Right and left symplectic keys, virtual keys and applications

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Basics

- Let $G = GL(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$.
- Fix $T \subseteq B \subseteq G$, T a maximal torus of G, B a Borel subgroup of G.
  B^- the opposite Borel subgroup to B, the unique Borel subgroup of G such that $B \cap B^- = T$.
- Example: $G = GL_n(\mathbb{C})$:
  T the subgroup of diagonal matrices.
  B the subgroup of upper triangular matrices.
  B^- the subgroup of lower triangular matrices.
- Let the Lie algebras of G and B be $\mathfrak{g}$ respectively $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$.
  Let $V(\lambda)$ be the irreducible G-module with highest weight λ a partition with at most n parts.
- Let W be the Weyl group of G, and $w \in W/W_\lambda \leftrightarrow W^\lambda$,
  The Demazure module (opposite) $V_w(\lambda) \subseteq V(\lambda)$ ($V^w(\lambda) \subseteq V(\lambda)$) is the B-submodule (B^--submodule)

$$V_w(\lambda) = \mathcal{U}(\mathfrak{b}).V(\lambda)_{w\lambda}, \quad V^w(\lambda) = \mathcal{U}(\mathfrak{b}^-).V(\lambda)_{w\lambda}.$$

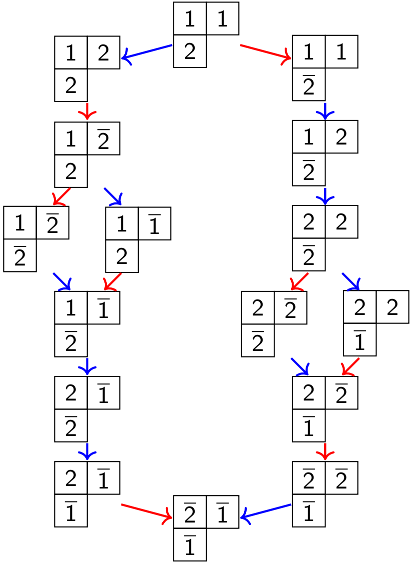
  $V(\lambda)_{w\lambda}$ is the one dimensional weight space of $V(\lambda)$ with extremal weight $w\lambda$.
- $V_{w_0}(\lambda) = V(\lambda) = V^e$, w_0 the longest element of W. If $G = Sp(2n, \mathbb{C})$, $w_0 = -Id$.

Kashiwara crystal and Demazure crystal

- The Kashiwara crystal $B(\lambda)$ is a combinatorial skeleton for the G-module $V(\lambda)$.
- Demazure characters are the characters of the B-submodules $V_w(\lambda)$.
- **Kashiwara, Littelmann 90's.** Demazure characters are generated by certain subsets $B_w(\lambda)$, $w \in W/W_\lambda$, in the crystal $B(\lambda)$, called Demazure crystals.
- For $w \in W/W_\lambda$, $B_w(\lambda)$ is a combinatorial skeleton of the Demazure module $V_w(\lambda)$.

- **Question**: Given $w \in W/W_\lambda$ and $b \in B(\lambda)$, how to check whether b is in the Demazure crystal $B_w(\lambda)$?

Symplectic C_2 crystal $B((2,1))$: $1 < 2 < \bar{2} < \bar{1}$



The type C_2 crystal graph $\mathcal{KN}((2,1),2)$ containing the A_1 crystal $\mathcal{SSYT}((2,1),2)$, consisting of the two top left most tableaux, as a subcrystal. The type C_2 lowering crystal operators are f_1, $\rightarrow$, and f_2, $\rightarrow$.

Demazure crystal and its opposite

- For $w = s_{i_\ell} \cdots s_{i_1} \in W^\lambda$ a reduced word in the Bruhat order $\le$ in W, Demazure crystal $B_w(\lambda) \subseteq B(\lambda)$

$$B_w(\lambda) := \{f_{i_\ell}^{k_\ell} \cdots f_{i_1}^{k_1}(b_\lambda) \mid (k_\ell, \dots, k_1) \in \mathbb{Z}_{\ge 0}^\ell\} \setminus \{0\},$$

  opposite Demazure crystal

$$B^{w_0 w}(\lambda) := \{e_{\theta(i_\ell)}^{k_\ell} \cdots e_{\theta(i_1)}^{k_1}(b_{w_0\lambda}) \mid (k_\ell, \dots, k_1) \in \mathbb{Z}_{\ge 0}^\ell\} \setminus \{0\} = \iota B_w(\lambda)$$

$$B^w(\lambda) = \iota(B_{w_0 w}(\lambda)), \quad \iota \text{ Lusztig-Schützenberger involution}$$

$$\theta \text{ Dynkin diagram automorphism.}$$

  $B_e(\lambda) = \{b_\lambda\} = B^{w_0}$, $B_{w_0}(\lambda) = B^e(\lambda)$.

- For $\rho \le w$ in W^λ, $B_\rho(\lambda) \subseteq B_w(\lambda) \Leftrightarrow B^\rho(\lambda) \supseteq B^w(\lambda)$
  Demazure crystal atom $\overline{B}_w(\lambda)$ and *opposite Demazure crystal atom* $\overline{B}^w(\lambda)$

$$\overline{B}_w(\lambda) = B_w(\lambda) \setminus \bigsqcup_{\substack{\rho \in W^\lambda \\ \rho < w}} B_\rho(\lambda) \qquad \overline{B}^w(\lambda) = \iota(\overline{B}_{w_0 w}(\lambda)).$$

- Decomposition into Demazure and opposite Demazure atoms

$$B_w(\lambda) = \bigsqcup_{\substack{\rho \in W^\lambda \\ \rho \le w}} \overline{B}_\rho(\lambda) \qquad B^w(\lambda) = \bigsqcup_{\substack{\rho \in W^\lambda \\ \rho \ge w}} \overline{B}^\rho(\lambda).$$

Schubert varieties and Demazure crystals

- G a simply-connected semisimple algebraic group over $\mathbb{C}$.
- Bruhat decomposition of G and (full) flag variety in G.
  The Bruhat decomposition describes the $B \times B$, respectively $B^- \times B$ orbits in G and are parameterized by W
  $$G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} B^- wB.$$
- $G/B = \{gB : g \in G\}$ the (full) flag variety in G.
- $$G/B = \bigsqcup_{w \in W} BwB/B = \bigsqcup_{w \in W} B^- wB/B.$$
- The Schubert cell C_w is $C_w = BwB/B = B\dot{w}$.
- The opposite Schubert cell C^w is $C^w = w_0 C_{w_0 w} = B^- wB/B = B^- \dot{w}$.
- The Schubert variety X_w, respectively the opposite Schubert variety X^w, in G/B
  $$X_w = \bigsqcup_{v \leq w} C_v, \;\; X^w = \bigsqcup_{u \geq w} C^u = w_0 X_{w_0 w} \subseteq G/B.$$

Relations among Schubert varieties/Demazure crystals

- For $w, w' \in W$,

$$X_w \subseteq X_{w'} \Leftrightarrow w \leq w' \Leftrightarrow X^w \supseteq X^{w'}.$$

- The Richardson variety X_α^β in G/B corresponding to the pair (α, β), $\alpha, \beta \in W$, is the (set theoretic) intersection

$$X_\alpha^\beta := X_\alpha \cap X^\beta = \bigsqcup_{\beta \leq v' \leq u' \leq \alpha} C_{u'} \cap C^{v'} \neq \emptyset \Leftrightarrow \beta \leq \alpha.$$

Let $u, v, x, y \in W^\lambda$ and $b \in B(\lambda)$. Then

1. $B_x(\lambda) \subseteq B_y(\lambda) \Leftrightarrow B^x(\lambda) \supseteq B^y(\lambda) \Leftrightarrow x \leq y$.
2. $B^u(\lambda) \cap B_v(\lambda) \neq \emptyset \Leftrightarrow u \leq v$.

Borel-Weil theorem, Demazure modules and Schubert varieties

- Let $\mathfrak{g}$ be the Lie algebra of G. Let $V(\lambda)$ be the irreducible highest weight G-module over $\mathbb{C}$ with highest weight λ, and let $B(\lambda)$ its combinatorial skeleton.
- Let L_λ be a line bundle on the flag variety G/B.
- By the Borel-Weil theorem the space $H^0(G/B, L_\lambda)$ of global sections is a G-module isomorphic to $V(\lambda)^*$ the dual of $V(\lambda)$,

$$H^0(G/B, L_\lambda) \simeq V(\lambda)^* = V(-w_0\lambda).$$

$$H^0(X_w, L_\lambda) \simeq V_w(\lambda)^* = V_w(-w_0\lambda) \quad H^0(X^w, L_\lambda) \simeq V^w(\lambda)^* = V^w(-w_0\lambda).$$

- Kashiwara constructed a specific $\mathbb{C}$-basis of $H^0(G/B, L_\lambda)$ via the quantized enveloping algebra associated to $\mathfrak{g}$, specialized at $q = 1$. This $\mathbb{C}$-basis, $\{G_\lambda^{up}(b) : b \in B(\lambda)\}$ the upper global basis (specialized at $q = 1$) is compatible with Schubert varieties $\{G_\lambda^{up}(b) : b \in B_w(\lambda)\}$ and opposite Schubert varieties $\{G_\lambda^{up}(b) : b \in B^w(\lambda)\}$.
- Associated to the combinatorial path model given by the crystal $\mathbf{B}(\lambda)$ of LS paths of shape λ, Lakshmibai and Littelmann constructed a basis for $H^0(G/B, L_\lambda)$ compatible with Schubert varieties and opposite Schubert varieties and their intersections, satisfying certain quadratic relations similar to the quadratic straightening relations.

Crystal of LS paths

- If $\pi = (\tau, \mathbf{a})$ is an LS path of shape λ, the sequence $\tau = (\tau_0, \ldots, \tau_r)$ is strictly decreasing in W^λ.
  The initial direction of the path π is $i(\pi) = \tau_0$, and
  the ending direction of the path π is $e(\pi) = \tau_r$.

- The LS path π in $\mathbf{B}(\lambda)$ is said to be *standard* on a Richardson variety $X_\tau^\kappa = X^\kappa \cap X_\tau$, $\kappa, \tau \in W^\lambda$ if
  $\kappa \leq e(\pi) \leq i(\pi) \leq \tau$ in the Bruhat order in W^λ

- $\mathbf{B}^\kappa(\lambda) = \{\pi \in \mathbf{B}(\lambda) : e(\pi) \geq \kappa\}$, opposite Demazure crystal, is the set of all L–S paths of shape λ, *standard* on the opposite Schubert variety X^κ.

- $\mathbf{B}_\tau(\lambda) = \{\pi \in \mathbf{B}(\lambda) : i(\pi) \leq \tau\}$, Demazure crystal, is the set of all L–S paths of shape λ, *standard* on the Schubert variety X_τ.

Demazure keys: Dilatation of crystals

- For any positive integer m, there exists a unique embedding of crystals

$$\psi_m : B(\lambda) \hookrightarrow B(m\lambda)$$

such that for $b \in B(\lambda)$ and any path $b = f_{i_1} \cdots f_{i_l}(b_\lambda)$ in $B(\lambda)$, we have

$$\psi_m(b) = f_{i_1}^m \cdots f_{i_l}^m(b_{m\lambda}).$$

- $b_\lambda^{\otimes m}$ is of highest weight $m\lambda$ in $B(\lambda)^{\otimes m} \Rightarrow B(b_\lambda^{\otimes m})$ is a realization of $B(m\lambda)$ in $B(\lambda)^{\otimes m}$ with highest weight vertex $b_\lambda^{\otimes m}$.
- This gives a canonical embedding

$$\theta_m : \begin{cases} B(b_\lambda) \hookrightarrow B(b_\lambda^{\otimes m}) \subset B(b_\lambda)^{\otimes m} \\ \quad b \longmapsto b_1 \otimes \cdots \otimes b_m \end{cases}$$

with important properties.

- For $\sigma \in W^\lambda$, $\theta_m(b_{\sigma\lambda}) = b_{\sigma\lambda}^{\otimes m}$.
- When m has sufficiently many factors, there exist elements $\sigma_1, \ldots, \sigma_m$ in W^λ such that $\theta_m(b) = \mathbf{b}_{\sigma_1}\lambda \otimes \cdots \otimes \mathbf{b}_{\sigma_m}\lambda$ tensor product of keys.
  - the elements $\mathbf{b}_{\sigma_1\lambda}$ and $\mathbf{b}_{\sigma_m\lambda}$ in $\theta_m(b)$ do not depend on m:
  - up to repetition, the sequence $(\sigma_1\lambda, \ldots, \sigma_m\lambda)$ in $\theta_m(b)$ does not depend on the realization of the crystal $B(\lambda)$ and we have $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$.

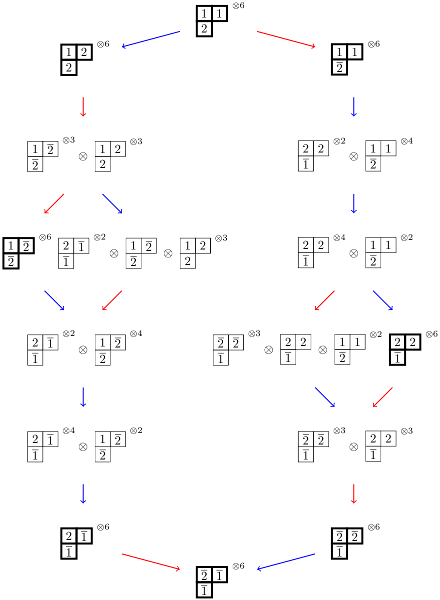
Demazure keys:right and left

- $O(\lambda) = \{f_{i_r}^{\max} \cdots f_{i_1}^{\max}(b(\lambda)) \mid i_1, \ldots, i_r \in [n],\ r \geq 0\}$.
- $O(\lambda) = \{b_{\sigma\lambda} : \sigma \in W^{\lambda}\}$ the set of keys in $B(\lambda)$.

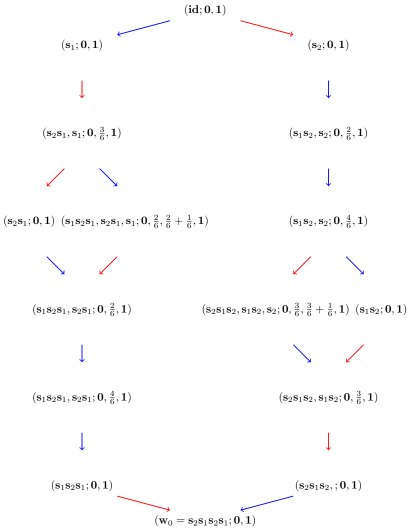
- For $b \in B(\lambda)$ and $\theta_m(b) = b_{\sigma_1\lambda} \otimes \cdots \otimes b_{\sigma_m\lambda}$:
  - The right key $K^+(b)$ and left key $K^-(b)$ of b are defined as follows:

$$K^+(b) = b_{\sigma_1\lambda} \text{ and } K^-(b) = b_{\sigma_m\lambda}.$$

    In particular, $K^+(b_{\sigma\lambda}) = K^-(b_{\sigma\lambda}) = b_{\sigma\lambda}$ for any $\sigma \in W^{\lambda}$.
  - $K^-(b) \leq K^+(b)$ for any $b \in B(\lambda)$, and
  - $K^-(b) = K^+(b)$ if and only if b is in $O(\lambda) = \{b_{\sigma\lambda} : \sigma \in W^{\lambda}\}$,

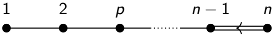


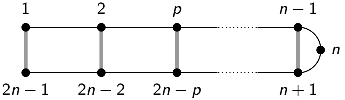
The dilatation of the crystal KN((2, 1), 2) , by $m = 6$, the least common multiple of the maximal i-string lengths, inside $\mathrm{KN}((12, 6), 2) \simeq B(K(2, 1)^{\otimes 6}, 2)$, exhibiting the right and left keys of each vertex of KN((2, 1), 2) as the leftmost respectively rightmost factor in each 6-fold tensor product of keys



The crystal $\mathbf{B}(2, 1)$ of Lakshmibai-Seshadri, L-S, paths, of shape $\lambda = \Lambda_2 + \Lambda_1 = (2, 1)$ obtained from the dilatation of the C_2 crystal $B(2, 1)$. The C_2 Weyl group $B_2 =< s_1, s_2 | s_1^2 = s_2^2 = 1, \ (s_1 s_2)^4 = 1 >$ with long element $\mathbf{w}_0 = s_2 s_1 s_2 s_1$.

Virtual symplectic keys

C_n 

$\longrightarrow$ A_{2n-1} folded 

W^C = hyperoctahedral group B_n embedded as a subgroup of $\mathfrak{S}_{2n} = \langle s_i^A : 1 \leq i < 2n \rangle$.
$W^C = B_n \simeq B_n^A := \langle \tilde{s}_i := s_i^A s_{2n-i}^A, \tilde{s}_n := s_n^A, 1 \leq i < n \rangle$ as a subgroup of $\mathfrak{S}_{2n}$.

- Baker crystal virtualization

$$\psi : \mathrm{KN}(\Lambda_i, n) \hookrightarrow \mathrm{SSYT}(\Lambda_{2n-i}^A + \Lambda_i^A, 2n) \subseteq SSYT(\Lambda_i^A, 2n) \otimes SSYT(\Lambda_{2n-i}^A, 2n)$$

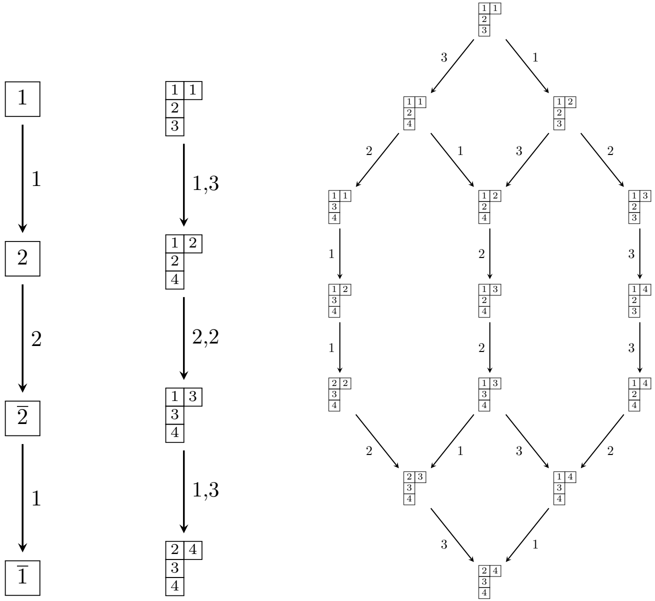
$$C = f_{i_1} \cdots f_{i_k}(\Lambda_i) \mapsto \psi(C) = f_{i_1}^A f_{2n-i_1}^A \cdots f_{i_k}^A f_{2n-i_k}^A (\Lambda_{2n-i}^A + \Lambda_i^A)$$

$$\mathrm{E} : \mathrm{KN}(\lambda, n) \longrightarrow \mathrm{SSYT}(\lambda^A, 2n) \subseteq \bigotimes_{i=1}^{k} SSYT(\Lambda_i^A + \Lambda_{2n-i}^A, 2n)$$

$$T = C_k \otimes \cdots \otimes C_1 \mapsto \mathrm{E}(T) = \psi(C_k) \otimes \cdots \otimes \psi(C_1)$$
$$= [\emptyset \leftarrow w(\psi(C_k)) \leftarrow \cdots \leftarrow w(\psi(C_1))]$$

- Embedding of symplectic keys into $\mathfrak{sl}_{2n}$ keys

$$O(\lambda) \quad \rightarrow \quad E(O(\lambda)) = O_{B_n}(\lambda^A) \subseteq O_{\mathfrak{S}_{2n}}(\lambda^A)$$
$$K(u) \quad \mapsto \quad K(u^A)$$



$$\psi : \mathsf{KN}(\square, 2) \hookrightarrow \mathsf{SSYT}(\Lambda_1^A \otimes \Lambda_3^A, 4) \subseteq \mathsf{SSYT}(\Lambda_1^A, 4) \otimes \mathsf{SSYT}(\Lambda_3^A, 4)$$

Baker crystal embedding and crystal dilatation commute

Proposition

For any positive integer $m > 0$*, the injections* θ_m *and* E *commute in* $KN(\lambda, n)$*:* $\theta_m E = E\theta_m$.

$$\begin{array}{ccc} KN(\lambda, n) & \xrightarrow{\theta_m} & KN(m\lambda, n) \\ \downarrow E & & \downarrow E \\ SSYT(\lambda^A, 2n) & \xrightarrow[\theta_m]{} & SSYT(m\lambda^A, 2n) \end{array}$$

Embedding of symplectic Demazure atoms and opposite into $\mathfrak{sl}_{2n}$ ones

Theorem

(a) *Let m be a positive integer such that, for each $T \in KN(\lambda, n)$, the m-dilatation map θ_m on $KN(\lambda, n)$ gives $\theta_m(T) = K(\sigma_1\lambda) \otimes \cdots \otimes K(\sigma_m\lambda)$ for some $\sigma_1 \geq \cdots \geq \sigma_m$ in B_n^λ. Then*

$$\theta_m(E(T)) = E(\theta_m(T)) = E(K(\sigma_1\lambda))\otimes\cdots\otimes E(K(\sigma_m\lambda)) = K(\tilde{\sigma}_1\lambda^A)\otimes\cdots\otimes K(\tilde{\sigma}_m\lambda^A),$$

where $E(K(\sigma_i\lambda)) = K(\tilde{\sigma}_i\lambda^A) \in O_{B_n^A}(\lambda^A)$, $1 \leq i \leq m$.

(b) *$E(K^+(T)) = K^+(E(T))$ and $E(K^-(T)) = K^-(E(T))$ for all $T \in KN(\lambda, n)$.*

(b) *For $\sigma \in W^\lambda$, $E(\overline{B}_\sigma(\lambda)) \hookrightarrow \overline{B}_{\tilde{\sigma}}(\lambda^A)$ and $E(\overline{B}^\sigma(\lambda)) \hookrightarrow \overline{B}^{\tilde{\sigma}}(\lambda^A)$.*

Example

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \bar{5} \\ \hline \bar{4} & \bar{3} \\ \hline \bar{3} \\ \cline{1-1}\end{array}, \quad \mathrm{wt}(T) = (1, 1, -1, -1, -1).$$

$$\psi(C_2) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \bar{5} \\ \hline 4 & \bar{3} \\ \hline \bar{5} \\ \cline{1-1} \bar{4} \\ \cline{1-1} \bar{3} \\ \cline{1-1} \bar{1} \\ \cline{1-1}\end{array}, \psi(C_1) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 5 & \bar{4} \\ \hline \bar{5} & \bar{2} \\ \hline \bar{4} \\ \cline{1-1} \bar{3} \\ \cline{1-1}\end{array}$$

$$\Rightarrow E(T) = [\emptyset \leftarrow w(\psi(C_2)) \leftarrow w(\psi(C_1))] = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 4 & \bar{5} \\ \hline 3 & \bar{5} & \bar{4} & \bar{3} \\ \hline 5 & \bar{4} & \bar{1} \\ \cline{1-3} \bar{5} & \bar{3} \\ \cline{1-2} \bar{4} & \bar{2} \\ \cline{1-2} \bar{3} \\ \cline{1-1}\end{array}.$$

$$
Q_\lambda = \begin{array}{cccc}
1 & 4 & 11 & 15 \\
2 & 5 & 12 & 16 \\
3 & 6 & 13 & 17 \\
7 & 14 & 18 & \\
8 & 19 & & \\
9 & 20 & & \\
10 & & &
\end{array}
\Rightarrow \psi(C_1') = \begin{array}{cc}
2 & 2 \\
4 & \overline{5} \\
\overline{5} & \overline{3} \\
\overline{4} & \overline{1} \\
\overline{3} & \\
\overline{1} &
\end{array}, \quad \psi(C_2') = \begin{array}{cc}
1 & 2 \\
2 & \overline{5} \\
4 & \overline{3} \\
\overline{5} & \\
\overline{4} & \\
\overline{3} & \\
\overline{1} &
\end{array}
$$

$$
\Longrightarrow K_+(T) = C_1' C_2' = \begin{array}{cc}
2 & 2 \\
\overline{5} & \overline{5} \\
\overline{3} & \overline{3} \\
\overline{1} &
\end{array}
$$