Key polynomials, Smith invariants and an action of the symmetric group on skew-tableaux

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Abstract. Lascoux and Schützenberger have defined a permutation on a Young tableau to associate to each Knuth class a right and left key which they have used to give a combinatorial description of a key polynomial. Introducing variants of *jeu de taquin*, we extend this action of the symmetric group to non congruent frank words running over tableaux having the same shape and weight. As a dual translation, we obtain an action of the symmetric group on words congruent with key-tableaux defined by reflection crystal operators based on nonstandard pairing of parentheses. This construction arises naturally as a combinatorial description of the Smith invariants of certain sequences of product of matrices, over a local principal ideal domain, under the action of the symmetric group by place permutation.

1. Introduction

We give a combinatorial description of the hexagons defined by the sequences of Smith classes arising from certain type of product of matrices, over a local principal ideal domain, under the action of the symmetric group by place permutation, and we show its relationship with the combinatorics developed by Lascoux and Schützenberger to give a combinatorial description of key polynomials. Key polynomials are certain polynomials in $\mathbb{Z}[x_1x_2\ldots]$ indexed by compositions and were combinatorially investigated by Lascoux and Schützenberger, in the case of the symmetric group, in [**16, 17**] (see also [**21**])

$$k_m = \sum_{\substack{T \text{ tableau of shape } \beta(m) \\ K_+(T)\leq K(m)}} x^T,$$

where m is a composition, $\beta(m)$ its decreasing rearrangement, $K(m)$ the key-tableau of weight m, $K_+(T)$ the right key of the Knuth class of T, and x^T the monomial defined by the weight of T. More specifically, we consider the ordered set $P[n]$ of sets of positive integers in $[n]$, and describe operations on pairs of comparable elements of $P[n]$ which generalize to non congruent frank words the action of the symmetric group defined by *jeu de taquin* slides on two-column frank words within a Knuth class. This action of the symmetric group has a dual translation on words congruent with key-tableaux defined by reflection crystal operators based on nonstandard pairing of parentheses. Crystal operators are due to Lascoux and Schützenberger [**15, 19**], and are equivalent to the ones coming from the theory of crystal graphs in the work of Kashiwara and Nakashima [**20**]. In particular, reflection crystal operators define an action of the symmetric group on Young tableaux [**15, 19**].

Given an n by n non-singular matrix A, with entries in a local principal ideal domain with prime p, we write $A \sim \Delta_\alpha$ to mean that by Gaußian elimination one can reduce A to a diagonal matrix Δ_α with diagonal entries $p^{\alpha_1}, \ldots, p^{\alpha_n}$, for unique nonnegative integers $\alpha_1 \geq \ldots \geq \alpha_n$, called the *Smith normal form* of A. The sequence $p^{\alpha_1}, \ldots, p^{\alpha_n}$ defines the *invariant factors* of A, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ the *invariant partition* or *Smith invariant* of A. The *Smith class* of α is $\{A : A \sim \Delta_\alpha\}$. In particular, the Smith class of the null

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partition, with n entries, is the set of n by n unimodular matrices $\{U : U \sim I_n\}$ where I_n denotes the identity matrix of order n. Smith classes are therefore objects indexed by partitions.

It is known that α, β, γ are respectively the Smith invariants of n by n nonsingular matrices A, B, and C such that $AB = C$ if and only if there exists a Littlewood-Richardson tableau T of type (α, β, γ), that is, a tableau of shape γ/α which rectifies to the key-tableau of weight β (Yamanouchi tableau of weight β) [**7, 8**]. In other words,

$$\exists U \sim I_n,\ \Delta_\alpha U \Delta_\beta \sim \Delta_\gamma \text{ if and only if } \ c^{\gamma}_{\alpha,\beta} \neq 0,$$

where $c^{\gamma}_{\alpha,\beta}$ is the Littlewood-Richardson coefficient of type (α, β, γ). The relationship between the invariant factors of a product of matrices and the product of Schur functions was noticed earlier by several authors, with different approaches, as P. Hall, J. A. Green, T. Klein, R. C. Thompson, *et al* [**1, 2, 11, 13, 24**]. (For an overview and other interconnectedness, see the survey by W. Fulton [**8**] as well as [**7, 9, 10**].)

To provide a matrix explanation of our combinatorial construction given in section 3, theorems 3.6 and 3.8, we have to show how matrices and tableaux are related in this paper. We adopt the French convention (the Cartesian coordinate system) for partition diagrams and tableaux. Let the symmetric group S_t act by place permutation on weak compositions with at most t first non null entries, *via* the left action $s_i m = (m_1, \ldots, m_{i+1}, m_i, \ldots, m_t)$ with s_i, $1 \leq i \leq t-1$, the simple transpositions of S_t. Let $\beta(m)$ be the unique partition in the orbit $S_t m$ and $\beta'(m)$ its conjugate. $K(m)$ denotes the key-tableau of weight m, that is, the tableau of weight m whose *column-shape* is $\beta'(m)$, and $\Delta_{[m_k]} := \Delta_{(1^{m_k})}$ the n by n diagonal matrix having the ith diagonal entry equals p whenever $i \in [m_k]$ and 1 otherwise. Indeed there is an obvious bijection between compositions and key-tableaux since $K(m)$ is the tableau whose first m_j columns contain the letter j, for all j [**21**]. Thus, we identify $K(m)$ with the sequence of diagonal matrices $(\Delta_{[m_1]}, \ldots, \Delta_{[m_t]})$ in the sense that the nested sequence of partitions $(1^{m_1}) \subseteq (1^{m_1}) + (1^{m_2}) \subseteq \ldots \subseteq (1^{m_1}) + \ldots + (1^{m_t}) = \beta'(m)$ defines the key $K(m)$ and, simultaneously, are the Smith invariants of the sequence of product of matrices $\Delta_{[m_1]}$, $\Delta_{[m_1]}\Delta_{[m_2]}$, $\ldots$, $\Delta_{[m_1]}\Delta_{[m_2]} \cdots \Delta_{[m_t]} = \Delta_{\beta'(m)}$. For instance,

$$K(10325) = \begin{array}{ccccc} 5 & & & & \\ 4 & 5 & & & \\ 3 & 4 & 5 & & \\ 1 & 3 & 3 & 5 & 5 \end{array} \qquad \text{is identified with } (\Delta_{[1]}, \Delta_{\emptyset}, \Delta_{[3]}, \Delta_{[2]}, \Delta_{[5]}).$$

Let T be a skew-tableau of weight m. T defines the pair of words (w, J) where $w = w^1 \cdots w^n$ is the column-reading word of T with w^i the column-word comprising the i-th column of T, left to right, $i \geq 1$, and $J = J_t \cdots J_2 J_1$ is the indexing-set word of T with J_k the column-word of length m_k defined by the set of column-indices of the letter k in T, left to right $k \geq 1$. (We identify a column-word with its underlying set. Thus the length of J_k as a word is its cardinality $|J_k| = m_k$ as a set.) There is an obvious duality between the *jeu de taquin* on the two-column skew-tableaux defined by the pair $J_{k+1}J_k$, and the k-th reflection crystal operator on the word w, for every $1 \leq k \leq t-1$. In fact, the *overlap* of the pair $J_{k+1}J_k$, defined by the length of the second column of the Schensted tableau $P(J_{k+1}J_k)$ of $J_{k+1}J_k$ (equivalently, the maximum number of rows of size two among the two-column skew-tableaux with first column J_{k+1} and second column J_k), is equal to the number of standard k-pairs in w [**22**]; and by *jeu de taquin* slides on the two-column skew-tableau $J_{k+1}J_k$ with maximum overlapping, we may swap the cardinal of the non overlapped entries of J_{k+1} with those of J_k. Such a duality is described by the dual Robinson-Schensted-Knuth correspondence [**14, 6**], a bijection from the set of sequences of column words $\{J_{t-i+1}\}_{1 \leq i \leq t}$, to tableau-pairs (Q, P) of conjugate shapes, where $Q = P(J)$ is the column insertion of the word $J_t \ldots J_2 J_1$ starting from the right end, and P records the insertion of letters in the factor J_k by conjugate place the letter k in P. By symmetry of the dual RSK correspondence, the row insertion of $w^1 w^2 \ldots w^n$ produces $P = P(w)$, and Q records the letters in the factor w^i by conjugate place the letter i in Q (see [**6**], Appendix A.4.3, Proposition 5). Indeed J is a frank word of column-shape the non null parts of the reverse of m if and only if $P = K(m)$ (see also [**18**]).

Let U be a n by n unimodular matrix. Put $\Delta_\alpha U K(m)$ for the sequence Δ_α, $\Delta_\alpha U\ \Delta_{[m_1]}$, $\Delta_\alpha\ U\ \Delta_{[m_1]}$ $\Delta_{[m_2]}$, $\cdots$, $\Delta_\alpha U \Delta_{[m_1]}\Delta_{[m_2]} \cdots \Delta_{[m_t]} = \Delta_\alpha U \Delta_{\beta'(m)}$. The nested sequence of Smith invariants $\alpha^0 = \alpha \subseteq \alpha^1 \subseteq \ldots \subseteq \alpha^t = \gamma$ defined by this sequence of matrices is such that α^{i+1}/α^i is a vertical strip of length m_{i+1}, for $i = 0, 1, \ldots, t-1$. Thus $\Delta_\alpha U K(m)$ is identified with the tableau T of type (α', m, γ') with indexing-set words $J_{k+1} = \{i : \alpha^{k+1}_i = \alpha^k_i + 1\}$, $1 \leq k \leq t-1$. It is shown in [**5**] that the column-reading word of T satisfies $P(w) = K(m)$ and thus J is a frank word of shape the non null parts of the reverse of m.

When we consider the action of the symmetric group S_t by place permutation on weak compositions m with at most t first non null entries, we are simultaneously defining an action of the symmetric group on sequences of matrices $\Delta_\alpha UK(m)$, where U is a fixed unimodular matrix and α a fixed partition, and therefore on tableaux of skew-shape whose rectifications are the key-tableaux $K(m)$. Thus, we obtain two families of hexagons which are dual translation of each other: one over non congruent frank words running over tableaux of column-shape $\beta(m)$ with the same weight; and, the other one on words congruent with key-tableaux $K(m)$, not necessarily sharing the same $\mathcal{Q}$-symbol. However in the first hexagon there is only one tableau and we may associate to it the right and left keys defined in the usual way by the the first and last columns of the vertices of the hexagon.

The first hexagon is obtained from a particular shuffle decomposition of a three-column tableau into row-words of length ≤ 3 and certain tableaux of column-shape $(2,1,1)$, defining a *variant of the jeu de taquin* on a two-column tableau or contretableau. This means that the second hexagon is obtained from a shuffle decomposition of a three-letter Yamanouchi word into column-words 321, 21, 1, and words 3121, giving rise to a *nonstandard reflection crystal operator* defined by nonstandard pairing of parentheses. The hexagons on frank words can be seen as shuffles of elementary hexagons defined by the *jeu de taquin* Lascoux-Schützenberger operator restricted to the Knuth class of a row-word, and to the frank words in the Knuth class of a certain tableau of column-shape $(2,1,1)$. In particular, it contains the one generated by the *jeu de taquin* operator acting on the full tableau.

The paper is organized as follows. In the next section we analyze the duality between reflection crystal operators on words over a two-letter alphabet and operators defined by *jeu de taquin slides* on two-column skew-tableaux. An important tool in this analysis is the dual RSK correspondence and its symmetry as well as its interpretation in terms of skew-tableaux. As an application of this duality, we study, in subsection 2.3, the sequence of Smith invariants, equivalently the skew-tableaux, associated with $\Delta_\alpha UK(m_1,m_2)$ and $\Delta_\alpha UK(m_2,m_1)$. For this, we have to define two-column frank word *variants* of *jeu de taquin* and to show its relationship with nonstandard pairing of parentheses on words congruent with two-letter key-tableaux. Moreover, variants of *jeu de taquin* and nonstandard reflection crystal operators have a full interpretation in this matrix context, given in theorem 2.6.

In section 3, we study the hexagon of skew-tableaux defined by $\{\Delta_\alpha UK(\sigma\beta(m)) : \sigma \in S_3\}$, with U a fixed unimodular matrix and α a fixed partition. We have two hexagons, one defined by the words of the skew-tableaux and the other one defined by the indexing-set frank words. Since these hexagons are generated by Smith invariants, the matrix setting imposes conditions on their vertices, given in lemma 3.2. Theorems 3.6 and 3.8 give a combinatorial interpretation of those conditions and characterize the variant *jeu de taquin* operators and the nonstandard reflection crystal operators closing those hexagons. Algorithm 3.11 shows that Lascoux-Schützenberger actions of the symmetric group on frank words and words congruent with key-tableaux are respectively obtained: from a particular shuffle decomposition of a three-column tableau; and from a shuffle decomposition of a three-letter Yamanouchi word. Again these actions of the symmetric group have a full interpretation in our matrix setting, given in theorem 3.14. Finally, in example 3.15, we exhibit two dual permutahedra in S_4, generated by variants of *jeu de taquin* operators and nonstandard reflection crystal operators.

2. Variants of two-column jeux de taquin, nonstandard reflection crystal operators and Smith invariants

2.1. Tableaux and dual RSK.

A composition $m = (m_1, \ldots, m_t)$ is a finite sequence of nonnegative integers. A partition is a weakly decreasing composition. It is convenient not to distinguish between two compositions which only differ by a string of zeros at the end. The Young diagram of a partition $\gamma = (\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0)$ is the set of ordered pairs of integers $Y(\gamma) = \{(i,j) : 1 \leq i \leq \gamma_j, 1 \leq j \leq n\}$. We identify a partition with its Young diagram. The conjugate partition $\gamma' = (\gamma'_1, \gamma'_2, \cdots)$ of γ is the transpose of $Y(\gamma)$. A skew-diagram γ/α is the set difference $Y(\gamma) - Y(\alpha)$ of Young diagrams of partitions. A (skew) tableau T over the alphabet $[t] = \{1, \ldots, t\}$ is a function $T : Y \longrightarrow [t]$ from a (skew) diagram Y to the positive integers in $[t]$ such that they are weakly increasing in each row and strictly increasing down each column. The shape $shape(T)$ of T is the domain of the tableau T. The *column-reading word* of the (skew) tableau T is $w^1w^2\ldots$ where w^i is the column-word (strictly decreasing word) over $[t]$ comprising the ith column of T, left to right. Define the column-shape of T to be the composition $colshape(T) = (|w^1|, |w^2|, \ldots,)$, where $|w|$ denotes the length of w. Similarly, the row-reading word of T is $\cdots u^2u^1$, where u^i is the row word (increasing word) over $[t]$ comprising the ith row of T, bottom to top, and the row-shape of T is the composition $(|u^1|, |u^2|, \ldots)$. A *vertical strip* is a skew-tableau with rows of length at most one. The *weight* of a word w (or tableau) is the sequence $(m_1, \ldots, m_t)$ where m_i is the number of occurrences of the letter i in w (or T). The column-indexing-set word of T is $J = J_t \cdots J_2 J_1$ where J_k is the column-word of length m_k defined by the set of column-indices of the letter k in T, left to right. An example of a Young tableau of skew-shape $(4,4,2,1)/(3,1)$ is

$$T = \begin{array}{cccc} 4 & & & \\ 2 & 2 & & \\ \bullet & 1 & 3 & 3 \\ \bullet & \bullet & \bullet & 2 \end{array}. \tag{2.1}$$

The column-reading word is $42\,21\,3\,32$, the column-indexing-set word is $J = 1\,43\,421\,2$ and the weight $(1,3,2,1)$.

A skew-tableau is said in the *compact form* if the overlap of any two consecutive columns is the number of rows of size two. Thus a skew-tableau T in the compact form is identified with its column-reading word seen as a sequence of column words. For instance the skew-tableau (2.1) is in the compact form.

The dual tableau $T^\#$ of T is the function $T^\# : \overline{Y(\alpha)} - \overline{Y(\gamma)} \longrightarrow [t]$ defined by $T^\#(i,j) = t + 1 - T(\gamma_1 - i + 1, n - j + 1)$, where $\overline{Y(\alpha)}$ and $\overline{Y(\gamma)}$ are, respectively, the complements of $Y(\alpha)$ and $Y(\gamma)$ in $[\gamma_1] \times [\gamma_1']$. The column-reading and indexing-set words of T are, respectively, $w^\# = (w^\#)^1 \cdots (w^\#)^n$ and $J^\# = (J^\#)_t \ldots (J^\#)_1$, where $(w^\#)^i$ is the reverse of the word obtained from w^{n-i+1} by complementing each letter in $[t]$, and $(J^\#)_i$ is the reverse of the word obtained from J_{t+1-i} by complementing each letter in $[\gamma_1]$. The dual tableau of (2.1) reading from the bottom to top and right to left, is

$$T^\# = \begin{array}{cccc} 1 & \bullet & \bullet & \bullet \\ 3 & 3 & \bullet & \bullet \\ & 4 & 2 & 2 \\ & & & 3 \end{array}. \tag{2.2}$$

with column-reading word $w^\# = 32\,2\,43\,31$, column-indexing-set word $J^\# = 3\,431\,21\,4$ and weight $(1,2,3,1)$.

A *standard tableau* is a tableau filled with the numbers $\{1, \ldots, n\}$ where n is the number of boxes in the (skew) Young diagram. The standardization $std(T)$ of a Young tableau T is the standard tableau obtained by simultaneously replacing the 1's in T from left to right by $1, 2, \ldots, m_1$, the $2's$ by $m_1 + 1, \ldots, m_1 + m_2$ etc, where $(m_1, \ldots, m_t)$ is the weight of T.

Let $[t]^*$ denote the free monoid over the alphabet $[t]$. The Knuth or plactic congruence $\equiv$ [**15, 6, 14, 19**] on the words over the alphabet $[t]$ is the congruence in $[t]^*$ defined by the transitive closure of the relations, where x, y and z are letters in $[t]$, and a e b are words in $[t]^*$:

$$axzyb \equiv azxyb, \quad x \leq y < z, \tag{2.3}$$

$$ayzxb \equiv ayxzb, \quad x < y \leq z. \tag{2.4}$$

Given the word $w = x_1 \cdots x_N$, write $(\emptyset \leftarrow w) = (P(w), \mathcal{Q}(w))$ to mean that the row insertion of w produces the pair of tableaux $P = P(w)$ and $\mathcal{Q}(w)$ of the same shape, with $P(w)$ the Schensted's tableau, that is, the unique tableau of partition shape whose column-reading word is Knuth equivalent to w, and $\mathcal{Q}(w)$ the $\mathcal{Q}$-symbol or row-insertion recording tableau, that is, the standard tableau of the same shape as $P(w)$ such that $shape(\mathcal{Q}(w)|_{[i]}) = shape(P(w_1 \cdots w_i))$, $1 \leq i \leq t$, where $\mathcal{Q}(w)|_{[i]}$ denotes the restriction of the tableau $\mathcal{Q}(w)$ to the letters in $[i]$. Two (skew) tableaux are said Knuth equivalent if their words are congruent, equivalently, one is obtained from the other one by *jeu de taquin* slides [**6, 23**].

Let $v = v^1 v^2 \ldots v^n$, with each v^i a column-word (some of the v^i may be the empty word), be a n-column factorization of v. The factorization $v^1 v^2 \ldots v^n$ may be identified with the skew-tableau in the compact form with column-reading word $v^1 v^2 \ldots v^n$. The column-shape of $v^1 v^2 \ldots v^n$, denoted by $colshape(v^1 v^2 \ldots v^n)$, is the column-shape of that skew-tableau. There is also an obvious bijection between column factorizations of v and skew-tableaux in the compact form with weight the reverse column-shape of the factorization since there is a unique skew-tableau in the compact form with $v^n v^2 \ldots v^1$ as indexing-set word. There exists a unique tableau P of weight $colshape(v^1 v^2 \ldots v^n)$ such that $\mathcal{Q}(v) = std(P)^t$. A factorization is maximal if the length of the nonempty words is maximal. We write $colshape(v)$ (unique up to zeros) to mean the column-shape of a maximal factorization of v. Lascoux and Schützenberger called v *frank* word [**17**] if it has a factorization of column-shape a permutation of the nonzero parts of the conjugate shape of $P(v)$. Then v is a frank word if and only if $\mathcal{Q}(v) = std(K(colshape(v)))^t$.

Let T be a tableau of partition shape over the alphabet $[t]$. The evacuation $ev(T)$ of T with respect to the alphabet $[t]$ is the unique tableau on the alphabet $[t]$ such that $shape(ev(T)|_{[i]}) = shape(P(T|_{[t+1-i,t]}))$ [**21**]. Clearly, $ev(T)$ has reverse weight and the same shape as T.

The dual RSK correspondence is a bijection from the set of finite sequences of column-words $(\cdots J_2J_1)$ to pairs of tableaux (Q, P) of conjugate shapes, defined by $Q = P(\cdots J_2J_1)$ and $shape(P|_{[i]}) = shape(P(J_i \cdots J_1))^t$ for all i, where ^{t} denotes transposition.

Let A and B be two totally ordered alphabets. We consider biwords of pairwise distinct biletters

$$\begin{pmatrix} u_1 & \cdots & u_k \\ v_1 & \cdots & v_k \end{pmatrix}, \tag{2.5}$$

with $u_i \in A$ and $v_j \in B$. The biword is said in anti-lexicographic order if its biletters $\binom{u_i}{v_i}$ satisfy $u_i < u_{i+1}$ or $u_i = u_{i+1}$ and $v_i > v_{i+1}$.

Let $J = \{J_{t-i+1}\}_{1 \le i \le t}$, be a sequence of column-words over the alphabet $[n]$ in dual RSK correspondence with the tableau-pair (Q, P) of conjugate shapes. Let m_i be the length of J_i, $1 \le i \le t$. An easy way to exhibit the symmetry of the dual RSK correspondence is to consider the biword with no two identical biletters,

$$\Sigma' = \begin{pmatrix} J_t & \cdots & J_1 \\ t^{m_t} & \cdots & 1^{m_1} \end{pmatrix}. \tag{2.6}$$

By increasing rearrangement of Σ' for the anti-lexicographic order with priority in the first row, we get

$$\Sigma = \begin{pmatrix} 1^{f_1} & \cdots & n^{f_n} \\ w^1 & \cdots & w^n \end{pmatrix}, \tag{2.7}$$

with w^i a column word of length f_i. Clearly the word $J_t \cdots J_1$ can be seen as the column-indexing-set word of a skew-tableau with column reading word $w^1 \cdots w^n$, and conversely $w^n \cdots w^1$ as the column-indexing-set word of a skew-tableau with column-reading word $J_1 \cdots J_t$. Therefore, the dual RSK correspondence defines also a bijection between skew-tableaux in the compact form and tableau-pairs of conjugate shapes. For instance, consider the skew-tableau (2.1). Construct Σ and Σ' accordingly. The bottom word of Σ is w and the top word of Σ' is J,

$$\Sigma = \begin{pmatrix} 11\,22\,3\,44 \\ 42\,21\,3\,32 \end{pmatrix} \longleftrightarrow \Sigma' = \begin{pmatrix} 1\,43\,421\,2 \\ 4\,33\,222\,1 \end{pmatrix}.$$

From this bijection and the definition of dual tableau the following statement follows easily.

THEOREM 2.1. [**15, 22**] *Let $\{J_{t-i+1}\}_{1 \le i \le t}$, be a sequence of column words in the alphabet $[n]$ in dual RSK correspondence with the tableau pair (Q, P). Then the sequence $\{(J^{\#})_{t-i+1}\}_{1 \le i \le t}$ is in dual RSK correspondence with the tableau-pair $(ev_{[n]}(Q), ev_{[t]}(P))$.*

Skew-tableaux (2.1), (2.2) are an illustration of this result.

We have from Proposition 5, the symmetry theorem, in [**6**], Appendix A.4.3.

THEOREM 2.2. *Given the sequence $J = \{J_{t-i+1}\}_{1 \le i \le t}$, of column-words over the alphabet $[n]$, there is one and only one word w over the alphabet $[t]$, with weight the reverse of $colshape(\{J_{t-i+1}\}_{1 \le i \le t})$ such that*
(a) the Schensted tableau $P(w) = P$ of w satisfies $std(ev\,P) = \mathcal{Q}(J)^t$;
(b) the $\mathcal{Q}$-symbol or recording tableau of w satisfies $\mathcal{Q}(w) = std(Q)^t$ with $Q = P(J)$.
Moreover J is a frank word of $colshape(\{J_{t-i+1}\}_{1 \le i \le t})$ if and only if $P = K(revcolshape(w))$.

As frank words, in a congruence class, are completely determined by their $\mathcal{Q}$-symbols, it follows that frank words, in a plactic class, are in bijection with the set of permutations of the nonzero parts of the conjugate shape of the tableau in that class [**17**] and we have an action of the symmetric group on frank words. Lascoux and Schützenberger have translated this action of the symmetric group on frank words in the language of *jeu de taquin* slides on two-column skew-tableaux either aligned in the bottom or in the top. We shall now extend this *jeu de taquin* operation to any two-column skew-tableaux in the compact form to define an action of the symmetric group on skew-tableaux.

2.2. Two-column word *jeux de taquin* and reflection crystal operators. The *jeu de taquin* slides exchanging the length of two consecutive columns i, $i+1$ of a t-column skew-tableau in the compact form, counting right to left, is translated, by the dual RSK-correspondence, into the i-th reflection crystal operator on all words over the alphabet $[t]$. In the particular case of frank words it is translated into the i-th reflection crystal operator on words congruent with keys. Define the operation Θ on a two-column skew-tableau $T = J_2J_1$ in the compact form and row-shape $(1^s, 2^q, 1^r)$, for some $q, r, s \geq 0$, as follows. If $r > s$ ($r < s$), perform *jeu de taquin* slides on the first $|r-s|$ inside (outside) corners of the skew-tableau T until they become outside (inside) corners in the second (first) column. In other words, we slide down (up) the first (second) column, maximally up to $|r-s|$ positions; then we exchange the east (west) neighbors with these corners. Then $\Theta T = T'$ is a two-column skew-tableau in the compact form with row-shape $(1^r, 2^q, 1^s)$. Obviously $\Theta T' = T$. In particular, when $r = 0$ or $s = 0$, Θ is the *jeu de taquin* on frank words. For instance, the *jeu de taquin* slides with respect to the corner $\blacksquare$ as below define the operation Θ on T and T'

$$T = \begin{matrix} 7 & \\ 3 & 6 \\ 2 & 5 \\ 1 & 4 \\ \blacksquare & 3 \\ & 2 \end{matrix} \quad \overset{\Theta}{\longleftrightarrow} \quad T' = \begin{matrix} 7 & \\ 6 & \blacksquare \\ 3 & 5 \\ 2 & 4 \\ 1 & 3 \\ & 2 \end{matrix}. \tag{2.8}$$

Let T be a t-column skew-tableau in the compact form. Define the operation Θ_i on T as follows: apply Θ to the columns i and $i+1$ of T, counting right to left, and put the outcome t-column skew-tableau in the compact form. As *jeu de taquin* preserves Knuth equivalence, we have $\Theta_i T \equiv T$.

Let $w = w_1 w_2 \ldots w_k$, $w_i \in [t]$, be a word. An r-pairing of w is a set of indexed pairs (called r-pairs) (w_i, w_j) such that $1 \leq i < j \leq k$, $w_i = r+1$, and $w_j = r$, and if (w_l, w_s) is another pair, then i, l, j, s are pairwise distinct. View each r (resp. $r+1$) as a left (resp. right) parenthesis and ignore the other letters. The r-pairs of w are precisely the matched parentheses. Furthermore the subword of unpaired $r's$ and $(r+1)'s$ must be a subword of w of the form $r^k(r+1)^l$. In general, not every r-pairing gives the maximal number of r-pairs of w, and if $\tilde{\theta}_r$ is the operation which replaces the word $r^k(r+1)^l$ of unpaired $r's$ and $(r+1)'s$ in w (in the corresponding positions) by $r^l(r+1)^k$, unless certain conditions are imposed on the r-pairing, the maximal number of r-pairs of $\tilde{\theta}_r w$ and w may be different. However, when either $k = 0$ or $l = 0$, although w and $\tilde{\theta}_r w$ may have different r-pairings, they have always the same maximal number of r-pairs. In this case, as we shall see in the next subsection, the operation $\tilde{\theta}_r$ can be reduced to a variant of *jeu de taquin* on two-column frank words.

The standard r-pairing on w is the particular r-pairing obtained in the following way. Start with the subword $w' = x_1 \cdots x_m$, where $x_i \in \{r, r+1\}$. Then, bracket every factor $r+1\, r$ of w'. The letters which are not bracket constitute a subword w'' of w'. Then, bracket every factor $r+1\, r$ of w''. Continue this procedure until it stops.

The reflection crystal operator, based on the standard r-pairing, denoted by θ_r, can be reduced to *jeu de taquin* slides on two-column words. The operations θ_i on words over the alphabet $[t]$, and Θ_i on t-column skew-tableaux $\{J_{t-i+1}\}_{1 \leq i \leq t}$, in the compact form are a translation of each other in the sense of the following commutative diagram

$$\begin{array}{ccc} \Sigma = \begin{pmatrix} J\uparrow \\ w \end{pmatrix} & \longleftrightarrow & \Sigma' = \begin{pmatrix} J \\ \cdots (i+1)^{q+s} i^{q+r} \cdots \end{pmatrix} \\ \updownarrow & & \updownarrow \\ \tilde{\Sigma} = \begin{pmatrix} J\uparrow \\ \theta_i w \end{pmatrix} & \longleftrightarrow & \tilde{\Sigma}' = \begin{pmatrix} \Theta_i J \\ \cdots (i+1)^{q+r} i^{q+s} \cdots \end{pmatrix} \end{array}, \tag{2.9}$$

where $J\uparrow$ indicates J by weakly increasing order.

Let $P[n]$ be the set of all subsets of $[n]$. We consider on $P[n]$ two orders, one by letting $B \leq B'$ whenever there is an increasing injection $i : B \rightarrow B'$, that is, $x \leq i(x)$, and the other one by putting $B \triangleright B'$ whenever there exists a decreasing injection $B \leftarrow B' : j$, that is, $j(x) \leq x$.

Let the columns i and $i+1$ of $\Theta_i J$, counting from right to left, define the skew-tableau in the compact form $\begin{matrix} A & \\ B & C \\ & D \end{matrix}$, where $A\cup B$, $C\cup D$ are columns such that $B\le C$, $|B|=|C|=q$, $|A|=s$, and $|D|=r$. Slide down maximally the column B along $C\cup D$ such that the row weakly increasing order is preserved, and define the subcolumn X of $C\cup D$ by the entries of $C\cup D$ with west adjacent neighbors. Then θ_i can be based in any pairing of parentheses defined by any increasing injection $j: B\to X$. We have the equivalence, $\Theta_i T\equiv T$ if and only if the operator θ_i preserves the Q-symbol, that is, $Q(w)=Q(\theta_i w)$. For instance, for the two-column tableau T (2.8), we have the following diagram

$$\begin{array}{ccc} \Sigma = \begin{pmatrix} 1\,2\,2\,3\,3\,4\,5\,6\,7 \\ 2\,2\,1\,2\,1\,1\,1\,1\,2 \end{pmatrix} & \longleftrightarrow & \Sigma' = \begin{pmatrix} T \\ 2^{3+1}\ 1^{3+2} \end{pmatrix} \\ \updownarrow & & \updownarrow \\ \widetilde{\Sigma} = \begin{pmatrix} 1\,2\,2\,3\,3\,4\,5\,6\,7 \\ 2\,2\,1\,2\,1\,1\,1\,2\,2 \end{pmatrix} & \longleftrightarrow & \widetilde{\Sigma}' = \begin{pmatrix} \Theta_1 T \\ 2^{3+2} 1^{3+1} \end{pmatrix} \end{array},$$

where $w=(2(21)(21)1)1^2 2\to\theta_1 w=(2(21)(21)1)12^2$.

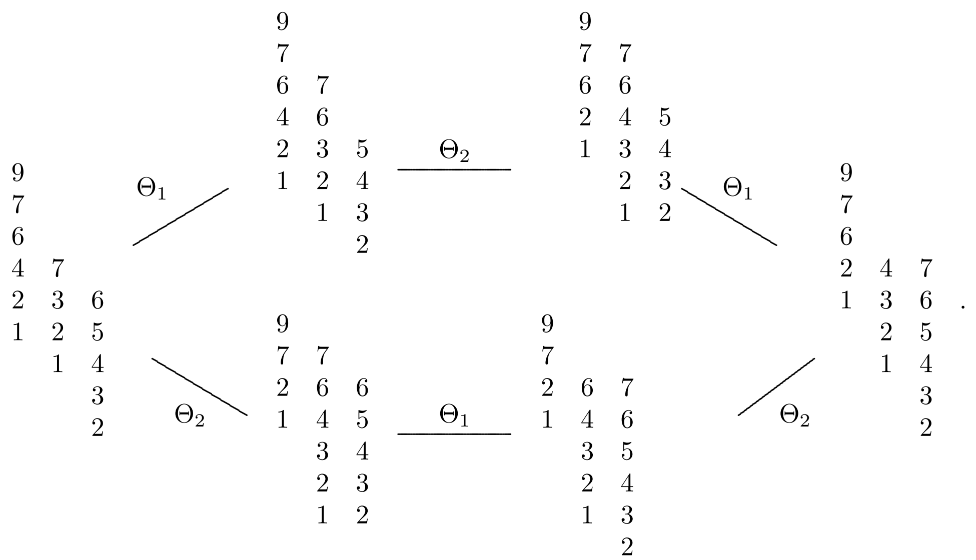
From the diagram (2.9), we have

THEOREM 2.3. *The following statements are equivalent:*

(*a*) *The operations Θ_i, $1\le i\le t-1$, define an action of the symmetric group S_t on the set of t-column words, equivalently, on the t-column skew-tableaux in the compact form. Moreover, $\Theta_i T\equiv T$, $1\le i\le t-1$.*

(*b*) [**15, 19**] *The operations θ_i, $1\le i\le t-1$, define an action of the symmetric group on all words over the alphabet $[t]$, and preserve the Q-symbol.*

EXAMPLE 2.4. An action of S_3 on three-column skew-tableaux in the compact form



2.3. Two-column frank word variants of *jeu de taquin*, pairing of parentheses and Smith invariants. In this section, we describe the Smith invariants, equivalently, the skew-tableaux on a two-letter alphabet, associated with the sequences $\Delta_\alpha UK(m)$ and $\Delta_\alpha UK(s_1 m)$ with $m=(m_1,m_2)$ and s_1 the elementary transposition $(1\,2)$. For this, we have to define *variants* of *jeu de taquin* on two-column frank words and to show its relationship with pairings of parentheses on words congruent with two-letter key-tableaux.

Restrict the *jeu de taquin* operation Θ to a two-column tableau or contre-tableau (a two-column skew-tableau such that the pair of columns is aligned at the top) $J_2\,J_1$ [**6, 23**], and denote by $\tilde\Theta$ a variant of Θ which runs as follows. If J_2J_1 is a contretableau (tableau), slide vertically the entries of the column J_2 (J_1)

along the column J_1 (J_2) such that the row weakly increasing order is preserved, and every common label to the two columns never has a vacant west (east) neighbor. Then exchange the vacant positions with the east (west) neighbors. In particular, when the first (second) column J_2 (J_1) is slid down (up) maximally such that the row weakly increasing order is preserved, we get the outcome of the *jeu de taquin* operation. For instance,

$$\Theta : \begin{array}{cc} 2 & 5 \\ 1 & 4 \\ \blacksquare & 3 \\ \blacksquare & 2 \end{array} \longleftrightarrow \begin{array}{cc} \blacksquare & 5 \\ \blacksquare & 4 \\ 2 & 3 \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 5 & \blacksquare \\ 4 & \blacksquare \\ 2 & 3 \\ 1 & 2 \end{array}, \tag{2.10}$$

$$\tilde{\Theta} : \begin{array}{cc} 2 & 5 \\ 1 & 4 \\ \blacksquare & 3 \\ \blacksquare & 2 \end{array} \longleftrightarrow \begin{array}{cc} \blacksquare & 5 \\ 2 & 4 \\ \blacksquare & 3 \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 5 & \blacksquare \\ 2 & 4 \\ 3 & \blacksquare \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 5 & \blacksquare \\ 3 & \blacksquare \\ 2 & 4 \\ 1 & 2 \end{array}. \tag{2.11}$$

Obviously, $\tilde{\Theta}(J_2J_1)$ and $\Theta(J_2J_1)$ are not congruent unless $\tilde{\Theta} = \Theta$, but $\tilde{\Theta}(J_2J_1)$ is a frank word with the same shape and weight as $\Theta(J_2J_1)$.

Suppose that w is congruent with the key-tableau of weight $(0^{r-1}, m_r, m_{r+1})$. Without loss of generality, assume $m_{r+1} \leq m_r$. Let $J_{r+1}J_r$ be a frank word of shape $(m_{r+1}, m_r, 0^{r-1})$, such that sorting the billeters of the biword $\Sigma' = \begin{pmatrix} J_{r+1}J_r \\ (r+1)^{m_{r+1}} r^{m_r} \end{pmatrix}$, by weakly increasing rearrangement of the billeters for the anti-lexicographic order with priority on the first row, we get $\Sigma = \begin{pmatrix} J_{r+1}J_r \uparrow \\ w \end{pmatrix}$. Consider an r-pairing in w defined by an increasing injection $i : J_{r+1} \longrightarrow J_r$ such that $J_r \cap J_{r+1} \subseteq i(J_{r+1})$. To perform $\tilde{\theta}_r w$ based on this r-pairing means to apply an operation $\tilde{\Theta}$ on $J_{r+1}J_r$ (denoted by $\tilde{\Theta}_r$) which exchanges the vacant entries of the first column with the correspondent east neighbors consisting of $J_r \setminus i(J_{r+1})$ in the second column J_r. Conversely, an operation $\tilde{\Theta}_r$ on $J_{r+1}J_r$ means an operation $\tilde{\theta}_r$ on w, where the r-pairing on w is defined by any increasing injection $i : J_{r+1} \longrightarrow J_r$ such that $\tilde{\Theta}J_{r+1}J_r = [J_{r+1} \cup (J_r \setminus B)]B$, where $J_r \cap J_{r+1} \subseteq i(J_{r+1}) = B$. When $\tilde{\Theta}_r = \Theta_r$ we get the standard pairing of parentheses on w and hence θ_r. Thus the operations $\tilde{\Theta}_r$, Θ_r and $\tilde{\theta}_r$, θ_r are respectively translated into each other, according the following commutative diagram,

$$\begin{array}{ccc} \Sigma = \begin{pmatrix} J_{r+1}J_r \uparrow \\ w \end{pmatrix} & \longleftrightarrow & \Sigma' = \begin{pmatrix} J_{r+1}J_r \\ (r+1)^{m_{r+1}} r^{m_r} \end{pmatrix} \\ \updownarrow & & \updownarrow \\ \tilde{\Sigma} = \begin{pmatrix} J_{r+1}J_r \uparrow \\ \tilde{\theta}_r w \end{pmatrix} & \longleftrightarrow & \tilde{\Sigma}' = \begin{pmatrix} \tilde{\Theta}(J_{r+1}J_r) \\ (r+1)^{m_r} r^{m_{r+1}} \end{pmatrix} \end{array}. \tag{2.12}$$

If $(\emptyset \leftarrow w) = (P, Q)$ then $(\emptyset \leftarrow \tilde{\theta}_r w) = (\theta_r P, Q')$, where Q and Q' are distinct unless $\tilde{\theta}_r = \theta_r$. As $\tilde{\Theta}_r$ runs out of the congruence class, $\tilde{\theta}_r$ does not preserve the Q-symbol but we have $\theta_r w \equiv \tilde{\theta}_r w$. For instance, in (2.10), any increasing injection $\{1, 2\} \rightarrow \{2, 3\}$ defines a standard pairing of parentheses, giving rise to $\theta_1 : (2(21)1)1 \rightarrow (2(21)1)2$; and in (2.11), any increasing injection $\{1, 2\} \rightarrow \{2, 4\}$ defines a pairing of parentheses, giving rise to $\tilde{\theta}_1 : (2(21)11) \rightarrow (2(21)21)$.

We are now in conditions to describe the invariant factors, equivalently, the skew-tableaux on a two-letter alphabet associated with the sequences $\Delta_\alpha UK(m)$ and $\Delta_\alpha UK(s_1 m)$.

LEMMA 2.5. [**4**] (a) *Let U be an n by n unimodular matrix. Then, there exists $\sigma \in \mathcal{S}_n$ such that $U = TP_\sigma QL$, where T is an n by n upper triangular matrix, with $1'$s along the main diagonal, Q is an n by n upper triangular matrix, with $1'$s along the main diagonal, and multiples of p above it, and L is an n by n lower triangular matrix, with units along the main diagonal.*

(b) *By elementary operations on the left and on the right, $\Delta_\alpha UK(m)$ may be considered equal to $\Delta_\alpha P_\sigma QK(m)$, with $\sigma \in \mathcal{S}_n$.*

(c) The Smith normal form of $\Delta_\alpha P_\sigma Q D_{[m_1]}$, with $\sigma \in \mathcal{S}_n$, is the diagonal matrix Δ_{α^1} where $\alpha \subseteq \alpha^1$ is a vertical strip of length m_1.

THEOREM 2.6. [**4**] *Let T and T' be respectively the tableaux defined by the sequences $\Delta_\alpha UK(m)$ and $\Delta_\alpha UK(s_1 m)$, with indexing-set words J_2J_1, $J_2'J_1'$ respectively, and words w, w' respectively. Then,*

(a) J_2J_1, $J_2'J_1'$ are frank words such that $\tilde{\Theta}_1 J_2J_1 = J_2'J_1'$.

(b) $w \equiv K(m)$ and $w' = \tilde{\theta}_1 w \equiv K(s_1 m)$.

Conversely, if T and T' are respectively tableaux with indexing-set frank words J_2J_1 and $J_2'J_1'$ satisfying $J_2'J_1' = \tilde{\Theta}_1 J_2J_1$, then there exist an unimodular matrix U such that $\Delta_\alpha UK(m)$ and $\Delta_\alpha U'K(s_1m)$ define the tableaux T and T' respectively.

EXAMPLE 2.7. Let $U = P_{4321}T_{14}(p)$, where P_{4321} is the permutation matrix associated with $4321 \in S_4$ and $T_{14}(p)$ is the elementary matrix obtained from the identity by placing the prime p in position $(1,4)$. With $\alpha = (2,1)$ the sequences $\Delta_\alpha UK(3,2)$ and $\Delta_\alpha UK(2,3)$ define, respectively, $T = \begin{array}{cccc} 2 & & & \\ \bullet & 1 & 2 & \\ \bullet & \bullet & 1 & 1 \end{array}$ and $T' = \begin{array}{cccc} 2 & & & \\ \bullet & 2 & 2 & \\ \bullet & \bullet & 1 & 1 \end{array}$. The words $w = 21211$ of T and $w' = 22211$ of T' satisfy $\tilde{\theta}_1 w = w' \equiv \theta_1 \omega$, where $\tilde{\theta}_1$ is the operation based on the parentheses matching (21(21)1). However, if we choose $U' = P_{3241}T_{24}(p)$, the sequences $\Delta_\alpha U'\ K(3,2)$ and $\Delta_\alpha U'\ K(2,3)$ define, respectively, T and $T'' = \begin{array}{cccc} 2 & & & \\ \bullet & 1 & 2 & \\ \bullet & \bullet & 1 & 2 \end{array}$. In this case, the word w'' of T'' satisfy $\theta_1 w = w''$. The corresponding operations on the indexing frank words are displayed as follows

$$\Theta: \begin{array}{cc} \blacksquare & 4 \\ 3 & 3 \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 4 & \blacksquare \\ 3 & 3 \\ 1 & 2 \end{array} \qquad \tilde{\Theta}: \begin{array}{cc} 3 & 4 \\ 1 & 3 \\ \blacksquare & 2 \end{array} \longleftrightarrow \begin{array}{cc} 3 & 4 \\ 1 & 3 \\ 2 & \blacksquare \end{array} \longleftrightarrow \begin{array}{cc} 3 & \blacksquare \\ 2 & 4 \\ 1 & 3 \end{array}. \tag{2.13}$$

The operations Θ_r (θ_r) can be extended to frank words with more than two columns (words on a t-letter alphabet, $t \geq 2$) [**15, 19**]. Under certain conditions, operations $\tilde{\Theta}_r$ ($\tilde{\theta}_r$) can be extended, as well, to frank words with more than two columns (words on a t-letter alphabet, $t \geq 2$). For this, we generalize a criterion, by Lascoux and Schützenberger in [**17**], to test whether the concatenation of a frank word with a column word is a frank word. Denote, respectively, by $L(J)$ and $R(J)$ the left and right columns of a frank word J.

THEOREM 2.8. [**17**] *The concatenation JJ' of two frank words J, J' is frank if and only if $R(H)L(H')$ is frank for any pair of frank words H, H' such that $H \equiv J$ and $H' \equiv J'$.*

Notice that when J, J' are column-words, JJ' is frank if and only if JJ' is a tableau or a contretableau. Therefore, we deduce the following criterion for the concatenation of a column with a frank word.

COROLLARY 2.1. *Let $J = J_k \cdots J_1$ be a frank word and J_{k+1} a column. Then, $J_{k+1}J$ is frank if and only if $J_{k+1}J_k$ and $\overline{J}_k J_{k-1} \cdots J_1$ are frank words, where $\overline{J}_{k+1}\,\overline{J}_k = \Theta_k(J_{k+1}J_k)$.*

The criterion given by this corollary can be generalized to operations $\tilde{\Theta}$.

COROLLARY 2.2. *Let $J = J_k \cdots J_1$ be a frank word and J_{k+1} a column. Then, $J_{k+1}J$ is frank if and only if $J_{k+1}J_k$ and $\widetilde{J}_k J_{k-1} \cdots J_1$ are frank words, where $\widetilde{J}_{k+1}\,\widetilde{J}_k = \tilde{\Theta}_k(J_{k+1}J_k)$ for some operation $\tilde{\Theta}_k$.*

PROOF. The necessary condition is a consequence of the previous corollary. Reciprocally, assume the existence of an operation $\tilde{\Theta}_k$ in the required conditions, and let $\overline{J}_{k+1}\,\overline{J}_k = \Theta_k(J_{k+1}J_k)$. Clearly, we have $\overline{J}_k \leq \widetilde{J}_k$, and also $\overline{J}_{k+1} \rhd \widetilde{J}_{k+1}$, since $|\overline{J}_k| = |\widetilde{J}_k|$. By the hypotheses, the product $\widetilde{J}_k L(H)$ is frank, for any frank word $H \equiv J_{k-1} \cdots J_1$. This means that either $\widetilde{J}_k \leq L(H)$, or $\widetilde{J}_k \rhd L(H)$. By transitivity, we find that either $\overline{J}_k \leq L(H)$, or $\overline{J}_k \rhd L(H)$, *i.e.*, $\overline{J}_k L(H)$ is frank. Thus, by theorem 2.8, the word $\overline{J}_k J_{k-1} \cdots J_1$ is frank, and therefore, by the previous corollary, $J_{k+1}J$ is frank. □

Using theorem 2.1, the previous criteria have a natural translation for the concatenation JJ_0 with J a frank word and J_0 a column word.

The following theorem was proved in [**5**]. Here we give a different proof based on indexing-set words.

THEOREM 2.9. *Let T be the tableau defined by $\Delta_\alpha U\ K(m)$, with word w and J the indexing set word. Then $P(w) = K(m)$ and J is a frank word of column-shape the reverse of m.*

PROOF. Let $J = J_t \ldots J_1$ with column-shape the reverse of m. We will prove, by induction on $t \geq 1$, that $J_t \cdots J_1$ is a frank word. When $t = 1$ the result is trivial, and the case $t = 2$ is a consequence of theorem 2.6 (see [**4**]). So, let $t > 2$ and let T be the tableau defined by $\Delta_\alpha UK(m_1, \ldots, m_t)$. By the inductive step, the word $J_{t-1} \cdots J_1$ is frank, since the sequence $\Delta_\alpha UK(m_1, \ldots, m_{t-1})$ defines the tableau T' with indexing-set word $J_{t-1} \ldots J_1$ and weight $(m_1, \ldots, m_{t-1})$.

By Smith normal form theorem, there is a partition $\bar{\alpha}$ and an unimodular matrix U' such that by elementary row operations, $\Delta_{\bar{\alpha}} UD_{[m_1]} \cdots D_{[m_{t-2}]}$ can be reduced to $\Delta_{\bar{\alpha}} U'$. The sequence $\Delta_{\bar{\alpha}} U'K(m_{t-1}, m_t)$ defines the tableau $\overline{T}$ with indexing-set word J_{t-1}, J_t, and weight (m_{t-1}, m_t). By the case $t = 2$, the word $J_t J_{t-1}$ is frank. Moreover, by theorem 2.6, we find that if $\overline{T}'$ is the tableau defined by the sequence $\Delta_{\bar{\alpha}} U, K(m_t, m_{t-1})$, the indexing sets $\overline{J}_{t-1}, \overline{J}_t$ of $\overline{T}'$ satisfy $\overline{J}_t \overline{J}_{t-1} = \tilde{\Theta}_{t-1}(J_t J_{t-1})$ for some operation $\tilde{\Theta}_{t-1}$.

Finally, notice that $\Delta_\alpha UK(m_1, \ldots, m_{t-2}, m_t)$ defines the tableau $\widetilde{T}$ with indexing-set word $\overline{J}_{t-1}\ J_{t-2} \ldots J_1$, and weight $(m_1, \ldots, m_{t-2},\ m_t)$. By the inductive step, $\overline{J}_{t-1} J_{t-2} \cdots J_1$ is a frank word. Thus, by corollary 2.2, the word $J_t \cdots J_1$ is frank, and therefore, $w \equiv K(m)$. $\square$

Let $U^- := det(U)(U^{-1})^t$, where det means determinant, ^{t} matrix transpose and U^{-1} the inverse of U. Clearly U^- is still unimodular. The following is a matrix interpretation of theorem 2.1.

COROLLARY 2.3. [**3**] *$\Delta_\alpha U\ K(m)$ defines the tableau T, with word w and indexing set frank word J if and only if $\Delta_{\alpha^\#} U^-\ K(m^\#)$ defines the tableau $T^\#$ with word $w^\#$ and indexing set frank word $J^\#$.*

3. An action of the symmetric group on Young tableaux of skew-shape

Let s_i denote the elementary transpositions $(i\, i+1)$ of S_t, $1 \leq i \leq t$. Let U be an n by n unimodular matrix and $\beta = (\beta_1, \beta_2, \beta_3)$ a partition. We consider the following hexagon

$$
\begin{array}{ccccccc}
 & & \Delta_\alpha UK(\beta_2,\beta_1,\beta_3) & \overset{s_2}{\text{———}} & \Delta_\alpha UK(\beta_2,\beta_3,\beta_1) & & \\
 & \overset{s_1}{\diagup} & & & & \overset{s_1}{\diagdown} & \\
\Delta_\alpha UK(\beta_1,\beta_2,\beta_3) & & & & & & \Delta_\alpha UK(\beta_3,\beta_2,\beta_1). \\
 & \underset{s_2}{\diagdown} & & & & \underset{s_2}{\diagup} & \\
 & & \Delta_\alpha UK(\beta_1,\beta_3,\beta_2) & \underset{s_1}{\text{———}} & \Delta_\alpha UK(\beta_3,\beta_1,\beta_2) & &
\end{array}
\tag{3.1}
$$

From the discussion in the introduction, we may look at (3.1) as an hexagon whose vertices are tableaux of skew-shape such that the words are congruent with a key-tableau $K(\beta_{i_1}, \beta_{i_2}, \beta_{i_3})$, and the indexing-set frank words have column-shape the reverse of $(\beta_{i_1}, \beta_{i_2}, \beta_{i_3})$ with $(\beta_{i_1}, \beta_{i_2}, \beta_{i_3})$ running over the orbit $S_3\beta$. Therefore, we have two hexagons, one defined by the words of the skew-tableaux and the other one defined by the indexing-set frank words. These hexagons are copies of each other since operations based on pairing of parentheses can be reduced to variations of the *jeu de taquin* on two-column frank words and *vice versa.* Taking into account theorems 2.6 and 2.9, the next statement follows from the hexagon above. Given $\sigma \in S_t$, *rev* denotes the longest permutation of S_t.

THEOREM 3.1. *Let $\sigma \in < s_1, s_2 >$, $\theta \in < \theta_1, \theta_2 >$ and $\Theta \in < \Theta_1, \Theta_2 >$ with the same reduced word. Let $T(\sigma)$ be the tableau defined by $\Delta_\alpha UK(\sigma\beta)$, with word σw and indexing-set frank word σJ of column-shape the reverse of $\sigma\beta$. Then $\{T(\sigma) : \sigma \in < s_1, s_2 >\}$ are the vertices of an hexagon such that*

(a) there exist $\tilde{\theta}_1$ and $\tilde{\theta}_2$ satisfying the Moore-Coxeter relations of the symmetric group S_3, where $\tilde{\theta} \in < \tilde{\theta}_1, \tilde{\theta}_2 >$, with the same reduced word as θ, verifies $\sigma w = \tilde{\theta} w \equiv \theta K(\beta) = K(\sigma\beta)$,

$$
\begin{array}{ccccccc}
 & & \tilde{\theta}_1 w & \text{———} & \tilde{\theta}_2\tilde{\theta}_1 w & & \\
 & \diagup & & & & \diagdown & \\
w & & & & & & \tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_1 w. \\
 & \diagdown & & & & \diagup & \\
 & & \tilde{\theta}_2 w & \text{———} & \tilde{\theta}_1\tilde{\theta}_2 w & &
\end{array}
\tag{3.2}
$$

(b) there exist $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ satisfying the Moore-Coxeter relations of the symmetric group S_3, where $\tilde{\Theta} \in < \tilde{\Theta}_1, \tilde{\Theta}_2 >$, with the same reduced word as Θ, verifies $\sigma J = \tilde{\Theta} J$,

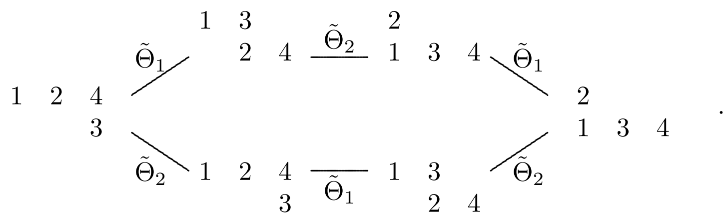
$$
\begin{array}{ccccc}
 & \tilde\Theta_1 J = J_3G_2G_1 & \text{——} & \tilde\Theta_2\tilde\Theta_1 J = F_3F_2G_1 & \\
J = J_3J_2J_1 & & & & \tilde\Theta_1\tilde\Theta_2\tilde\Theta_1 J = F_3XH_1. \\
 & \tilde\Theta_2 J = L_3L_2J_1 & \text{——} & \tilde\Theta_1\tilde\Theta_2 J = L_3H_2H_1 &
\end{array}
\tag{3.3}
$$

Our aim is to describe explicitly the operators $\tilde\theta_i$ and $\tilde\Theta_i$ closing the hexagons (3.2) and (3.3).

In fact the hexagon (3.1) and, hence, hexagon (3.3), obey the following conditions. (The translation of these conditions to hexagon (3.2) will be done later.)

Lemma 3.2. [**4**] *The operators closing hexagons (3.1) and (3.3) obey the following conditions.*
(a) *If L_3L_2 and F_3F_2 are, respectively, the indexing-set frank words of $\Delta_\alpha UK(\beta_1,\beta_3)$ and $\Delta_\alpha UK(\beta_2,\beta_3)$, it holds $F_2 \le L_2$.*
(b) *If L_3H_2 and J_3G_2 are, respectively, the indexing-set frank words of $\Delta_\alpha UK(\beta_3,\beta_1)$ and $\Delta_\alpha UK(\beta_2,\beta_1)$, it holds $G_2 \le H_2$.*
(c) *The operators $\tilde\Theta_1$ and $\tilde\Theta_2$ defining the hexagon (3.3) are such that $\tilde\Theta_2[\tilde\Theta_1 J] = F_3F_2G_1$ with $F_2 \le L_2$, and $\tilde\Theta_1[\tilde\Theta_2 J] = L_3H_2H_1$ with $G_2 \le H_2$.*

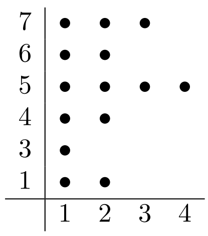
Remark 3.3. The condition (c) in the previous lemma, imposed on the operators of the hexagon (3.3) do not come from the braid relations of the operators $\tilde\Theta_i$. As can be seen in the example below, there are operators $\tilde\Theta_1$ and $\tilde\Theta_2$ which close the hexagon and do not satisfy the conditions in (c). For instance,



We start to analyze the hexagon (3.3) under the conditions in (c) of the previous lemma. The Knuth class of a key-tableau over a three-letter alphabet as well as any frank word with three columns can be characterized in terms of the shuffling operation. This characterization gives a combinatorial explanation of our hexagons (3.1), (3.2) and (3.3). Indeed by Greene's theorem [**12**], the set of all shuffles of the columns of a key-tableau are contained in its the Knuth class. However under certain conditions we have equality.

Theorem 3.4. [**5**] *Let K be a key-tableau with first column A. Then, the Knuth class of K is equal to the set of all shuffles of its columns if and only if each of its column is either an interval of A or is obtained from an interval of A by removing a single letter.*

This criterion can be easily applied considering the planar representation of the weight of the key-tableau. For instance $K(2,0,1,2,4,2,3)$ is the shuffle of its columns, since each column in the planar representation of the weight $(2,0,1,2,4,2,3)$,



has at most, one gap of size 1. Each column is either an interval of $A = \{1,3,4,5,6,7\}$ or is obtained from an interval of A removing one letter.

Corollary 3.1. *The following statements are equivalent:*
(a) *The Knuth class of a key-tableau over a three-letter alphabet is the set of all shuffles of its columns.*
(b) *J is a three-column frank word if and only if J has one of the following forms*

$$\text{(I)}\ \begin{array}{ccc} A_1^1 & & \\ A_1^2 & A_2^2 & \\ A_1^3 & A_2^3 & A_3^3 \end{array},\quad \text{(II)}\ \begin{array}{ccc} & A_2^1 & \\ A_1^2 & A_2^2 & \\ A_1^3 & A_2^3 & A_3^3 \end{array},\quad \text{(III)}\ \begin{array}{ccc} & & A_3^1 \\ A_1^2 & & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array},$$

$$\text{(IV)}\ \begin{array}{ccc} & & A_3^1 \\ & A_2^2 & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array},\quad \text{(V)}\ \begin{array}{ccc} A_1^1 & & \\ A_1^2 & & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array},\quad \text{(VI)}\ \begin{array}{ccc} & A_2^1 & \\ & A_2^2 & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array},$$

where $A_1^3 \le A_2^3 \le A_3^3$, *with* $|A_1^3| = |A_2^3| = |A_3^3|$; $A_i^r \cap A_i^s = \emptyset$, *for* $r \neq s$, $i = 1, 2, 3$, *and* $A_1^2 \le A_2^2$, $A_1^2 \le A_3^2$, $A_2^2 \le A_3^2$, *with* $|A_1^2| = |A_2^2| = |A_3^2|$.

Proof. This follows either using criterion in corollary 2.2 and evacuation or previous theorem. □

Every three-column frank word is a shuffle of row words of length ≤ 3. Clearly the operators $\tilde{\Theta}_i$ closing hexagons of row words of length ≤ 3, satisfy conditions (c) of lemma 3.2, and, in this case, we have $\tilde{\Theta}_i = \Theta_i$ coinciding with the identity. But if the column-shape of the frank word is any permutation of $(1, 1, 2)$, then the action of the operator Θ_i is not always split into an action on row words of length 3 and on row words of length 1.

Let $J = \begin{array}{ccc} c^2 & b^4 & a^4 \\ & & a^2 \end{array}$ be a three-column contretableau. We have to distinguish three situations. Either we have $c^2 \le a^2 < b^4 \le a^4$ or $a^2 < c^2 \le b^4 \le a^4$ or $c^2 \le b^4 \le a^2 < a^4$.

If $c^2 \le a^2 < b^4 \le a^4$, we have the hexagon closing in the Knuth class of J, that is $\tilde{\Theta}_i = \Theta_i$, satisfying conditions (c) of lemma 3.2, but the operators do not split into operators acting on row words. When $c^2 = a^2$ the following is the only hexagon satisfying conditions (c) of lemma 3.2,

$$\begin{array}{ccccccc} & \overset{\Theta_1}{\nearrow} & \begin{array}{ccc} & b^4 & a^4 \\ c^2 & a^2 & \end{array} & \xrightarrow{\ \Theta_2\ } & \begin{array}{ccc} b^4 & & a^4 \\ c^2 & a^2 & \end{array} & \overset{\Theta_1}{\searrow} & \\ J = \begin{array}{ccc} & b^4 & a^4 \\ c^2 & & a^2 \end{array} & & & & & & \begin{array}{ccc} b^4 & & a^4 \\ c^2 & a^2 & \end{array} \\ & \underset{\Theta_2}{\searrow} & \begin{array}{ccc} & b^4 & a^4 \\ c^2 & & a^2 \end{array} & \xrightarrow[\ \Theta_1\ ]{} & \begin{array}{ccc} & b^4 & a^4 \\ c^2 & a^2 & \end{array} & \underset{\Theta_2}{\nearrow} & \end{array}.$$

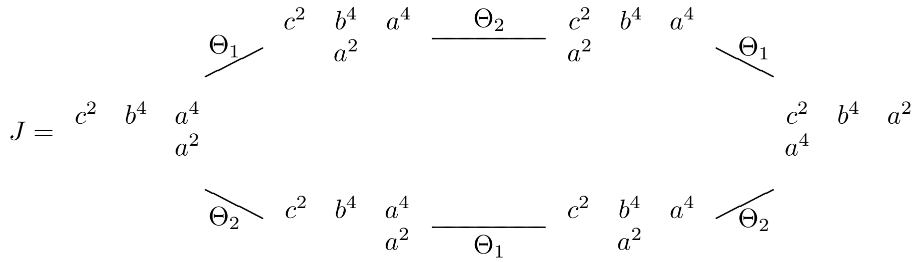
When $c^2 < a^2$ there is still another hexagon (obviously not closing in the Knuth class) in the conditions (c) of lemma 3.2,

$$\begin{array}{ccccccc} & \overset{\Theta_1}{\nearrow} & \begin{array}{ccc} & b^4 & a^4 \\ c^2 & a^2 & \end{array} & \xrightarrow{\ \tilde{\Theta}_2\ } & \begin{array}{ccc} c^2 & & a^4 \\ a^2 & b^4 & \end{array} & \overset{\Theta_1}{\searrow} & \\ J = \begin{array}{ccc} & b^4 & a^4 \\ c^2 & & a^2 \end{array} & & & & & & \begin{array}{ccc} c^2 & & a^4 \\ a^2 & b^4 & \end{array} \\ & \underset{\Theta_2}{\searrow} & \begin{array}{ccc} & b^4 & a^4 \\ c^2 & & a^2 \end{array} & \xrightarrow[\ \Theta_1\ ]{} & \begin{array}{ccc} & b^4 & a^4 \\ c^2 & a^2 & \end{array} & \underset{\tilde{\Theta}_2}{\nearrow} & \end{array}.$$

If $c^2 \le b^4 \le a^2 < a^4$, we have two hexagons closing in the conditions (c) of lemma 3.2: one closing in the Knuth class where the operators $\tilde{\Theta}_i = \Theta_i$ can be split by their action on the row words $c^2 \le b^4 \le a^2$ and a^4,

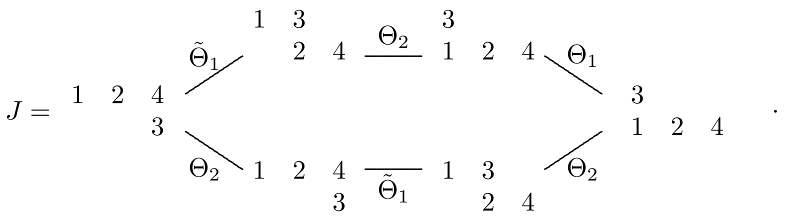
$$\begin{array}{ccccccc} & \overset{\Theta_1}{\nearrow} & \begin{array}{ccc} c^2 & b^4 & a^2 \\ & a^4 & \end{array} & \xrightarrow{\ \Theta_2\ } & \begin{array}{ccc} c^2 & b^4 & a^2 \\ a^4 & & \end{array} & \overset{\Theta_1}{\searrow} & \\ J = \begin{array}{ccc} c^2 & b^4 & a^2 \\ & & a^4 \end{array} & & & & & & \begin{array}{ccc} c^2 & b^4 & a^2 \\ a^4 & & \end{array} \\ & \underset{\Theta_2}{\searrow} & \begin{array}{ccc} c^2 & b^4 & a^2 \\ & & a^4 \end{array} & \xrightarrow[\ \Theta_1\ ]{} & \begin{array}{ccc} c^2 & b^4 & a^2 \\ & a^4 & \end{array} & \underset{\Theta_2}{\nearrow} & \end{array},$$

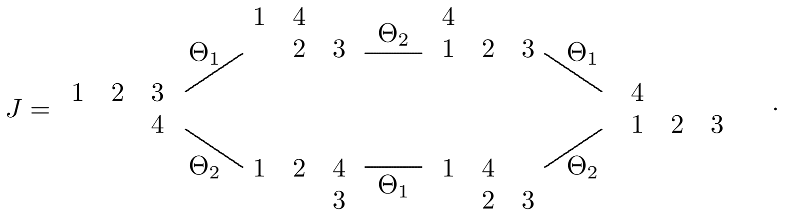
and the other one not closing in the Knuth class where the operators $\tilde{\Theta}_i$ are split by their action on the row words $c^2 \leq b^4 \leq a^4$ and a^2,



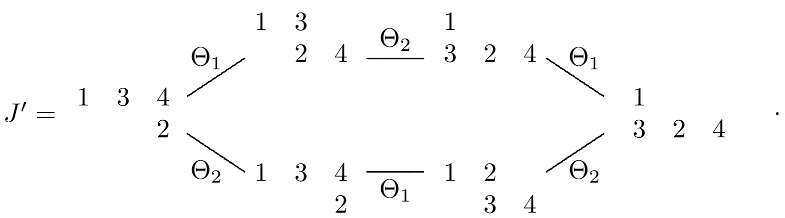
If $a^2 < c^2 \leq b^4 \leq a^4$, there is only one hexagon satisfying conditions (c) of lemma 3.2, which split over the rows $c^2 \leq b^4 \leq a^4$ and a^2.

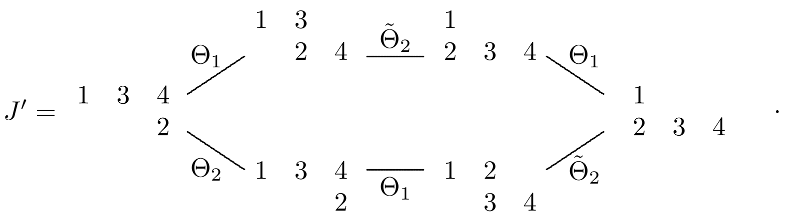
Example 3.5. From the discussion above, there are only two hexagons in the conditions (c) of lemma 3.2 having the contretableau $J = \begin{array}{ccc} 1 & 2 & 4 \\ & & 3 \end{array}$ as a vertex. The second one gives the frank words in the Knuth class.





In the case of the contretableau $J' = \begin{array}{ccc} 1 & 3 & 4 \\ & & 2 \end{array}$ where $1 < 2 < 3 < 4$, we have also only two hexagons in the conditions (c) of lemma 3.2. The first hexagon gives the frank words in the Knuth class.





The meaning of remark 3.3 becomes now clear. As any three-column frank word is a shuffle of rows of length ≤ 3, and frank words of column-shape a permutation of $(2, 1, 1)$, then, given a three-column frank word, it is always possible to construct an hexagon satisfying conditions (c) of lemma 3.2 and having that frank word as a vertex. Define such an hexagon as a *shuffle* of those above. Our next main theorem makes this *shuffle* precise and shows more, any hexagon on three-column frank words satisfying conditions (c) of lemma 3.2 is exactly a shuffle of those hexagons.

THEOREM 3.6. *Let $J = J_3J_2J_1$ be a three-column contretableau. The following assertions are equivalent.*

(a) *There exist $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ defining the hexagon (3.3) such that $\tilde{\Theta}_2[\tilde{\Theta}_1 J] = F_3F_2G_1$ with $F_2 \le L_2$, and $\tilde{\Theta}_1[\tilde{\Theta}_2 J] = L_3H_2H_1$ with $G_2 \le H_2$.*

(b) *The contretableau J has a decomposition, as below, giving rise to the hexagon of frank words with the same weight, and T a tableau,*

(3.4)

$$J=\begin{array}{ccc} A_3^5 & A_2^5 & A_1^5\\ & A_2^4 & A_1^4\\ & A_2^3 & A_1^3\\ A_3^2 & & A_1^2\\ & & A_1^1\end{array}
\xrightarrow{\tilde{\Theta}_1}
\begin{array}{ccc} A_3^5 & A_2^5 & A_1^5\\ & A_2^4 & A_1^4\\ & A_2^3 & A_1^3\\ A_3^2 & A_1^2 & \\ & A_1^1 & \end{array}
\xrightarrow{\tilde{\Theta}_2}
\begin{array}{ccc} A_3^5 & A_2^5 & A_1^5\\ A_2^4 & & A_1^4\\ A_2^3 & & A_1^3\\ A_3^2 & A_1^2 & \\ A_1^1 & & \end{array}
\xrightarrow{\tilde{\Theta}_1}
T=\begin{array}{ccc} A_3^5 & A_2^5 & A_1^5\\ A_2^4 & & A_1^4\\ A_2^3 & A_1^3 & \\ A_3^2 & A_1^2 & \\ A_1^1 & & \end{array},$$

$$J\xrightarrow{\tilde{\Theta}_2}
\begin{array}{ccc} A_3^5 & A_2^5 & A_1^5\\ & A_2^4 & A_1^4\\ A_2^3 & & A_1^3\\ A_3^2 & & A_1^2\\ & & A_1^1\end{array}
\xrightarrow{\tilde{\Theta}_1}
\begin{array}{ccc} A_3^5 & A_2^5 & A_1^5\\ & A_2^4 & A_1^4\\ A_2^3 & A_1^3 & \\ A_3^2 & A_1^2 & \\ & A_1^1 & \end{array}
\xrightarrow{\tilde{\Theta}_2} T$$

where the sets A_i^j are pairwise disjoint in each column J_i, $A_{i+1}^j \le A_i^j$, with $|A_{i+1}^j| = |A_i^j|$,

$$A_3^2 \le A_1^2 < A_2^4 \le A_1^4,$$

$|A_3^2| = |A_1^2| = |A_2^4| = |A_1^4|$, *and* $J_1 \cap A_2^5 \subseteq A_1^5$, $(J_1 \setminus A_1^5) \cap A_2^4 \subseteq A_1^4$, $[J_1 \setminus (A_1^5 \cup A_1^4)] \cap A_2^3 \subseteq A_1^3$, $[J_2 \cup (A_1^2 \cup A_1^1)] \cap A_3^2 \subseteq A_1^2$, *and* $[J_2 \cup (A_1^2 \cup A_1^1)] \cap A_3^5 \subseteq A_2^5$, *where $<$ means $\le$ without common elements.*

PROOF. $(b) \Rightarrow (a)$ By previous corollary, the vertices of the hexagon (3.4) are frank words with the same weight, and clearly satisfy (c) of lemma 3.2.

$(a) \Rightarrow (b)$ The frank words $J_3J_2J_1$ and $J_3G_2G_1$ are, respectively, in the conditions (IV) and (II) of corollary 3.1 and satisfy $\tilde{\Theta}_1 J_3J_2J_1 = J_3G_2G_1$. Then

$$\begin{aligned}&G_1 \subseteq J_1,\ |G_1| = |J_2|,\ J_2 \le G_1,\ J_1 \cap J_2 \subseteq G_1 \text{ and}\\ &G_2 = J_2 \cup (J_1 \setminus G_1),\ J_3 \le G_2.\end{aligned} \tag{3.5}$$

Since the frank word $\tilde{\Theta}_2(J_3J_2J_1) = L_3L_2J_1$ satisfy conditions (III) of corollary 3.1 we have $L_2 \subsetneq J_2$, $|L_2| = |J_3|$, $J_3 \le L_2 \le J_1$ $J_2 \cap J_3 \subseteq L_2$ and $L_3 = J_3 \cup (J_2 \setminus L_2)$. Again the frank word $F_3F_2G_1 = \tilde{\Theta}_2(J_3G_2G_1)$ satisfy (V) of corollary 3.1. Then

$$\begin{aligned}&F_2 \subseteq G_2,\ |F_2| = |J_3|,\ J_3 \le F_2 \le G_1,\ G_2 \cap J_3 \subseteq F_2 \text{ and}\\ &F_3 = J_3 \cup (G_2 \setminus F_2).\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we have $F_2 \subseteq G_2 = J_2 \cup (J_1 \setminus G_1)$. Thus, we may write $F_2 = A_2^5 \cup A_1^2$, with $A_2^5 \subseteq J_2$ and $A_1^2 \subseteq J_1 \setminus G_1$. Moreover, since $J_3 \le F_2$, we may also write $J_3 = A_3^5 \cup A_3^2$, where $A_3^5 \le A_2^5$ e $A_3^2 \le A_1^2$ satisfy $|A_3^5| = |A_2^5|$, $|A_3^2| = |A_1^2|$, $G_2 \cap A_3^5 \subseteq A_2^5$ and $G_2 \cap A_3^2 \subseteq A_1^2$. We define $A_1^1 = J_1 \setminus (G_1 \cup A_1^2)$, therefore $J_1 \setminus G_1 = A_1^1 \cup A_1^2$.

The frank word $F_3XH_1 = \tilde{\Theta}_1 F_3F_2G_1$ satisfy (I) of corollary 3.1. Then

$$\begin{aligned}&H_1 \subseteq G_1,\ |H_1| = |F_2|,\ F_2 \le H_1,\ F_2 \cap G_1 \subseteq H_1 \text{ and}\\ &F_3 \rhd X = F_2 \cup (G_1 \setminus H_1) \rhd H_1.\end{aligned} \tag{3.7}$$

Since $F_2 = A_2^5 \cup A_1^2 \leq H_1$, we can define $A_1^5 = \min\{Z \subseteq H_1 : |Z| = |A_2^5| \text{ and } A_2^5 \leq Z\}$, where the minimum is taken with respect to $\leq$, and $A_1^4 = H_1 \setminus A_1^5$. As $H_1 \subseteq G_1$, put $A_1^3 = G_1 \setminus H_1$. We have $H_1 = A_1^5 \cup A_1^4$ and $X = A_2^5 \cup A_1^2 \cup A_1^3$. From $F_2 \leq H_1$ and the definition of A_1^5, we get $A_3^5 \leq A_2^5 \leq A_1^5$ and $A_3^2 \leq A_1^2 < A_1^4$, where $A_1^2 < A_1^4$ means that $A_1^2 \leq A_1^4$ and $A_1^2 \cap A_1^4 = \emptyset$. Note that from (3.5) and (3.7), we obtain $J_1 \cap A_2^5 \subseteq A_1^5$. By lemma 3.2

$$F_2 \leq L_2. \tag{3.8}$$

Now we consider the bottom edges of our hexagon (3.3). Since the frank word $L_3H_2H_1 = \tilde{\Theta}_1(L_3L_2J_1)$ satisfy (II) of corollary 3.1 we have

$$\begin{aligned} &H_1 \subseteq J_1,\ |H_1| = |L_2|,\ L_2 \leq H_1,\ L_2 \cap J_1 \subseteq H_1 \text{ and} \\ &L_3 \leq H_2 = L_2 \cup (J_1 \setminus H_1) \triangleright H_1. \end{aligned} \tag{3.9}$$

By lemma 3.2, (c), we have

$$G_2 \leq H_2. \tag{3.10}$$

Finally, since $F_3XH_1 = \tilde{\Theta}_2(L_3H_2H_1)$ we have $X \subseteq H_2$, $|X| = |L_3|$, $L_3 \leq X$, $H_2 \cap L \subseteq X$ and $F_3 = L_3 \cup (H_2 \setminus X)$.

By (3.9) and $A_2^5 \cup A_1^2 \cup A_1^3 = X_2 \subseteq H_2 = L_2 \cup A_1^1 \cup A_1^2 \cup A_1^3$, we conclude that $A_2^5 \subseteq L_2 \cup A_1^1$. Since A_2^5 and A_1^1 are disjoint sets, it follows $A_2^5 \subseteq L_2$. Define $A_2^4 = L_2 \setminus A_2^5$ and $A_2^3 = J_2 \setminus L_2$. As $|L_2| = |H_1|$, we also have $|A_1^4| = |A_2^4|$, $|A_1^3| = |A_2^3|$, $(J_1 \setminus A_1^5) \cap A_2^4 \subseteq A_1^4$ and $(J_1 \setminus (A_1^5 \cup A_1^4)) \cap A_2^3 \subseteq A_1^3$. Moreover from the inequality $L_2 \leq H_1$, we get $A_2^4 \leq A_1^4$. By (3.8) and (3.5), we get $A_1^2 < A_2^4$ and by (3.10), we have $A_2^3 \leq A_1^3$. □

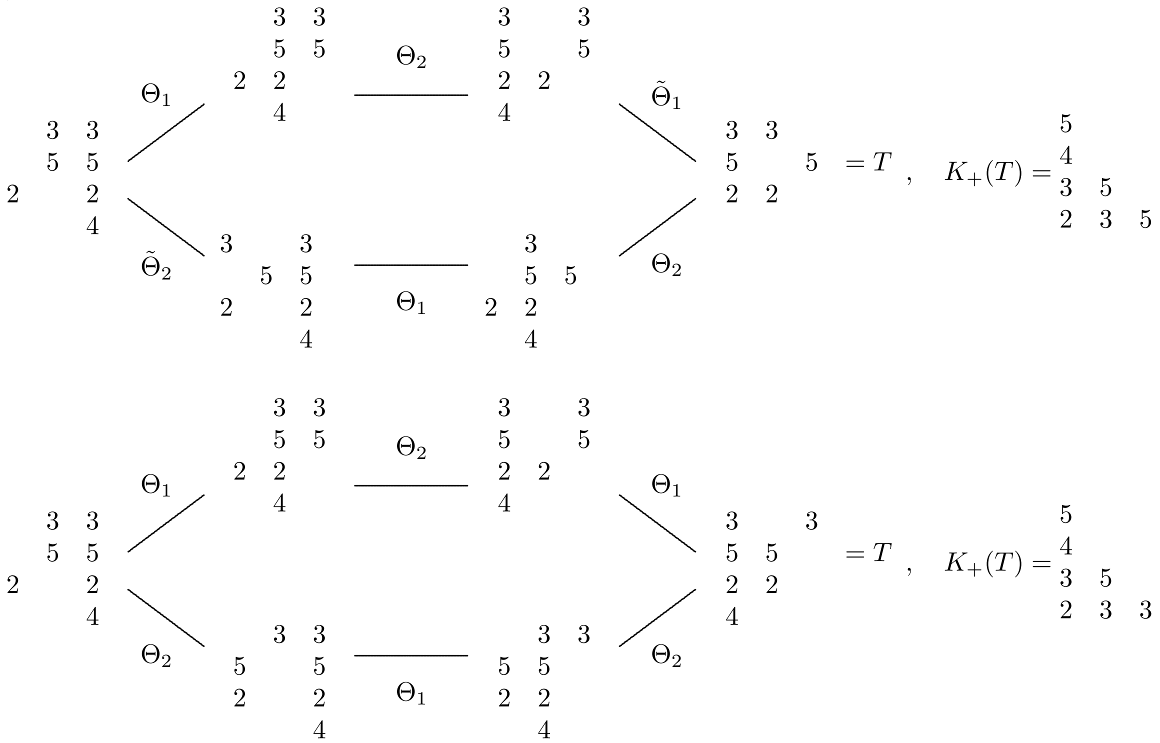
Considering all tableaux of a given shape and weight, this theorem defines an action of the symmetric group on frank words running on the Knuth classes of these tableaux.

¿From the hexagon (3.4) we get, respectively, a right key of T, $\widetilde{K}_+(T) = \begin{array}{lll} A_1^5 & A_1^5 & A_1^5 \\ A_1^4 & A_1^4 & A_1^4 \\ A_1^3 & A_1^3 & \\ A_1^2 & & \\ A_1^1 & & \end{array}$, and a left key of T, $\widetilde{K}_-(T) = \begin{array}{lll} A_3^5 & A_3^5 & A_3^5 \\ A_2^4 & & \\ A_2^3 & A_2^3 & \\ A_3^2 & A_3^2 & A_3^2 \\ A_1^1 & & \end{array}$, with $\widetilde{K}_+(T) \geq \widetilde{K}_-(T)$.

A right (left) key $\widetilde{K}_+(T)$ ($\widetilde{K}_-(T)$) of the tableau T is the key-tableau of the same shape as T whose j-th column is the first (last) column of any skew-tableau in a hexagon (3.4) with the following property: its first (last) column has the same length as the j-th column of T.

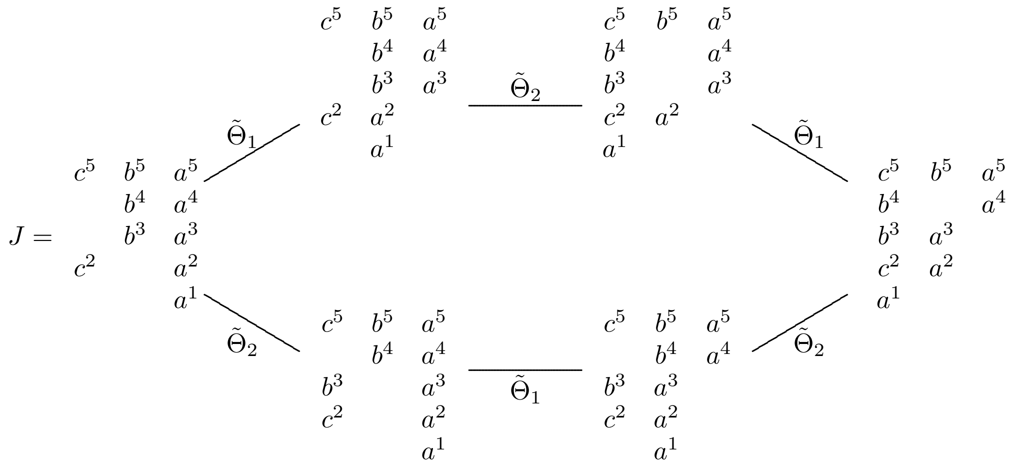
Example 3.7. Below we give two decompositions of the tableau $T = \begin{array}{lll} 5 & & \\ 4 & & \\ 3 & 3 & \\ 2 & 2 & 5 \end{array}$ leading to different left and right keys. The second hexagon gives the frank words in the Knuth class of T.

(3.11)



We may now describe the hexagon (3.2). Without loss of generality, we may consider the hexagon (3.4) in the simplified form in the sense that the sets A_i^j are singular,

(3.12)



with $c^5 \le b^5 \le a^5$, $b^3 \le a^3$, and $c^2 \le a^2 < b^4 \le a^4$. The contretableau J is therefore split into the frank word $Y_1 = c^2b^4a^4a^2$ of shape $(1,1,2)$, and row words $X_2 = c^5b^5a^5, X_3 = b^3a^3$, and $X_4 = a^1$. Let $X_1 = c^2a^2b^4a^4$. We consider the biwords with pairwise distinct biletters

$$\Sigma' = \begin{pmatrix} J_3 & J_2 & J_1 \\ 3^2 & 2^3 & 1^5 \end{pmatrix} \longleftrightarrow \Pi = \begin{pmatrix} c^2a^2b^4a^4 & c^5b^5a^5 & b^3a^3 & a^1 \\ 3\ 1\ 2\ 1 & 3\ 2\ 1 & 2\ 1 & 1 \end{pmatrix} \longleftrightarrow \Sigma = \begin{pmatrix} (J_3J_2J_1)\uparrow \\ w \end{pmatrix}, \tag{3.13}$$

where Π is obtained by sorting the billeters of Σ', and Σ is obtained by sorting the billeters of Π in weakly increasing rearrangement for the anti-lexicographic order with priority on the first row. Since $(J_3J_2J_1)\uparrow$ is a shuffle of the biwords X_1, X_2, X_3 and X_4, then w is a shuffle of 3121, 321, 21 and 1 such that the biword Σ is a shuffle of $\begin{pmatrix} X_1 \\ 3121 \end{pmatrix}$, $\begin{pmatrix} X_2 \\ 321 \end{pmatrix}$, $\begin{pmatrix} X_3 \\ 21 \end{pmatrix}$ and $\begin{pmatrix} X_4 \\ 1 \end{pmatrix}$. Therefore the hexagon (3.2) is a "*shuffle*" of four elementary hexagons,

$$\begin{array}{ccccccc} & & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 2\ 1 \end{pmatrix} & \overset{\theta_2}{\text{———}} & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 3\ 1 \end{pmatrix} & & \\ & \theta_1 \diagup & & & & \diagdown \theta_1 & \\ \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 1\ 2\ 1 \end{pmatrix} & & & & & & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 3\ 1 \end{pmatrix} \\ & \theta_2 \diagdown & & & & \diagup \theta_2 & \\ & & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 1\ 2\ 1 \end{pmatrix} & \underset{\theta_1}{\text{———}} & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 2\ 1 \end{pmatrix} & & \end{array} \tag{3.14}$$

$$\begin{array}{ccccccc} & & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} & \overset{\theta_2}{\text{———}} & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} & & \\ & \theta_1 \diagup & & & & \diagdown \theta_1 & \\ \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} & & & & & & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \\ & \theta_2 \diagdown & & & & \diagup \theta_2 & \\ & & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} & \underset{\theta_1}{\text{———}} & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} & & \end{array} \tag{3.15}$$

$$\begin{array}{ccccccc} & & \begin{pmatrix} b^3a^3 \\ 2\ 1 \end{pmatrix} & \overset{\theta_2}{\text{———}} & \begin{pmatrix} b^3a^3 \\ 3\ 1 \end{pmatrix} & & \\ & \theta_1 \diagup & & & & \diagdown \theta_1 & \\ \begin{pmatrix} b^3a^3 \\ 2\ 1 \end{pmatrix} & & & & & & \begin{pmatrix} b^3a^3 \\ 3\ 2 \end{pmatrix} \\ & \theta_2 \diagdown & & & & \diagup \theta_2 & \\ & & \begin{pmatrix} b^3a^3 \\ 3\ 1 \end{pmatrix} & \underset{\theta_1}{\text{———}} & \begin{pmatrix} b^3a^3 \\ 3\ 2 \end{pmatrix} & & \end{array} \tag{3.16}$$

$$\begin{array}{ccccccc} & & \begin{pmatrix} a^1 \\ 2 \end{pmatrix} & \overset{\theta_2}{\text{———}} & \begin{pmatrix} a^1 \\ 3 \end{pmatrix} & & \\ & \theta_1 \diagup & & & & \diagdown \theta_1 & \\ \begin{pmatrix} a^1 \\ 1 \end{pmatrix} & & & & & & \begin{pmatrix} a^1 \\ 3 \end{pmatrix} \\ & \theta_2 \diagdown & & & & \diagup \theta_2 & \\ & & \begin{pmatrix} a^1 \\ 1 \end{pmatrix} & \underset{\theta_1}{\text{———}} & \begin{pmatrix} a^1 \\ 2 \end{pmatrix} & & \end{array} . \tag{3.17}$$

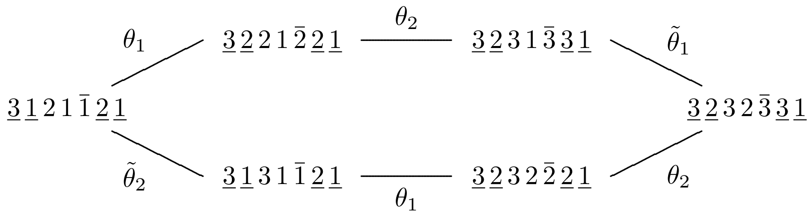
A Yamanouchi tableau is a key-tableau whose shape and weight coincide. A Yamanouchi word is a word congruent to a Yamanouchi tableau. By corollary 3.1, every three-letter Yamanouchi word w is a shuffle of $k \geq 0$ words 3121, $l_1 \geq 0$ words 321, $l_2 \geq 0$ words 21 and $l_3 - k \geq 0$ words 1, that, by abuse of notation, we shall write $w = shuffle((3121)^k, (321)^{l_1}, (21)^{l_2}, 1^{l_3-k})$.

Theorem 3.8. *The hexagon* (3.2) *is a "shuffle" of the hexagons defined by the bottom rows of the four hexagons* (3.14), (3.15), (3.16) *and* (3.17) *with the appropriate multiplicities. That is, there exist a shuffle of* $k \geq 0$ *words* 3121, $l_1 \geq 0$ *words* 321, $l_2 \geq 0$ *words* 21 *and* $l_3 - k \geq 0$ *words* 1, $w = shuffle((3121)^k, (321)^{l_1}, (21)^{l_2}, 1^{l_3-k})$, *such that*

(a) $\tilde{\theta}_i w = shuffle((\theta_i 3121)^k, (\theta_i 321)^{l_1}, (\theta_i 21)^{l_2}, (\theta_i 1)^{l_3-k})$, $i = 1, 2$;
(b) $\tilde{\theta}_i \tilde{\theta}_j w = shuffle((\theta_i\theta_j 3121)^k, (\theta_i\theta_j 321)^{l_1}, (\theta_i\theta_j 21)^{l_2}, (\theta_i\theta_j 1)^{l_3-k})$, $1 \leq i \neq j \leq 2$;
(c) $\tilde{\theta}_1 \tilde{\theta}_2 \tilde{\theta}_1 w = shuffle((\theta_1\theta_2\theta_1 3121)^k, (\theta_1\theta_2\theta_1 321)^{l_1}, (\theta_1\theta_2\theta_1 21)^{l_2}, (\theta_1\theta_2\theta_1 1)^{l_3-k})$.

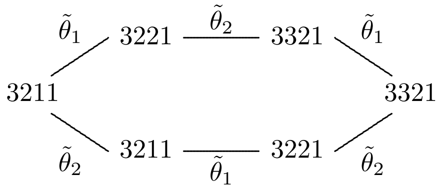
Example 3.9. The hexagon (3.11) gives rise to the hexagon, below, where the operators are based on nonstandard pairing of parentheses

(3.18)



(the bared letters indicate the subwords 3121 and 1 in the shuffle).

Remark 3.10. The following example is the translation of the previous remark to hexagon (3.2). The hexagon



is not a shuffle of the two hexagons (3.15) and (3.17).

Next we show that the family of actions of S_3 defined by the operators $\tilde{\theta}_i$ $(\tilde{\Theta}_i)$, $i = 1, 2$, based on shuffle decompositions of a three-letter Yamanouchi word w (three-column tableau) as shown in the previous theorems, includes the action defined by the operators θ_i (Θ_i), $i = 1, 2$. This is achieved in the following algorithm, where a special shuffle decomposition for three-letter Yamanouchi word w is exhibited. Using (3.13), it follows that the hexagon (3.4) contains, in particular, the action defined by the operators Θ_i. That is, Lascoux-Schützenberger actions of the symmetric group on frank words are obtained from a particular shuffle decomposition of a three-column tableau into rows of length ≤ 3 and tableaux of column-shape $(2, 1, 1)$, and on words congruent with keys are obtained from a shuffle decomposition of a three-letter Yamanouchi word into words 3121 and column-words 321, 21, 1.

We denote by $w_{|A}$ the subword of w obtained by suppressing the letters not in A. If $X \subseteq [l]$ with l the length of w, then $w|X$ is the subword of w defined by the letters of w in positions X. If $X, Y \subseteq [l]$ with $X \cap Y = \emptyset$, then $w|(X, Y)$ is the shuffle of the subwords $w|X$ and $w|Y$ defined by the letters of w in positions $X \cup Y$. By induction, we define $w|(X_1, \ldots, X_k)$, for any $k \geq 0$, putting the empty word for $k = 0$.

Algorithm 3.11. Let w be a Yamanouchi word over a three-letter alphabet. Our algorithm is presented as a three step definition.

Step 1. Consider the subword $w_{|\{2,1\}}$ and bracket every factor 21 of $w_{|\{2,1\}}$. The letters which are not bracketed constitute a subword of $w_{|\{2,1\}}$. Then bracket every factor 21 of this subword. Again, the letters which are not bracketed constitute a subword. Continue this procedure until it stops, that is, until we get a word consisting of l_1 non bracketed letters $1's$ in w. This bracketing process enables us to decompose w as

$$w|(I_1, \ldots, I_{l_3+l_2}, J_1, \ldots, J_{l_3}, K_1, \ldots, K_{l_1}), \tag{3.19}$$

where $w|I_l = 21$, $l \in [l_3 + l_2]$, $w|J_l = 3$, $l \in [l_3]$, and $w|K_l = 1$, $l \in [l_1]$.

Step 2. Let w' be the subword of w obtained by removing all letters 1 belonging to the factors $w|I_l$, for all $l \in [l_3 + l_2]$. As in the previous step, we bracket all the successive factors 32 and 31 of w'. We get a new decomposition (3.19), by making the unions of k sets J_l with k sets K_l, for some integer $0 \leq q \leq \min\{l_3, l_1\}$, and making the unions of the remaining $l_3 - q$ sets J_l with $l_3 - q$ sets I_l:

$$w|(F_1, \ldots, F_q, G_1, \ldots, G_{l_3-q}, I_1, \ldots, I_{l_2+q}, K_1, \ldots, K_{l_1-q}),$$

where $w|F_l = 31$, $l \in [q]$, $w|G_l = 321$, $l \in [l_3 - q]$, $w|I_l = 21$, $l \in [l_2 + q]$, and $w|K_l = 1$, $l \in [l_1 - q]$ (reordering the sets I_i's, J_j's and K_l's in (3.19) if necessary).

Step 3. Finally, let w'' be the subword of w obtained by removing the subwords $w|G_l = 321$ and $w|K_l = 1$, for all $l \geq 1$. As before, we bracket all the successive factors 3121 of w''. This operator consists of the union of the q sets F_l with q sets I_l. The decomposition of w obtained in this way, is denoted by $w|(I_1^*, \ldots, I_{l_3+l_2+l_1-q}^*)$, where $w|I_l^* = 3121$, $l \in [q]$, $w|I_l^* = 321$, $l \in [q+1, l_3]$, $w|I_l^* = 21$, $l \in [l_3 + 1, l_3 + l_2]$, and $w|I_l^* = 1$, $l \in [l_3 + l_2 + 1, l_3 + l_2 + l_1 - q]$.

Next example illustrates the application of the previous algorithm.

EXAMPLE 3.12. Let $w = 33121121 \equiv K(4,2,2)$. Following the first step of algorithm 3.11, we bracket all the successive factors 21 of $w_{|\{1,2\}}$, that is, 331(21)1(21), obtaining in this way the decomposition

$$w = w|(\{4,5\},\{7,8\},\{1\},\{2\},\{3\},\{6\}),$$

where $w|\{4,5\} = w|\{7,8\} = 21$, $w|\{1\} = w|\{2\} = 3$ and $w|\{3\} = w|\{6\} = 1$. Next, let $w' = 3312-12-$ (where $-$ indicates the place of the suppressed letters of w) be the subword of w obtained by removing the letters 1 belonging to $w|\{4,5\}$ and $w|\{7,8\}$, and bracket all the successive factors 31 and 32 of w'. Thus, we have $w' = 3(31)2-12-$, with the letters 3 and 1 belonging to $w|\{2\}$ and $w|\{3\}$, respectively; and then, we have $w'_1 = (3--2)-12-$, with the letters 3 and 2 of this factor belonging to $w|\{1\}$ and $w|\{4,5\}$, respectively. Then, we get the decomposition

$$w = w|(\{1,4,5\},\{7,8\},\{2,3\},\{6\}),$$

with $w|\{1,4,5\} = 321$, $w|\{7,8\} = 21$, $w|\{2,3\} = 31$ and $w|\{6\} = 1$. Finally, let $w'' = -31---21$ be the subword of w obtained by removing the subwords $w|(\{1,4,5\} = 321$ and $w|\{6\} = 1$. This word has only one factor 3121 and thus we get the decomposition

$$w = w|(\{2,3,7,8\}^*,\{1,4,5\}^*,\{6\}^*) = \underline{3}\,\overline{3\,1}\,\underline{2\,1}\,1\,\overline{2\,1},$$

where the underlined letters define 3121, the overline letters define 321 and the remaining letter define the shuffle component 1. It is easy to check that the parenthesis matching operations induced by this decomposition are the standard ones:

$$\begin{array}{ccccccc}
 & \theta_1 \nearrow & \underline{3}\,\overline{3\,2}\,\underline{2\,1}\,2\,\overline{2\,1} & \xrightarrow{\theta_2} & \underline{3}\,\overline{3\,2}\,\underline{2\,1}\,3\,\overline{3\,1} & \searrow \theta_1 & \\
\underline{3}\,\overline{3\,1}\,\underline{2\,1}\,1\,\overline{2\,1} & & & & & & \underline{3}\,\overline{3\,2}\,\underline{2\,1}\,3\,\overline{3\,1} \\
 & \theta_2 \searrow & \underline{3}\,\overline{3\,1}\,\underline{2\,1}\,1\,\overline{2\,1} & \xrightarrow[\theta_1]{} & \underline{3}\,\overline{3\,2}\,\underline{2\,1}\,2\,\overline{2\,1} & \nearrow \theta_2 &
\end{array} \quad . \tag{3.20}$$

THEOREM 3.13. *Let w be a Yamanouchi word over a three-letter alphabet and consider the decomposition $w|(I_1^*,\dots,I_q^*)$ given by algorithm 3.11. Then, $\theta_i\theta_j\cdots\theta_k(w) = w^{ij\cdots k}$, for all $ij\cdots k \in [2]^*$, where $w^{ij\cdots k}|I_l = \theta_i\theta_j\cdots\theta_k(w|I_l)$, for all $l = 1,\dots,q$.*

PROOF. By the construction of $w|(I_1^*,\dots,I_q^*)$, it is clear that $\theta_i\theta_j\cdots\theta_k(w) = w^{ij\cdots k}$, for $ij\cdots k \in \{1,21,121\}$. For the computation of $\theta_2(w)$, we must match all successive factors 32 of $w_{|\{2,3\}}$, until we get the subword 2^{l_2}, for some nonnegative integer l_2. Each matched pair 32 belongs either to a component 321, or 3121, while the letters 2 of the subword 2^{l_2} belong to components 21. The word $\theta_2(w)$ is then obtained replacing in w the subword 2^{l_2} by 3^{l_2}. Since $\theta_2(3121) = 3121$, $\theta_2(321) = 321$ and $\theta_2(21) = 31$, it follows that $\theta_2(w) = w^2$.

Finally, consider the subword $\theta_1\theta_2(w)$, obtained by matching the successive factors 21 of $\theta_2(w)_{|\{1,2\}}$. By the construction of $w|(I_1^*,\dots,I_q^*)$, each one of the matched pairs 21 belongs either to a component 321 or 3121. On the other hand, each letter of the subword $1^{l_2+l_1}$, obtained by removing the matched pairs, is itself a component or it represents the leftmost letter 1 of a component 3121. Since $\theta_1\theta_2(3121) = 3221$ and $\theta_1\theta_2(1) = 2$, we get $\theta_1\theta_2(w) = w^{12}$. □

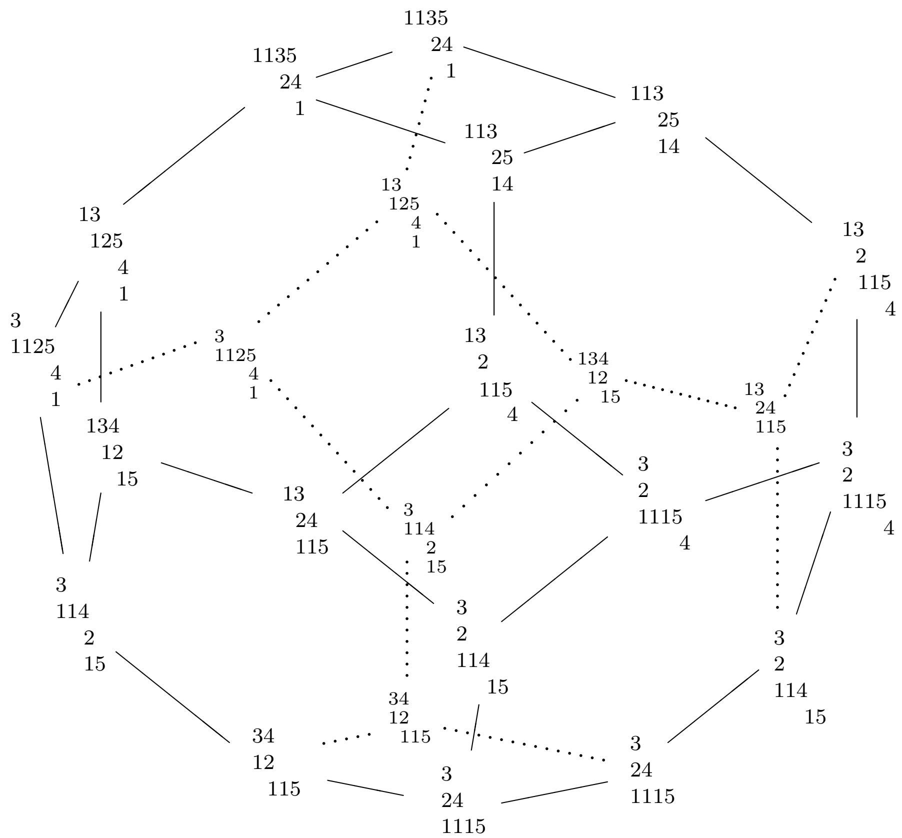
Finally to each hexagon (3.4) corresponds an hexagon (3.1).

THEOREM 3.14. [**4**] *To each hexagon* (3.4) *corresponds an hexagon* (3.1). *That is, given an hexagon* (3.4), *$\exists U \sim I$ such that, for some partition α, $\{\Delta_\alpha UK(\sigma\beta) : \sigma \in S_3\}$ is an hexagon whose indexing-set frank words are those of* (3.4).

EXAMPLE 3.15. Dual permutahedra in S_4 generated by *variants of jeu de taquin* and *non-standard reflection crystal operators.*

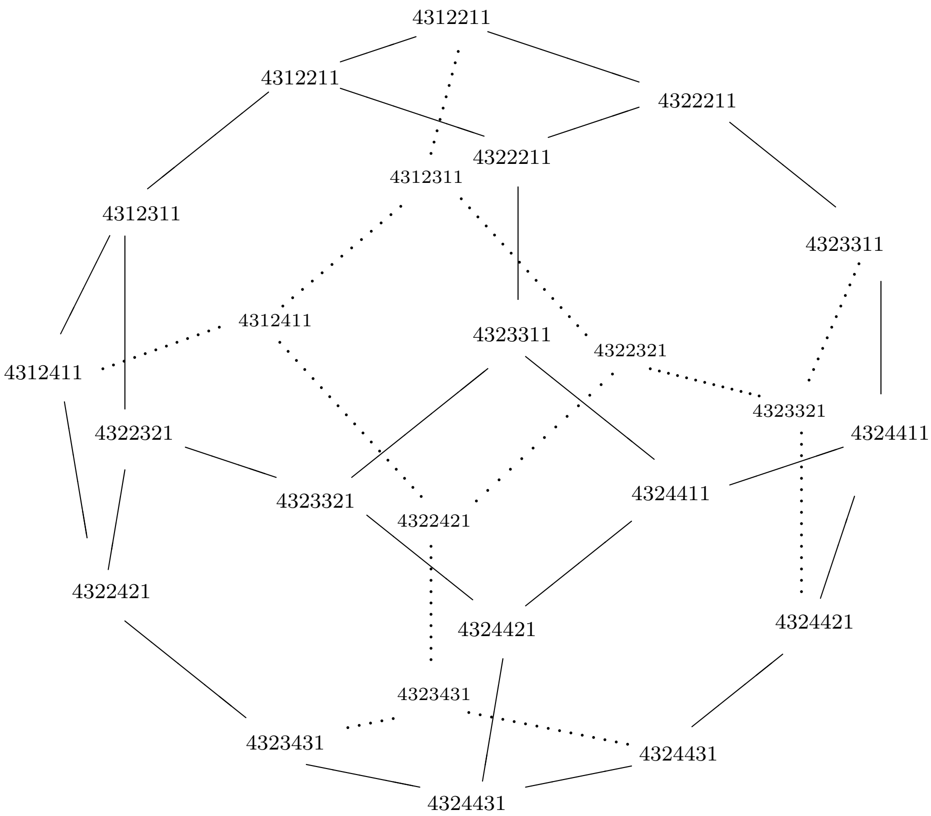
Consider the Yamanouchi word $w = 4312211 \in [4]^*$ and the contretableau $J = 1\,1\,32\,451$. The biwords $\Sigma = \begin{pmatrix} 1112345 \\ 4312211 \end{pmatrix}$ and $\Sigma' = \begin{pmatrix} 1\,1\,32\,451 \\ 4\,3\,22\,111 \end{pmatrix}$ correspond by the dual RSK to the pair (K,P), with $K = 4321\,21\,1$ the Yamanouchi tableau of shape $(3,2,1,1)$, and $P = 321\,51\,1\,4$.

The vertices of the following permutahedron in S_4 contains the contretableau $J = 1\,1\,32\,541 \equiv P = 321\,51\,1\,4$ and the tableau $T = 321\,41\,1\,5$ of the same shape and weight as P. The remain ones are frank words in the Knuth classes of P and T.



(3.21)

The vertices of the corresponding dual permutahedron in S_4 contain the Yamanouchi words $4312211 \equiv K(3211)$ and $4324431 \equiv K(1123)$ with $\mathcal{Q}$-symbols $Q = \begin{array}{lll} 6 & & \\ 3 & & \\ 2 & 7 & \\ 1 & 4 & 5 \end{array} = (std\,P)^t$ and $Q' = \begin{array}{lll} 7 & & \\ 3 & & \\ 2 & 6 & \\ 1 & 4 & 5 \end{array} = (std\,T)^t$ respectively. The $\mathcal{Q}$-symbols of the remain ones are either Q or Q'.



(3.22)

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