An analogue of the Robinson–Schensted–Knuth correspondence and non-symmetric Cauchy kernels for truncated staircases

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Abstract

We prove a restriction of an analogue of the Robinson–Schensted–Knuth correspondence for semi-skyline augmented fillings, due to Mason, to multisets of cells of a staircase possibly truncated by a smaller staircase at the upper left end corner, or at the bottom right end corner. The restriction to be imposed on the pairs of semi-skyline augmented fillings is that the pair of shapes, rearrangements of each other, satisfies an inequality in the Bruhat order, w.r.t. the symmetric group, where one shape is bounded by the reverse of the other. For semi-standard Young tableaux the inequality means that the pair of their right keys is such that one key is bounded by the Schützenberger evacuation of the other. This bijection is then used to obtain an expansion formula of the non-symmetric Cauchy kernel, over staircases or truncated staircases, in the basis of Demazure characters of type A, and the basis of Demazure atoms. The expansion implies Lascoux expansion formula, when specialised to staircases or truncated staircases, and make explicit, in the latter, the Young tableaux in the Demazure crystal by interpreting Demazure operators via elementary bubble sorting operators acting on weak compositions.

Keywords: Young tableau, semi-skyline augmented filling, analogue of the Robinson–Schensted–Knuth correspondence, isobaric divided differences, Demazure character, Demazure atom, Demazure crystal graph, nonsymmetric Cauchy kernels.
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1. Introduction and statement of results

The purpose of this paper is to give a bijective proof via the standard Robinson–Schensted–Knuth (RSK)-type bijection [21] for the truncated staircase shape version

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of the Cauchy identity, due to Lascoux, where Schur polynomials are replaced by Demazure characters and Demazure atoms [26, 7]. To this aim we build on the interesting analogue of the RSK bijection, recently given by Mason [35], where semi-standard tableaux (SSYTs) are replaced by semi-skyline augmented fillings (SSAFs). The later combinatorial objects are coming from the Haglund–Haiman–Loehr formula for non-symmetric Macdonald polynomials [10]. The RSK correspondence is an important combinatorial bijection between two line arrays of positive integers (or non-negative integer matrices) and pairs of SSYTs of the same shape with applications to the representation theory of the Lie algebra $\mathfrak{gl}_n$, and to the theory of symmetric functions among others. Mason's bijection has the advantage of giving information about the filtration of irreducible representations of $\mathfrak{gl}_n$ by Demazure modules, which is detected by the key of a SSYT, after Lascoux and Schützenberger [23, 24], and manifested in the shape of a SSAF [34]. Although a general Ferrers shape version of the Cauchy identity had been given by Lascoux in [26], aside from the staircase shape, the characterization of the pairs of SSYTs is expressed in a less explicit way. Regarding to the shapes to be considered here, our expansions are explicit. Lascoux's proof in [26] and Fu–Lascoux's proof in [7] (in type A case) are different from the standard bijective proof based on an RSK-type correspondence, which is precisely what is done here.

1.1. Crystals, Demazure crystals and keys

Given the general Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, and its quantum group $U_q(\mathfrak{gl}_n)$ - the q-analogue of the universal enveloping algebra $U(\mathfrak{gl}_n)$ - finite-dimensional representations of $U_q(\mathfrak{gl}_n)$ are also classified by the highest weight. Let λ be a dominant integral weight (i.e. a partition), and $V(\lambda)$ the integrable representation with highest weight λ, and u_λ the highest weight vector. For a given permutation w in the symmetric group $\mathfrak{S}_n$, the shortest in its class modulo the stabiliser of λ, the Demazure module is defined to be $V_w(\lambda) := U_q(\mathfrak{g})^{>0}.u_{w\lambda}$, and the Demazure character is the character of $V_w(\lambda)$. (We refer the reader to [12, 22] for details.) In the early 90's Kashiwara [16, 17] has associated with λ a crystal $\mathfrak{B}^\lambda$, which can be realised in type A as a coloured directed graph having vertices all SSYTs of shape λ with entries $\leq n$, and arrows $P \xrightarrow{i} P'$ if and only $f_iP = P'$, for each crystal (coplactic) operator f_i, $1 \leq i < n$. The coloured directed graph $\mathfrak{B}^\lambda$ reflects the combinatorial structure of the given integrable representation $V(\lambda)$ and the relationship between $\mathfrak{B}^\lambda$ and the module $V(\lambda)$ can be made precise using the notion of crystal basis for $V(\lambda)$ [16, 17]. Littelmann conjectured [31] and Kashiwara proved [18] that the intersection of a crystal basis of V_λ with $V_w(\lambda)$ is a crystal basis for $V_w(\lambda)$. The resulting subset $\mathfrak{B}_{w\lambda} \subseteq \mathfrak{B}^\lambda$ is called Demazure crystal, and the Demazure character corresponding to λ and w, is the polynomial combinatorially expressed by the SSYTs in the Demazure crystal $\mathfrak{B}_{w\lambda}$. These polynomials are the key polynomials corresponding to λ and w in Reiner-Shimozono's work [39].

The irreducible representations of $\mathfrak{gl}_n$ have then a filtration by Demazure modules, compatible with the Bruhat order of $\mathfrak{S}_n$ and the crystal structure. That is, $\mathfrak{B}_{w'\lambda} \subset \mathfrak{B}_{w\lambda}$ whenever $w' < w$ in the Bruhat order on the classes modulo the stabiliser of λ and $\mathfrak{B}^\lambda = \bigcup_{w\in\mathfrak{S}_n} \mathfrak{B}_{w\lambda}$. In particular, if ω is the longest permutation of $\mathfrak{S}_n$, $\mathfrak{B}^\lambda = \mathfrak{B}_{\omega\lambda}$. Given that the Schur polynomial s_λ is expressed combinatorially by all SSYTs of shape λ and entries $\leq n$, in the late 80's, Lascoux and Schützenberger [23, 24] identified the SSYTs contributing to the key polynomial corresponding to λ and w via a condition in the Bruhat order involving their right keys. That is, the key polynomial is decomposed into a sum of Demazure atoms [34] (standard bases [24]) which is equivalent to a decomposition of the Demazure crystal $\mathfrak{B}_{w\lambda}$.

1.2. Demazure characters and Demazure operators

The Demazure character (or key polynomial) κ_α and the Demazure atom $\widehat{\kappa}_\alpha$, with $\alpha \in \mathbb{N}^n$ (a rearrangement of λ), are also generated recursively by the application of Demazure operators (or isobaric divided differences [28]) π_i and $\hat{\pi}_i := \pi_i - 1$, for $1 \leq i < n$, respectively, to the monomial x^λ. Such operators are defined for each simple reflection of $\mathfrak{S}_n$ and satisfy the braid relations of a Coxeter group. See Section 5 for precise definitions, recursive rules and combinatorial descriptions. They were introduced by Demazure [5] for all Weyl groups and were studied combinatorially, in the case of $\mathfrak{S}_n$, by Lascoux and Schützenberger [23, 24] who produce a crystal structure by providing a combinatorial version for Demazure operators in terms of crystal (or coplactic) operators [27].

1.3. Combinatorics of nonsymmetric Macdonald polynomials and RSK analogue

Non symmetric Macdonald polynomials $E_\alpha(X, q, t)$, with $\alpha \in \mathbb{N}^n$ (we are assuming zero in $\mathbb{N}$), form a basis of $\mathbb{C}(q, t)[x_1, \dots, x_n]$, and were introduced and studied by Opdam [36], Cherednik [4], and Macdonald [32]. Their representational-theoretical nature in connection with Demazure characters has been investigated by Sanderson [40] and Ion [14]. In 2004, Haglund, Haiman and Loher gave a combinatorial formula for non symmetric Macdonald polynomials [10]. Specialising the Haglund–Haiman–Loehr formula for the nonsymmetric Macdonald polynomial $E_\alpha(x; q^{-1}; t^{-1})$, [10, Corollary 3.6.4], by letting $q, t \to 0$, implies that $E_\alpha(x; \infty; \infty)$ is combinatorially expressed by all SSAFs of shape α. See Section 3 for details on SSAFs. These polynomials are also a decomposition of the Schur polynomial s_λ, with λ the decreasing rearrangement of α. Semi-skyline augmented fillings are in bijection with SSYTs so that the content is the same and the right key is given by the shape of the SSAF [35]. The Demazure atom $\widehat{\kappa}_\alpha$ and $E_\alpha(x; \infty; \infty)$ are then equal [10, 34]. An interesting analogue of the RSK bijection was given by Mason [35], where SSYTs are replaced by SSAFs which manifest the keys.

1.4. Our results

We consider the following Ferrers diagram, in the French convention, $\lambda = (m^{n-m+1}, m-1, \ldots, n-k+1)$, $1 \leq m \leq n$, $1 \leq k \leq n$, $n+1 \leq m+k$, shown in green colour in Figure 1. Theorem 3, in Section 4, exhibits an RSK-type bijection between multisets

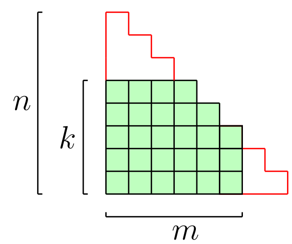


Figure 1: The truncated Ferrers shape λ, in green, fitting the k by m rectangle so that the staircase of size n is the smallest containing λ. If $k \leq m$, $(k, k-1, \ldots, 1)$ is the biggest staircase inside of λ.

of cells of λ and pairs of SSAFs where the image is described by a Bruhat inequality between the keys of the recording and the insertion fillings. When $m+k = n+1$ then λ is a rectangle and it reduces to the ordinary RSK correspondence in the sense that the inequality on the right keys is relaxed. This bijection is used in Section 6, Theorem 6, to give an expansion of the non-symmetric Cauchy kernel $\prod_{(i,j)\in\lambda}(1 - x_i y_j)^{-1}$ in the basis of Demazure characters, and the basis of Demazure atoms. The kernel expands

$$\prod_{\substack{(i,j)\in\lambda \\ k\leq m}} (1 - x_i y_j)^{-1} = \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\alpha)}(y), \tag{1}$$

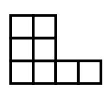
with $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ such that, for each $i = k, \ldots, 1$, the entry α_i is the maximum element among the last $\min\{i, n-m+1\}$ entries of μ in reverse order, after deleting α_j, for $i < j \leq k$. If $m < k$, the formula is symmetrical, swapping in (1) x with y, and k with m. If λ is a rectangle, α is a partition and the classical Cauchy identity in the basis of Schur polynomials is recovered; and if λ is the staircase of length n, α is the reverse of μ, and Lascoux's expansion in Theorem 6 of [26], and in [7], is also recovered. For truncated staircases, the expansion (1) implies Lascoux's formula in Theorem 7 of [26], and makes explicit the SSYTs of the Demazure crystal.

Our paper is organised in six sections. In Section 2, we first recall the definitions of compositions, Young tableaux, and key tableaux, then the Bruhat orders of the symmetric group $\mathfrak{S}_n$, their several characterizations, and their conversions to a $\mathfrak{S}_n$-orbit. In Section 3, we review the necessary terminology and theory of SSAFs, in particular, the RSK analogue for SSAFs along with useful properties for the next section. Then, in Section 4, we give our main result, Theorem 3, and an illustration of the bijection described in this theorem. Section 5 is devoted to the necessary theory of crystal graphs in type A in connection with the combinatorial descriptions of Demazure operators and the two families of key polynomials to be used in the last section, in particular, in Lemma 3. Finally, in the last section, we apply the bijection provided in Theorem 3 to obtain expansions of Cauchy kernels over truncated stair cases as described in Theorem 6.

2. Weak compositions, key tableaux and Bruhat orders on $\mathfrak{S}_n$ and orbits

2.1. Young tableaux and key tableaux

Let $\mathbb{N}$ denote the set of non-negative integers. Fix a positive integer n, and define $[n]$ as the set $\{1, \ldots, n\}$. A weak composition $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a vector in $\mathbb{N}^n$. If $\gamma_i = \cdots = \gamma_{i+k-1}$, for some $k \geq 1$, then we also write $\gamma = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_i^k, \gamma_{i+k}, \ldots, \gamma_n)$. We often concatenate weak compositions $\alpha \in \mathbb{N}^r$ and $\beta \in \mathbb{N}^s$, with $r+s = n$, to form the weak composition $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s) \in \mathbb{N}^n$. A weak composition γ whose entries are in weakly decreasing order, that is, $\gamma_1 \geq \cdots \geq \gamma_n$, is said to be a partition. Every weak composition γ determines a unique partition γ^+ obtained by arranging the entries of γ in weakly decreasing order. A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is identified with its Young diagram (or Ferrers shape) $dg(\lambda)$ in French convention, an array of left-justified cells with λ_i cells in row i from the bottom, for $1 \leq i \leq n$. The cells are located in the diagram $dg(\lambda)$ by their row and column indices (i, j), where $1 \leq i \leq n$ and $1 \leq j \leq \lambda_i$. The number $\ell(\lambda)$ of rows in the Young diagram $dg(\lambda)$ with a positive number of cells is said to be the length of the partition λ. For instance, for $n = 4$, if $\lambda = (4, 2, 2, 0)$, $\ell(\lambda) = 3$, and the Young diagram of $\lambda = (4, 2, 2, 0)$, is



A filling of shape λ (or a filling of $dg(\lambda)$), in the alphabet $[n]$, is a map $T : dg(\lambda) \to [n]$. A semi-standard Young tableau (SSYT) T of shape $sh(T) = \lambda$, in the alphabet $[n]$, is a filling of $dg(\lambda)$ which is weakly increasing in each row from left to right and strictly increasing up in each column. Let SSYT_n denote the set of all semi-standard Young tableaux with entries $\leq n$. The column word of $T \in \mathrm{SSYT}_n$ is the word, over the alphabet $[n]$, which consists of the entries of each column, read top to bottom and left to right. The content or weight of $T \in \mathrm{SSYT}_n$ is the content or weight of its column word in the alphabet $[n]$, which is the weak composition $c(T) = (\alpha_1, \ldots, \alpha_n)$ such that α_i is the multiplicity of i in the column word of T. For instance, a SSYT of shape $\lambda = (4, 2, 2, 0)$, in the alphabet $[4]$, with $col(T) = 32143123$ and content $c(T) = (2, 2, 3, 1)$ is

$$T = \begin{array}{|c|c|cc}\cline{1-2} 3 & 4 & & \\ \cline{1-2} 2 & 3 & & \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

A key tableau is a semi-standard Young tableau such that the set of entries in the $(j+1)^{th}$ column is a subset of the set of entries in the j^{th} column, for all j. There is a bijection [39] between weak compositions in $\mathbb{N}^n$ and keys in the alphabet $[n]$ given by $\gamma \to key(\gamma)$, where $key(\gamma)$ is the key such that for all j, the first γ_j columns contain the letter j. Any key tableau is of the form $key(\gamma)$ with γ its content and γ^+ the shape. (See Example 1.) When $\gamma = \gamma^+$ one obtains the key of shape and content γ, called Yamanouchi tableau of shape γ. Note that $evac(key(\gamma_1, \ldots, \gamma_n)) = key(\gamma_n, \ldots, \gamma_1)$ where "*evac*" denotes the Schützenberger's evacuation on SSYTs [41, 8, 42].

2.2. Bruhat orders on $\mathfrak{S}_n$ and orbits

The symmetric group $\mathfrak{S}_n$ is generated by the simple transpositions s_i which exchanges i with $i+1$, $1 \leq i < n$, and they satisfy the Coxeter relations

$$s_i^2 = 1, \quad s_i s_j = s_j s_i, \text{ for } |i - j| > 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. \tag{2}$$

Given $\sigma \in \mathfrak{S}_n$, if $\sigma = s_{i_N} \cdots s_{i_1}$ is a decomposition of σ into simple transpositions, where the number N is minimised, we say that we have a reduced decomposition of σ, and N is called its length $\ell(\sigma)$. In this case, we say that the sequence of indices $(i_N, \ldots, i_1)$ is a reduced word for σ. The unique element of maximal length in $\mathfrak{S}_n$ is denoted by ω. It is a well known fact that any two reduced decompositions for σ are connected by a sequence of the last two Coxeter relations (2), called commutation and braid relations, respectively. Recall that a pair (i, j), with $i < j$, is said to be an inversion of σ if $\sigma(i) > \sigma(j)$. The number of inversions of σ is the same as $\ell(\sigma)$ [33].

The (strong) Bruhat order in $\mathfrak{S}_n$ is a partial order in $\mathfrak{S}_n$ compatible with the length of a permutation. For any θ in $\mathfrak{S}_n$ and t a transposition, we write

$$\theta < t\theta \text{ if and only if } \ell(\theta) < \ell(t\theta). \tag{3}$$

The transitive closure of these relations is said to be the (strong) Bruhat order in $\mathfrak{S}_n$. Regarding θ as the linear array $\theta_1\theta_2 \ldots \theta_n$ with $\theta(i) = \theta_i$, the Bruhat order says that $\theta < t\theta$ with t a transposition if and only if t exchanges θ_i and θ_j with $\theta_i < \theta_j$ for some $i < j$. It can be shown that if $t\theta = \sigma$, there is also a transposition t' such that $\theta t' = \sigma$ [3]. We recall the subword property of the (strong) Bruhat order in a Coxeter group.

Proposition 1. [3] *Let* θ, σ *in* $\mathfrak{S}_n$ *and* $(i_N, \ldots, i_1)$ *a reduced word for* σ, *then* $\theta \leq \sigma$ *if and only if there exists a subsequence of* $(i_N, \ldots, i_1)$ *which is a reduced word for* θ.

The maximal length element ω is the maximal element of the Bruhat order, $\sigma \leq \omega$, for any $\sigma \in \mathfrak{S}_n$, and it satisfies $\omega^2 = 1$. Besides, its left and right translations $\sigma \to \omega\sigma$ and $\sigma \to \sigma\omega$ are anti automorphisms for the Bruhat order.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition, and $\mathfrak{S}_n\lambda$ the $\mathfrak{S}_n$-orbit of λ. The $\mathfrak{S}_n$-stabiliser of λ, $stab_\lambda := \{\sigma \in \mathfrak{S}_n : \sigma\lambda = \lambda\}$, is the parabolic subgroup generated by $\{s_i, 1 \leq i < n : s_i\lambda = \lambda\}$. For each coset $wstab_\lambda$ in $\mathfrak{S}_n/stab_\lambda$ we may always choose w to be the shortest permutation in $wstab_\lambda$ [3]. This allows $\mathfrak{S}_n/stab_\lambda$, with cardinality $n!/|stab_\lambda|$, to be identified with $\mathfrak{S}_n\lambda$ [2, 3, 13]. The Bruhat order on $\mathfrak{S}_n/stab_\lambda$ is the restriction to the minimal coset representatives, [3, 43, 44], and can be converted to an ordering of $\mathfrak{S}_n\lambda$ by taking the transitive closure of the relations

$$\gamma < t\gamma, \text{ if } \gamma_i > \gamma_j,\ i < j, \text{ and } t \text{ the transposition } (i\,j)\ (\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathfrak{S}_n\lambda). \quad (4)$$

We call this ordering the Bruhat order on $\mathfrak{S}_n\lambda$. If we replace, in (3), t with the simple transposition s_i, the transitive closure of a such relations defines the left weak Bruhat order on $\mathfrak{S}_n$. Its restriction to $\mathfrak{S}_n/stab_\lambda$ is then converted to an ordering in $\mathfrak{S}_n\lambda$ by replacing, in (4), t with s_i. This conversion of the left weak Bruhat order on $\mathfrak{S}_n$ to $\mathfrak{S}_n\lambda$ is interestingly described by elementary bubble sorting operators [11]. The elementary bubble sorting operation π_i, $1 \leq i < n$, on words $\gamma_1\gamma_2\cdots\gamma_n$ of length n (or weak compositions in $\mathbb{N}^n$), sorts the letters in positions i and $i+1$ in weakly increasing order, that is, it swaps γ_i and γ_{i+1} if $\gamma_i > \gamma_{i+1}$, or fixes $\gamma_1\gamma_2\cdots\gamma_n$ otherwise. Define now the partial order on $\mathfrak{S}_n\lambda$ by taking the transitive closure of the relations $\gamma < \pi_i\gamma$ when $\gamma_i > \gamma_{i+1}$, ($\gamma \in \mathfrak{S}_n\lambda$ and $1 \leq i < n$). It can also be proved that the elementary bubble sorting operations π_i, $1 \leq i < n$, satisfy the relations

$$\pi_i^2 = \pi_i,\ \pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}, \text{ and } \pi_i\pi_j = \pi_j\pi_i, \text{ for } |i-j| > 1. \quad (5)$$

The set of minimal length coset representatives of $\mathfrak{S}_n/stab_\lambda$ may be described as $\{\sigma \in \mathfrak{S}_n : \ell(\sigma s_i) > \ell(\sigma),\ s_i \in stab_\lambda\}$ [2, 3, 13]. We now recall a construction of the minimal length coset representatives for $\mathfrak{S}_n/stab_\lambda$, due to Lascoux, in [26], where the notion of key tableau is used. This allows to convert the tableau criterion for the Bruhat order in $\mathfrak{S}_n$ to a tableau criterion for the Bruhat order (4) in $\mathfrak{S}_n\lambda$. Recall that the bijection between staircase keys of shape $(n, n-1, \ldots, 1)$ and permutations in $\mathfrak{S}_n$ gives the well-known tableau criterion for the Bruhat order in $\mathfrak{S}_n$ [6, 33].

Proposition 2. [33] *Let* σ, $\beta \in \mathfrak{S}_n$, *we have* $\sigma \leq \beta$ *if and only if* $key(\sigma(n, \ldots, 1)) \leq key(\beta(n, \ldots, 1))$ *for the entrywise comparison.*

In [26], Lascoux constructs the shortest permutation w in the coset $wstab_\lambda$ such that $w\lambda = \gamma \in \mathbb{N}^n$ using the key tableau of γ as follows: firstly, add the complete column $[n \dots 1]$ as the left most column of $key(\gamma)$, if γ has an entry equal to zero; secondly, write the elements of the right most column of $key(\gamma)$ in increasing order then the new elements that appear in the column next to the last in increasing order and so on until the first column. The resulting word is the desired permutation w in $\mathfrak{S}_n$.

Example 1. *Let $\gamma = (1,3,0,0,1)$ and its $\mathfrak{S}_5$-stabiliser $stab_{(3,1,1,0,0)} =< s_2, s_4 >$, the parabolic subgroup generated by the simple transpositions of $\mathfrak{S}_5$ that leave γ invariant. Let*

$key(\gamma) = \begin{array}{|c|lll} \cline{1-1} 5 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array}$. *First add the complete column $[5,4,3,2,1]$, to get* $\begin{array}{|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-1} 4 \\ \cline{1-2} 3 & 5 \\ \cline{1-2} 2 & 2 \\ \hline 1 & 1 & 2 & 2 \\ \hline \end{array}$ *Hence, $w = 21534 = s_1 s_4 s_3$ is the shortest permutation in the coset $w\, stab_{(3,1,1,0,0)}$.*

Theorem 1. *Let α_1 and α_2 be in the $\mathfrak{S}_n\lambda$. Then*
(a) $\alpha_1 \leq \alpha_2$ if and only if $key(\alpha_1) \leq key(\alpha_2)$.
(b) $\alpha_1 \leq \alpha_2$ if and only if $evac(key(\alpha_2)) \leq evac(key(\alpha_1))$.

Proof. (a) Let σ_1 and σ_2 be the shortest length representatives of $\mathfrak{S}_n/stab_\lambda$ such that $\sigma_1\lambda = \alpha_1$, $\sigma_2\lambda = \alpha_2$. Then, $\alpha_1 \leq \alpha_2$ if and only if $\sigma_1 \leq \sigma_2$ in Bruhat order, and, by Proposition 2, this means $key(\sigma_1(n,\dots,1)) \leq key(\sigma_2(n,\dots,1))$. Using the constructions of σ_1 and σ_2 explained above this is equivalent to say that $key(\alpha_1) \leq key(\alpha_2)$.

(b) Recall that $evac(key(\alpha)) = key(\omega\alpha)$. □

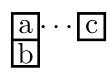
3. Semi-skyline augmented fillings

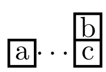
3.1. Definitions and properties

We follow closely the conventions and terminology in [9, 10] and [34, 35]. A weak composition $\gamma = (\gamma_1, \dots, \gamma_n)$ is visualised as a diagram consisting of n columns, with γ_j boxes in column j, for $1 \leq j \leq n$. Formally, the *column diagram* of γ is the set $dg'(\gamma) = \{(i,j) \in \mathbb{N}^2 : 1 \leq j \leq n, 1 \leq i \leq \gamma_j\}$ where the coordinates are in French convention, i indicates the vertical coordinate, indexing the rows, and j the horizontal coordinate, indexing the columns. (The prime reminds that the components of γ are the columns.) The number of cells in a column is called the height of that column and a cell a in a column diagram is denoted $a = (i,j)$, where i is the row index and j is the column index. The *augmented diagram* of γ, $\widehat{dg}(\gamma) = dg'(\gamma) \cup \{(0,j) : 1 \leq j \leq n\}$, is the column diagram with n extra cells adjoined in row 0. This adjoined row is called the *basement* and it always contains the numbers 1 through n in strictly increasing order. The shape of $\widehat{dg}(\gamma)$ is defined to be γ. For example, column diagram and the augmented diagram for $\gamma = (1, 0, 3, 0, 1, 2, 0)$ are

$dg'(\gamma)$ $\widehat{dg}(\gamma)$

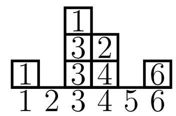
An augmented filling F of an augmented diagram $\widehat{dg}(\gamma)$ is a map $F : \widehat{dg}(\gamma) \to [n]$, which can be pictured as an assignment of positive integer entries to the non-basement cells of $\widehat{dg}(\gamma)$. Let $F(i)$ denote the entry in the i^{th} cell of the augmented diagram encountered when F is read across rows from left to right, beginning at the highest row and working down to the bottom row. This ordering of the cells is called the reading order. A cell $a = (i, j)$ precedes a cell $b = (i', j')$ in the reading order if either $i' < i$ or $i' = i$ and $j' > j$. The reading word of F is obtained by recording the non-basement entries in reading order. The content of an augmented filling F is the weak composition $c(F) = (\alpha_1, \ldots, \alpha_n)$ where α_i is the number of non-basement cells in F with entry i, and n is the number of basement elements. The standardization of F is the unique augmented filling that one obtains by sending the i^{th} occurrence of j in the reading order to $i + \sum_{m=1}^{j-1} \alpha_m$.

Let $a, b, c \in \widehat{dg}(\gamma)$ three cells situated as follows, 

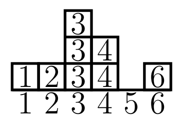
, where a and c are in the same row, possibly the first row, possibly with cells between them, and the height of the column containing a and b is greater than or equal to the height of the column containing c. Then the triple a, b, c is an inversion triple of type 1 if and only if after standardization the ordering from smallest to largest of the entries in cells a, b, c induces a counterclockwise orientation. Similarly, consider three cells $a, b, c \in \widehat{dg}(\gamma)$ situated as follows, 

where a and c are in the same row (possibly the basement) and the column containing b and c has strictly greater height than the column containing a. The triple a, b, c is an inversion triple of type 2 if and only if after standardization the ordering from smallest to largest of the entries in cells a, b, c induces a clockwise orientation.

Define a semi-skyline augmented filling (SSAF) of an augmented diagram $\widehat{dg}(\gamma)$ to be an augmented filling F such that every triple is an inversion triple and columns are weakly decreasing from bottom to top. The shape of the semi-skyline augmented filling is γ and denoted by $sh(F)$. The picture below is an example of a semi-skyline augmented filling with shape $(1, 0, 3, 2, 0, 1)$, reading word 1321346 and content $(2, 1, 2, 1, 0, 1)$,



The entry of a cell in the first row of a SSAF is equal to the basement element where it sits and, thus, in the first row the cell entries strictly increase from left to the right. For any weak composition γ in $\mathbb{N}^n$, there is a unique SSAF, with shape and content γ, by putting γ_i cells with entries i in the top of the basement element i. We call it key SSAF of shape γ. The following is the key SSAF of shape $(1, 1, 3, 2, 0, 1)$,



(6)

In [35] a sequence of lemmas provides several conditions on triples of cells in a SSAF. In particular, we recall Lemma 2.6 in [35] which characterises completely the relative values of the entries in the cells of a type 2 inversion triple in a SSAF. This property of type 2 inversion triples will be used in the proof of our main theorem. Given a cell a in SSAF F define $F(a)$ to be the entry in a.

Lemma 1. [35] *If a, b, c is a type 2 inversion triple in F, as defined above, then $F(a) < F(b) \leq F(c)$.*

3.2. An analogue of Schensted insertion and RSK for SSAFs

The fundamental operation of the Robinson–Schensted–Knuth (RSK) algorithm is Schensted insertion which is a procedure for inserting a positive integer k into a semi-standard Young tableau T. In [35], Mason defines a similar procedure for inserting a positive integer k into a SSAF F, which is used to describe an analogue of the RSK algorithm. If F is a SSAF of shape γ, we set $F := (F(j))$, where $F(j)$ is the entry in the j^{th} cell in reading order, with the cells in the basement included, and j goes from 1 to $n + \sum_{i=1}^{n} \gamma_i$. If $\hat{j}$ is the cell immediately above j and the cell is empty, set $F(\hat{j}) := 0$. The operation $k \to F$, for $k \leq n$, is defined as follows.

Procedure. The insertion $k \to F$:

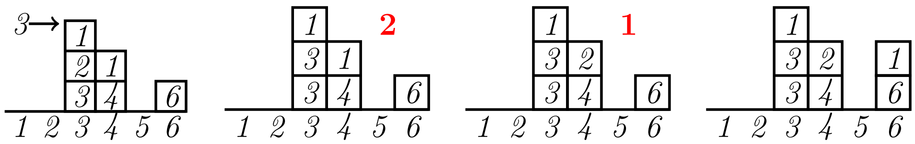
1. Set $i := 1$, set $x_1 := k$, and set $j := 1$.

2. If $F(j) < x_i$ or $F(\hat{j}) \geq x_i$, then increase j by 1 and repeat this step. Otherwise, set $x_{i+1} := F(\hat{j})$ and set $F(\hat{j}) := x_i$.

3. If $x_{i+1} \neq 0$ then increase i by 1, increase j by 1, and repeat step 2. Otherwise, terminate the algorithm.

The procedure terminates in finitely many steps and the result is a SSAF.

Example 2. *Insertion 3 to the SSAF.*



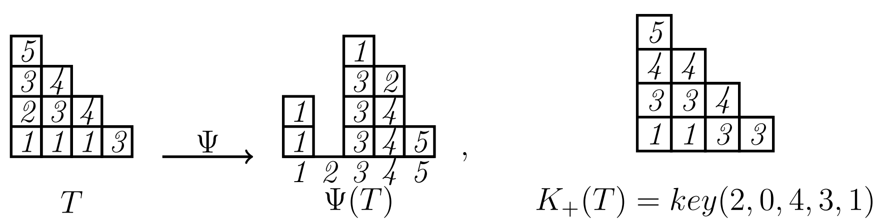
Let SSAF_n be the set of all semi-skyline augmented fillings with basement $[n]$. Based on this Schensted insertion analogue, Mason gives a weight preserving and shape rearranging bijection Ψ between SSYT_n and SSAF_n. The bijection Ψ is defined to be the insertion, from right to left, of the column word of a SSYT into the empty SSAF with basement $1, \ldots, n$. The shape of $\Psi(T)$ provides the right key of T, $K_+(T)$, a notion due to Lascoux and Schützenberger [23, 24]. There are now several ways to describe the right key of a tableau [24, 8, 30, 34, 45]. For our purpose we consider the following Mason's theorem as the definition of right key of T.

Theorem 2 (Mason [34])**.** *Given an arbitrary SSYT T, let γ be the shape of $\Psi(T)$. Then $K_+(T) = key(\gamma)$.*

Given the partition $\lambda \in \mathbb{N}^n$, let $\mathfrak{B}^\lambda$ denote the set of all semi-standard Young tableaux in SSYT_n of shape λ. This theorem decompose $\mathfrak{B}^\lambda$ into a disjoint union of semi-standard Young tableaux w.r.t. to their right keys:

$$\mathfrak{B}^\lambda = \biguplus_{\gamma \in \mathfrak{S}_n \lambda} \{T \in SSYT_n : K_+(T) = key(\gamma)\}.$$

Example 3. *One has $sh(\Psi(T)) = (2, 0, 4, 3, 1)$,*



Given the alphabet $[n]$, the RSK algorithm is a bijection between biwords in lexicographic order and pairs of SSYT of the same shape over $[n]$. Equipped with the Schensted insertion analogue, Mason finds in [35] an analogue Φ of the RSK yielding a pair of SSAFs with shapes a rearranging of each other. This bijection has an advantage over the classical RSK because the pair of SSAFs comes along with the extra pair of right keys.

The two line array $w = \begin{pmatrix} i_1 & i_2 & \cdots & i_l \\ j_1 & j_2 & \cdots & j_l \end{pmatrix}$, $i_r < i_{r+1}$, or $i_r = i_{r+1}$ & $j_r \leq j_{r+1}$, $1 \leq i, j \leq l-1$, with $i_r,\ j_r \in [n]$, is called a biword in lexicographic order over

the alphabet $[n]$. The map Φ defines a bijection between the set $\mathbb{A}_n$ of all biwords w in lexicographic order in the alphabet $[n]$, and pairs of SSAFs with shapes in the same $\mathfrak{S}_n$-orbit, and the contents are those of the second and first rows of w, respectively.
Procedure. The map $\Phi : \mathbb{A}_n \longrightarrow SSAF_n \times SSAF_n$. Let $w \in \mathbb{A}_n$.

1. Set $r := l$, where l is the number of biletters in w. Let $F = \emptyset = G$, where $\emptyset$ is the empty $SSAF$.

2. Set $F := (j_r \to F)$. Let h_r be the height of the column in $(j_r \to F)$ at which the insertion procedure $(j_r \to F)$ terminates.

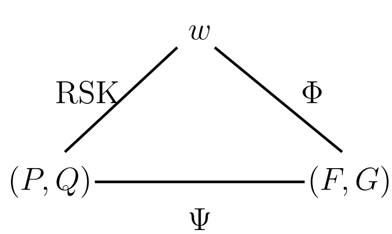
3. Place i_r on top of the leftmost column of height $h_r - 1$ in G such that doing so preserves the decreasing property of columns from bottom to top. Set G equal to the resulting figure.

4. If $r - 1 \neq 0$, repeat step 2 for $r := r - 1$. Else terminate the algorithm.

Remark 1. 1. *The entries in the top row of the biword are weakly increasing when read from left to right. Henceforth, if $h_r > 1$, placing i_r on top of the leftmost column of height $h_r - 1$ in G preserves the decreasing property of columns. If $h_r = 1$, the i_r^{th} column of G does not contain an entry from a previous step. It means that the number i_r sits on the top of basement i_r.*
2. *Let h be the height of the column in F at which the insertion procedure $(j \to F)$ terminates. Lemma 1 implies that there is no column of height $h+1$ in F to the right.*
3. *Steps 2 and 3 guarantee that, at each stage of the map Φ procedure, the shapes of the pair of SSAFs are a rearrangement of each other.*

Corollary 1 (Mason [34, 35])**.** *The RSK algorithm commutes with the above analogue Φ. That is, if (P, Q) is the pair of SSYTs produced by RSK algorithm applied to biword w, then $(\Psi(P), \Psi(Q)) = \Phi(w)$, and $K_+(P) = key(sh(\Psi(P)))$, $K_+(Q) = key(sh(\Psi(Q)))$.*

This result is summarised in the following scheme from which, in particular, it is clear the RSK analogue Φ also shares the symmetry of RSK,



$$c(P) = c(F),\ c(Q) = c(G),$$
$$sh(F)^+ = sh(G)^+ = sh(P) = sh(Q),$$
$$K_+(P) = key(sh(F)),\ K_+(Q) = key(sh(G)).$$

4. Main Theorem

We prove a restriction of the bijection Φ to multisets of cells in a staircase or truncated staircase of length n, such that the staircases of length $n-k$ on the upper left corner, or of length $n-m$ on the bottom right corner, with $1 \leq m \leq n$, $1 \leq k \leq n$ and $k+m \geq n+1$, are erased. The restriction to be imposed on the pairs of SSAFs is that the pair of shapes in a same $\mathfrak{S}_n$-orbit, satisfy an inequality in the Bruhat order, where one shape is bounded by the reverse of the other. Equivalently, pairs of SSYTs whose right keys are such that one is bounded by the evacuation of the other. The following lemma gives sufficient conditions to preserve the Bruhat order relation between two weak compositions when one box is added to a column of their diagrams.

Lemma 2. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be in the same $\mathfrak{S}_n$-orbit, with key$(\beta) \leq$ key(α). Given $k \in \{1, \ldots, n\}$, let $k' \in \{1, \ldots, n\}$ be such that $\beta_{k'}$ is the left most entry of β satisfying $\beta_{k'} = \alpha_k$. Then if $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k + 1, \ldots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \beta_2, \ldots, \beta_{k'} + 1, \ldots, \beta_n)$, it holds key$(\tilde{\beta}) \leq$ key$(\tilde{\alpha})$.*

Proof. Let $k,\ k' \in \{1, \ldots, n\}$ as in the lemma, and put $\alpha_k = \beta_{k'} = m \geq 1$. (The proof for $m = 0$ is left to the reader. The case of interest for our problem is $m > 0$ which is related with the procedure of map Φ.) This means that k appears exactly in the first m columns of *key*(α), and k' is the smallest number that does not appear in column $m+1$ of *key*(β) but appears exactly in the first m columns. Let t be the row index of the cell with entry k' in column m of *key*(β). Every entry less than k' in column m of *key*(β) appears in column $m+1$ as well, and since in a key tableau each column is contained in the previous one, this implies that the first t rows of columns m and $m+1$ of *key*$(\tilde{\beta})$ are equal. The only difference between *key*$(\tilde{\beta})$ and *key*(β) is in columns $m+1$, from row t to the top. Similarly if z is the row index of the cell with entry k in column $m+1$ of *key*$(\tilde{\alpha})$, the only difference between *key*$(\tilde{\alpha})$ and *key*(α) is in columns $m+1$ from row z to the top. To obtain column $m+1$ of *key*$(\tilde{\beta})$, shift in the column $m+1$ of *key*(β) all the cells with entries $> k'$ one row up, and add to the position left vacant (of row index t) a new cell with entry k'. The column $m+1$ of *key*$(\tilde{\alpha})$ is obtained similarly, by shifting one row up in the column $m+1$ of *key*(α) all the cells with entries $> k$ and adding a new cell with entry k in the vacant position.

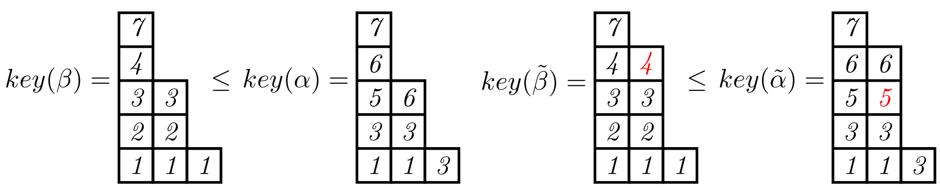
Put $p := \min\{t, z\}$ and $q := \max\{t, z\}$. We divide the columns $m+1$ in each pair of tableaux *key*(β), *key*$(\tilde{\beta})$ and *key*(α), *key*$(\tilde{\alpha})$ into three parts: the first, from row one to row $p-1$; the second, from row p to row q; and the third, from row $q+1$ to the top row. The first parts of column $m+1$ of *key*$(\tilde{\beta})$ and *key*(β) are the same, equivalently, for *key*$(\tilde{\alpha})$ and *key*(α). The third part of column $m+1$ of *key*$(\tilde{\beta})$ consists of row q plus the third part of *key*(β), equivalently, for *key*$(\tilde{\alpha})$ and *key*(α). As the columns $m+1$ of *key*(β) and *key*(α) are entrywise comparable, the same happens to the first and third parts of columns $m+1$ in *key*$(\tilde{\beta})$ and *key*$(\tilde{\alpha})$. It remains to analyse the second parts of the pair *key*$(\tilde{\beta})$, *key*$(\tilde{\alpha})$ which we split into two cases according to the relative magnitude of p and q.

Case 1. $p = t < q = z$. Let $k' < b_t < \cdots < b_{z-1}$ and $d_t < \cdots < d_{z-1} < k$ be the entries of the second parts of columns $m+1$ in $key(\tilde{\beta})$ and $key(\tilde{\alpha})$, respectively. By construction $k' < b_t \leq d_t < d_{t+1}$, $b_i < b_{i+1} \leq d_{i+1}$, $t < i < z-2$, and $b_{z-1} \leq d_{z-1} < k$, and, therefore, the second parts are also comparable.

Case 2. $p = z \leq q = t$. In this case, the assumption on k' implies that the first q rows of columns m and $m+1$ of $key(\tilde{\beta})$ are equal. On the other hand, since column m of $key(\beta)$ is less or equal than column m of $key(\alpha)$, which is equal to the column m of $key(\tilde{\alpha})$ and in turn is less or equal to column $m+1$ of $key(\tilde{\alpha})$, forces by transitivity that the second part of column $m+1$ of $key(\tilde{\beta})$ is less or equal than the corresponding part of $key(\tilde{\alpha})$. □

We illustrate the lemma with

Example 4. *Let* $\beta = (3, 2^2, 1, 0^2, 1)$, $\alpha = (2, 0, 3, 0, 1, 2, 1)$, $\tilde{\beta} = (3, 2^3, 0^2, 1)$, *and* $\tilde{\alpha} = (2, 0, 3, 0, 2^2, 1)$,



We are now ready to state and prove the main theorem.

Theorem 3. *Let w be a biword in lexicographic order in the alphabet $[n]$, and let $\Phi(w) = (F, G)$. For each biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w one has $i + j \leq n + 1$ if and only if $key(sh(G)) \leq key(\omega sh(F))$, where ω is the longest permutation of $\mathfrak{S}_n$. Moreover, if the first row of w is a word in the alphabet $[k]$, with $1 \leq k \leq n$, and the second row is a word in the alphabet $[m]$, with $1 \leq m \leq n$, the shape of G has the last $n-k$ entries equal to zero, and the shape of F the last $n-m$ entries equal to zero.*

Proof. We describe $\Phi(w) = (F, G)$ in terms of a Bruhat relation between the shapes of F and G, when the billeters are cells in a staircase of size n.

Only if part. We prove, by induction on the number of biletters of w, that if $\Phi(w) = (F, G)$, where the billeters are cells in a staircase of size n, then $sh(G) \leq \omega sh(F)$. If w is the empty word then F and G are the empty semi-skyline augmented filling, the shapes are null vectors, and there is nothing to prove. Let $w' = \begin{pmatrix} i_{p+1} & i_p & \cdots & i_1 \\ j_{p+1} & j_p & \cdots & j_1 \end{pmatrix}$ be a biword in lexicographic order such that $p \geq 0$ and $i_t + j_t \leq n + 1$ for all $1 \leq t \leq p + 1$, and $w = \begin{pmatrix} i_p \cdots i_1 \\ j_p \cdots j_1 \end{pmatrix}$ such that $\Phi(w) = (F, G)$.

Let $F' := (j_{p+1} \to F)$ and h the height of the column in F' at which the insertion procedure terminates. There are two possibilities for h which the third step of the algorithm procedure of Φ requires to consider.

• $h = 1$. It means j_{p+1} is sited on the top of the basement element j_{p+1} in F and therefore i_{p+1} goes to the top of the basement element i_{p+1} in G. Let G' be the semi-skyline augmented filling obtained after placing i_{p+1} in G. See Figure 2.

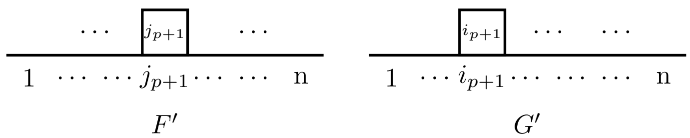


Figure 2: The pair (F', G') of SSAFs exhibiting the columns of height one, with basements j_{p+1} and i_{p+1}, where $i_{p+1} \leq n - j_{p+1} + 1$. G' is empty to the left of i_{p+1}.

As $i_{p+1} \leq i_t$, for all t, i_{p+1} is the bottom entry of the first column in $key(sh(G'))$ whose remaining entries constitute the first column of $key(sh(G))$. Suppose $n+1-j_{p+1}$ is added to the row z of the first column in $key(\omega sh(F))$ by shifting all the entries above it one row up. Let $i_{p+1} < a_1 < \cdots < a_z < a_{z+1} < \cdots < a_l$ be the entries in the first column of $key(sh(G'))$ and $b_1 < b_2 < \cdots < b_{z-1} < n + 1 - j_{p+1} < b_z < \cdots < b_l$ be the entries in the first column of $key(\omega sh(F'))$, where $a_1 < \cdots < a_z < \cdots < a_l$ and $b_1 < \cdots < b_z < \cdots < b_l$ are the entries in the corresponding first columns of $key(sh(G))$ and $key(\omega sh(F))$. If $z = 1$, as $i_{p+1} \leq n + 1 - j_{p+1}$ and $a_i \leq b_i$ for all $1 \leq i \leq l$, then $key(sh(G')) \leq key(\omega sh(F'))$. If $z > 1$, as $i_{p+1} < a_1 \leq b_1 < b_2$, we have $i_{p+1} \leq b_1$ and $a_1 \leq b_2$. Similarly $a_i \leq b_i < b_{i+1}$, and $a_i \leq b_{i+1}$, for all $2 \leq i \leq z - 2$. Moreover $a_{z-1} \leq b_{z-1} < n + 1 - j_{p+1}$, therefore $a_{z-1} \leq n + 1 - j_{p+1}$. Also $a_i \leq b_i$ for all $z \leq i \leq l$. Hence $key(sh(G')) \leq key(\omega sh(F'))$.

• $h > 1$. Place i_{p+1} on the top of the leftmost column of height $h-1$. This means, by Lemma 2, $key(sh(G')) \leq key(\omega sh(F'))$. See Figure 3.

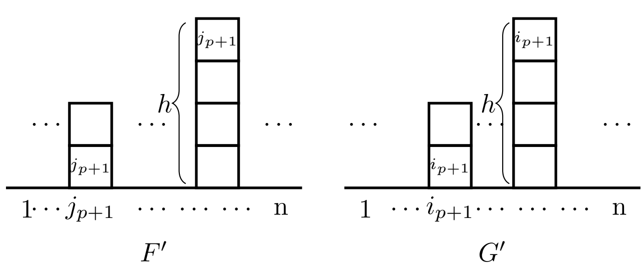


Figure 3: The pair (F', G') of SSAFs exhibiting the columns of height $h > 1$.

If part. It is enough to show the following. *If there exists a biletter* $\binom{i}{j}$ *in* w *such* *that $i+j>n+1$, then at least one entry of key$(sh(G))$ is strictly bigger than the corresponding entry of key$(\omega sh(F))$.*

This means that if not all biletters satisfy $i+j\le n+1$ then either the shapes $sh(G)$ and $\omega sh(F)$ are not comparable in the Bruhat order, or $sh(G)>\omega sh(F)$. (Example 5.2. and 5.3. show that both situations may happen.) Therefore, $key(sh(G))\not\le \omega sh(F)$.

Before delving into the proof, for the reader's convenience, we give an outline of it. Let $w=\begin{pmatrix} i_p\cdots i_1 \\ j_p\cdots j_1\end{pmatrix}$, $p\ge 1$, be a biword in lexicographic order on the alphabet $[n]$, and $\binom{i_t}{j_t}$, $t\ge 1$, the first biletter in w, from right to left, with $i_t+j_t>n+1$. Set $F_0=G_0:=\emptyset$, and for $d\ge 1$, let $(F_d,G_d):=\Phi\begin{pmatrix} i_d\cdots i_1 \\ j_d\cdots j_1\end{pmatrix}$. After inserting j_t in F_{t-1} and placing i_t in G_{t-1} where $i_t>n+1-j_t$, it will be easily seen that the letters i_t and $n+1-j_t$ appear as bottom entries in the first column of *key*$(sh(G_t))$ and in the first column of *key*$(\omega sh(F_t))$, respectively. We call to this pair of letters $(i_t,n+1-j_t)$ where $i_t>n+1-j_t$ a *problem* in the key-pair $(key(sh(G_t)),key(\omega sh(F_t)))$. Further insertions of letters j_d in F_{d-1} and placements of i_d in G_{d-1}, for $d>t$, either corresponding to cells below, or above the staircase of size n, will not solve the *problem* of a pair of letters $(i_t,n+1-j_t)$ such that $i_t>n+1-j_t$, in some row of homologous columns in the key-pair $(key(sh(G_{d-1})),key(\omega sh(F_{d-1})))$. The *problem* $i_t>n+1-j_t$ will always appear in some row of a pair of homologous columns in the key-pair $(key(sh(G_d)),key(\omega sh(F_d)))$ for any $d\ge t$. To show this, we keep track of the *problem* in a sequence of four claims with the aim to *locate* the *problem* at any stage of the insertion. The *locus* of $i_t>n+1-j_t$ in a row of homologous columns in the key-pair $(key(sh(G_d)),key(\omega sh(F_d)))$ with $d\ge t$, will be called the classification of the *problem.*

We now embark on the details. First apply the map Φ to the biword $\begin{pmatrix} i_{t-1}\cdots i_1 \\ j_{t-1}\cdots j_1\end{pmatrix}$ to obtain the pair (F_{t-1},G_{t-1}) of SSAFs whose right keys satisfy, by the "only if part" of the theorem, $key(sh(G_{t-1}))\le key(\omega sh(F_{t-1}))$. Now insert j_t to F_{t-1}. As $i_k+j_k\le n+1$, for $1\le k\le t-1$, $i_k+j_k\le n+1<i_t+j_t$, and $i_t\le i_k$, $1\le k\le t-1$, then $j_t>j_k$, $1\le k\le t-1$, and, since w is in lexicographic order, this implies $i_t<i_{t-1}$. Therefore, j_t sits on the top of the basement element j_t in F_{t-1} and i_t sits on the top of the basement element i_t in G_{t-1}. Since the column with basement j_t, in the insertion filling, and the column with basement i_t, in the recording filling, play an important role in what follows they will be denoted by J and I, respectively. See Figure 4.

It means that $n+1-j_t$ is added to the first row and first column of $key(\omega sh(F_{t-1}))$ and all entries in this column are shifted one row up. Similarly, i_t is added to the first

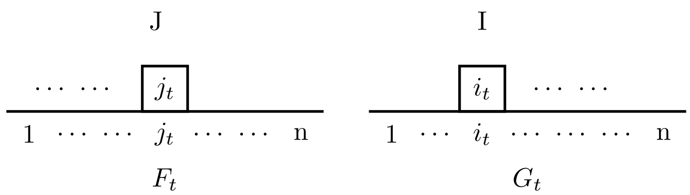


Figure 4: The pair (F_t, G_t) of SSAFs after inserting j_t and placing i_t, where $i_t > n+1-j_t$, in (F_{t-1}, G_{t-1}) . The columns J and I, with basements j_t and i_t, respectively, are both of height one. The SSAF F_t is empty to the right of the column J, and G_t is empty to the left of the column I.

row and first column of $key(sh(G_{t-1}))$, and all the entries in this column are shifted one row up. As $i_t > n+1-j_t$ then the first column of $key(sh(G_t))$ and the first column of $key(\omega sh(F_t))$ are not entrywise comparable, and we have a "*problem*" in the key-pair $(key(sh(G_t)), key(\omega sh(F_t)))$. See Figure 5.

$$\begin{array}{ccc} \vdots & \leq & \vdots \\ & & \\ a_1 & \leq & b_1 \\ i_t & > & n+1-j_t \end{array}$$

Figure 5: The bottom entry in the first column of $key(sh(G_t))$ and the bottom entry in the first column of $key(\omega sh(F_t))$ satisfy $i_t > n+1-j_t$. The key-pair $(key(sh(G_t)), key(\omega sh(F_t))$ is not comparable when $t > 1$.

From now on, we shall see that the "*problem*" $i_t > n+1-j_t$ remains in some row of a pair of homologous columns in the key-pair $(key(sh(G_d)), key(\omega sh(F_d)))$, with $d \geq t$. Given $d \geq t$, J still will denote the column with basement j_t in F_d, and I the column with basement i_t in G_d. Their heights will be denoted by $|J|$ and $|I|$, respectively, and they are ≥ 1. For each $i \geq 1$, let r_i and k_i denote the number of columns of height $\geq i$, to the right of J and to the left of I, respectively. See Figure 6. The fillings (not their basements) of the columns I and J as their heights depend indeed on d but we shall avoid cumbersome notation as long as there is no danger of confusion. However, we put the superscript d and $d+1$ on k_i and r_i to distinguish between (F_d, G_d) and (F_{d+1}, G_{d+1}), whenever clarity of presentation makes this necessary.

The classification of the "*problem*" will follow from a sequence of four claims as follows. The first is used to prove the second. The second is used to prove the third. Finally, the last claim complements the third.

Claim 1: *Let* (F_d, G_d), *with* $d \geq t$. *Then* $k_i \geq r_i \geq 0$, *for all* $i \geq 1$.

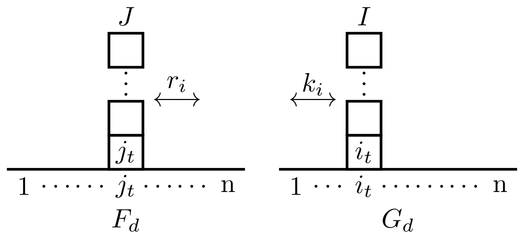


Figure 6: The pair of SSAFs (F_d, G_d), $d \geq t$, with the nonempty columns I and J. For each $i \geq 1$, r_i and k_i count the number of columns of height $\geq i$, to the right of J and to the left of I, respectively. If $d = t$, as in Figure 4, one has $r_i = k_i = 0$ for all i.

Proof. By induction on $d \geq t$. For $d = t$, one has, $k_i = r_i = 0$, for all $i \geq 1$. See Figure 4. Let $d \geq t$, and suppose (F_d, G_d) satisfies $k_i \geq r_i \geq 0$, for all $i \geq 1$. Let us prove for (F_{d+1}, G_{d+1}). If the insertion of j_{d+1} terminates in a column of height l to the left or on the top of J, then $r_i^{d+1} = r_i^d$, for all i, $k_i^{d+1} = k_i^d$, for all $i \neq l+1$, and $k_{l+1}^{d+1} = k_{l+1}^d + 1$ or k_{l+1}^d. Thus, $k_i^{d+1} \geq r_i^{d+1}$, for all $i \geq 1$. On the other hand, if the insertion of j_{d+1} terminates to the right of J, then in F_d one has $r_l^d > r_{l+1}^d$, and two cases have to be considered for placing i_{d+1} in G_d. First, i_{d+1} sits on the left of I and, hence, $k_{l+1}^{d+1} = k_{l+1}^d + 1 \geq r_{l+1}^d + 1 = r_{l+1}^{d+1}$, $k_i^{d+1} = k_i^d \geq r_i^d = r_i^{d+1}$, for $i \neq l+1$. Second, either i_{d+1} sits on the top of I or to the right of I, in both cases, (F_d, G_d) satisfy $k_{l+1}^d = k_l^d \geq r_l^d > r_{l+1}^d$, and, therefore, $k_{l+1}^d > r_{l+1}^d$. This implies for (F_{d+1}, G_{d+1}), $r_{l+1}^{d+1} = r_{l+1}^d + 1$, and $k_{l+1}^{d+1} = k_{l+1}^d \geq r_{l+1}^{d+1}$, $k_i^{d+1} = k_i^d \geq r_i^d = r_i^{d+1}$, for $i \neq l+1$. See Figure 7. $\square$

j_{d+1}
- left or top of J
- right of J : i_{d+1}
  - left of I
  - top or right of I

Figure 7: All the possible places for terminating the insertion of j_{d+1} and placing i_{d+1} with respect to the columns I and J.

Using Claim 1, it is shown next that the number of columns of height $|I| < i \leq |J|$, to the right of J is strictly bigger than the number of columns of height $|I| < i \leq |J|$, to the left of I, whenever the height of J is strictly bigger than the height of I.

Claim 2. *Let (F_d, G_d), with $d \geq t$, and $|J| > |I|$. Then $k_i > r_i \geq 0$, $i = |I|+1, \ldots, |J|$.*

Proof. Since, for $d = t$, it holds $|I| = |J|$, and there is a $d > t$ where for the first time one has $|J| = |I| + 1$. We assume that, for some $d > t$, one has (F_d, G_d) with $|J| - |I| \geq 1$. Then, either (F_{d-1}, G_{d-1}) has $|I| = |J|$ or $|J| > |I|$. In the first case, it means that the insertion of j_d has terminated on the top of J and the cell i_d sits on the left of I on a column of height $|J| = |I|$. (Otherwise, it would sit on the top of I and again one would have (F_d, G_d) with new columns J and I satisfying $|J| = |I|$. Absurd.) Then, by the previous claim, $k^d_{|J|+1} = k^{d-1}_{|J|+1} + 1 > r^{d-1}_{|J|+1} = r^d_{|J|+1}$. See Figure 8.

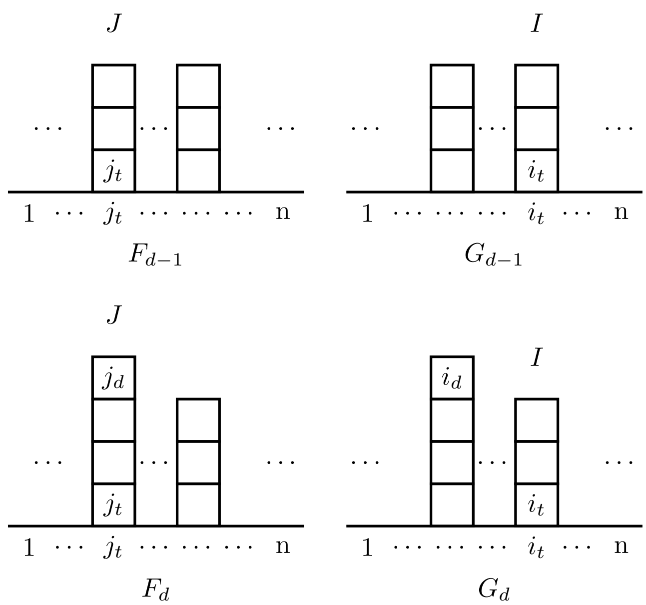


Figure 8: The pairs (F_{d-1}, G_{d-1}), (F_d, G_d) of SSAFs when $|I| = |J|$ in (F_{d-1}, G_{d-1}). It holds $r^{d-1}_{|J|+1} = r^d_{|J|+1}$ and $k^d_{|J|+1} = k^{d-1}_{|J|+1} + 1$.

Having in mind the first case, we suppose in the second that (F_{d-1}, G_{d-1}) satisfies $k^{d-1}_i > r^{d-1}_i \geq 0$, for $i = |I| + 1, \ldots, |J|$. Put $z := |I| < h := |J|$ for I and J in (F_{d-1}, G_{d-1}). Let us prove that, if (F_d, G_d) has $|J| > |I|$, then still $k^d_i > r^d_i \geq 0$, for $i = |I| + 1, \ldots, |J|$. If the insertion of j_d terminates in a column of height l $(\neq h - 1)$ to the left of J then $r^d_i = r^{d-1}_i$, for all $i \geq 1$, $k^d_{l+1} = k^{d-1}_{l+1} + 1$, or k^{d-1}_{l+1} and $k^d_i = k^{d-1}_i$, for $i \neq l + 1$, and $z \leq |I| < |J| = h$, $|J| - |I| \geq 1$. Therefore, $k^d_i > r^d_i \geq 0$, for $i = |I| + 1, \ldots, |J|$. If the insertion terminates on the top of J, then $|J| = h + 1$, $|I| = z$, $r^d_i = r^{d-1}_i$, for all $i \geq 1$, $k^d_i = k^{d-1}_i$, for $i = z + 1, \ldots, h$, and $k^d_{h+1} = k^{d-1}_{h+1} + 1 > r^d_{h+1}$ or $k^d_{h+1} = k^{d-1}_h > r^d_h \geq r^d_{h+1}$. Again $k^d_i > r^d_i$, for $i = |I| + 1, \ldots, |J| = h + 1$. Finally, if the insertion terminates to the right of J, $|J| = h$ and three cases for the height l have to be considered. When $l < z$, or $l \geq h$, $r^d_i = r^{d-1}_i < k^{d-1}_i = k^d_i$, for $i = z + 1, \ldots, |J|$; when $l = z$, then either $|I| = z$ and $k^d_{z+1} = k^{d-1}_{z+1} + 1 > r^{d-1}_{z+1} + 1 = r^d_{z+1}$, $k^d_i = k^{d-1}_i > r^{d-1}_i = r^d_i$, $z < i \leq |J|$ or $z + 1 = |I| \leq |J|$, and $k^d_i = k^{d-1}_i > r^{d-1}_i = r^d_i$, $i = z + 2, \ldots, |J|$; and when

$z < l < h$, then $|I| = z$ and $r_i^d = r_i^{d-1}$, $i \neq l+1$, and either $k_{l+1}^d = k_{l+1}^{d-1} + 1 > r_{l+1}^{d-1} + 1 = r_{l+1}^d$ or $k_{l+1}^d = k_{l+1}^{d-1} = k_l^{d-1} > r_l^{d-1} \geq r_{l+1}^{d-1} + 1 = r_{l+1}^d$. Henceforth $k_i^d > r_i^d$, for $i = |I|+1, \ldots, |J|$. See Figure 9. □

2nd case: j_d
- left of J : i_d
  - top or right of I
  - left of I
- top of J : i_d
  - right of I
  - left of I
- right of J : i_d height l
  - $l < z$ or $l \geq h$
  - $l = z$
    - left of I
    - top of I
  - $z < l < h$
    - left of I
    - right of I

Figure 9: All the possible places for i_d depending on the terminating place for the insertion of j_d in (F_{d-1}, G_{d-1}) under the condition $h = |J| > z = |I|$, and the new columns J and I still satisfy $|J| > |I|$.

Claim 3: *Let (F_d, G_d), with $d \geq t$, be such that, for some $s \geq 1$, one has $|I|, |J| \geq s$ and $k_s = r_s > 0$. Then, for (F_{d+1}, G_{d+1}) there exists also an $s \geq 1$ with the same properties.*

Proof. Observe that, from the previous claim, $k_{s+1}^d = r_{s+1}^d$ and $|J| \geq s+1$ only if $|I| \geq s+1$. If the insertion of j_{d+1} terminates on the top of a column of height $l \neq s-1$, then still $|I|, |J| \geq s$ and $k_s^{d+1} = r_s^{d+1} > 0$. It remains to analyse when $l = s-1$ which means that the insertion of j_{d+1} either terminates to the left or to the right of J. In the first case, (F_d, G_d) satisfies $|J| \geq s+1$ (using Remark 1), $r_s^d = r_{s+1}^d$, and, therefore, $k_{s+1}^d \geq r_{s+1}^d = r_s^d = k_s^d \geq k_{s+1}^d$. It implies for (F_{d+1}, G_{d+1}) that $k_{s+1}^{d+1} = r_{s+1}^{d+1} > 0$, $|J|, |I| \geq s+1$, and thus the claim is true for $s+1$. In the second case, (F_d, G_d) satisfies $k_{s-1}^d \geq r_{s-1}^d > r_s^d = k_s^d$ and thus $k_{s-1}^d > k_s^d$. Thereby the cell i_{d+1} sits to the left of I and $r_s^{d+1} = r_s^d + 1 = k_s^d + 1 = k_s^{d+1}$, with $|I|, |J| \geq s$. The claim is true for s. See Figure 10. □

j_{d+1}
- height: $l = s-1$
  - left of J
  - right of J
- height: $l \neq s-1$

Figure 10: All the possibilities for the insertion of j_{d+1} in F_d.

Next claim describes the pair (F_d, G_d) of SSAFs, for $d \geq t$, when it does not fit the conditions of Claim 3.

Claim 4. *Let (F_d, G_d), with $d \geq t$, be a pair of SSAFs such that, for all $i = 1, \dots, \min\{|I|, |J|\}$, $k_i = r_i > 0$ never holds. Then, $|J| \leq |I|$ and, there is $1 \leq f \leq |J|$, such that $k_i > r_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$.*

Proof. We show by induction on $d \geq t$ that (F_d, G_d) either satisfy the conditions of the Claim 3 or, otherwise, $|J| \leq |I|$ and, there is $1 \leq f \leq |J|$, such that $r_i < k_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$. For $d = t$, we have $|I| = |J| = 1$, and $k_i^d = r_i^d = 0$, $i \geq 1$. Put $f := 1$. Let (F_d, G_d), with $d \geq t$. If (F_d, G_d) fits the conditions of Claim 3, then (F_{d+1}, G_{d+1}) does it as well. Otherwise, assume for (F_d, G_d), $|J| \leq |I|$, and, there exists $1 \leq f \leq |J|$, such that $r_i^d < k_i^d$, for $1 \leq i < f$, and $k_i^d = r_i^d = 0$, for $i \geq f$. We show next that (F_{d+1}, G_{d+1}) either fits the conditions of the previous Claim 3, or, otherwise, it is as described in the present claim. If the insertion of j_{d+1} terminates to the left of J, and i_{d+1} sits on the top or to the right of I, still $|I| \geq |J|$ and there is nothing to prove. If i_{d+1} sits on the top of a column of height l, to the left of I, then, since $k_f^d = 0$, one has $l < f$, and two cases have to be considered. When $l = f - 1$, it implies $|I| \geq |J| \geq f + 1$ (using Remark 1), $r_f^{d+1} = 0$ and $k_f^{d+1} = 1$, and (F_{d+1}, G_{d+1}) satisfies the claim for $f + 1$; in the case of $l < f - 1$, $r_{l+1}^{d+1} = r_{l+1}^d < k_{l+1}^d + 1 = k_{l+1}^{d+1}$ and still, for the same f, $k_i^{d+1} > r_i^{d+1}$, $1 \leq i < f$, $k_i^{d+1} = r_i^{d+1} = 0$, $i \geq f$. If the insertion of j_{d+1} terminates on the top of J, since $k_{|J|}^d = 0$ and $|I| \geq |J|$, then i_{d+1} either sits on the top of I when $|I| = |J|$, and still for the same f, $k_i^{d+1} > r_i^{d+1}$, $1 \leq i < f$, $k_i^{d+1} = r_i^{d+1} = 0$, $i \geq f$, or sits to the right of I, when $|I| > |J|$, and still $|I| \geq |J| + 1$, and, for the same f, $k_i^{d+1} > r_i^{d+1}$, $1 \leq i < f$, $k_i^{d+1} = r_i^{d+1} = 0$, $i \geq f$. If the insertion of j_{d+1} terminates to the right of J on the top of a column of height $l < f$ (recall that $r_f^d = 0$), then, since $|I| > f$, i_{d+1} either sits on the left of I or to the right of I. In the first case, if $l = f - 1$, one has $r_f^{d+1} = r_f^d + 1 = k_f^d + 1 = k_f^{d+1} = 1$, and, therefore, we are in the conditions of Claim 3, with $s = f < |J| \leq |I|$; if $l < f - 1$, still $r_{l+1}^{d+1} = r_{l+1}^d + 1 < k_{l+1}^d + 1 = k_{l+1}^{d+1}$, so $k_i^{d+1} > r_i^{d+1}$, for $1 \leq i < f$ and $r_i^{d+1} = k_i^{d+1} = 0$, for $i \geq f$. In the second case, it means $k_{l+1}^{d+1} = k_{l+1}^d = k_l^d > r_l^d \geq r_{l+1}^d + 1 = r_{l+1}^{d+1}$ and hence $k_{l+1}^{d+1} > r_{l+1}^{d+1}$, with $l + 1 < f$. Note that $k_f = 0$ and $k_{f-1} > r_{f-1}$, so $k_{f-1} \neq 0$ therefore $l \neq f - 1$. Similarly, $k_i^{d+1} > r_i^{d+1}$, for $1 \leq i < f$ and $k_i^{d+1} = r_i^{d+1} = 0$, $i \geq f$. See Figure 11. □

Recall that for any $d \geq t$, i_t appears in the $|I|$ first columns of the $key(sh(G_d))$ and j_t appears in the $|J|$ first columns of the $key(\omega sh(F_d))$.

Classification of the "problem": For each $d \geq t$, either there exists $s \geq 1$ such that $|J|, |I| \geq s$, $r_s = k_s > 0$; or $1 \leq |J| \leq |I|$, and there exists $1 \leq f \leq |J|$, such that $k_i > r_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$. In the first case, one has a "*problem*" in the $(r_s + 1)^{th}$ rows of the s^{th} columns in the key-pair $(key(sh(G_d)), key(\omega sh(F_d)))$.

- j_{d+1}
  - left of J : i_{d+1}
    - top or right of I
    - left of I with height l:
      - $l = f - 1$
      - $l < f - 1$
  - top of J : i_{d+1}
    - top of I
    - right of I
  - right of J : i_{d+1}
    - left of I with height l:
      - $l = f - 1$
      - $l < f - 1$
    - right of I

Figure 11: All the possibilities for inserting j_{d+1} in F_d and placing i_{d+1} in G_d.

In the second case, one has a "*problem*" in the bottom of the $|J|^{th}$ columns. See Example 5.2. and Figure 12.

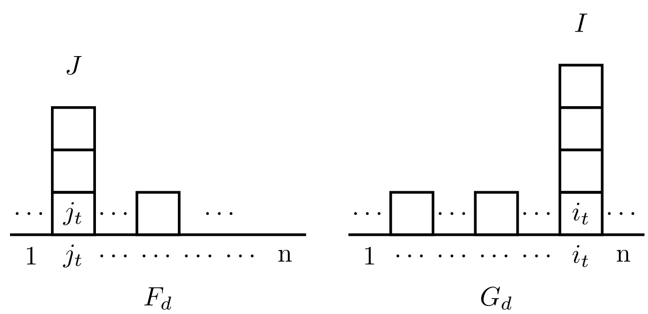


Figure 12: $1 \leq |J| = 3 \leq |I| = 4$, and there exists $1 \leq f = 2 \leq |J| = 3$, such that $k_i > r_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$. The entry at the bottom of the 3th column of $key(\omega sh(F_d))$ is $n - j_t + 1$ while the corresponding entry in the 3th column of $key(sh(G_d))$ is equal to i_t and $n - j_t + 1 < i_t$.

Finally, if the second row of w is over the alphabet $[m]$, there is no cell on the top of the basement of F greater than m. Therefore, the shape of F has the last $n - m$ entries equal to zero and thus its decreasing rearrangement is a partition of length $\leq m$. Using the symmetry of Φ, the other case is similar. □

Remark 2. *1. Given $\nu \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^n$ such that $\beta \leq \omega\nu$, recalling the definition of key SSAF,* (6)*, there exists always a pair (F, G) of SSAFs with shapes ν and β respectively. For instance, the corresponding key SSAF pair.*

2. If the rows in w are swapped and rearranged in lexicographic order, one obtains the biword $\tilde{w}$ such that $\Phi(\tilde{w}) = (G, F)$ with $key(sh(F)) \leq key(\omega sh(G))$.

3. If $k + m \leq n + 1$, the biletters of w are cells of a rectangle (m^k) inside the staircase of length n, and $sh(G) \leq \omega sh(F)$ is trivially satisfied. Notice that $key(sh(G))$ has entries $\leq k$ while $key(\omega sh(F))$ has entries $\geq n - m + 1 \geq k$.

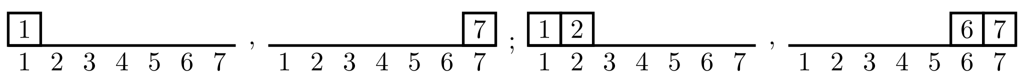
4. Example 5.2., below, shows that if w consists both of biletters above and inside the staircase of size n, then we may have either $sh(G)$ and $\omega sh(F)$ not comparable or $sh(G) > \omega sh(F)$.

Using the bijection Ψ between $SSYT_n$ and $SSAF_n$, one has,

Corollary 2. *Let w be a biword in lexicographic order in the alphabet $[n]$, and let $w \xrightarrow{RSK} (P, Q)$. For each biletter $\binom{i}{j}$ in w we have $1 \le j \le m \le n$, $1 \le i \le k \le n$, and $i + j \le n + 1$ if and only if Q has entries $\le k$, P has entries $\le m$, and $K_+(Q) \le evac(K_+(P))$.*

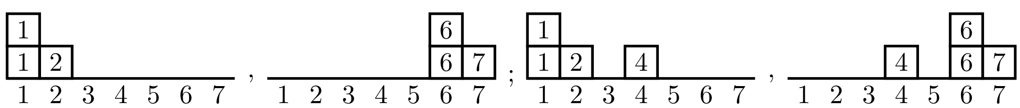
Two examples are given to illustrate Theorem 3.

Example 5. 1. Given $w = \begin{pmatrix} 4 & 6 & 6 & 7 \\ 4 & 1 & 2 & 1 \end{pmatrix}$, $\Phi(w)$ and its key-pair satisfying $key(sh(G)) \le key(\omega sh(F))$ are calculated.



$sh(F_1) = (1, 0^6)$ $\quad sh(G_1) = (0^6, 1)$ $\quad sh(F_2) = (1^2, 0^5)$ $\quad sh(G_2) = (0^5, 1^2)$

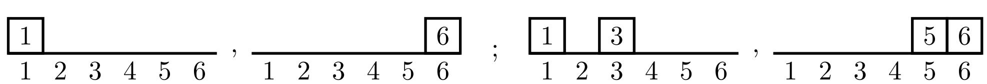
$key(sh(G_1)) = 7 = key(\omega sh(F_1))$ $\qquad key(sh(G_2)) = 67 = key(\omega sh(F_2))$



$sh(F_3) = (2, 1, 0^5)$ $\quad sh(G_3) = (0^5, 2, 1)$ $\quad sh(F_4) = (2, 1, 0, 1, 0^3)$ $\quad sh(G_4) = (0^3, 1, 0, 2, 1)$

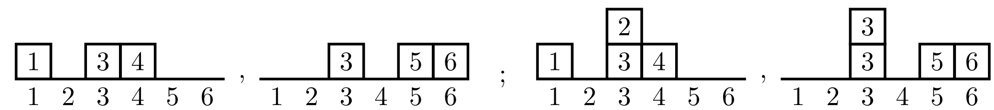
$key(sh(G_3)) = \begin{matrix} 7 & \\ 6 & 6 \end{matrix} \le \begin{matrix} 7 & \\ 6 & 7 \end{matrix} = key(\omega sh(F_3))$ $\qquad key(sh(G_4)) = \begin{matrix} 7 & \\ 6 & \\ 4 & 6 \end{matrix} \le \begin{matrix} 7 & \\ 6 & \\ 4 & 7 \end{matrix} = key(\omega sh(F_4))$

2. Let $w = \begin{pmatrix} 1 & 2 & 3 & 3 & 5 & 6 \\ 6 & 3 & 2 & 4 & 3 & 1 \end{pmatrix}$, $n = 6$, $i_2 = 5 > 6 + 1 - 3 = 6 + 1 - j_2 = 4$. We calculate $\Phi(w)$ whose key-pair $(key(sh(G)) \not\le key(\omega sh(F)))$.



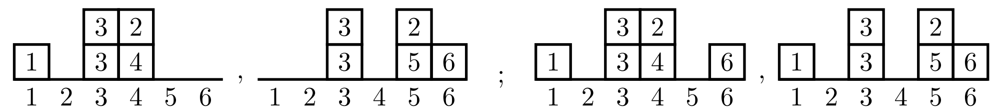
$sh(F_1) = (1, 0^5)$ $\quad sh(G_1) = (0^5, 1)$ $\quad sh(F_2) = (1, 0, 1, 0^3)$ $\quad sh(G_2) = (0^4, 1^2)$

$key(sh(G_1)) = 6 = key(\omega sh(F_1))$ $\qquad key(sh(G_2)) = \begin{matrix} 6 \\ 5 \end{matrix} > \begin{matrix} 6 \\ 4 \end{matrix} = key(\omega sh(F_2))$



$sh(F_3) = (1, 0, 1^2, 0^2)$, $sh(G_3) = (0^2, 1, 0, 1^2)$; $sh(F_4) = (1, 0, 2, 1, 0^2)$, $sh(G_4) = (0^2, 2, 0, 1^2)$

$$key(sh(G_3)) = \begin{matrix} 6 \\ {\color{red}5} \\ 3 \end{matrix} > \begin{matrix} 6 \\ {\color{red}4} \\ 3 \end{matrix} = key(\omega sh(F_3)) \qquad key(sh(G_4)) = \begin{matrix} 6 & \\ {\color{red}5} & \\ 3 & 3 \end{matrix} \begin{matrix} \\ \not> \\ \not< \end{matrix} \begin{matrix} 6 & \\ {\color{red}4} & \\ 3 & 4 \end{matrix} = key(\omega sh(F_4))$$



$sh(F_5) = (1, 0, 2^2, 0^2)$, $sh(G_5) = (0^2, 2, 0, 2, 1)$; $sh(F_6) = (1, 0, 2^2, 0, 1)$, $sh(G_6) = (1, 0, 2, 0, 2, 1)$

$$key(sh(G_5)) = \begin{matrix} 6 & \\ {\color{red}5} & {\color{red}5} \\ 3 & 3 \end{matrix} > \begin{matrix} 6 & \\ {\color{red}4} & {\color{red}4} \\ 3 & 3 \end{matrix} = key(\omega sh(F_5)) \qquad key(sh(G_6)) = \begin{matrix} 6 & \\ {\color{red}5} & \\ 3 & {\color{red}5} \\ 1 & 3 \end{matrix} > \begin{matrix} 6 & \\ {\color{red}4} & \\ 3 & {\color{red}4} \\ 1 & 3 \end{matrix} = key(\omega sh(F_6))$$

3. If the biword w, in 2., is restricted to the last 4 biletters, $sh(G)$ and $\omega sh(F)$ are not comparable. However, when the two first are added, $sh(G) > \omega sh(F)$ holds, as one sees above.

5. Isobaric divided differences and crystal graphs

We review the main results on Demazure operators with an eye on their combinatorial interpretations either as bubble sorting operators acting on weak compositions or their combinatorial version in terms of crystal (coplactic) operators to be used in the last section.

5.1. Isobaric divided differences, and the generators of the 0-Hecke algebra

The action of the simple transpositions $s_i \in \mathfrak{S}_n$ on weak compositions in $\mathbb{N}^n$, induces an action of $\mathfrak{S}_n$ on the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ by considering weak compositions $\alpha \in \mathbb{N}^n$ as exponents of monomials $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ [28], and defining $s_i x^\alpha := x^{s_i \alpha}$ as the transposition of x_i and x_{i+1} in the monomial x^α. If $f \in \mathbb{Z}[x_1, \dots, x_n]$, $s_i f$ indicates the result of the action of s_i in each monomial of f. For $i = 1, \dots, n-1$, one defines the linear operators π_i, $\widehat{\pi}_i$ on $\mathbb{Z}[x_1, \dots, x_n]$ by

$$\pi_i f = \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}, \qquad \widehat{\pi}_i f = (\pi_i - 1) f = \pi_i f - f, \tag{7}$$

where 1 is the identity operator on $\mathbb{Z}[x_1, \dots, x_n]$. These operators are called isobaric divided differences [28], and the first is the Demazure operator [5] for the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. Isobaric divided difference operators π_i and $\widehat{\pi}_i$, $1 \le i < n$, (7), have an equivalent definition

$$\pi_i(x_i^a x_{i+1}^b m) = \begin{cases} x_i^a x_{i+1}^b m + (\sum_{j=1}^{a-b} x_i^{a-j} x_{i+1}^{b+j})m, & \text{if } a > b, \\ x_i^a x_{i+1}^b m, & \text{if } a = b, \\ x_i^a x_{i+1}^b m - (\sum_{j=0}^{b-a-1} x_i^{a+j} x_{i+1}^{b-j})m, & \text{if } a < b, \end{cases} \tag{8}$$

where m is a monomial not containing x_i nor x_{i+1}. It follows from the definition that $\pi_i(f) = f$ and $\widehat{\pi}_i(f) = 0$ if and only if $s_i f = f$. They both satisfy the commutation and the braid relations (2) of $\mathfrak{S}_n$, and this guarantees that, for any permutation $\sigma \in \mathfrak{S}_n$, there exists a well defined isobaric divided difference $\pi_\sigma := \pi_{i_N} \cdots \pi_{i_2}\pi_{i_1}$ and $\hat{\pi}_\sigma := \hat{\pi}_{i_N} \cdots \hat{\pi}_{i_2}\hat{\pi}_{i_1}$, where $(i_N, \dots, i_2, i_1)$ is any reduced word of $\mathfrak{S}_n$. In addition, they satisfy the quadratic relations $\pi_i^2 = \pi_i$ and $\hat{\pi}_i^2 = -\hat{\pi}_i$.

The 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$, a deformation of the group algebra of $\mathfrak{S}_n$, is an associative $\mathbb{C}$-algebra generated by $T_1, \dots, T_{n-1}$ satisfying the commutation and the braid relations of the symmetric group $\mathfrak{S}_n$, and the quadratic relation $T_i^2 = T_i$ for $1 \le i < n$. Setting $\widehat{T}_i := T_i - 1$, for $1 \le i < n$, one obtains another set of generators of the 0-Hecke algebra $H_n(0)$. The sets $\{T_\sigma : \sigma \in \mathfrak{S}_n\}$ and $\{\hat{T}_\sigma : \sigma \in \mathfrak{S}_n\}$ are both linear bases for $H_n(0)$, where $T_\sigma = T_{i_N} \cdots T_{i_2}T_{i_1}$ and $\hat{T}_\sigma := \hat{T}_{i_N} \cdots \hat{T}_{i_2}\hat{T}_{i_1}$, for any reduced expression $s_{i_N} \cdots s_{i_2}s_{i_1}$ in $\mathfrak{S}_n$ [3]. Since Demazure operators π_i (7) or bubble sort operators (5) satisfy the same relations as T_i, and similarly for isobaric divided difference operators $\widehat{\pi}_i$ (7) and $\widehat{T}_i$, the 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$ may be viewed as an algebra of operators realised either by any of the two isobaric divided differences (7), or by bubble sort operators (5), swapping entries i and $i+1$ in a weak composition α, if $\alpha_i > \alpha_{i+1}$, and doing nothing, otherwise. Therefore, the two families $\{\pi_\sigma : \sigma \in \mathfrak{S}_n\}$ and $\{\hat{\pi}_\sigma : \sigma \in \mathfrak{S}_n\}$ are both linear bases for $H_n(0)$, and from the relation $\widehat{\pi}_i = \pi_i - 1$, the change of basis from the first to the second is given by a sum over the Bruhat order in $\mathfrak{S}_n$, precisely [25, 38],

$$\pi_\sigma = \sum_{\theta \le \sigma} \hat{\pi}_\theta. \tag{9}$$

5.2. Demazure characters, Demazure atoms and sorting operators

Let $\lambda \in \mathbb{N}^n$ be a partition and α a weak composition in the $\mathfrak{S}_n$-orbit of λ. Write $\alpha = \sigma\lambda$, where σ is a minimal length coset representative of $\mathfrak{S}_n/stab_\lambda$. The key polynomial [24, 39] or Demazure character [5, 15] in type A, corresponding to the dominant weight λ and permutation σ, is the polynomial in $\mathbb{Z}[x_1, \dots, x_n]$ indexed by the weak composition $\alpha \in \mathbb{N}^n$, defined by

$$\kappa_\alpha := \pi_\sigma x^\lambda, \tag{10}$$

and the standard basis [23, 24] or Demazure atom [34] is defined similarly,

$$\widehat{\kappa}_\alpha := \widehat{\pi}_\sigma x^\lambda. \tag{11}$$

Due to (9), the identity (11) consists of all monomials in κ_α which do not appear in κ_β for any $\beta < \alpha$. Thereby, key polynomials (10) are decomposed into Demazure atoms [24, 28],

$$\kappa_\alpha = \sum_{\beta \leq \alpha} \widehat{\kappa}_\beta. \tag{12}$$

Key polynomials $\{\kappa_\alpha : \alpha \in \mathbb{N}^n\}$ and Demazure atoms $\{\hat{\kappa}_\alpha : \alpha \in \mathbb{N}^n\}$ form linear $\mathbb{Z}$-basis for $\mathbb{Z}[x_1, \ldots, x_n]$ [39]. The change of basis from the first to the second is expressed in (12). The operators π_i act on key polynomials κ_α via elementary bubble sorting operators on the entries of the weak composition α [39],

$$\pi_i \kappa_\alpha = \begin{cases} \kappa_{s_i\alpha} & \text{if } \alpha_i > \alpha_{i+1} \\ \kappa_\alpha & \text{if } \alpha_i \leq \alpha_{i+1} \end{cases}. \tag{13}$$

This suggests the following recursive definition of key polynomials [28]. For $\alpha \in \mathbb{N}^n$, the key polynomial κ_α (resp. $\hat{\kappa}_\alpha$) is $\kappa_\alpha = \hat{\kappa}_\alpha = x^\alpha$, if α is a partition. Otherwise, $\kappa_\alpha = \pi_i \kappa_{s_i\alpha}$ (resp. $\hat{\kappa}_\alpha = \hat{\pi}_i \hat{\kappa}_{s_i\alpha}$), if $\alpha_{i+1} > \alpha_i$. The key polynomial κ_α is symmetric in x_i and x_{i+1} if and only if $\alpha_{i+1} \geq \alpha_i$. Thus it lifts the Schur polynomial $s_{\alpha^+}(x)$, $\kappa_\alpha = s_{\alpha^+}(x)$, when $\alpha_1 \leq \cdots \leq \alpha_n$. Henceforth, (12) contains, as a special case, the decomposition of a Schur polynomial, into Demazure atoms,

$$s_{\alpha^+}(x) = \sum_{\alpha \in \mathfrak{S}_n \alpha^+} \widehat{\kappa}_\alpha. \tag{14}$$

5.3. Crystals and combinatorial descriptions of Demazure operators

In [24] Lascoux and Schützenberger have given a combinatorial version for Demazure operators π_i and $\hat{\pi}_i$ in terms of crystal (or coplactic) operators f_i, e_i to produce a crystal graph on $\mathfrak{B}^\lambda$, the set of SSYTs with entries $\leq n$ and shape λ [16, 17, 27]. A SSYT can be uniquely recovered from its column word. The action of the crystal operators f_i and e_i, $1 \leq i < n$, on $T \in \mathfrak{B}^\lambda$ is described by the usual parentheses matching on the column word of T, and we refer the reader for details to [22, 27]. For convenience, we extend f_i and e_i to $\mathfrak{B}^\lambda \cup \{0\}$ by setting them to map 0 to 0.

Kashiwara and Nakashima [17, 19] have given to $\mathfrak{B}^\lambda$ a $U_q(\mathfrak{gl}_n)$- crystal structure. We view crystals as special graphs. The crystal graph on $\mathfrak{B}^\lambda$ is a coloured directed graph whose vertices are the elements of $\mathfrak{B}^\lambda$, and the edges are coloured with a colour i, for each pair of crystal operators f_i, e_i, such that there exists a coloured i-arrow from the vertex T to T' if and only if $f_i(T) = T'$, equivalently, $e_i(T') = T$. We refer to [20, 12, 29] for details. Start with the Yamanouchi tableau $Y := key(\lambda)$ and apply all the crystal operators f_i's until each unmatched i has been converted to $i+1$, for $1 \leq i < n$ [17, 20]. See Example 6. The resulting set is $\mathfrak{B}^\lambda$ whose elements index basis vectors for the representation of the quantum group $U_q(\mathfrak{gl}_n)$ with highest weight λ. From the definition of this graph, in each vertex there is at most one incident arrow of colour i, and at most one outgoing arrow of colour i. Hence, for any i, $1 \leq i < n$, the crystal graph on $\mathfrak{B}^\lambda$ decompose $\mathfrak{B}^\lambda$ into disjoint connected components of colour i, $P_1 \xrightarrow{i} \cdots \xrightarrow{i} P_k$, called i-strings, having lengths $k-1 \geq 0$. A SSYT P_1, satisfying $e_i(P_1) = 0$, is said to be the head of the i-string, and, in the case of $f_i(P_k) = 0$, P_k is called the end of the i-string. Given α in the $\mathfrak{S}_n\lambda$, the Demazure crystal $\mathfrak{B}_\alpha$ is viewed as a certain subgraph of the crystal $\mathfrak{B}^\lambda$ which can be defined inductively [18, 31] as $\mathfrak{B}_\alpha = \{Y\}$ if $\alpha = \lambda$, otherwise

$$\mathfrak{B}_\alpha = \{f_i^k(T) : T \in \mathfrak{B}_{s_i\alpha}, k \geq 0, e_i(T) = 0\} \setminus \{0\}, \quad \text{if } \alpha_{i+1} > \alpha_i. \tag{15}$$

(When α is the reverse of λ, one has $\mathfrak{B}_{\omega\lambda} = \mathfrak{B}^\lambda$.) The vertices of this subgraph index basis vectors of the Demazure module $V_\sigma(\lambda)$ where σ is a minimal length coset representative modulo the stabiliser of λ, such that $\sigma\lambda = \alpha$. In fact $\mathfrak{B}_\alpha$ (15) is well defined, it does not depend on the reduced expression for σ. More generally, write $\alpha = s_{i_N} \dots s_{i_2} s_{i_1}\lambda$, with $(i_N, \dots, i_2, i_1)$ a reduced word, then apply the crystal operator f_{i_1} to Y until each unmatched i_1 has been converted to i_1+1, then apply similarly f_{i_2} to each of the previous Young tableaux until each unmatched i_2 has been converted to i_2+1, and continue this procedure with $f_{i_3}, \dots, f_{i_N}$. Therefore, $\mathfrak{B}_\alpha = \{f_{i_N}^{m_N} \dots f_{i_1}^{m_1}(Y) : m_k \geq 0\} \setminus \{0\}$.

Let $T \in \mathfrak{B}^\lambda$, and $f_{s_i}(T) := \{f_i^m(T) : m \geq 0\} \setminus \{0\}$. (If $f_i(T) = 0$, $f_{s_i}(T) = \{T\}$.) If P is the head of an i-string $S \subseteq \mathfrak{B}^\lambda$, $S = f_{s_i}(P)$. We abuse notation and say the Demazure operator π_i (8) sends the head of an i-string to the sum of all elements of the string [24, 18],

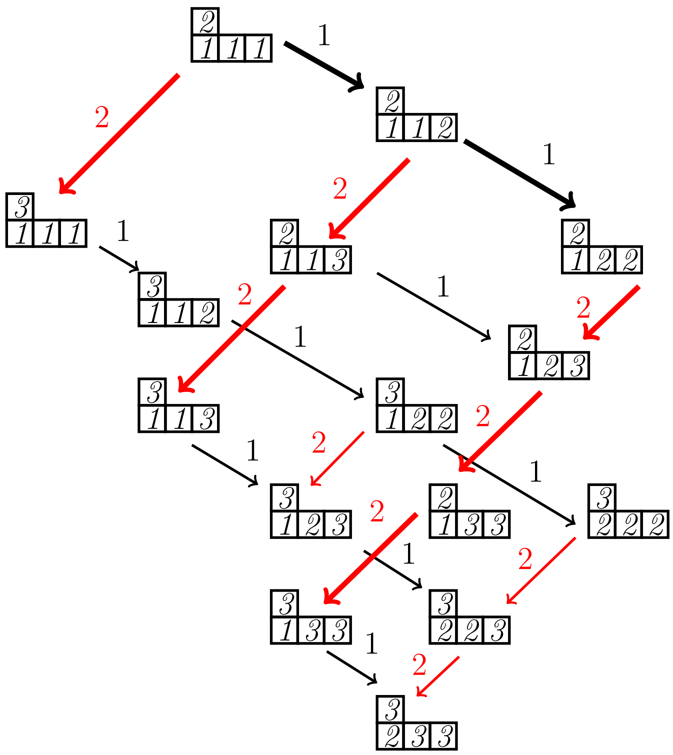
$$\pi_i(x^P) = \sum_{T \in S} x^T, \text{ and } \pi_i\left(\sum_{T \in S} x^T\right) = \pi_i(x^P). \tag{16}$$

If $\beta \leq \alpha$, then $\mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$. Let $s_i\alpha < \alpha$, equivalently, $\alpha_i < \alpha_{i+1}$. For any i-string $S \subseteq \mathfrak{B}^\lambda$, either $\mathfrak{B}_{s_i\alpha} \cap S = \mathfrak{B}_\alpha \cap S$ is empty, or $\mathfrak{B}_{s_i\alpha} \cap S = \mathfrak{B}_\alpha \cap S = S$, or $\mathfrak{B}_{s_i\alpha} \cap S$ is only the head of S in which case $S \subseteq \mathfrak{B}_\alpha$. Since $\mathfrak{B}^\lambda$ is the disjoint union of i-strings, from these string properties, and (16), one has for any i-string S

$$\sum_{T \in \mathfrak{B}_\alpha \cap S} x^T = \pi_i\left(\sum_{T \in \mathfrak{B}_{s_i\alpha} \cap S} x^T\right); \text{ and } \sum_{T \in \mathfrak{B}_\alpha} x^T = \pi_i\left(\sum_{T \in \mathfrak{B}_{s_i\alpha}} x^T\right). \tag{17}$$

Henceforth, $\kappa_\alpha = \pi_i \kappa_{s_i\alpha}$, if $\alpha_i < \alpha_{i+1}$, and $\kappa_\alpha = \pi_{i_N} \cdots \pi_{i_1} x^\lambda$ for any reduced word $(i_N, \ldots, i_1)$ such that $s_{i_N} \cdots s_{i_1}\lambda = \alpha$.

Example 6. *The Demazure crystal $\mathfrak{B}_{s_2s_1\lambda}$ with $\lambda = (3,1,0)$ is shown with thick edges while the rest of the crystal graph $\mathfrak{B}^\lambda$ is shown with thinner lines. The 1 and 2-strings are represented in black and red colours respectively. The key polynomial $\kappa_{(1,0,3)}$ is calculated using the thick coloured strings in the Demazure crystal graph $\mathfrak{B}_{s_2s_1(3,1,0)}$,* $\kappa_{(1,0,3)} = \pi_2\pi_1 x^{(3,1,0)} = \pi_2(x^{(3,1,0)+x^{(2,2,0)}+x^{(1,3,0)}}) = x^{(3,1,0)} + x^{(2,2,0)} + x^{(1,3,0)} + x^{(3,0,1)} + x^{(2,1,1)} + x^{(2,0,2)} + x^{(1,2,1)} + x^{(1,1,2)} + x^{(1,0,3)}$.



Set $\widehat{\mathfrak{B}}_\alpha := \mathfrak{B}_\alpha \setminus \bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. Then $\mathfrak{B}_\alpha = \biguplus_{\beta \le \alpha} \widehat{\mathfrak{B}}_\beta$. In Example 6, with $\alpha = (1,0,3) = s_2s_1(3,1,0)$, the component $\widehat{\mathfrak{B}}_{s_2s_1(3,1,0)} = \mathfrak{B}_{s_2s_1(3,1,0)} \setminus (\mathfrak{B}_{s_1(3,1,0)} \cup \mathfrak{B}_{s_2(3,1,0)})$ consists of the two lowest thick red strings, starting in the thick black string, minus their heads. Lascoux and Schützenberger have characterised the SSYTs in $\widehat{\mathfrak{B}}_\alpha$ [24] as those whose right key is $key(\alpha)$, precisely the unique key tableau in $\widehat{\mathfrak{B}}_\alpha$. The Demazure crystal $\mathfrak{B}_\alpha$ consists of all Young tableaux in $\mathfrak{B}^\lambda$ with right key bounded by $key(\alpha)$.

Theorem 4 (Lascoux, Schützenberger [23, 24])**.** *The Demazure atom $\widehat{\kappa}_{\sigma\lambda} = \hat{\pi}_\sigma x^\lambda$ is the sum of the weight monomials of all SSYTs with entries $\le n$ whose right* *key is equal to key*$(\sigma\lambda)$*, with* σ *a minimal length coset representative modulo the stabiliser of* λ*.*

We may put together the three combinatorial interpretations of Demazure characters and Demazure atoms

$$\hat{\kappa}_\alpha = \sum_{T\in\widehat{\mathfrak{B}}_\alpha} x^T = \sum_{\substack{T\in SSYT_n \\ K_+(T)=key(\alpha)}} x^T = \sum_{\substack{F\in SSAF_n \\ sh(F)=\alpha}} x^F,$$

$$\kappa_\alpha = \sum_{T\in\mathfrak{B}_\alpha} x^T = \sum_{\substack{T\in SSYT_n \\ K_+(T)\le key(\alpha)}} x^T = \sum_{\substack{F\in SSAF_n \\ sh(F)\le\alpha}} x^F.$$

In particular, the sum of the weight monomials over all crystal graph $\mathfrak{B}^\lambda$ gives the Schur polynomial s_λ, and thus Demazure atoms decompose Schur polynomials in $\mathbb{Z}[x_1,\dots,x_n]$.

6. Expansions of Cauchy kernels over truncated staircases

6.1. Cauchy identity and Lascoux's non-symmetric Cauchy kernel expansions

Given $n\in\mathbb{N}$ positive, let m and k be fixed positive integers where $1\le m\le n$ and $1\le k\le n$. Let $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ be two sequences of indeterminates. The well-known Cauchy identity expresses the Cauchy kernel $\prod_{i=1}^k\prod_{j=1}^m(1-x_iy_j)^{-1}$, symmetric in x_i and y_j separately, as a sum of products of Schur polynomials s_{μ^+} in $(x_1,x_2,\dots,x_k)$ and $(y_1,y_2,\dots,y_m)$,

$$\prod_{(i,j)\in(m^k)}(1-x_iy_j)^{-1} = \prod_{i=1}^{k}\prod_{j=1}^{m}(1-x_iy_j)^{-1} = \sum_{\mu^+} s_{\mu^+}(x_1,\dots,x_k)s_{\mu^+}(y_1,\dots,y_m), \quad (18)$$

over all partitions μ^+ of length $\le\min\{k,m\}$. Using either the RSK correspondence [21] or the Φ correspondence, the Cauchy formula (18) can be interpreted as a bijection between monomials, on the left hand side, and pairs of SSYTs or SSAFs on the right. As the basis of key polynomials lifts the Schur polynomials w.r.t. the same list of indeterminates, and key polynomials decompose into Demazure atoms (14), the expansion (18) can also be expressed in the two bases of key polynomials. Assuming $k\le m$, we may write (18) as

$$\sum_{\mu^+\in\mathbb{N}^k} s_{\mu^+}(x_1,\dots,x_k)s_{(\mu^+,0^{m-k})}(y_1,\dots,y_m) = \sum_{\mu^+\in\mathbb{N}^k}\sum_{\mu\in\mathfrak{S}_k\mu^+}\widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\omega\mu^+)}(y),$$

$$= \sum_{\mu\in\mathbb{N}^k}\widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\omega\mu^+)}(y). \quad (19)$$

(Since we are dealing with two sequences of indeterminates x and y, it is convenient to write $\kappa_\alpha(x)$ and $\kappa_\alpha(y)$ instead of κ_α. Similarly for Demazure atoms.)

We now replace in the Cauchy kernel the rectangle (m^k) by the truncated staircase $\lambda = (m^{n-m+1}, m-1, \ldots, n-k+1)$, with $1 \le m \le n$, $1 \le k \le n$, and $n+1 \le m+k$, as shown in Figure 1. If $n+1 = m+k$, we recover the rectangle shape (m^k). When $m = n = k$, one has the staircase partition $\lambda = (n, n-1, \ldots, 2, 1)$, that is, the cells (i,j) in the NW-SE diagonal of the square diagram (n^n) and below it, and thus $(i,j) \in \lambda$ if and only if $i + j \le n+1$. Lascoux has given in [26], and with Fu, in [7], the following expansion for the non-symmetric Cauchy kernel over staircases,

$$\prod_{\substack{i+j \le n+1 \\ 1 \le i,j \le n}} (1 - x_i y_j)^{-1} = \sum_{\nu \in \mathbb{N}^n} \widehat{\kappa}_\nu(x) \kappa_{\omega\nu}(y), \tag{20}$$

where $\hat{\kappa}$ and κ indicate the two families of key polynomials in x and y respectively, and ω is the longest permutation of $\mathfrak{S}_n$.

In [26], Lascoux extends (20) to an expansion of $\prod_{(i,j)\in\lambda}(1 - x_i y_j)^{-1}$, over any Ferrers shape λ, as follows. Given a Ferrers shape λ, consider $\rho := (t, t-1, \ldots, 1)$, the biggest staircase contained in λ, and a pair of permutations $\sigma(\lambda, NW)$ and $\sigma(\lambda, SE)$ encoding the cells in a NW and SE parts of the skew-diagram λ/ρ, the diagram consisting of the cells in λ not in ρ. To define such a pair of permutations, one takes an arbitrary cell in the staircase $(t+1, t, \ldots, 1)$ which does not belong to λ. The SW-NE diagonal passing through this cell cuts the skew-diagram of λ/ρ, into two pieces that are called the North-West (NW) part and the South-East (SE) part of λ/ρ. Fill each cell of row $r \ge 2$ of the NW part with the number $r-1$. Similarly, fill each cell of column $c \ge 2$ of the SE part with the number $c-1$. Reading the columns of the NW part, from right to left, top to bottom, and interpreting r as the simple transposition s_r, gives a reduced decomposition of the permutation $\sigma(\lambda, NW)$; similarly, reading rows of the SE part, from top to bottom, and from right to left, gives the permutation $\sigma(\lambda, SE)$.

Theorem 5 (Lascoux, [26, Theorem 7])**.** *Let λ be a partition in $\mathbb{N}^n$, $\rho(\lambda) = (t, t-1, \ldots, 1)$ the maximal staircase contained in the diagram of λ, and $\sigma(\lambda, NW)$, $\sigma(\lambda, SE)$ the two permutations obtained by cutting the diagram of λ/ρ as explained above. Then*

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\mu \in \mathbb{N}^t} (\pi_{\sigma(\lambda,NW)} \widehat{\kappa}_\mu(x))(\pi_{\sigma(\lambda,SE)} \kappa_{\omega\mu}(y)). \tag{21}$$

For our truncated staircase shape λ, Figure 1, if $1 \le k \le m \le n$, $\lambda = (m^{n-m+1}, m-1, \ldots, n-k+1)$, where $n-k \le m-1$, $\rho = (k, k-1, \ldots, 1)$, and the cell $(k+1, 1)$, on the top of the first column of λ, does not belong to λ. In this case, the NW piece of λ/ρ is empty, thus $\sigma(\lambda, NE) = id$, and the SE piece consists of all cells in λ/ρ. In Figure 13, the row reading word, top to bottom and right to left, defines the reduced word

$$\sigma(\lambda, SE) = \prod_{i=1}^{k-(n-m)-1} (s_{i+n-k-1} \dots s_i) \prod_{i=0}^{n-m} (s_{m-1} \dots s_{k-(n-m)+i}). \tag{22}$$

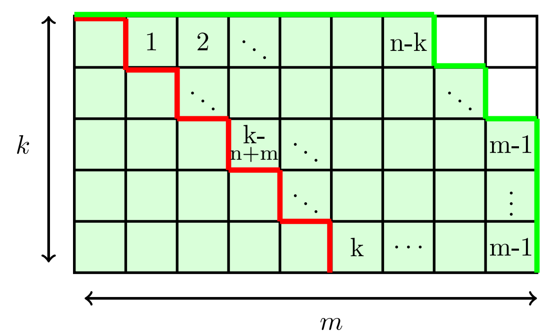


Figure 13: The labels in λ/ρ indicate the column index of λ minus one. The reading word, from right to left and from the top to bottom, defines the reduced word $\sigma(\lambda, SE)$ (22).

Similarly, in Figure 1, if $n \geq k \geq m \geq 1$, then $\rho = (m, m-1, \dots, 1)$, and the cell $(1, m+1)$ immediately after to the end of the first row of λ, does not belong to λ. Thus the SE piece of λ/ρ is empty, $\sigma(\lambda, SE) = id$, and the NW piece consists of all cells in λ/ρ. Recall that $\overline{\lambda}$, the conjugate partition of λ, is the transpose of the Ferrers diagram λ, and notice that $\sigma(\overline{\lambda}, SE) = \sigma(\lambda, NW)$. Therefore, the formula (21) is translated to

$$\prod_{\substack{(i,j)\in\lambda \\ k\leq m}} (1 - x_i y_j)^{-1} = \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)(\pi_{\sigma(\lambda,SE)}\kappa_{\omega\mu}(y)); \tag{23}$$

$$\prod_{\substack{(i,j)\in\lambda \\ m\leq k}} (1 - x_i y_j)^{-1} = \sum_{\mu\in\mathbb{N}^m} (\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\mu(x))\kappa_{\omega\mu}(y). \tag{24}$$

Indeed (24) is just (23), with x and y swapped, followed by the change of basis (12) from Demazure characters to Demazure atoms, where we also use the linearity of Demazure operators. Then we have

$$\prod_{\substack{(i,j)\in\lambda \\ m\leq k}} (1 - x_i y_j)^{-1} = \prod_{\substack{(j,i)\in\overline{\lambda} \\ m\leq k}} (1 - x_i y_j)^{-1} = \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\overline{\lambda},SE)}\kappa_{\omega\mu}(x)$$

$$
\begin{aligned}
&= \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\kappa_{\omega\mu}(x) = \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)} \sum_{\beta\leq\omega\mu} \widehat{\kappa}_\beta(x)\\
&= \sum_{\mu\in\mathbb{N}^m} \sum_{\substack{\beta\in\mathbb{N}^m\\ \beta\leq\omega\mu}} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x) = \sum_{\beta\in\mathbb{N}^m} \sum_{\substack{\mu\in\mathbb{N}^m\\ \mu\leq\omega\beta}} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x)\\
&= \sum_{\beta\in\mathbb{N}^m} \pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x) \sum_{\substack{\mu\in\mathbb{N}^m\\ \mu\leq\omega\beta}} \widehat{\kappa}_\mu(y) = \sum_{\beta\in\mathbb{N}^m} \pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x)\kappa_{\omega\beta}(y).
\end{aligned}
\tag{25}
$$

Next we give a bijective proof of (23), which amounts to computing the indexing weak composition of the Demazure character $\pi_{\sigma(\lambda,SE)}\ \kappa_{\omega\mu}(y)$, by making explicit the Young tableaux in the Demazure crystal.

6.2. Our expansions

We now use the bijection in Theorem 3 to give an expansion of the non-symmetric Cauchy kernel for the shape $\lambda = (m^{n-m+1}, m-1, \ldots, n-k+1)$, where $1 \leq m \leq n$, $1 \leq k \leq n$, and $n+1 \leq m+k$, which includes, in particular, the rectangle (18), the staircase (20), and implies the truncated staircases (23).

The generating function for the multisets of ordered pairs of positive integers $\{(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)\}$, $r \geq 0$, where $(a_i, b_i) \in \lambda$, that is, $a_i + b_i \leq n+1$, $1 \leq a_i \leq k$, $1 \leq b_i \leq m$, $1 \leq i \leq r$, weighted by the contents $((\alpha, 0^{n-k});\ (\delta, 0^{n-m})) \in \mathbb{N}^k \times \mathbb{N}^m$, with α_j the number of i's such that $a_i = j$, and δ_j the number of i's such that $b_i = j$, is

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\substack{\{(a_i,b_i)\}_{i=1}^r\\ r\geq 0}} x_{a_1}y_{b_1}\cdots x_{a_r}y_{b_r} = \sum_{\substack{\{(a_i,b_i)\}_{i=1}^r\\ r\geq 0}} x^\alpha y^\delta.$$

Each multiset $\{(a_1, b_1),\ (a_2, b_2),\ \ldots,\ (a_r, b_r)\}$, $r \geq 0$, and, hence, each monomial $x_{a_1}\ y_{b_1} \cdots\ x_{a_r} y_{b_r}$, $r \geq 0$, is in one-to-one correspondence with the lexicographically ordered biword $\left(\begin{smallmatrix} a_r & \cdots & a_1\\ b_r & \cdots & b_1\end{smallmatrix}\right)$ in the product of alphabets $[k] \times [m]$. In turn, each biword is bijectively mapped by Φ into the pair (F, G) of SSAFs such that G has entries in $\{a_1, \ldots, a_r\}$, F has entries in $\{b_1, \ldots, b_r\}$, and their shapes $sh(G) = \mu \in \mathbb{N}^k$, and $sh(F) = \beta \in \mathbb{N}^m$, in a same $\mathfrak{S}_n$-orbit, satisfy $(\beta, 0^{n-m}) \leq (0^{n-k}, \omega\mu)$ with ω the longest permutation in $\mathfrak{S}_k$. (For $r = 0$, put $F = G = \emptyset$.) Thereby, $x_{a_1}y_{b_1}\cdots x_{a_r}y_{b_r} = y^F x^G$, for all $r \geq 0$. Assume $k \leq m$. Since $(\mu, 0^{n-k})$, $(\beta, 0^{n-m})$ are in a same $\mathfrak{S}_n$-orbit, $(\mu^+, 0^{m-k}) = \beta^+ \in \mathbb{N}^m$. We then may write

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} \ = \ \sum_{\mu\in\mathbb{N}^k} \ \sum_{\substack{F,G\,\in SSAF_n\\ sh(F)=\beta\in\mathbb{N}^m,\ sh(G)=\mu\\ (\beta,0^{n-m})\leq(0^{n-k},\omega\mu)}} y^F x^G$$

$$= \sum_{\mu\in\mathbb{N}^k}\left(\sum_{\substack{G\in SSAF_n\\ sh(G)=\mu}} x^G\right)\left(\sum_{\substack{\beta\in\mathbb{N}^m\\ (\beta,0^{n-m})\leq(0^{n-k},\omega\mu)}}\ \sum_{\substack{F\in SSAF_n\\ sh(F)=\beta}} y^F\right)$$

$$\begin{aligned} &= \sum_{\mu\in\mathbb{N}^k}\left(\sum_{\substack{Q\in SSYT_n\\ sh(Q)=\mu^+\\ K_+(Q)=key(\mu)}} x^Q\right)\left(\sum_{\substack{\beta\in\mathbb{N}^m\\ (\beta,0^{n-m})\leq(0^{n-k}\omega\mu)}}\ \sum_{\substack{P\in SSYT_n\\ sh(P)=\mu^+\\ K_+(P)=key(\beta)}} y^P\right)\\ &= \sum_{\mu\in\mathbb{N}^k}\widehat{\kappa}_\mu(x)\sum_{\substack{P\in\mathfrak{B}_{(0^{n-k},\omega\mu)}\\ entries\leq m}} y^P. \end{aligned} \tag{26}$$

Given $\mu\in\mathbb{N}^k$, since $m\geq k$, put $\nu:=(\mu,0^{m-k},0^{n-m})$. Then $\omega\nu=(0^{m-k},0^{n-m},\omega\mu)$. Recall that $\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}=\mathfrak{B}^{(\mu^+,0^{m-k})}$ is the crystal graph consisting of all SSYTs with shape $(\mu^+,0^{m-k})$ and entries less or equal than m. (The size of the longest permutation ω should be understood from the context.) Therefore the arrows are $P\overset{i}{\to}P'$ for each crystal operator f_i such that $f_i(P)=P'$, $1\leq i<m$. Henceforth, one has

$$\sum_{\substack{P\in\mathfrak{B}_{\omega\nu}\\ entries\leq m}} y^P=\sum_{P\in\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}} y^P, \tag{27}$$

the weight polynomial of all SSYTs in the $\mathfrak{B}_{\omega\nu}$ with entries less or equal than m, equivalently, of all SSYTs with entries $\leq m$ and shape $\mu^+\in\mathbb{N}^k$ whose right key is bounded by $key(0^{n-k},\omega\mu)$. It is also equivalent to consider all SSAFs such that the shape has at most k nonzero entries with zeros in the last $n-m$ entries, and is bounded by $\omega\nu$. At this point we can say that a SSYT, in the intersection of the two Demazure crystals $\mathfrak{B}_{(0^{n-m},0^{m-k},\omega\mu)}\cap\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}$, has shape in $\mathbb{N}^k$, entries $\leq m$, and necessarily its right key is specified by a vector $\zeta\in\mathbb{N}^m$ with at most k non zero entries and satisfying $\zeta\leq(0^{m-k},\omega\mu^+)$. On the other hand, one also has $(\zeta,0^{n-m})\leq(0^{n-m},0^{m-k},\omega\mu)$ despite that $\omega\mu\leq\omega\mu^+$. Indeed $\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}\subseteq\mathfrak{B}_{(0^{m-k},\alpha,0^{n-m})}$ for some $\alpha\in\mathbb{N}^k$ and $\alpha\leq\omega\mu^+$. Next, we determine the optimal $\alpha\in\mathbb{N}^k$ so that the Demazure crystal $\mathfrak{B}_{(0^{m-k},\alpha,0^{n-m})}=\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}$. This shows that (27) is a key polynomial and simultaneously describes its indexing weak composition. See Example 7.

Lemma 3. *Let $\gamma\in\mathbb{N}^n$ such that $\gamma^+=(\eta,0^{n-m})$ is a partition of length $\leq m\leq n$. Consider a sequence of positive integers $1\leq i_M,\dots,i_1<n$ (not necessarily a reduced word of $\mathfrak{S}_n$) such that $\kappa_\gamma(y)=\pi_{i_M}\cdots\pi_{i_1}y^{(\eta,0^{n-m})}$. If $j_s,\dots,j_1$ is the subsequence*

consisting of all elements $\geq m$, it holds

$$\sum_{\substack{P\in\mathfrak{B}_\gamma\\ entries\leq m}} y^P = \sum_{P\in\mathfrak{B}_\gamma\cap\mathfrak{B}_{(\omega\eta,0^{n-m})}} y^P = \pi_{i_M}\cdots\tilde{\pi}_{j_s}\cdots\tilde{\pi}_{j_1}\cdots\pi_{i_1}y^{(\eta,0^{n-m})}, \tag{28}$$

where the tilde "$\sim$" means omission, and ω is the longest permutation of $\mathfrak{S}_m$.

Proof. Notice that from the recursive definition of key polynomial or (13), $\mathfrak{B}_\gamma\subseteq\mathfrak{B}^{\gamma^+}$. Also $\mathfrak{B}_{(\omega\eta,0^{n-m})}=\mathfrak{B}^\eta\subseteq\mathfrak{B}^{\gamma^+}$, and $\mathfrak{B}_\gamma\cap\mathfrak{B}_{(\omega\eta,0^{n-m})}=\mathfrak{B}_\gamma\cap\mathfrak{B}^\eta$. If $n=m$ or γ has the last $n-m$ entries equal to zero, then $\gamma\leq(\omega\eta,0^{n-m})$, $\mathfrak{B}_\gamma\subseteq\mathfrak{B}^\eta$, and $1\leq i_M,\dots,$ $i_1<m$. Henceforth, in this case, $\sum_{\substack{P\in\mathfrak{B}_\gamma\\ entries\leq m}} y^P=\sum_{P\in\mathfrak{B}_\gamma} y^P=\kappa_\gamma(y)$. Otherwise, the intersection of the two graphs $\mathfrak{B}_\gamma\cap\mathfrak{B}^\eta$ is the graph obtained from $\mathfrak{B}_\gamma$ by deleting all the vertices consisting of SSYTs with entries $>m$, and, therefore, all i-edges incident on them (either getting in or out), in particular, those with $i\geq m$. This means that all i-strings, with $i\geq m$, in $\mathfrak{B}_\gamma$, are deleted, while just the heads remain, in the case of $i=m$. Furthermore, every i-string with $i<m$ whose head has an entry $>m$ is ignored. In conclusion, $\mathfrak{B}_\gamma\cap\mathfrak{B}^\eta$ consists of the i-strings in $\mathfrak{B}_\gamma$ with $i<m$ whose heads have entries $\leq m$. From the combinatorial interpretation of Demazure operators π_i in terms of the i-strings of a crystal graph, (15), (16), (17), this means we are deleting in $\pi_{i_M}\cdots\pi_{i_2}\pi_{i_1}y^{(\eta,0^{n-m})}$ the action of the Demazure operators π_i for $i\geq m$, and, thanks to (13), one still has a key polynomial, precisely, (28). □

We now calculate the indexing weak composition of the key polynomial (28) in the case $\eta=(\mu^+,0^{m-k})$ and $\gamma=\omega\nu=\omega(\mu,0^{m-k},0^{n-m})$, and, therefore, the key polynomial (27).

Proposition 3. *Let $1\leq k\leq m\leq n$, and $n-m+1\leq k$. Given $\mu\in\mathbb{N}^k$, let $\alpha=(\alpha_1,\dots,\alpha_k)\in\mathbb{N}^k$ such that for each $i=k,\dots,1$, the entry α_i is the maximum element among the last* $\min\{i,n-m+1\}$ *entries of $\omega\mu$ after deleting α_j, for $i<j\leq k$. Then, if $\nu=(\mu,0^{m-k},0^{n-m})$,*

1.
$$\begin{aligned}\sum_{\substack{P\in\mathfrak{B}_{\omega\nu}\\ entries\leq m}} y^P &= \sum_{P\in\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}} y^P=\sum_{P\in\mathfrak{B}_{(0^{m-k},\alpha,0^{n-m})}} y^P\\ &= \pi_{\sigma(\lambda,SE)}\kappa_{(\omega\mu,0^{n-k})}(y)=\kappa_{(0^{m-k},\alpha,0^{n-m})}(y).\end{aligned} \tag{29}$$

2. $\mathfrak{B}_{\omega(\mu,0^{m-k},0^{n-m})}\cap\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}=\mathfrak{B}_{(0^{m-k},\alpha,0^{n-m})}$ *and $\omega\mu\leq\alpha\leq\omega\mu^+$.*
In particular, when $m=n$, then $\alpha=\omega\mu$; and when $m+k=n+1$, then $\alpha=\omega\mu^+$ and $\kappa_{(0^{m-k},\omega\mu^+,0^{n-m})}(y)=s_{(\mu^+,0^{m-k})}(y_1,\dots,y_m)$ is a Schur polynomial.

Proof. 1. Recalling the action of Demazure operators π_i on key polynomials via bubble sorting operators on their indexing weak compositions (13), and since $\omega\nu = (0^{n-k}, \omega\mu)$, one may write,

$$\kappa_{\omega\nu}(y) = \prod_{i=1}^{k}(\pi_{i+n-k-1}\dots\pi_i)\kappa_{(\omega\mu,0^{n-k})}(y) \tag{30}$$

$$= \prod_{i=1}^{k-(n-m)-1}(\pi_{i+n-k-1}\dots\pi_i) \tag{31}$$

$$\bullet \prod_{i=0}^{n-m}(\pi_{m-1+i}\dots\pi_{k-(n-m)+i})\kappa_{(\omega\mu,0^{n-k})}(y). \tag{32}$$

The Demazure operators in (30) act as bubble sorting operators on the weak composition $(\omega\mu, 0^{n-k})$, shifting $n-k$ times to the right each of the k entries of $\omega\mu$. This can be done by shifting, first, the last $n-m+1$ entries of $\omega\mu$ (32) and then (31) the remaining first $k-(n-m)-1 \geq 0$ entries. From Lemma 3, with $\eta = (\mu^+, 0^{m-k})$ and $\gamma = \omega\nu$, omitting in (32) the operators with indices $\geq m$, one has

$$\sum_{\substack{P\in\mathfrak{B}_{\omega\nu}\\ entries\leq m}} y^P = \sum_{P\in\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}(\mu^+,0^{m-k})} y^P = \pi_{\sigma(\lambda,SE)}\kappa_{(\omega\mu,0^{n-k})}(y)$$

$$= \prod_{i=1}^{k-(n-m)-1}(\pi_{i+n-k-1}\dots\pi_i) \tag{33}$$

$$\bullet \prod_{i=0}^{n-m}(\pi_{m-1}\dots\pi_{k-(n-m)+i})\kappa_{(\omega\mu,0^{n-k})}(y) \tag{34}$$

$$= \kappa_{(0^{m-k},\alpha,0^{n-m})}(y). \tag{35}$$

The Demazure operators in (34) act as bubble sorting operators on the weak composition $(\omega\mu, 0^{m-k}, 0^{n-m})$, shifting $m-k$ times to the right the last $n-m+1$ entries of $\omega\mu$, and sorting them in ascending order. Next, the operators (33) act similarly on the resulting vector ignoring the entry m, then ignoring the entry $m-1$, and so on. Thus the weak composition indexing the new key polynomial $\kappa_{(0^{m-k},\alpha,0^{n-m})}$ (35) is such that $\alpha = (\alpha_1, \dots, \alpha_k)$, where for each $i = k, \dots, 1$, α_i is the maximum element of the last $\min\{i, n-m+1\}$ entries of $\omega\mu$ after deleting α_j, for $i < j \leq k$. (After some point, the number of remaining entries in $\omega\mu$ is less than $n-m+1$ and just the i remaining entries are considered.)

2. It is a consequence of 1, recalling that, in Section 2.2, the left Bruhat order (implies Bruhat order) in $\mathfrak{S}_k\mu$ is described by bubble sorting operators. An alternative

proof comes from the construction of α, provided $\omega\mu$, and using the Bruhat order characterization (4) in an orbit. Start with $\alpha^0 := \omega\mu$. Next put, for $i = 0, \ldots, k-1$, α^{i+1} equal to the result of swapping in α^i the $i+1$-th last entry of α^i with the maximum among the last $min\{k-i, n-m+1\}$ entries in α^i, after ignoring the i last entries. Eventually, one obtains α. In each step, one has $\alpha^i \leq \alpha^{i+1}$, for $i \geq 0$, and finally $\omega\mu \leq \alpha$. □

Example 7 illustrates this proposition.

Theorem 6. *Let $\lambda = (m^{n-m+1}, m-1, \ldots, n-k+1)$, where $1 \leq k, m \leq n$, and $n+1 \leq m+k$, be the Ferrers shape in Figure 1. Then we have the following explicit expansions in the SSYTs in the Demazure crystal*

1. *If $1 \leq k \leq m$,*

$$\prod_{\substack{(i,j)\in\lambda\\ k\leq m}} (1-x_iy_j)^{-1} = \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)\pi_{\sigma(\lambda,SE)}\kappa_{\omega\mu}(y)$$
$$= \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\alpha)}(y), \tag{36}$$

where $\alpha \in \mathbb{N}^k$ is defined in Proposition 3 for each $\mu \in \mathbb{N}^k$.

2. *If $1 \leq m \leq k$,*

$$\prod_{\substack{(i,j)\in\lambda\\ m\leq k}} (1-x_iy_j)^{-1} = \sum_{\mu\in\mathbb{N}^m} \pi_{\sigma(\lambda,NW)}\hat{\kappa}_\mu(x)\kappa_{\omega\mu}(y)$$
$$= \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\kappa_{\omega\mu}(x)$$
$$= \sum_{\mu\in\mathbb{N}^m} \kappa_{(0^{k-m},\alpha')}(x)\widehat{\kappa}_\mu(y), \tag{37}$$

where $\alpha' \in \mathbb{N}^m$ is defined similarly, swapping k with m in Proposition 3, for each $\mu \in \mathbb{N}^m$.

Proof. 1. Identity (36) follows from (26) and Proposition 3.

2. Considering $\overline{\lambda}$, the conjugate of λ, and the expansion (36), one has

$$\prod_{\substack{(i,j)\in\lambda\\ m\leq k}} (1-x_iy_j)^{-1} = \prod_{\substack{(j,i)\in\overline{\lambda}\\ m\leq k}} (1-x_iy_j)^{-1} = \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\overline{\lambda},SE)}\kappa_{\omega\mu}(x)$$
$$= \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\kappa_{\omega\mu}(x) = \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\kappa_{(0^{k-m},\alpha')}(x), \tag{38}$$

where $\alpha' \in \mathbb{N}^m$ is defined, swapping k with m, in Proposition 3, for each $\mu \in \mathbb{N}^m$. On the other hand, using the change of basis (12), one has (25), which together with (38) gives

$$\prod_{\substack{(i,j)\in\lambda \\ m\leq k}} (1-x_iy_j)^{-1} = \sum_{\mu\in\mathbb{N}^m} \pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\mu(x)\kappa_{\omega\mu}(y)$$

$$= \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\kappa_{(0^{k-m},\alpha')}(x).$$

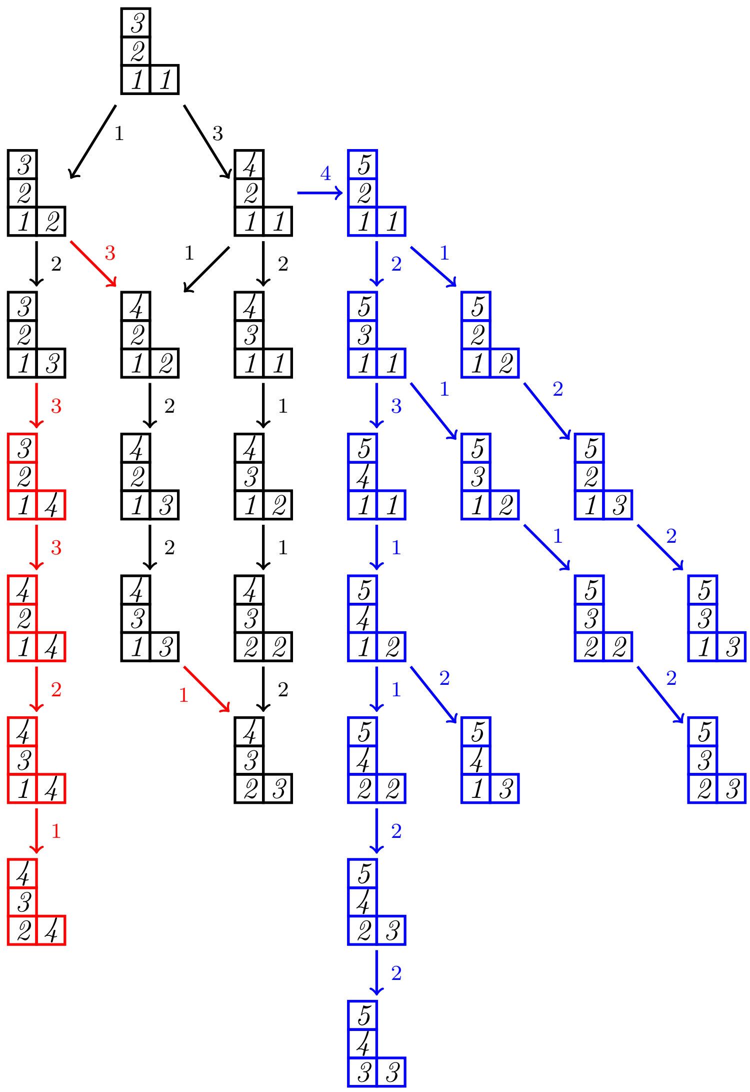
□

In Figure 1, if $m = n$, $\lambda = (n, n-1, \ldots, n-k+1)$, with $1 \leq k \leq n$, and from identity (36) and Proposition 3, one has

$$\prod_{(i,j)\in\lambda} (1-x_iy_j)^{-1} = \sum_{\substack{\mu\in\mathbb{N}^k \\ \nu=(\mu,0^{n-k})}} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y).$$

(Similarly, for $k = n$, in identity (37).) In particular, if $m = n = k$ ($\lambda = \overline{\lambda}$), we recover (20) from both previous identities. When $n+1 = m+k$, from Proposition 3, identity (36) becomes (19), and hence we recover identity (18) with $k \leq m$. Similarly, (37) leads to (18) with $m \leq k$.

Example 7. *Let $n = 5$, $m = 4 \geq k = 3$, $\mu = (1,1,2)$, and $\nu = (1,1,2,0,0)$. The black and blue tableaux constitute the vertices of the Demazure crystal $\mathfrak{B}_{\omega\nu} = \mathfrak{B}_{(0,0,2,1,1)} = \mathfrak{B}_{s_2s_1s_3s_2s_4s_3(2,1,1,0,0)}$. One has $\pi_2\pi_1\pi_3\pi_2\pi_3x^{(2,1,1,0,0)} = \pi_2\pi_1\pi_2\pi_3x^{(2,1,1,0,0)} = \kappa_{(0,1,2,1,0)}(x)$. (The shortest element in the coset $s_2s_1s_3s_2s_3 < s_2 >$ is $s_2s_1s_2s_3$.) The black and the red tableaux are the vertices of the crystal $\mathfrak{B}_{(0,\omega\mu^+,0)} = \mathfrak{B}_{(0,1,1,2,0)} = \mathfrak{B}_{s_1s_2s_3s_2s_1\nu^+}$. The intersection $\mathfrak{B}_{\omega\nu} \cap \mathfrak{B}_{(0,\omega\mu^+,0)}$ consists of the black tableaux which constitute the vertices of the Demazure crystal $\mathfrak{B}_{(0,\alpha,0)} = \mathfrak{B}_{(0,1,2,1,0)} = \mathfrak{B}_{s_2s_1s_2s_3(2,1,1,0,0)}$, with $\alpha = (1,2,1)$ defined in Proposition 3. (Note that the crystal graph does not have all the edges represented. Only those referring to the words under consideration.)*



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