Skew RSK and coincidence of Littlewood-Richardson commutors

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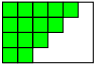
Littlewood-Richardson coefficients as structure constants

- *The ring of symmetric polynomials: the product of Schur polynomials*. Let $x = (x_1, \ldots, x_d)$. Schur polynomials $s_\lambda(x)$ for all partitions with $\ell(\lambda) \leq d$ form a $\mathbb{Z}$-linear basis for the ring $\Lambda_d := \mathbb{Z}[x]^{\mathfrak{S}_d}$ of symmetric polynomials in x,

$$s_\mu s_\nu = \sum_{\substack{\lambda \\ \ell(\lambda) \leq d}} c^\lambda_{\mu\,\nu} s_\lambda, \quad c^\lambda_{\mu\,\nu} \in \mathbb{Z}^+_{\geq 0}.$$

- *Schubert calculus of Grassmannians: the product in the cohomology of Grassmannians*. Schur polynomials $s_\lambda(x)$ with λ inside a rectangle $d \times (n-d)$ $(0 < d < n)$ may be interpreted as representatives of Schubert classes σ_λ

.

Schubert classes $\{\sigma_\lambda\}_{\lambda \subseteq n \times (n-d)}$ form a $\mathbb{Z}$-linear basis for the cohomology ring $H^*(G(d,n))$ of the Grassmannian $G(d,n)$ (the set of all complex d-dimensional linear subspaces of $\mathbb{C}^n$), and

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq d \times (n-d)} c^\lambda_{\mu\,\nu} \sigma_\lambda.$$

Littlewood-Richardson coefficients

- These numbers $c^{\lambda}_{\mu,\nu}$ also arise as
  - *tensor product multiplicities*. Schur polynomials $s_\lambda(x)$ may be interpreted as irreducible characters of the general linear group $GL_d(\mathbb{C})$. The decomposition of the tensor product of two irreducible polynomial representations V^μ and V^ν of the general linear group $GL_d(\mathbb{C})$ into irreducible representations of $GL_d(\mathbb{C})$, is given by

$$V^\mu \otimes V^\nu = \bigoplus_{\ell(\lambda)\leq d} V^{\lambda^{\oplus c^{\lambda}_{\mu\,\nu}}}.$$

- *Positivity of Littlewood-Richardson coefficients in existence problems*. There exist $d \times d$ non singular matrices A, B and C, over a *local principal ideal domain*, with Smith invariants μ, ν and λ respectively, such that $AB = C$ iff $c^{\lambda}_{\mu\,\nu} > 0$.

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- *The commutativity of Littlewood-Richardson coefficients*

$$c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$$

Our structure coefficients are combinatorial numbers

- The structure coefficient $c^{\lambda}_{\mu,\nu}$ is

  the cardinality of an explicit set of combinatorial objects.

- It is possible to determine $c^{\lambda}_{\mu,\nu} > 0$ without determining its exact value.
- Exhibiting the commutation symmetry of these combinatorial objects means:

  Fix $\mu, \nu \subset \lambda$ with $|\mu| + |\nu| = |\lambda|$, and your set $\mathbb{LR}_{\mu,\nu,\lambda}$ of combinatorial objects counted by $c^{\lambda}_{\mu,\nu}$. An LR commutor is any bijection

$$\mathbb{LR}_{\mu,\nu,\lambda} \to \mathbb{LR}_{\nu,\mu,\lambda}$$

$$c^{\lambda}_{\mu\,\nu} = c^{\lambda}_{\nu\,\mu}.$$

Coincidence of LR commutors

- **Problem** (Pak, Vallejo'04). Show that all LR commutors are involutions and **coincide**.

One has several LR commutors and all are involutions.

One involution needs to be included in the coincidence list :

Goal: Internal insertion LR commutor coincides with switching LR commutor.

- The Littlewood-Richardson (LR) rule (*D.E. Littlewood and A. Richardson 34; M.-P. Schützenberger 77; G.P. Thomas 74*). In the linear expansion of a product of two Schur polynomials

$$s_\mu(x)\ s_\nu(x) = \sum_\lambda\ c^\lambda_{\mu\nu}\ s_\lambda(x)$$

the coefficients are given by

$$c^\lambda_{\mu\nu} = \#\{\text{ballot SSYT of shape } \lambda/\mu \text{ and content } \nu\}.$$

The coefficients $c^\lambda_{\mu\nu}$ are known as Littlewood–Richardson (LR) coefficients, and the ballot SSYT's are also known as Littlewood-Richardson tableaux.

- LR tableau, ballot tableau The content of each initial segment of the reading word, right to left across rows and top to bottom, is a partition.

		1	1	2
	1	2		
1	3			

2112131

		1	1	1
	2	2		
1	3			

1112231

		1	1	1
	1	2		
2	3			

1112132

one has 3 candidates 1,2,3 each receiving 4,2,1 votes respectively. A particular ordering of the votes is then a *ballot sequence* of length 7 where at any stage candidate 1 has at least many votes as candidate 2, and candidate 2 has at least many votes as candidate 3.

Tableau Switching (Benkart-Sotille-Stroomer)

- A *perforated tableau* U of shape λ is a filling of some of the boxes in λ with integers satisfying some restrictions: $\begin{matrix}\mathbf{x}' & \\ & \mathbf{x}\end{matrix}$, $\;\mathbf{x}'\;\mathbf{x}\;\; x \geq x'$; and $\begin{matrix}\mathbf{x}' \\ \mathbf{x}\end{matrix}\;\; x > x'$.

  Switch an integer with the neighbour empty box $\blacksquare$ to the south, east, north or west, in a perforated tableau, so that the result is still a perforated tableau:

$$\text{contracts } U \quad \begin{matrix}\blacksquare \\ \mathbf{x}\end{matrix} \underset{s}{\rightarrow} \begin{matrix}x \\ \blacksquare\end{matrix} \qquad \blacksquare\; x \underset{s}{\rightarrow} x\; \blacksquare$$

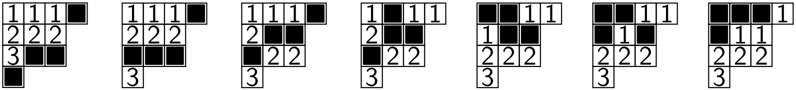
$$\text{expands } U \quad \begin{matrix}\mathbf{x} \\ \blacksquare\end{matrix} \underset{s}{\rightarrow} \begin{matrix}\blacksquare \\ \mathbf{x}\end{matrix} \qquad \mathbf{x}\; \blacksquare \underset{s}{\rightarrow} \blacksquare\; \mathbf{x}\,.$$

  A *perforated tableau pair* $U \cup V$ of shape λ is the superimposing of two perforated tableaux U and V of shape λ, so that together they completely fill λ and no two letters are in a same box.

- If **u** and **v** are vertically or horizontally adjacent letters from U and V respectively, then an interchanging of **u** with **v** is a *switch*, written $\mathbf{u} \underset{s}{\leftrightarrow} \mathbf{v}$, provided it produces a new perforated tableau pair,

$$\begin{matrix}\mathbf{u} \\ \mathbf{v}\end{matrix} \underset{s}{\leftrightarrow} \begin{matrix}\mathbf{v} \\ \mathbf{u}\end{matrix} \qquad \mathbf{u}\;\mathbf{v} \underset{s}{\leftrightarrow} \mathbf{v}\;\mathbf{u}\,.$$

- *switching* in a perforated tableau: local moves preserve the perforated condition.



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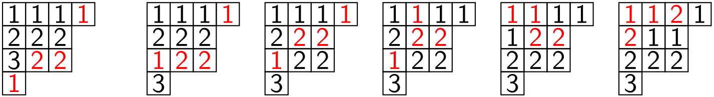
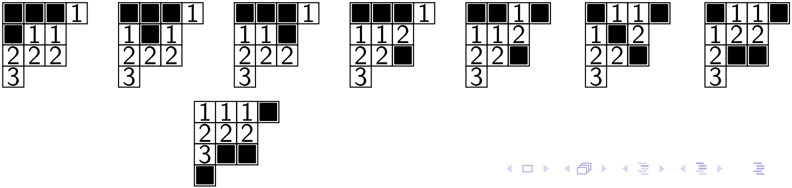


- *Jeu de taquin* in a skew tableau is a particular case of *switching* where an order of the switches has been imposed. The game can be reduced to the local moves:



Switching LR commutor ρ_1

- $\rho_1^{(n)} : \mathbb{LR}_{\mu,\nu,\lambda} \to \mathbb{LR}_{\nu,\mu,\lambda},\ \ell(\lambda) \leq n.$

$$Y_{32} \cup T = \begin{array}{|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & {\color{red}1} & 1 \\ \hline {\color{red}2} & {\color{red}2} & 1 & 2 \\ \hline 1 & 2 & 3 \\ \cline{1-3}\end{array} \underset{s}{\to} \begin{array}{|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & {\color{red}1} & 1 \\ \hline 1 & {\color{red}2} & 1 & 2 \\ \hline {\color{red}2} & 2 & 3 \\ \cline{1-3}\end{array} \underset{s}{\to} \begin{array}{|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & 1 & {\color{red}1} \\ \hline 1 & 1 & 2 & {\color{red}2} \\ \hline {\color{red}2} & 2 & 3 \\ \cline{1-3}\end{array} \underset{s}{\to} \begin{array}{|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & 1 & {\color{red}1} \\ \hline 1 & 1 & 2 & {\color{red}2} \\ \hline 2 & 3 & {\color{red}2} \\ \cline{1-3}\end{array} \underset{s}{\to}$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} \\ \hline {\color{red}1} & {\color{red}1} & 2 & {\color{red}2} \\ \hline 2 & 3 & {\color{red}2} \\ \cline{1-3}\end{array} \underset{s}{\to} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} \\ \hline 2 & {\color{red}1} & 2 & {\color{red}2} \\ \hline {\color{red}1} & 3 & {\color{red}2} \\ \cline{1-3}\end{array} \underset{s}{\to} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} \\ \hline 2 & 2 & {\color{red}1} & {\color{red}2} \\ \hline {\color{red}1} & 3 & {\color{red}2} \\ \cline{1-3}\end{array} \underset{s}{\to} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} \\ \hline 2 & 2 & {\color{red}1} & {\color{red}2} \\ \hline 3 & {\color{red}1} & {\color{red}2} \\ \cline{1-3}\end{array} = Y_{321} \cup H,\ H \equiv T.$$

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$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} \\ \hline {\color{red}1} & {\color{red}1} & 2 & {\color{red}2} \\ \hline 2 & 3 & {\color{red}2} \\ \cline{1-3}\end{array} \underset{s}{\to} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} \\ \hline 2 & {\color{red}1} & 2 & {\color{red}2} \\ \hline {\color{red}1} & 3 & {\color{red}2} \\ \cline{1-3}\end{array} \underset{s}{\to} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} \\ \hline 2 & 2 & {\color{red}1} & {\color{red}2} \\ \hline {\color{red}1} & 3 & {\color{red}2} \\ \cline{1-3}\end{array} \underset{s}{\to} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} \\ \hline 2 & 2 & {\color{red}1} & {\color{red}2} \\ \hline 3 & {\color{red}1} & {\color{red}2} \\ \cline{1-3}\end{array} = Y_{321} \cup H,\ H \equiv T.$$

- Tableau sliding LR commutor (Thomas-Yong infusion). *To each entry of the inner shape (in red) associate a* jeu de taquin *slide.*

$$Y_{32} \cup T = \begin{array}{|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & {\color{red}1} & 1 \\ \hline {\color{red}2} & {\color{red}2} & 1 & 2 \\ \hline 1 & 2 & 3 \\ \cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & {\color{red}1} & 1 \\ \hline {\color{red}2} & 1 & {\color{red}2} & 2 \\ \hline 1 & 2 & 3 \\ \cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & {\color{red}1} & 1 \\ \hline {\color{red}2} & 1 & 2 & {\color{red}2} \\ \hline 1 & 2 & 3 \\ \cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & {\color{red}1} & 1 \\ \hline 1 & 1 & 2 & {\color{red}2} \\ \hline {\color{red}2} & 2 & 3 \\ \cline{1-3}\end{array} \to$$

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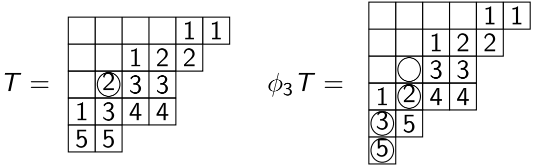
$$\to \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} \\ \hline 2 & 2 & {\color{red}1} & {\color{red}2} \\ \hline 3 & {\color{red}1} & {\color{red}2} \\ \cline{1-3}\end{array} = Y_{321} \cup H,\ H \equiv T.$$

Internal and External insertion on skew-tableaux

For skew tableaux there are two types of row insertion: external and internal. External insertion is similar to Schensted's original procedure.

Sagan-Stanley internal insertion operator ϕ_i on T a skew-tableau:



- Definition of internal insertion operator ϕ_i on T. Need $(i, \mu_i + 1)$ to be an inner corner of T:
  - ϕ_i bumps the entry, say x, in the inner corner cell $(i, \mu_i + 1)$ of T, and then inserts (externally as in the Schensted procedure) the bumped element x in the subtableau consisting of the last $n - i$ rows of T.

Special case of Sagan-Stanley skew RSK correspondence

- *Internal row insertion correspondence*. Fix partitions $\mu \subseteq \alpha, \beta$. There is a bijection, defined below,

$$\begin{array}{ccc} YT(\alpha/\mu) \times YT(\beta/\mu) & \longrightarrow & \displaystyle\bigcup_{\substack{\lambda \\ |\lambda|=|\alpha|+|\beta|-|\mu|}} YT(\lambda/\beta) \times YT(\lambda/\alpha) \\ (T, U) & \longrightarrow & (P, Q), \end{array}$$

where $P \equiv T$ and $Q \equiv U$. The P-tableau P is the internal row insertion of T whose sequence of inner corners containing the entries of T to be internally inserted is dictated by the entries of U in the standard order.

- $T = \begin{array}{|c|c|c|} \hline & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$, $\quad U = \begin{array}{|c|c|c|c|} \hline & 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$ $stdU = \begin{array}{|c|c|c|c|} \hline & 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$.

- internal insertion order word $R(U) = 12211 = R(stdU)$.

- $P = \phi_{R(U)} T = \phi_{R(stdU)} T = \phi_{12211} T = \phi_1 \phi_2 \phi_2 \phi_1 \phi_1 T$.

P-tableau in the Internal row insertion correspondence

- $\phi_{R(U)} = \phi_{R(stdU)} = \phi_{12211} = \phi_1\phi_2\phi_2\phi_1\phi_1$,

$$P = \phi_{12211}\,T = \phi_{12211} \begin{array}{|c|c|c|} \hline & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} = \phi_{1221} \begin{array}{|c|c|c|} \hline & & 3 \\ \hline 1 & 3 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} = \phi_{122} \begin{array}{|c|c|c|} \hline & & \\ \hline 1 & 3 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$= \phi_{12} \begin{array}{|c|c|c|} \hline & & \\ \hline & 3 & 3 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} = \phi_{1} \begin{array}{|c|c|c|} \hline & & \\ \hline & & 3 \\ \hline 1 & 3 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & 3 \\ \cline{1-3} 1 & 3 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array}$$

- Under which conditions do we have coincidence of the P-tableau?

$$U = \begin{array}{|c|c|} \hline & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \text{ and } V = \begin{array}{|c|c|} \hline & 3 \\ \hline 1 \\ \cline{1-1} \end{array},\ \phi_2\phi_1 T = \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & 3 \\ \cline{1-2} 1 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \neq \phi_1\phi_2 T = \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & 1 \\ \cline{1-2} 2 & 3 \\ \cline{1-2} \end{array}.$$

$$U' = \begin{array}{|c|c|} \hline & 2 \\ \hline 1 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \text{ and } V' = \begin{array}{|c|c|} \hline & 3 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array},\ R(U') = 312 \equiv R(V') = 132, \text{ and}$$

$$\phi_{312}\,T = \phi_{132}\,T = \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & 1 \\ \cline{1-2} & 3 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array}.$$

The Knuth class of an internal insertion order word

- Elementary Knuth transformations on words:

$$kij \equiv kji = \begin{array}{|c|c|}\hline i & j \\ \hline k & \\ \cline{1-1}\end{array}\,, \quad i < k \leq j, \quad ijk \equiv jik = \begin{array}{|c|c|}\hline i & k \\ \hline j & \\ \cline{1-1}\end{array}\,, \quad i \leq k < j.$$

- Sufficient conditions for the coincidence of the P-tableau in the internal insertion correspondence.

Theorem

Knuth commutation of internal insertion operators A. (2016). Let u be an internal insertion order word of T and $v \equiv u$. Then

(a) v is an internal insertion order word of T.

(c) $\phi_u T = \phi_v T = P$.

Extension of Internal insertion operators

Sagan-Stanley internal insertion operator $\bar{\phi}_i$ on $Y \cup T$, Y Yamanouchi tableau, T a skew-tableau:

$Y \cup T =$

1	1	1	1	1	1
2	2	1	2	2	
3	2	3	3		
1	3	4	4		
5	5				

$\bar{\phi}_3(Y \cup T) =$

1	1	1	1	1	1
2	2	1	2	2	
3	3	3	3		
1	2	4	4		
3	5				
5					

Internal insertion LR commutor A.'99; A., King, Terada'16

- $Y_{32} \cup T = \begin{array}{|c|c|c|c|}\hline \color{red}{1} & \color{red}{1} & \color{red}{1} & 1 \\ \hline \color{red}{2} & \color{red}{2} & 1 & 2 \\ \hline 1 & 2 & 3 & \\ \cline{1-3} \end{array}$ $\qquad \bar{\phi}_{12}\bar{\phi}_1$

- $\emptyset \underset{\chi_1^3\omega_1^1}{\longrightarrow} \begin{array}{|c|c|c|c|}\hline 1 & \color{red}{1} & \color{red}{1} & \color{red}{1} \\ \hline \end{array} \underset{\chi_2^2\bar{\phi}_1\omega_2^1}{\longrightarrow} \begin{array}{|c|c|c|c|}\hline 1 & 1 & \color{red}{1} & \color{red}{1} \\ \hline 2 & \color{red}{1} & \color{red}{2} & \color{red}{2} \\ \hline \end{array} \underset{\chi_3^0\bar{\phi}_1\bar{\phi}_2\omega_3^1}{\longrightarrow} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & \color{red}{1} \\ \hline 2 & 2 & \color{red}{1} & \color{red}{2} \\ \hline 3 & \color{red}{1} & \color{red}{2} & \\ \cline{1-3} \end{array} = Y_{321} \cup H$

$$\chi_3^0\bar{\phi}_{12}\omega_3^1[\chi_2^2\bar{\phi}_1\omega_2^1[\chi_1^3\omega_1^1(\emptyset)]] = Y_{321} \cup H$$

Internal insertion LR commutor A.'99; A., King, Terada'16

- $Y_{32} \cup T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 \\ \hline 1 & 2 & 3 & \\ \cline{1-3} \end{array} \qquad \bar{\phi}_{12}\bar{\phi}_1$

- $\emptyset \xrightarrow[\chi_1^3\omega_1^1]{} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline \end{array} \xrightarrow[\chi_2^2\bar{\phi}_1\omega_2^1]{} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 2 & 2 \\ \hline \end{array} \xrightarrow[\chi_3^0\bar{\phi}_1\bar{\phi}_2\omega_3^1]{} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 \\ \hline 3 & 1 & 2 & \\ \cline{1-3} \end{array} = Y_{321} \cup H$

$$\chi_3^0\bar{\phi}_{12}\omega_3^1[\chi_2^2\bar{\phi}_1\omega_2^1[\chi_1^3\omega_1^1(\emptyset)]] = Y_{321} \cup H$$

- $(Y \cup T)^{(1)} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline \end{array} \xrightarrow[s]{} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline \end{array} = \rho_1^{(1)}[(Y \cup T)^{(1)}] = \chi_1^3\omega_1^1\rho_1^{(0)}[\emptyset]$

Internal insertion LR commutor A.'99; A., King, Terada'16

- $Y_{32} \cup T = \begin{array}{|c|c|c|c|}\hline \color{red}1 & \color{red}1 & \color{red}1 & 1 \\ \hline \color{red}2 & \color{red}2 & 1 & 2 \\ \hline 1 & 2 & 3 \\ \cline{1-3}\end{array}$ $\qquad \bar{\phi}_{12}\bar{\phi}_1$

- $$\emptyset \underset{\chi_1^3\omega_1^1}{\longrightarrow} \begin{array}{|c|c|c|c|}\hline 1 & \color{red}1 & \color{red}1 & \color{red}1 \\ \hline\end{array} \underset{\chi_2^2\bar{\phi}_1\omega_2^1}{\longrightarrow} \begin{array}{|c|c|c|c|}\hline 1 & 1 & \color{red}1 & \color{red}1 \\ \hline 2 & \color{red}1 & \color{red}2 & \color{red}2 \\ \hline\end{array} \underset{\chi_3^0\bar{\phi}_1\bar{\phi}_2\omega_3^1}{\longrightarrow} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & \color{red}1 \\ \hline 2 & 2 & \color{red}1 & \color{red}2 \\ \hline 3 & \color{red}1 & \color{red}2 \\ \cline{1-3}\end{array} = Y_{321} \cup H$$

$$\chi_3^0\bar{\phi}_{12}\omega_3^1[\chi_2^2\bar{\phi}_1\omega_2^1[\chi_1^3\omega_1^1(\emptyset)]] = Y_{321} \cup H$$

- $$(Y \cup T)^{(1)} = \begin{array}{|c|c|c|c|}\hline \color{red}1 & \color{red}1 & \color{red}1 & 1 \\ \hline\end{array} \underset{s}{\rightarrow} \begin{array}{|c|c|c|c|}\hline 1 & \color{red}1 & \color{red}1 & \color{red}1 \\ \hline\end{array} = \rho_1^{(1)}[(Y \cup T)^{(1)}] = \chi_1^3\omega_1^1\rho_1^{(0)}[\emptyset]$$

$$(Y \cup T)^{(2)} = \begin{array}{|c|c|c|c|}\hline \color{red}1 & \color{red}1 & \color{red}1 & 1 \\ \hline \color{red}2 & \color{red}2 & 1 & 2 \\ \hline\end{array} \underset{s}{\rightarrow} \begin{array}{|c|c|c|c|}\hline 1 & 1 & \color{red}1 & \color{red}1 \\ \hline 2 & \color{red}1 & \color{red}2 & \color{red}2 \\ \hline\end{array} = \rho_1^{(2)}[(Y \cup T)^{(2)}]$$

$$= \chi_2^2\bar{\phi}_1\omega_2^1\rho_1^{(1)}[(Y \cup T)^{(1)}]$$

Internal insertion LR commutor A.'99; A., King, Terada'16

- $Y_{32} \cup T = \begin{array}{|c|c|c|c|}\hline {\color{red}1}&{\color{red}1}&{\color{red}1}&1\\\hline {\color{red}2}&{\color{red}2}&1&2\\\hline 1&2&3\\\cline{1-3}\end{array}$ $\qquad \bar\phi_{12}\bar\phi_1$

- $\emptyset \xrightarrow[\chi_1^3\omega_1^1]{} \begin{array}{|c|c|c|c|}\hline 1&{\color{red}1}&{\color{red}1}&{\color{red}1}\\\hline\end{array} \xrightarrow[\chi_2^2\bar\phi_1\omega_2^1]{} \begin{array}{|c|c|c|c|}\hline 1&1&{\color{red}1}&{\color{red}1}\\\hline 2&{\color{red}1}&{\color{red}2}&{\color{red}2}\\\hline\end{array} \xrightarrow[\chi_3^0\bar\phi_1\bar\phi_2\omega_3^1]{} \begin{array}{|c|c|c|c|}\hline 1&1&1&{\color{red}1}\\\hline 2&2&{\color{red}1}&{\color{red}2}\\\hline 3&{\color{red}1}&{\color{red}2}\\\cline{1-3}\end{array} = Y_{321}\cup H$

$$\chi_3^0\bar\phi_{12}\omega_3^1[\chi_2^2\bar\phi_1\omega_2^1[\chi_1^3\omega_1^1(\emptyset)]] = Y_{321}\cup H$$

- $(Y\cup T)^{(1)} = \begin{array}{|c|c|c|c|}\hline {\color{red}1}&{\color{red}1}&{\color{red}1}&1\\\hline\end{array} \xrightarrow[s]{} \begin{array}{|c|c|c|c|}\hline 1&{\color{red}1}&{\color{red}1}&{\color{red}1}\\\hline\end{array} = \rho_1^{(1)}[(Y\cup T)^{(1)}] = \chi_1^3\omega_1^1\rho_1^{(0)}[\emptyset]$

$$(Y\cup T)^{(2)} = \begin{array}{|c|c|c|c|}\hline {\color{red}1}&{\color{red}1}&{\color{red}1}&1\\\hline {\color{red}2}&{\color{red}2}&1&2\\\hline\end{array} \xrightarrow[s]{} \begin{array}{|c|c|c|c|}\hline 1&1&{\color{red}1}&{\color{red}1}\\\hline 2&{\color{red}1}&{\color{red}2}&{\color{red}2}\\\hline\end{array} = \rho_1^{(2)}[(Y\cup T)^{(2)}]$$

$$= \chi_2^2\bar\phi_1\omega_2^1\rho_1^{(1)}[(Y\cup T)^{(1)}]$$

$$(Y\cup T)^{(3)} = \begin{array}{|c|c|c|c|}\hline {\color{red}1}&{\color{red}1}&{\color{red}1}&1\\\hline {\color{red}2}&{\color{red}2}&1&2\\\hline 1&2&3\\\cline{1-3}\end{array} \xrightarrow[s]{} \begin{array}{|c|c|c|c|}\hline 1&1&1&{\color{red}1}\\\hline 2&2&{\color{red}1}&{\color{red}2}\\\hline 3&{\color{red}1}&{\color{red}2}\\\cline{1-3}\end{array} = Y_{321}\cup H = \rho_1^{(3)}[(Y\cup T)^{(3)}]$$

$$= \chi_3^0\bar\phi_{12}\omega_3^1\rho_1^{(2)}(Y\cup T)^{(2)}.$$

Internal insertion and switching LR commutors coincide

Theorem

(A., 2017)

Let $n \geq 1$ and $Y \cup T \in \mathcal{LR}^{(n)}$ with $Y = Y_\mu$ Yamanouchi tableau and T a ballot tableau of shape λ/μ and weight ν. Consider the nth row word of T where F is the row subword restricted to the entries in $[n-1]$, and ν_n is the number of entries equal to n. Then

$$\rho_1^{(n)}(Y \cup T) = \chi_n^{\mu_n} \bar{\phi}_F \omega_n^{\nu_n} \rho_1^{(n-1)}[(Y \cup T)^{(n-1)}] = \chi_n^{\mu_n} \omega_n^{\nu_n} \bar{\phi}_F \rho_1^{(n-1)}[(Y \cup T)^{(n-1)}],$$

with $\rho_1^{(0)}(\emptyset) := \emptyset$. In particular, all bumping routes of $\bar{\phi}_F$ are pairwise disjoint and terminate in the nth row.

$(Y \cup T)^{(n-1)}$ consists of the first $n-1$ rows of T.

A bit more: Sketch of the proof (main steps)

- Reduce to the case $\mu = (\mu_1, \dots, \mu_{n-1}, 0)$. By induction on $n \geq 1$ and $|F| \geq 0$.
- $n = 1$, trivial,

$$\rho_1^{(1)}(\emptyset \cup Y_{(\nu_1)}) = \omega_1^{\nu_1}\emptyset = \omega_1^{\nu_1}\rho_1^{(0)}[\emptyset] = Y_{(\nu_1)} \cup \emptyset.$$

$$\emptyset \cup Y_{(\nu_1)} = \emptyset \cup \boxed{1}\boxed{1}\boxed{1} \rightarrow \boxed{1}\boxed{1}\boxed{1} \cup \emptyset = Y_{(\nu_1)} \cup \emptyset$$

Sketch of the proof

- Assume the statement true for n and prove for $n+1$. Detach the $n+1$th row

$$Y \cup T = F(n+1)^{\nu_{n+1}} * (Y \cup T)^{(n)},$$

F a word on the alphabet $[n]$.

We claim

$$\rho_1^{(n+1)}(Y \cup T) = \omega_{n+1}^{\nu_{n+1}} \bar{\phi}_F \rho_1^{(n)}[(Y \cup T)^{(n)}].$$

  - $|F| = 0$. Trivial. Switching on $Y \cup T$ reduces to $(Y \cup T)^{(n)}$, the first n rows of $Y \cup T$, and $\bar{\phi}_F = id$,

$$\rho_1^{(n+1)}(Y \cup T) = \omega_{n+1}^{\nu_{n+1}} \rho_1^{(n)}[(Y \cup T)^{(n)}].$$

Sketch of the proof

- Let $|F| \geq 1$. Apply switching to decompose $Y \cup T$ and reduce $|F|$

$$Y \cup T \underset{s}{\rightarrow} [Y' \cup S] \cup [D * Q^{(n)}] = [\widehat{F}(n+1)^{\nu_{n+1}} * (Y' \cup S)^{(n)}] \cup [D * Q^{(n)}],$$

$Y' = Y_{(\mu_1,\dots,\mu_{d-1})}$, $\mu_d > 0$, $S \equiv T$, $D * Q^{(n)} \equiv Y_{(\mu_d,\dots,\mu_{n-1})}$, $\widehat{F}$ strict subword of F and $D = d^{|D|}$, $|D| > 0$,

$$|\widehat{F}| + |D| = |F|.$$

  - $0 \leq |\widehat{F}| < |F|$. By induction on $|F|$,

$$\rho_1^{(n+1)}(Y' \cup S) = \omega_{n+1}^{\nu_{n+1}} \bar{\phi}_{\widehat{F}} \rho_1^{(n)}[(Y' \cup S)^{(n)}],$$

  all $\bar{\phi}_{\widehat{F}}$-*bumping routes* terminate in the $(n+1)$th row.

- One has so far

$$\rho_1^{(n+1)}(Y \cup T) = \omega_{n+1}^{\nu_{n+1}} \bar{\phi}_{\widehat{F}} \rho_1^{(n)}[(Y' \cup S)^{(n)}] \cup (D * Q^{(n)}),$$

all $\bar{\phi}_{\widehat{F}}$-*bumping routes* terminate in the $(n+1)$th row.

- We claim

$$\rho_1^{(n+1)}(Y \cup T) = \omega_{n+1}^{\nu_{n+1}} \bar{\phi}_F \rho_1^{(n)}[(Y \cup T)^{(n)}].$$

- $Y \cup T \underset{s}{\rightarrow} [Y' \cup S] \cup [D * Q^{(n)}] = [\widehat{F}(n+1)^{\nu_{n+1}} * (Y' \cup S)^{(n)}] \cup [D * Q^{(n)}]$.

  Detach the nth row

$$(Y' \cup S)^{(n)} = G\,C n^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)},$$

G and C row words in the alphabet $[n-1]$, $|G| = |\widehat{F}| + |D| = |F|$.

$[\widehat{F}(n+1)^{\nu_{n+1}} \cup D] * [G\,C n^{\hat{\nu}_n} \cup X] =$

A	B	C	n^*	$\cdots$	n^*	X
$\widehat{F}$	$n{+}1^*$	$\cdots$	$n{+}1^*$	D		

$\overset{s}{\leftrightarrow}$

G_1	F_2	G_3	F_4	$\cdots$	G_{k-1}	F_k	C	n^*	$\cdots$	n^*	X
F_1	D_2	F_3	D_4	$\cdots$	F_{k-1}	D_k	$n{+}1^*$	$\cdots$	$n{+}1^*$		

$\overset{s}{\leftrightarrow}$

G_1	D_2	G_3	D_4	$\cdots$	G_{k-1}	D_k	C	n^*	$\cdots$	n^*	X
F_1	F_2	F_3	F_4	$\cdots$	F_{k-1}	F_k	$n{+}1^*$	$\cdots$	$n{+}1^*$		

$\widehat{G} := G_1 G_3 \ldots G_{k-1}$ a subword of $G := AB$, $|\widehat{G}| = |\widehat{F}|$, and

$\widehat{F} G \equiv F \widehat{G}$ Knuth equivalent.

- $Y \cup T \underset{s}{\to} [Y' \cup S] \cup [D * Q^{(n)}] = [\widehat{F}(n+1)^{\nu_{n+1}} * (Y' \cup S)^{(n)}] \cup [D * Q^{(n)}]$.
  Detach the nth row

$$(Y' \cup S)^{(n)} = G\,Cn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)},$$

G and C row words in the alphabet $[n-1]$, $|G| = |\widehat{F}| + |D| = |F|$.

$[\widehat{F}(n+1)^{\nu_{n+1}} \cup D] * [G\,Cn^{\hat{\nu}_n} \cup X] =$

A	B	C	n^*	$\cdots$	n^*	X
$\widehat{F}$	$n{+}1^*$	$\cdots$	$n{+}1^*$	D		

$\overset{s}{\leftrightarrow}$

G_1	F_2	G_3	F_4	$\cdots$	G_{k-1}	F_k	C	n^*	$\cdots$	n^*	X
F_1	D_2	F_3	D_4	$\cdots$	F_{k-1}	D_k	$n{+}1^*$	$\cdots$	$n{+}1^*$		

$\overset{s}{\leftrightarrow}$

G_1	D_2	G_3	D_4	$\cdots$	G_{k-1}	D_k	C	n^*	$\cdots$	n^*	X
F_1	F_2	F_3	F_4	$\cdots$	F_{k-1}	F_k	$n{+}1^*$	$\cdots$	$n{+}1^*$		

$\widehat{G} := G_1 G_3 \ldots G_{k-1}$ a subword of $G := AB$, $|\widehat{G}| = |\widehat{F}|$, and
$\widehat{F}G \equiv F\widehat{G}$ Knuth equivalent.

- $(Y \cup T)^{(n)} \underset{s}{\to} [\widehat{G}\,Cn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] \cup (DX * Q^{(n-1)})$

- By induction on n
  - $$\begin{aligned}\rho_1^{(n)}(Y' \cup S)^{(n)} &= \rho_1^{(n)}[GCn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] \\ &= \bar{\phi}_G \rho_1^{(n)}(Cn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}),\end{aligned}$$

    all $\bar{\phi}_G$-*bumping routes* terminate in the nth row.
  - $$\begin{aligned}\rho_1^{(n)}[(Y \cup T)^{(n)}] &= \rho_1^{(n)}[\widehat{G}Cn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] \cup (DX * Q^{(n-1)}) \\ &= \bar{\phi}_{\widehat{G}} \rho_1^{(n)}[Cn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] \cup (DX * Q^{(n-1)}).\end{aligned}$$

    all $\bar{\phi}_{\widehat{G}}$-*bumping routes* terminate in the nth row.

- $$\widehat{F}G \equiv F\widehat{G} \Rightarrow \bar{\phi}_{\widehat{F}G} = \bar{\phi}_{F\widehat{G}}.$$

$$\rho_1^{(n)}[(Y' \cup S)^{(n)}] = \bar{\phi}_G \rho_1^{(n)}[Cn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}]$$
$$\rho_1^{(n)}[(Y \cup T)^{(n)}] = \bar{\phi}_{\widehat{G}} \rho_1^{(n)}[Cn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] \cup (DX * Q^{(n-1)})$$

- $$\begin{aligned} \rho_1^{(n+1)}(Y \cup T) &= \omega_{n+1}^{\nu_{n+1}} \bar{\phi}_{\widehat{F}} \rho_1^{(n)}[(Y' \cup S)^{(n)}] \cup (D * Q^{(n)}), \\ &= \omega_{n+1}^{\nu_{n+1}} \bar{\phi}_{\widehat{F}} \bar{\phi}_G \rho_1^{(n)}[Cn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] \cup (D * Q^{(n)}) \\ &= \omega_{n+1}^{\nu_{n+1}} \bar{\phi}_F \bar{\phi}_{\widehat{G}} \rho_1^{(n)}[Cn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] \cup (D * X * Q^{(n-1)}) \\ &= \omega_{n+1}^{\nu_{n+1}} \bar{\phi}_F [\bar{\phi}_{\widehat{G}} \rho_1^{(n)}(Cn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}) \cup (DX * Q^{(n-1)})] \\ &= \omega_{n+1}^{\nu_{n+1}} \bar{\phi}_F \rho_1^{(n)}[(Y \cup T)^{(n)}]. \end{aligned}$$