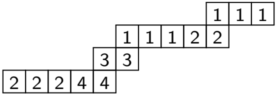
Ribbon tableaux

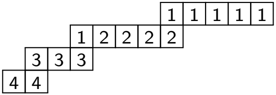
- A semistandard tableau of straight shape (partition) ν is a filling T of the ν with positive integers,

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2}\end{array} \mapsto x^T = x_1^2 x_2^0 x_3^3 x_4^3 \text{ monomial weight of } T$$

weight $(2, 0, 3, 3)$, $|\nu| = 8$.

- A ribbon semistandard tableau is a semistandard filling R of a ribbon shape R_α





$x_1^6 x_2^5 x_3^2 x_4^2$ $\qquad$ $x_1^6 x_2^4 x_3^3 x_4^2$

$\alpha = (3525)$ $\qquad$ $\alpha^+ = (5532)$

Ribbon Schur functions

- Given a partition ν the **Schur function** s_ν in the variables $x = (x_1, x_2, \dots)$ is the generating function of all semi standard tableaux T of shape ν

$$s_\nu = \sum_T x^T.$$

  The Schur functions s_ν for all ν, form a linear basis of the ring Λ of (homogeneous) symmetric functions in the components of x.

- Given the composition α, the **ribbon-Schur function** s_{R_α} in the variables $x = (x_1, x_2, \dots)$ is the generating function of all ribbons tableaux R of shape R_α,

$$s_{R_\alpha} = \sum_R x^R.$$

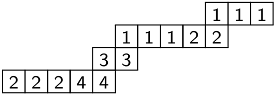
- s_{R_α} is a symmetric function

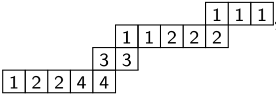
$$s_{R_\alpha} = \sum_{\nu \in [\alpha^+, (|\alpha| - \ell(\alpha) + 1, \ell(\alpha) - 1)]} c^\nu_{R_\alpha} s_\nu, \quad c^\nu_{R_\alpha} \in \mathbb{Z}_{\geq 0},$$

  $c^\nu_{R_\alpha}$ are called **Littlewood-Richardson coefficients**.

LR coefficients as numbers that count

- Given a ribbon R_α and ν a partition of $|\alpha|$,
- $c^{\nu}_{R_\alpha} = \#$ribbon **LR tableaux** of shape R_α
  - ribbon semistandard tableaux of shape R_α and weight ν with;
  - the word Yamanouchi condition.
- $\alpha = (3, 5, 2, 5)$, $\nu = (6522)$





, $c^{\nu}_{R_\alpha} = 2$,

111221113344222

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LR coefficients as numbers that count

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- $\alpha = (3, 5, 2, 5)$, $\nu = (6522)$

									1	1	1
					1	1	1	2	2		
				3	3						
2	2	2	4	4							

									1	1	1
					1	1	2	2	2		
				3	3						
1	2	2	4	4							

, $c^{\nu}_{R_\alpha} = 2$,

111221113344222 111222113344221

- The **companion tableau** of shape ν and content α of each of the two ribbon LR tableaux of shape R_α and weight ν

1	1	1	2	2	2
2	2	4	4	4	
3	3				
4	4				

1	1	1	2	2	4
2	2	2	4	4	
3	3				
4	4				

Companions of ribbon LR tableaux and descent sets

- How to detect the companion tableau of a ribbon LR tableau?
  The descent set does it!

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- $\alpha = (3,5,2,5),\ \nu = (6522)$

									1	1	1
					1	1	1	2	2		
				3	3						
2	2	2	4	4							

									1	1	1
					1	1	2	2	2		
				3	3						
1	2	2	4	4							

- The companion tableau of a ribbon LR tableau of shape R_α and weight ν is a SSYT of shape ν and weight α and **descent set** $S(\alpha) = \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{\ell(\alpha)-1}\}$.

1	1	1	2	2	2
2	2	4	4	4	
3	3				
4	4				

1	1	1	2	2	4
2	2	2	4	4	
3	3				
4	4				

1	2	3	6	7	8
4	5	13	14	15	
9	10				
11	12				

1	2	3	7	8	15
4	5	6	13	14	
9	10				
11	12				

descent set $\{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3\} = \{3, 8, 10\}$

LR coefficients as structure coefficients

- Let α be any composition and R_α the corresponding connected ribbon shape. Then

$$s_{R_\alpha} = \sum_{\nu \in [\alpha^+, \hat{\alpha} = (|\alpha| - \ell(\alpha)+1, \ell(\alpha)-1)]} c^{\nu}_{R_\alpha} s_\nu,$$

where

$$c^{\nu}_{R_\alpha} = \#\text{ SYT of shape } \nu \text{ and descent set}, \qquad c^{\alpha^+}_{R_\alpha} = c^{\hat{\alpha}}_{R_\alpha} = 1.$$

$$S(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell(\alpha)-1}\}.$$

LR coefficients and crystals: multiplicity numbers

- B_μ and B_ν crystals of type A with highest weights μ and ν.

$$B_\mu \otimes B_\nu \cong \bigoplus_\lambda B_\lambda^{c^\nu_{\lambda/\mu}},$$

$$c^\nu_{\lambda/\mu} = \#\text{highest weight elements of weight } \lambda \text{ in } B_\mu \otimes B_\nu$$

- **How to detect the highest weight elements of weight λ in $B_\mu \otimes B_\nu$?**

  Each crystal connected component in $B_\mu \otimes B_\nu$ has highest weight element

$$Y_\mu \otimes T_\nu$$

  of weight λ, where T_ν is the companion tableau of the LR tableau T of shape λ/μ and weight ν.

Schur support and Schur support equality

- The **Schur support** $[R_\alpha]$ of the ribbon shape R_α is a subset of the Schur interval of α^+,

$$[R_\alpha] := \{\nu : c^\nu_{R_\alpha} > 0\} \subseteq [\alpha^+, \hat{\alpha} = (|\alpha| - \ell(\alpha) + 1, \ell(\alpha) - 1)].$$

$$\alpha^+, \hat{\alpha} = (|\alpha| - \ell(\alpha) + 1, \ell(\alpha) - 1) \in [R_\alpha]$$

  The ribbon R_α is said to have **full Schur support** when the support coincides with the Schur interval.

- Let α and β be compositions, rearrangements of each other. The ribbons R_α and R_β are said to be **Schur support equivalent** if their supports coincide, $[R_\alpha] = [R_\beta]$.

  The ribbon R_{α^+} is said to have **support full equivalence class** if $[R_{\alpha^+}] = [R_\alpha]$, for any rearrangement α of the entries of α^+.

Schur support equality and symmetries of LR coefficients

- LR coefficients satisfy a number of symmetries:

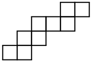
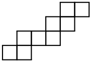
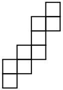
$$c^{\nu}_{\lambda/\mu} = c^{\mu}_{\lambda/\nu},\ c^{\nu}_{\lambda/\mu} = c^{\nu}_{(\lambda/\mu)^\circ},\ c^{\nu}_{\lambda/\mu} = c^{\nu'}_{\lambda'/\mu'},$$

  $(\lambda/\mu)^\circ$ is the π-rotation of λ/μ, and

- $[\lambda/\mu] = [(\lambda/\mu)^\circ]$ and $[(\lambda/\mu)'] = [\lambda/\mu]'$ and

$$s_{\lambda/\mu} = s_{(\lambda/\mu)^\circ} \ \text{ and } \ s_{\lambda'/\mu'} = \sum_{\nu \in [\mathrm{r}(\lambda/\mu), \mathrm{c}(\lambda/\mu)']} c^{\nu}_{\lambda/\mu} s_{\nu'}.$$

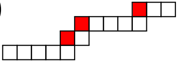
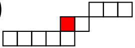
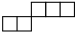
- The full support of one of the shapes λ/μ, $(\lambda/\mu)'$ or $(\lambda/\mu)^\circ$ implies the full support of any of the others

(2322)  (2232)  

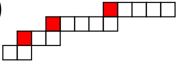
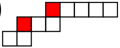
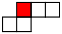
- **How monotone ribbons with full support and ribbons with full equivalence class are related**?

Obstructions for the full Schur support

- Given α a composition, the *overlapping partition* $p = (p_1, \ldots, p_{\ell(\alpha)-1}, 0)$ of the connected ribbon α

$\alpha = (3525)$  (325)  (32)  (2) 

$$p = (3, 1, 0, 0)$$

$\alpha^+ = (5532)$  (532)  (32)  (2) 

$$p = (3, 2, 1, 0)$$

- Schur support of R_{α^+} does not decrease with rearrangements of the row lengths of α^+?

- combinatorial interpretation of dominance order on partitions:

  $\lambda \preceq \nu \Leftrightarrow$ Young diagram of ν is obtained by *lifting*

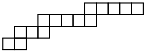
  at least one box in the Young diagram of λ

- $$\lambda \preceq \nu \Rightarrow \nu_i \leq \sum_{q=i}^{\ell(\lambda)} \lambda_q = \lambda_i + \sum_{q=i+1}^{\ell(\lambda)} \lambda_q,\ 1 \leq i \leq \ell(\lambda)$$

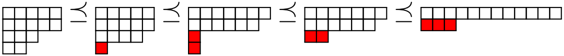
- **Theorem**. (A., Mamede) Given $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)})$ a partition with parts ≥ 2, let ν be a partition of $|\alpha|$, and $p = (p_1, \dots, p_{\ell(\alpha)-1}, 0)$, with $p_i = \ell(\alpha) - i$, for $i = 1, \dots, \ell(\alpha)$. Then

$$c^{\nu}_{R_\alpha} > 0 \Leftrightarrow \begin{cases} \nu \in [\alpha, (|\alpha| - p_1, p_1)] \Leftrightarrow \alpha \preceq \nu \preceq (|\alpha| - p_1, p_1), \\ \nu_i \leq \sum_{q=i}^{\ell(\alpha)} \alpha_q - p_i = \alpha_i + \sum_{q=i+1}^{\ell(\alpha)} \alpha_q - p_i, \ \text{ for } 1 \leq i \leq \ell(p). \end{cases}$$

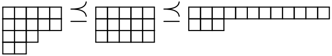
- **Example** $\alpha = (5532)$, $[R_\alpha] \subseteq [(5532), (15-3, 3)]$



$p = (3, 2, 1, 0)$



$$\Rightarrow (5541), (7611), (762) \in [R_\alpha]$$



$\nu_3 = 5 > 3 + 2 - 1 \Rightarrow (555) \notin [R_\alpha]$

$$c^{(7611)}_{R_\alpha}, c^{(5541)}_{R_\alpha}, c^{(552)}_{R_\alpha} > 0, \quad c^{(555)}_{R_\alpha} = 0$$

Classification of full Schur support monotone ribbons

- **Definition** Let α be a partition with parts ≥ 2 with overlapping partition p. For each $i \in \{1,\ldots,\ell(p)-1\}$, put

$$\varrho_i := \sum_{q=i+1}^{\ell(\alpha)} \alpha_q - p_{i+1} + 1 > 0.$$

- **Theorem**. (A., Mamede) Let α be a partition with parts ≥ 2, and overlapping partition p. $R_\alpha = [\alpha, (|\alpha| - p_1, p_1)]$ if and only if

$$\ell(\alpha) = 2 \text{ or } \ell(\alpha) \geq 3 \text{ and } \sum_{\substack{1 \leq j \leq i \\ \alpha_j < \varrho_i}}^{i} (\varrho_i - \alpha_j) \geq p_{i+1}, \; 1 \leq i \leq \ell(p) - 1.$$

  A weaker but simpler version: α has full support $\Rightarrow$

$$\alpha_i < \varrho_i \Leftrightarrow \alpha_i \leq \sum_{q=i+1}^{\ell(\alpha)} \alpha_q - p_{i+1}, \quad 1 \leq i \leq \ell(p) - 1.$$

- $R_{(5532)}$ has not full support: $\alpha_2 = 5 > \varrho_2 = \alpha_3 + \alpha_4 - 1 = 3 + 2 - 1$

Necessary condition for full equivalence support

- **Theorem: Gaetz-Hardt-Sridhar necessary condition** (2017). Let α be a partition with parts ≥ 2 and R_α a connected ribbon. If R_α has full equivalence class then

$$N_j := max\{k : \sum_{\substack{1\leq i\leq j \\ \alpha_i<k}} (k-\alpha_i) \leq \ell(\alpha)-j-2\} < \varrho_j,\ 1 \leq j \leq \ell(\alpha)-2.$$

- **Lemma**(A., Mamede, 2018) For all $j \in \{1, \ldots, \ell(\alpha)-2\}$,

$$N_j := max\{k : \sum_{\substack{1\leq i\leq j \\ \alpha_i<k}} (k-\alpha_i) \leq \ell(\alpha)-j-2\} < \varrho_j \Leftrightarrow$$

$$\Leftrightarrow \sum_{\substack{1\leq i\leq j \\ \alpha_i<\varrho_j}} (\varrho_j-\alpha_i) \geq \ell(\alpha)-j-1.$$

- **Theorem** . (A., Mamede, 2018) Let α be a partition with parts ≥ 2 and R_α a connected ribbon. If R_α has full equivalence class then R_α has full Schur support. When $\ell(\alpha) \leq 4$, R_α has full equivalence class if and only if R_α has full Schur support.

Examples

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$$\ell(\alpha) = 3: \ \alpha \text{ full equivalence class} \Leftrightarrow [R_\alpha] = [\alpha, (|\alpha| - 2, 2)]$$
$$\Leftrightarrow \ \alpha_1 < \sum_{q=2}^{3} \alpha_q.$$
$$\ell(\alpha) = 4: \ \alpha \text{ full equivalence class} \Leftrightarrow [R_\alpha] = [\alpha, (|\alpha| - 3, 3)]$$
$$\Leftrightarrow \ \alpha_1 < \sum_{q=2}^{4} \alpha_q - 2, \ \ \alpha_2 < \sum_{q=3}^{4} \alpha_q.$$

Towards to a coincidence between partitions with full Schur support and full equivalence Schur support

- **Theorem**

  Let α be a partition with parts ≥ 2, and R_{α_π} a connected ribbon with overlapping partition p^π, with $\pi \in \sum_{\ell(\alpha)}$. Let $\nu \in [\alpha, (|\alpha| - \ell(\alpha) + 1, \ell(\alpha) - 1)]$. Then

$$\nu \in [R_{\alpha_\pi}](c^{\nu}_{R_{\alpha_\pi}} > 0) \Rightarrow \nu_i \leq \sum_{q=i}^{\ell(\alpha)} \alpha_q - p_i^\pi, \;\; 1 \leq i \leq \ell(p). \qquad (1)$$

- Assuming that inequalities (1) are sufficient for $\nu \in [R_{\alpha_\pi}]$,

$$\nu \in [R_\alpha] \Rightarrow \nu_i \leq \sum_{q=i}^{\ell(\alpha)} \alpha_q - (\ell(\alpha) - i) \leq \sum_{q=i}^{\ell(\alpha)} \alpha_q - p_i^\pi, \;\; 1 \leq i \leq \ell(p)$$

$$\Rightarrow \nu \in [R_{\alpha_\pi}].$$

  $[R_\alpha] \subseteq [R_{\alpha_\pi}]$, for any $\pi \in \sum_{\ell(\alpha)}$.

  R_α has full Schur support $\Rightarrow [R_{\alpha_\pi}] = [R_\alpha]$, for any $\pi \in \sum_{\ell(\alpha)}$, and R_α has full equivalence class.

Horn-Klyachko linear inequalities

- Let $N = \{1, 2, \ldots, n\}$, then for fixed d, with $1 \leq d \leq n$, let $I = \{i_1 > i_2 > \cdots > i_d\} \subseteq N$.

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- Let $I, J, K \subseteq N$ with $\#I = \#J = \#K = d$ and ordered decreasingly. One defines the partitions

$$\alpha(I) = I - (d, \ldots, 2, 1),$$

$$\beta(J) = J - (d, \ldots, 2, 1),$$

$$\gamma(K) = K - (d, \ldots, 2, 1).$$

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- Let T_d^n be the set of all triples (I, J, K) with $I, J, K \subseteq N$ and $\#I = \#J = \#K = d$ such that $c_{\alpha(I),\beta(J)}^{\gamma(K)} > 0$.

- $c^{\lambda}_{\mu,\nu} > 0$ if and only if the Horn-Klyachko inequalities are satisfied

$$\sum_{k=1}^{n} \lambda_k = \sum_{i=1}^{n} \mu_i + \sum_{j=1}^{n} \nu_j$$

$$\sum_{k \in K} \lambda_k \leq \sum_{i \in I} \mu_i + \sum_{j \in J} \nu_j$$

for all triples $(I, J, K) \in T^n_d$ with $d = 1, \ldots, n-1$.

- *Lidskii- Wielandt inequalities and the dominance order*

$$\sum_{i \in I} \lambda_i \leq \sum_{i \in I} \mu_i + \sum_{i \leq d} \nu_i, \text{ for all } I \subseteq \{1, \ldots, n\} \text{ with } \#I = d$$

$$\Leftrightarrow \sum_{i \in I} (\lambda_i - \mu_i) \leq \sum_{i \leq d} \nu_i, \text{ for all } I \subseteq \{1, \ldots, n\} \text{ with } \#I = d$$

$$\Leftrightarrow \sum_{i \leq d} (\lambda_i - \mu_i)^+ \leq \sum_{i \leq d} \nu_i.$$