Crystals, and the Berenstein-Kirillov, cacti and related groups

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The cactus group $J_{\mathfrak{g}}$

- Let $\mathfrak{g}$ be a finite dimensional, complex, semisimple Lie algebra and
  - I its Dynkin diagram, $\Delta = \{\alpha_i\}_{i \in I}$ the simple roots:

$$A_n,\ B_n,\ C_n,\ D_n,\ G_2,\ F_4,\ E_6,\ E_7,\ E_8$$

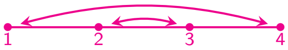
  - $W_{\mathfrak{g}}$ the Weyl group, $w_0 \in W_{\mathfrak{g}}$ the longest element.
  - $\theta : I \to I$ the Dynkin diagram automorphism of I defined by

$$\alpha_{\theta(i)} = -w_0.\alpha_i,\ i \in I.$$

Example: For $\mathfrak{g} = \mathfrak{gl}_n$, Cartan type A_{n-1}: $I = [n-1]$, $\Delta = \{\alpha_i = \mathbf{e_i} - \mathbf{e_{i+1}}\}_{\mathbf{i \in [n-1]}}$,
$W_{\mathfrak{g}} = \mathfrak{S}_n =< r_1, \dots, r_{n-1} : R1, R2, R3 >$

$$\begin{array}{lll} R1: & r_i^2 = 1, & 1 \leq i \leq n, \\ R2: & (r_i r_j)^2 = 1, & |i-j| > 1, \\ R3: & (r_i r_{i+1})^3 = 1, & 1 \leq i \leq n-2, \end{array}$$

$n = 5$ $\quad A_4$



$\alpha_1 = (1,-1,0,0,0) \to \alpha_4 = (0,0,0,1,-1) = -w_0\alpha_1,\ \theta(i) = n-i$

  - $\theta_J : J \to J$ the Dynkin diagram automorphism of a connected subdiagram $J \subseteq I$, defined by

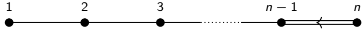
$$\alpha_{\theta_J(j)} = -w_0^J.\alpha_j,\ j \in J,$$

w_0^J the long element of the parabolic subgroup $W_{\mathfrak{g}}^J \subseteq W_{\mathfrak{g}}$.

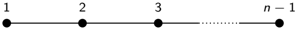
Example: $\mathfrak{gl}_5$, $J = [1,2]$, $J = \{2\}$, $J = \{3\}$: $A_2,\ A_1,\ A_1,$

Example: For $\mathfrak{g} = \mathfrak{sp}(2n)$, Cartan type C_n

C_n



A_{n-1}

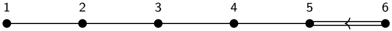


- Cartan type C_n: $I = [n]$, $\Delta = \{\alpha_i = \mathbf{e_i} - \mathbf{e_{i+1}}\}_{\mathbf{i \in [n-1]}} \cup \{\alpha_\mathbf{n} = \mathbf{2e_n}\}$,
  $W = B_n =< r_1, \ldots, r_{n-1}, r_n : R1, R2, R3, R4 >$

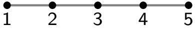
$$\begin{array}{llr} R1: & r_i^2 = 1, & 1 \leq i \leq n, \\ R2: & (r_i r_j)^2 = 1, & |i-j| > 1, \\ R3: & (r_i r_{i+1})^3 = 1, & 1 \leq i \leq n-2, \\ R4: & (r_{n-1} r_n)^4 = 1 & , \end{array}$$

$\alpha_{\theta(i)} = -w_0.\alpha_i = -(-\alpha_i) = \alpha_i, \quad \theta(i) = i.$

C_6



$J : A_5$



$J : A_2, \ A_1, \ A_1, \ \ C_2$

$\theta_{[1,5]}(i) = 5 - i, \quad \theta_{[5,6]} = 1$

- [Henriques-Kamnitzer 2006] The *cactus group* $J_{\mathfrak{gl}_n} = J_n$.
- [Halacheva 2016]. The *cactus group* $J_{\mathfrak{g}}$ corresponding to $\mathfrak{g}$ is the group defined by:
  - **Generators**: s_J, $J \subseteq I$ running over all connected subdiagrams of the Dynkin diagram I of $\mathfrak{g}$, and
  - **Relations**:

    1 $\mathfrak{g}$. $s_J^2 = 1$, for all $J \subseteq I$,

    2 $\mathfrak{g}$. $s_J s_{J'} = s_{J'} s_J$, for all $J, J' \subseteq I$ such that $J \sqcup J'$ is not connected,

    $J = [1, 2],\ J' = \{4\},\ J \sqcup J'$ is not connected diagram

    3 $\mathfrak{g}$. $s_J s_{J'} = s_{\theta_J(J')} s_J$, for all $J' \subseteq J \subseteq I$.

The pure cactus and the (cousin) braid group

- The cactus group $J_{\mathfrak{g}}$ surjects into the Weyl group $W_{\mathfrak{g}}$

$$\pi : J_{\mathfrak{g}} \to W_{\mathfrak{g}}, s_J \mapsto w_0^J$$

$$w_0^J w_0^{J'} w_0^J = w_0^{\theta_J(J')},\ J' \subseteq J;\ w_0^J w_0^{J'} = w_0^{J'} w_0^J,\ J \sqcup J' \text{ disconnected; } (w_0^J)^2 = 1$$

-
  - $Ker(\pi)$ contains the elements $(s_{\{i\}} s_{\{j\}})^{m_{ij}}$ such that $(r_i r_j)^{m_{ij}} = 1$ in $W_{\mathfrak{g}}$ as a Coxeter group.
  - $W_{\mathfrak{g}} \cong J_{\mathfrak{g}} / Ker(\pi)$
    The pure cactus group is $PJ_{\mathfrak{g}} := Ker(\pi)$.
    The pure cactus group PJ_n is the *fundamental group* of $\overline{M}_{(0,n+1)}(\mathbb{R})$ real locus of the Deligne-Mumford moduli space of rational curves with $n+1$ marked points.

- The Braid group $Br_{\mathfrak{g}}$ also surjects into the Weyl group $W_{\mathfrak{g}}$:

$$\mathfrak{g} = \mathfrak{gl}_n,\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \quad \text{for } i, j \text{ connected in } I$$

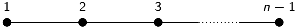
$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{for } i, j \text{ disconnected in } I$$

$$\sigma : Br_{\mathfrak{g}} \to W_{\mathfrak{g}},\ \sigma_i \mapsto w_0^{\{i\}}$$

  - $Ker(\sigma)$ contains the elements σ_i^2 and $W_{\mathfrak{g}} \cong Br_{\mathfrak{g}} / Ker(\sigma)$
    The pure braid group is $PBr_{\mathfrak{g}} := Ker(\sigma)$.

The cactus $J_n := J_{\mathfrak{gl}_n}$

A_{n-1}



1A. $s_J^2 = 1, J \subseteq [n-1]$,

2A. $s_J s_{J'} = s_{J'} s_J$, for all $J, J' \subseteq [n-1]$ such that $J \cup J'$ is disconnected.

3A. $s_{[p,q]} s_{[k,l]} = s_{[p+q-l, p+q-k]} s_{[p,q]}$ for $[k,l] \subset [p,q] \subseteq [n-1]$.

- J_k is a subgroup of J_n , for $1 \leq k \leq n$.

Alternative $n-1$ generators for J_n,

- $s_{[1,k]}$, $1 \leq k \leq n-1$,

$$s_{[i,j]} = s_{[1,j]} s_{[1,j-i+1]} s_{[1,j]}.$$

- $s_{[i,n-1]}, 1 \leq i \leq n-1$,

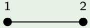
$$s_{[1,n-1]} s_{[i,n-1]} s_{[1,n-1]} = s_{[1,n-i]}.$$

Example

- A_1 

  $J_2 =< s_1 : s_1^2 = 1 >= \mathbb{Z}_2,\ W = \mathfrak{S}_2,$

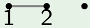
  $PJ_2 =< 1 >.$

- A_2 

  $J_3 =< s_1, s_2, s_{[1,2]} : s_1^2 = s_2^2 = s_{[1,2]}^2 = 1, s_{[1,2]} s_1 s_{[1,2]} = s_2 >$

  $J_3 \simeq < s_1, s_{[1,2]} : s_1^2 = s_{[1,2]}^2 = 1 > \simeq < s_2, s_{[1,2]} : s_2^2 = s_{[1,2]}^2 = 1 > .$

$$J = [1,2],\ J = \{1\},\ J = \{2\},$$

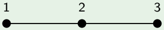


$$W = \mathfrak{S}_3$$

$$PJ_3 =< s_{[1,2]} s_1 s_2 s_1 = s_2 s_1 s_2 s_{[1,2]} > = < (s_1 s_{[1,2]})^3 > \ni (s_1 s_2)^3, \quad PJ_3 \simeq \mathbb{Z}.$$

$$J_3 \subseteq J_n,\ n \geq 3.$$

Example (Bellingeri-Chemin-Lebed, 2022)

- A_3 

$$J_4 = \left\langle s_1, s_2, s_3, s_{[1,2]}, s_{[1,3]}, s_{[2,3]} \;\middle|\; \begin{array}{l} s_i^2 = s_{[1,i]}^2 = s_{[2,3]}^2 = 1, 1 \leq i \leq 3, \\ (s_1 s_3)^2 = 1, \\ s_{[1,2]} s_1 s_{[1,2]} = s_{[1,3]} s_2 s_{[1,3]} = s_2, \\ s_{[1,3]} s_1 s_{[1,3]} = s_{[2,3]} s_2 s_{[2,3]} = s_3, \\ s_{[1,3]} s_{[1,2]} s_{[1,3]} = s_{[1,3]} s_{[2,3]} s_{[1,3]} = s_{[2,3]} \end{array} \right\rangle$$

$$J_4 \simeq < s_1, s_{[1,2]}, s_{[1,3]} : s_1^2 = s_{[1,2]}^2 = s_{[1,3]}^2 = 1,\ (s_1 s_{[1,3]})^4 = 1,\ (s_{[1,3]} s_{[1,2]} s_1 s_{[1,2]})^2 = 1 >$$

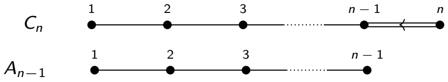
The cacti $J_n := J_{\mathfrak{gl}_n}$ and $J_{\mathfrak{sp}_{2n}}$

- The cactus group $J_{\mathfrak{sp}_{2n}}$ is the group defined by
  - Generators: s_J, J connected subdiagrams of the C_n Dynkin diagram,
  - Relations:

    1C. $s_J^2 = 1$, $J \subseteq [n]$,

    2C. $s_J s_{J'} = s_{J'} s_J$, $\quad J, J' \subseteq [n]$ such that $J \sqcup J'$ is not connected,

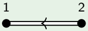
    3C. 1. $s_{[p,q]} s_{[k,l]} = s_{[p+q-l,p+q-k]} s_{[p,q]}$, $[k,l] \subset [p,q] \subseteq [n-1]$.
        2. $s_{[p,n]} s_{[q,l]} = s_{[q,l]} s_{[p,n]}$, $[q,l] \subset [p,n] \subseteq [n]$,



- $J_n = J_{\mathfrak{gl}_n} \subseteq J_{\mathfrak{sp}_{2n}} \subseteq J_{\mathfrak{sp}_{2m}}$, $m \geq n$.
- Alternative $n-1$ generators for J_n,
  $s_{[1,p]}$, $1 \leq p \leq n-1$, or
  $s_{[p,n-1]}, 1 \leq p \leq n-1$.
- Alternative $2n-1$ generators for $J_{\mathfrak{sp}_{2n}}$: $s_{[1,p]}$, $1 \leq p \leq n-1$, $\quad s_{[p,n]}$, $1 \leq p \leq n$.

Example

- $C_1 = A_1,\ W = B_1 = \mathfrak{S}_2$.
  $J_{\mathfrak{sp}_2} = J_2 =< s_1 : s_1^2 = 1 >= \mathbb{Z}_2$,
  $PJ_2 =< 1 >$.

- C_2 

  $W = B_2 =< r_1, r_2 : r_1^2 = r_2^2 = (r_1 r_2)^4 = 1 >,\ w_0 = (r_1 r_2)^2$.

$$J_{\mathfrak{sp}_4} =< s_1, s_2, s_{[1,2]} : s_1^2 = s_2^2 = s_{[1,2]}^2 = 1, (s_1 s_{[1,2]})^2 = (s_2 s_{[1,2]})^2 = 1 >$$

$$J = [1,2],\ J = \{1\},\ J = \{2\},$$

$$PJ_{\mathfrak{sp}_4} \ni s_{[1,2]}(s_1 s_2)^2,\ (s_1 s_2)^4.$$

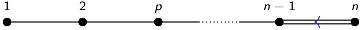
$$J_3 \subset J_{\mathfrak{sp}_6}$$

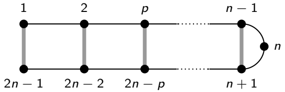
$$J_3 \subseteq J_4 \subset J_{\mathfrak{sp}_8}$$

Further directions...

Embedding of $J_{\mathfrak{sp}(2n)}$ into J_{2n}

- Dynkin diagram folding $C_n \hookrightarrow A_{2n-1}$

C_n 

$\longrightarrow$ A_{2n-1} folded 

- $J_n \subseteq J_{\mathfrak{sp}_{2n}} \hookrightarrow J_{2n}$ [A–Tarighat–Torres, 22].

$$\begin{array}{rcll} \tilde{\iota} : J_{\mathfrak{sp}_{2n}} & \hookrightarrow & J_{2n} & \\ s_{[p,q]} & \mapsto & \tilde{s}_{[p,q]} := s_{[p,q]} s_{[2n-q,2n-p]} = s_{[2n-q,2n-p]} s_{[p,q]}, & [p,q] \subseteq [1, n-1], \\ s_{[p,n]} & \mapsto & s_{[p,2n-p]}, & [p,n] \subseteq [1,n]. \end{array}$$

- $J_n \subseteq J_{\mathfrak{sp}_{2n}} \cong \tilde{J}_{2n} := \tilde{\iota}(J_{\mathfrak{sp}_{2n}}) \subseteq J_{2n}$.
- $\tilde{J}_{2n}$ is the virtual symplectic cactus group (of $J_{\mathfrak{sp}_{2n}}$)
  - generators: $\tilde{s}_{[p,q]}$, $[p,q] \subseteq [1, n-1]$, and $s_{[p,2n-p]}$, $[p,n] \subseteq [1,n]$,
  - $J_{\mathfrak{sp}_{2n}}$ symplectic cactus relations:

$$s'^{2}_{[p,q]} = 1, \; s^2_{[p,2n-p]} = 1,$$

$$s'_{[p,q]} s'_{[k,l]} = s'_{[k,l]} s'_{[p,q]}, \; [p,q] \sqcup [k,l] \subseteq [n-1] \text{ disconnected},$$

$$s_{[p,2n-p]} s'_{[k,l]} = s'_{[k,l]} s_{[p,2n-p]}, \; [p,n] \sqcup [k,l] \subset [1,n] \text{ disconnected},$$

$$s'_{[p,q]} s'_{[k,l]} = s'_{[p+q-l,p+q-k]} s'_{[p,q]}, \; [k,l] \subseteq [p,q] \subseteq [1,n-1],$$

$$s_{[p,2n-p]} s'_{[k,l]} = s'_{[k,l]} s_{[p,2n-p]}, \; [k,l] \subset [p,n], \quad s_{[p,2n-p]} s_{[k,2n-k]} = s_{[k,2n-k]} s_{[p,2n-p]}, \; [k,n] \subseteq [p,n],$$

Normal Crystals

- A $\mathfrak{g}$-crystal is a finite set B along with maps

$$\mathrm{wt} : \mathrm{B} \to \Lambda, \quad e_i, f_i : \mathrm{B} \to \mathrm{B} \cup \{0\}, \, \varepsilon_i, \varphi_i : \mathrm{B} \to \mathbb{Z},$$

  obeying the following axioms for any $b, b' \in \mathrm{B}$ and $i \in I$,
  - $b' = e_i(b)$ if and only if $b = f_i(b')$,
  - if $f_i(b) \neq 0$ then $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$;
    if $e_i(b) \neq 0$, then $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$, and
  - $\varepsilon_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : e_i^a(b) \neq 0\}$ and $\varphi_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : f_i^a(b) \neq 0\}$.
  - $\varphi_i(b) - \varepsilon_i(b) = \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$,

  where $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$ are the coroots.

- A_{n-1}, $I = [n-1]$, and C_n, $I = [n]$: $\Lambda = \mathbb{Z}^n$; $\alpha_i^\vee = \alpha_i$, $1 \leq i < n$, $\alpha_n^\vee = e_n$.

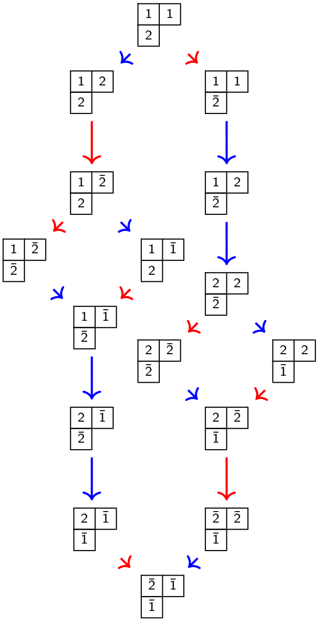
$$1 \xrightarrow{1} 2 \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$$

$$112 \xrightarrow{2} 11\overline{2} \qquad 112 \xrightarrow{1} 211$$

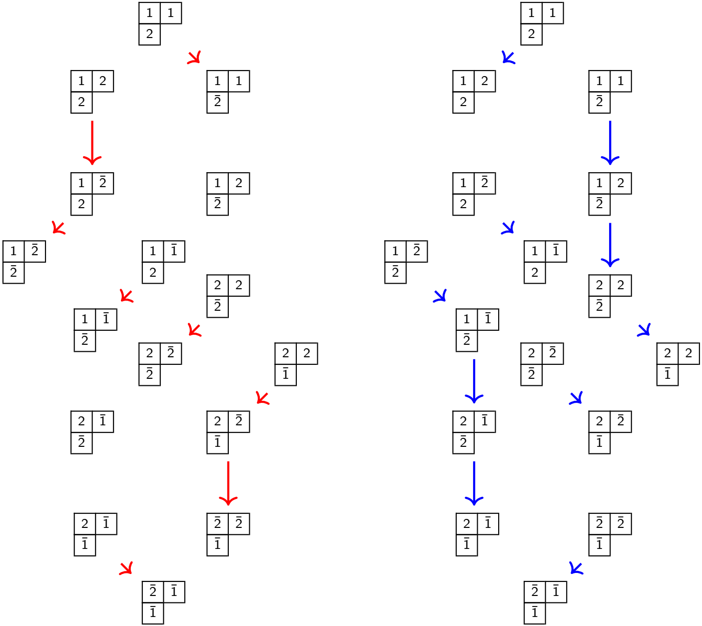
$$+(+-) \xrightarrow{1} -(+-)$$

$$\begin{matrix} 1 & \overline{2} & + \\ 2 & \overline{1} & - \end{matrix}$$

Normal crystals: C_2 crystal $\mathrm{B} = \mathrm{KN}(\lambda, 2)$, $\lambda = (2, 1)$, Kashiwara-Nakashima tableaux



Levi restrictions for $J \subseteq I$: $\mathrm{KN}_{\{2\}}(\lambda, 2)$ and $\mathrm{KN}_{\{1\}}(\lambda, 2)$



Schützenberger–Lusztig involution on crystals

- $B(\lambda)$ $\mathfrak{g}$-normal crystal with h.w. λ and u_λ^{high} and u_λ^{low}.
- The *Schützenberger–Lusztig involution* $\xi : B(\lambda) \to B(\lambda)$ is the unique set involution such that, for all $b \in B(\lambda)$, and $i \in I$,
  - $e_i \xi(b) = \xi f_{\theta(i)}(b)$
  - $f_i \xi(b) = \xi e_{\theta(i)}(b)$
  - $\text{wt}(\xi(b)) = w_0 \text{wt(b)}$

  where w_0 is the long element of the Weyl group W.

- Let $b = f_{j_r} \cdots f_{j_1}(u_\lambda^{high})$, for $j_r, \ldots, j_1 \in I$. Then
  - type A_{n-1}, $\xi(b) = e_{n-j_r} \cdots e_{n-j_1}(u_\lambda^{\text{low}})$, and $\text{wt}(\xi(b)) = w_0 \text{wt}(b)$, $w_0 \in \mathfrak{S}_n$.

    On $SSYT(\lambda, n)$, ξ coincides with *Schützenberger evacuation*.
  - type C_n, $\xi(b) = e_{j_r} \cdots e_{j_1}(u_\lambda^{\text{low}})$, and $\text{wt}(\xi(b)) = -\text{wt}(b)$.

    On $KN(\lambda, n)$, ξ coincides with *Santos symplectic evacuation*, 2021.

The Weyl group action on a normal $\mathfrak{g}$-crystal

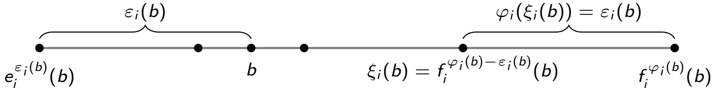
- The *partial Schützenberger–Lusztig involution* ξ_J is the Schützenberger–Lusztig involution ξ on the normal crystal B_J, for J any sub-diagram of I.

- When $J = \{i\}$, ξ_i is the Schützenberger–Lusztig involution on the i-strings $B_{\{i\}}$ and coincides with the Weyl group $W_{\mathfrak{g}}$ action on the i-strings $B_{\{i\}}$:

  - ξ_i, $i \in I$, satisfy the Weyl group relations.

$$A_{n-1}\colon\ \xi_i^2 = 1,\ (\xi_i\xi_j)^2 = 1,\ |i-j| > 1,\ (\xi_i\xi_{i+1})^3 = 1,\ 1 \le i < n-1$$

$$C_n\colon\ \xi_i^2 = 1,\ (\xi_i\xi_j)^2 = 1,\ |i-j| > 1,\ (\xi_i\xi_{i+1})^3 = 1,\ 1 \le i < n-1,\ (\xi_n\xi_{n-1})^4 = 1$$



- A_{3-1} crystal SSYT$((2,1,0),3)$: $\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \xrightarrow{1} \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \xrightarrow{2} \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \xrightarrow{2} \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} \xrightarrow{1} \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$

$$\xi_1 \begin{array}{|c|c|}\hline \mathbf{1}&\mathbf{1}\\\hline \mathbf{2}\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline \mathbf{1}&\mathbf{2}\\\hline \mathbf{2}\\\cline{1-1}\end{array} \qquad \xi_2 \begin{array}{|c|c|}\hline 1&\mathbf{2}\\\hline \mathbf{2}\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&\mathbf{3}\\\hline \mathbf{3}\\\cline{1-1}\end{array} \qquad \xi_2 \begin{array}{|c|c|}\hline 1&\mathbf{3}\\\hline \mathbf{2}\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&3\\\hline \mathbf{2}\\\cline{1-1}\end{array}$$

- C_5 crystal KN$((2,2,2,1,0),5)$:

$$T = \begin{array}{|c|c|}\hline 1&1\\\hline \mathbf{3}&\bar{5}\\\hline \bar{4}&\bar{\mathbf{3}}\\\hline \bar{\mathbf{3}}\\\cline{1-1}\end{array} \xrightarrow{2} \begin{array}{|c|c|}\hline 1&1\\\hline \mathbf{3}&\bar{5}\\\hline \bar{4}&\bar{\mathbf{3}}\\\hline \bar{\mathbf{2}}\\\cline{1-1}\end{array} = \xi_2^C(T)$$

The Weyl group and the $J_{\mathfrak{g}}$-cactus action on a normal $\mathfrak{g}$-crystal

Theorem

Halacheva, 2016 (Henriques–Kamnitzer $\mathfrak{g} = \mathfrak{gl}_n$, 2006) The map $s_J \mapsto \xi_J$, for all $J \subseteq I$ connected Dynkin sub-diagrams of I, defines an action of the cactus group $J_{\mathfrak{g}}$ on the set $B(\lambda)$; that is, the involutions ξ_J in B satisfy the $J_{\mathfrak{g}}$ cactus relations, and the following is a group homomorphism

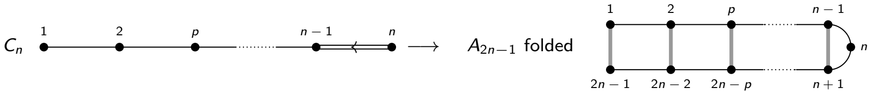
$$\begin{array}{rccc} \Phi_{\mathfrak{g}}: & J_{\mathfrak{g}} & \to & \mathfrak{S}_B \\ & s_J & \mapsto & \xi_J. \end{array}$$

In addition, when $J = \{i\}$, the ξ_i satisfy the Weyl group $W_{\mathfrak{g}}$ relations.

- On $SSYT(\lambda, n)$, ξ_J is realized by J-partial Benkart-Sottile-Stroomer-reversal,1999.
- On $KN(\lambda, n)$, ξ_J, $J = [p, n]$, is realized by the colourful J-partial symplectic reversal, a generalization of Santos' symplectic evacuation for symplectic skew-tableaux, A.-Tarighat-Torres, 2022.

Baker, 2006, virtualization of KN tableau crystals

- Dynkin diagram folding $C_n \hookrightarrow A_{2n-1}$



- Baker virtualization is an injective map

$$\mathsf{E} : \mathsf{KN}(\lambda, n) \hookrightarrow \mathsf{SSYT}(\lambda^A, 2n)$$

$$n = 5, \quad T = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \bar{5} \\ \hline \bar{4} & \bar{3} \\ \hline \bar{3} \\ \cline{1-1} \end{array} \quad \mapsto \quad \mathsf{E}(T) = [\emptyset \leftarrow w(\psi(C_2)) \leftarrow w(\psi(C_1))] = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 4 & 6 \\ \hline 3 & 6 & 7 & 8 \\ \hline 5 & 7 & 9 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} 7 & 9 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$$

such that $\mathsf{E}(\mathsf{K}(\lambda, n))$ has crystal structure with $f_i^E = f_i^A f_{2n-i}^A$, $i < n$, and $f_n^E = (f_n^A)^2$, isomorphic to $\mathsf{K}(\lambda, n)$ such that

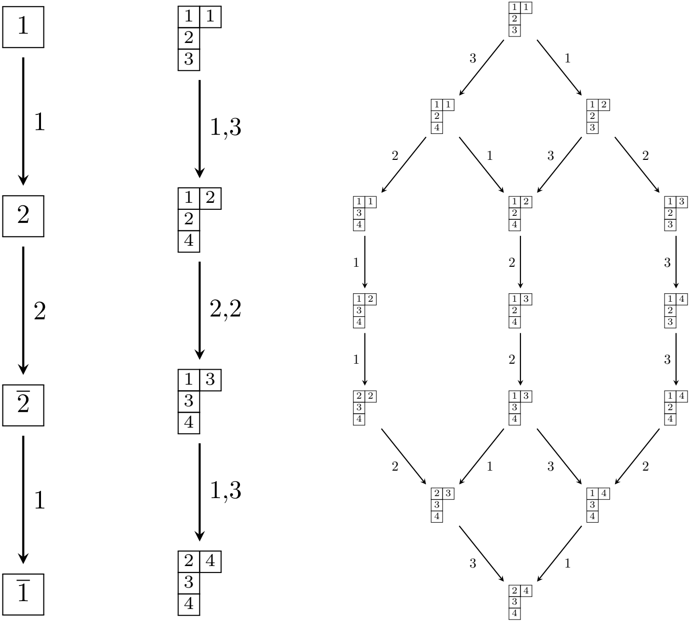
$$\mathsf{E} f_i(T) = f_i^E \mathsf{E}(T), \text{ for } T \in \mathsf{KN}(\lambda, n), 1 \leq i \leq n.$$

- Robinson-Schensted-Knuth correspondence (RSK): $(\mathsf{E}(T), Q_\lambda) = \mathrm{RSK} \circ \psi(T) = (P(w_T), Q_\lambda)$ and

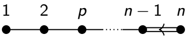
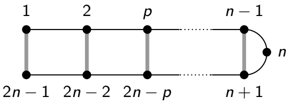
$$\mathsf{E}^{-1} = \psi^{-1} \mathrm{RSK}^{-1}_{|\mathsf{E}(\mathsf{K}(\lambda,n)) \times \{Q_\lambda\}}$$

where $\mathrm{RSK}^{-1}_{|\mathsf{K}(\lambda,n) \times \{Q_\lambda\}}$ denotes the inverse of RSK restricted to $\mathsf{E}(\mathsf{K}(\lambda, n)) \times \{Q_\lambda\}$.

$KN((1),2)$ embedded in $SSYT((2,1,1),4)$



Virtualization of the symplectic cactus action on KN tableau crystals

- C_n  $\longrightarrow$ A_{2n-1} folded 

The virtualization map E behaves very nicely with respect to Levi restriction!

$$\begin{array}{lcll} \mathrm{KN}_{[1,p]}(\lambda, n) & \underset{E}{\hookrightarrow} & \mathrm{SSYT}_{[1,p]\sqcup[2n-p,2n-1]}(\lambda^A, 2n), & p < n, \\ \mathrm{KN}_{[p,n]}(\lambda, n) & \underset{E}{\hookrightarrow} & \mathrm{SSYT}_{[p,2n-p]}(\lambda^A, 2n), & p \leq n \end{array}$$

$$\begin{array}{ccc} \mathrm{KN}(\lambda, n) & \xrightarrow{\ \mathrm{E}\ } & \mathrm{SSYT}(\lambda^A, 2n) \\ \xi^{C_n}_{[p,n]} \Big\downarrow \xi^{C_n}_{[1,p]} & & \xi^{A_{2n-1}}_{[p,2n-p]} \Big\downarrow \xi^{A_{2n-1}}_{[1,p]}\, \xi^{A_{2n-1}}_{[2n-p,2n-1]} \\ \mathrm{KN}(\lambda, n) & \xrightarrow[\ \mathrm{E}\ ]{} & \mathrm{SSYT}(\lambda^A, 2n) \end{array}$$

- $E\xi^{C_n}_{[1,p]}(T) = \xi^{A_{2n-1}}_{[1,p]}\xi^{A_{2n-1}}_{[2n-p,2n-1]}E(T),\ T \in \mathrm{KN}(\lambda, n).$

- Virtualization of the symplectic cactus action of $J_{\mathfrak{sp}(2n)}$ on the crystal $\mathsf{KN}(\lambda, n)$

$$\begin{array}{ccc} J_{\mathfrak{sp}(2n)} & \xrightarrow{\Phi_{\mathfrak{sp}(2n,\mathbb{C})}} & \mathfrak{S}_{\mathsf{KN}(\lambda,n)} \\ \tilde{\imath}\downarrow & & \imath\downarrow \\ \tilde{J}_{2n} & \xrightarrow[\widetilde{\Phi}^{E}_{\mathfrak{gl}(2n,\mathbb{C})}]{} & \mathfrak{S}_{E(\mathsf{KN}(\lambda,n))} \end{array} \qquad \widetilde{\Phi}^{E}_{\mathfrak{gl}(2n,\mathbb{C})}\tilde{\imath} = \imath\Phi_{\mathfrak{sp}(2n,\mathbb{C})}$$

- $\xi^A_{[1,p]}\xi^A_{[2n-p,2n-1]}$, $1 \le p < n$, and $\xi^A_{n-p,2n-p}$, $1 \le p \le n$ satisfy the $\tilde{J}_{2n}$ relations.
- $\xi^A_{[p]}\xi^A_{[2n-p]}$, $1 \le p < n$, and ξ^A_n satisfy the the Weyl group B_n relations

$$(\xi^A_{n-1}\xi^A_{n+1}\xi^A_n)^4 = 1.$$

The Berenstein–Kirillov group

The *Berenstein–Kirillov group* $\mathcal{BK}$ (*Gelfand-Tsetlin group*) [Berenstein, Kirillov, 1995], is the free group generated by the Bender-Knuth involutions t_i, for $i > 0$, modulo the relations they satisfy on `straight shaped semistandard Young tableaux`.

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline & & 2 & 2 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array} \xrightarrow{t_2} \begin{array}{|c|c|c|c|c|c|c|}\hline & & 2 & 2 & 3 & 3 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}$$

$$\xi_1 \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} = t_1 \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \xi_2 \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \neq t_2 \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Proposition

[Berenstein–Kirillov, 1995] *Let $\mathcal{BK}_n$ be the subgroup of $\mathcal{BK}$ generated by $t_1, \dots, t_{n-1}$.*

- *The elements $q_{[1,1]}, \dots, q_{[1,n-1]}$ are generators of $\mathcal{BK}_n$, $q_{[1,i]} = \xi_{[1,i]}$, $i \geq 1$.*
- $t_1 = q_{[1,1]}$, $\quad t_i = q_{[1,i-1]} q_{[1,i]} q_{[1,i-1]} q_{[1,i-2]}$, *for $i \geq 2$,* $\quad q_{[1,0]} := 1$.

- The following are group epimorphisms from J_n to $\mathcal{BK}_n$.
  1. $s_{[i,j]} \mapsto q_{[i,j]}$ [Chmutov–Glick–Pylyavskii 2016, 2020].
  2. $s_{[1,j]} \mapsto q_{[1,j]}$ [Halacheva 2016, 2020].

  The group $\mathcal{BK}_n$ is isomorphic to a quotient of J_n.

$$\mathcal{BK}_n \cong J_n / Ker$$

The known relations for the $\mathcal{BK}_n$ group

$$
\begin{aligned}
t_i^2 &= 1, \text{ for } i \geq 1, \\
t_i t_j &= t_j t_i, \text{ for } |i-j| > 1, \\
(t_1 q_{[1,i]})^4 &= 1, \text{ for } i > 2, \\
(t_1 t_2)^6 &= 1, \\
(t_i q_{[j,k-1]})^2 &= 1, \text{ for } i+1 < j < k,
\end{aligned}
$$

where

$$
\begin{aligned}
q_{[1,i]} = \xi_{[1,i]} &= t_1(t_2 t_1)\cdots(t_i t_{i-1}\cdots t_1), \text{ for } i \geq 1, \\
q_{[j,k-1]} &:= q_{[1,k-1]} q_{[1,k-j]} q_{[1,k-1]}, \text{ for } j < k.
\end{aligned}
$$

The type C_n Berenstein–Kirillov group $\mathcal{BK}^{C_n}$

Definition (A–Tarighat–Torres 2022)

The *symplectic Berenstein–Kirillov group* $\mathcal{BK}^{C_n}$, $n \geq 1$, is the free group generated by the $2n-1$ symplectic partial Schützenberger-Lusztig involutions

$$q^C_{[1,i]} =: \xi^{C_n}_{[1,i]},\ 1 \leq i < n, \ \text{ and } \ q^C_{[i,n]} =: \xi^{C_n}_{[i,n]},\ 1 \leq i \leq n,$$

on straight shaped KN tableaux on the alphabet $[\pm n]$ modulo the relations they satisfy on those tableaux.

- [A–Tarighat–Torres 2022] The following is a group epimorphism from $J_{\mathfrak{sp}_{2n}}$ to $\mathcal{BK}^{C_n}$:

$$s_{[1,j]} \mapsto q^{C_n}_{[1,j]},\ 1 \leq j < n, \qquad s_{[j,n]} \mapsto q^C_{[j,n]},\ 1 \leq j \leq n.$$

$\mathcal{BK}^{C_n}$ is isomorphic to a quotient of $J_{\mathfrak{sp}_{2n}}$.

The symplectic Bender Knuth involutions

- [A–Tarighat–Torres 2022] For $n \geq 1$, the *symplectic Bender–Knuth involutions* $t_i^{C_n}$, $1 \leq i \leq 2n-1$, on straight shaped KN tableaux on the alphabet $[\pm n]$, are defined as

$$t_i^{C_n} := q_{[1,i-1]}^{C_n} q_{[1,i]}^{C_n} q_{[1,i-1]}^{C_n} q_{[1,i-2]}^{C_n} = E^{-1} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} E, \quad 1 \leq i \leq n-1,$$

$$\tilde{t}_{2n-i}^{A_{2n-1}} := q_{[1,2n-1]}^{A_{2n-1}} t_i^{A_{2n-1}} q_{[1,2n-1]}^{A_{2n-1}} \quad 1 \leq i \leq n-1,$$

$$t_{n-1+i}^{C_n} := q_{[n-i+1,n]}^{C_n} q_{[n-i+2,n]}^{C_n} = E^{-1} q_{[n-(i-1),n+(i-1)]}^{A_{2n-1}} q_{[n-(i-2),n+(i-2)]}^{A_{2n-1}} E, \quad 1 \leq i \leq n.$$

  The symplectic Bender-Knuth involutions $t_i^{C_n}$, $1 \leq i \leq 2n-1$ also generate $\mathcal{BK}^{C_n}$.

- $q_{[1,n-1]}^{C_n} = t_1^{C_n}(t_2^{C_n} t_1^{C_n}) \cdots (t_{n-1}^{C_n} t_{n-2}^{C_n} \cdots t_1^{C_n}), \qquad q_{[1,n]}^{C_n} = t_{2n-1}^{C_n} t_{2n-2}^{C_n} \cdots t_n^{C_n}.$

The virtual symplectic Bender Knuth involutions

$$Et_i^{C_n} = t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} E, \quad 1 \leq i \leq n-1,$$

$$\tilde{t}_{2n-i}^{A_{2n-1}} = \mathrm{evac}^A t_i^{A_{2n-1}} \mathrm{evac}, \quad 1 \leq i \leq n-1,$$

$$Et_{n-1+i}^{C_n} = q_{[n-(i-1),n+(i-1)]}^{A_{2n-1}} q_{[n-(i-2),n+(i-2)]}^{A_{2n-1}} E, \quad 1 \leq i \leq n.$$

Example

$$n = 5, \; T = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \overline{5} \\ \hline \overline{4} & \overline{3} \\ \hline \overline{3} \\ \cline{1-1}\end{array} \text{ and } E(T) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 4 & 6 \\ \hline 3 & 6 & 7 & 8 \\ \hline 5 & 7 & 9 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} 7 & 9 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}, \text{ and } \tilde{t}_7 E(T) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 4 & 6 \\ \hline 3 & 6 & \mathbf{7} & \mathbf{9} \\ \hline 5 & \mathbf{7} & \mathbf{8} \\ \cline{1-3} 6 & \mathbf{8} \\ \cline{1-2} \mathbf{7} & \mathbf{9} \\ \cline{1-2} \mathbf{8} \\ \cline{1-1}\end{array},$$

$$t_3^A \tilde{t}_7 E(T) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & \mathbf{3} & 6 \\ \hline \mathbf{4} & 6 & \mathbf{7} & \mathbf{9} \\ \hline 5 & \mathbf{7} & \mathbf{8} \\ \cline{1-3} 6 & \mathbf{8} \\ \cline{1-2} \mathbf{7} & \mathbf{9} \\ \cline{1-2} \mathbf{8} \\ \cline{1-1}\end{array}$$

$$Et_3^C(T) = t_3^A \tilde{t}_7 E(T) \neq E\xi_3^C(T) = \xi_3^A \xi_7^A E(T) = E(T)$$

Relations satisfied by $\mathcal{BK}^{C_n}$

Proposition (A–Tarighat–Torres 2022)

The symplectic Bender–Knuth involutions $t_i^{C_n} = 1$, $i = 1, \dots, 2n-1$, *satisfy the following relations:*

1. $(t_i^{C_n})^2 = 1$, $i = 1, \dots, 2n-1$.
2. $(t_{n+i-1}^{C_n} t_{n+j-1}^{C_n})^2 = 1$, $1 \le i, j \le n$.
3. $(t_i^{C_n} t_j^{C_n})^2 = 1$, $|i-j| > 1$, $1 \le i, j < n$.
4. $(t_i^{C_n} t_{n+j-1}^{C_n})^2 = 1$, $i < n-j$.
5. $(t_i^{C_n} q_{[j,k-1]}^{C_n})^2 = 1$, $i+1 < j < k \le n$.
6. $(t_i^{C_n} q_{[j,n]}^{C_n})^2 = 1$, $i+1 < j \le n$.
7. $(t_{n+i-1}^{C_n} q_{[j,n]}^{C_n})^2 = 1$, $1 \le i, j \le n$.
8. $(t_{n+i-1}^{C_n} q_{[j,k-1]}^{C_n})^2 = 1$, $n-i+1 < j < k \le n$.
9. $(t_1^{C_n} t_2^{C_n})^6 = 1$, $n \ge 3$.
10. $(t_{n-1}^{C_n} \cdots t_2^{C_n} t_1^{C_n} t_2^{C_n} \cdots t_{n-1}^{C_n} t_n^{C_n})^4 = 1$.