Key Polynomials, invariant factors and an action of the symmetric group on Young tableaux

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ABSTRACT

Lascoux and Schützenberger have defined a permutation on a Young tableau to associate to each Knuth class a right and left key which they have used to give a combinatorial description of a key polynomial

$$k_m = \sum_{\substack{T \text{ of shape } \beta(m)\\ K_+(T) \le K(m)}} x^T.$$

We extend this action of the symmetric group to Young tableaux of the same shape and weight. More precisely, the vertices of the hexagons are frank words running over tableaux, with the same shape and weight, rather than only on the frank words within a Knuth class. As a dual translation, we obtain an action of the symmetric group on words congruent with key-tableaux defined by reflection crystal operators based on non- standard pairing of parentheses. This construction arises naturally as a combinatorial description of the invariant factors associated with certain type of sequences of product of matrices, over a local principal ideal domain, under the action of the symmetric group by place permutation.

1 Introduction

Partitions parameterize different objects and by developing the combinatorics of one of them one naturally obtain facts about other theories. In this paper, the objects are Smith classes and rather than a coincidence, one obtain a generalization.

- **The Smith classes of nonsingular matrices over a local principal ideal domain with prime p are parameterized by partitions**

$$A \sim \Delta_\alpha = \begin{pmatrix} p^{\alpha_1} & & \\ & \ddots & \\ & & p^{\alpha_n}\end{pmatrix}, \qquad \alpha = (\alpha_1 \ge \ldots \ge \alpha_n) = \underbrace{\begin{matrix}\bullet\\ \bullet\ \bullet\\ \bullet\ \bullet\ \bullet\\ \bullet\ \bullet\ \bullet\ \bullet\end{matrix}}_{\alpha_1} \qquad \alpha' = (\alpha'_1 \ge \ldots) = \underbrace{\begin{matrix}\bullet\ \bullet\\ \bullet\ \bullet\\ \bullet\ \bullet\ \bullet\\ \bullet\ \bullet\ \bullet\ \bullet\end{matrix}}_{\alpha'_1}$$

- **Littlewood-Richardson rule for the product of Smith classes**

$$\exists\, U \sim I, \qquad \Delta_\alpha U \Delta_\beta \sim \Delta_\gamma, \qquad \Longleftrightarrow \qquad c^\gamma_{\alpha,\beta} \neq 0.$$

- **Key-tableaux are are in bijection with compositions**

$$D_{[2]}, D_{[2]}D_{[3]} = \begin{pmatrix} p^1 & & & \\ & p^1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}, \begin{pmatrix} p^1 & & & \\ & p^1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix} p^1 & & & \\ & p^1 & & \\ & & p^1 & \\ & & & 1\end{pmatrix} \longleftrightarrow ((1^2),(1^2)+(1^3)) = \begin{matrix}\bullet\\ \bullet\ \bullet\\ \bullet\ \bullet\end{matrix} \longleftrightarrow \begin{matrix}\bullet\ \bullet\\ \bullet\ \bullet\ \bullet\end{matrix}$$

$$(1032) \qquad \longleftrightarrow \qquad K(1032) = \begin{matrix}4\\ 3\ 4\\ 1\ 3\ 3\end{matrix} \qquad \longleftrightarrow (D_{[1]},\ D_{[1]}D_\emptyset,\ D_{[1]}D_\emptyset D_{[3]},\ D_{[1]}D_\emptyset D_{[3]}D_{[2]})$$

- **Dual RSK correspondence, symmetry and invariant factors**

$$T = \begin{matrix}4\\ 3\ 4\\ \bullet\ 1\ 3\\ \bullet\ \bullet\ \bullet\ 3\end{matrix} \qquad \Sigma = \underbrace{\begin{pmatrix} 11 & 22 & 3 & 4\\ 43 & 41 & 3 & 3\end{pmatrix}}_{w} \longleftrightarrow \Sigma' = \overbrace{\begin{pmatrix} 21 & 431 & 2\\ 44 & 333 & 1\end{pmatrix}}^{J} \qquad J = J_3J_2J_1 = \begin{matrix} & 2 & 4 & 2\\ 1 & 3 & & \\ & 1 & & \end{matrix}$$

$$J = \begin{Bmatrix} 2, & 4, & 2\\ 1 & 3 & \\ & 1 & \end{Bmatrix} \longrightarrow (P(J), P(w)) \qquad w = \begin{Bmatrix} 4, & 4, & 3, & 3\\ 3 & 1 & & \end{Bmatrix} \longrightarrow (P(w), P(J))$$

$Q(J) = std(P(w^\#)^t$, $\qquad Q(w) = std(P(J))^t$

J is a frank word of shape the reverse of $m \iff P(w) = K(m)$.

- $U \sim I$, $\quad \Delta_\alpha U D_{[l]} \sim \Delta_{\alpha^1}$, $\quad \alpha^1 = \alpha + \sum_{\substack{i \in J \subseteq [n]\\ |J| = l}} e_i$
- $\Delta_\alpha UK(m) := (\Delta_\alpha, \Delta_\alpha U D_{[m_1]}, \cdots, \Delta_\alpha U D_{[m_1]} \cdots D_{[m_l]}) \sim (\Delta_\alpha, \Delta_{\alpha^1}, \cdots, \Delta_{\alpha^t}) \leftrightarrow T:\ \alpha^0 = \alpha \subseteq \alpha^1 \subseteq \cdots \subseteq \alpha^t,\ \alpha^k_i \le \alpha^{k+1}_i \le \alpha^k_i + 1$
- $\Delta_\alpha UK(m)$ **defines a tableau** $T \equiv K(m)$

2 Variants of the *jeu de taquin* on two-column frank words, pairing of parentheses and invariant factors

- **Variants of the *jeu de taquin* on two-column frank words**

Θ (*jeu de taquin*): slide down (up) maximally the first (second) column
$\tilde\Theta$ (variant of *jeu de taquin*): slide down (up) the first (second) column

$$\Theta: \begin{matrix}3&5\\1&4\\\blacksquare&3\\\blacksquare&2\end{matrix} \longleftrightarrow \begin{matrix}\blacksquare&5\\\blacksquare&4\\3&3\\1&2\end{matrix} \longleftrightarrow \begin{matrix}5&\blacksquare\\4&\blacksquare\\3&3\\1&2\end{matrix}, \qquad \tilde\Theta: \begin{matrix}3&5\\1&4\\\blacksquare&3\\\blacksquare&2\end{matrix} \longleftrightarrow \begin{matrix}\blacksquare&5\\3&4\\1&3\\\blacksquare&2\end{matrix} \longleftrightarrow \begin{matrix}5&\blacksquare\\3&4\\1&3\\2&\blacksquare\end{matrix} \longleftrightarrow \begin{matrix}5&\blacksquare\\3&\blacksquare\\2&4\\1&3\end{matrix}. \tag{0.1}$$

$\tilde\Theta(J_2J_1)$ is a frank word with the same shape and weight as $\Theta(J_2J_1)$
$\tilde\Theta(J_2J_1) \equiv J_2J_1$ if and only if $\tilde\Theta = \Theta$

- **Duality between reflection crystal operators and *jeux de taquin***

θ : reflection crystal operator based on standard pairing of parentheses
$\tilde\theta$: variant of *reflection crystal operator* based on non- standard pairing of parentheses

$$\Sigma = \begin{pmatrix}1&2&33&4&5\\2&1&21&1&1\end{pmatrix} \longleftrightarrow \Sigma' = \begin{pmatrix}31&5432\\22&11111\end{pmatrix}, \quad \theta \updownarrow \quad \updownarrow \Theta, \quad \hat\Sigma = \begin{pmatrix}1&2&33&4&5\\2&1&21&2&2\end{pmatrix} \longleftrightarrow \hat\Sigma' = \begin{pmatrix}5431&32\\2222&11\end{pmatrix},$$

$$\Sigma = \begin{pmatrix}1&2&33&4&5\\2&1&21&1&1\end{pmatrix} \longleftrightarrow \Sigma' = \begin{pmatrix}31&5432\\22&1112\end{pmatrix}, \quad \tilde\theta \updownarrow \quad \updownarrow \tilde\Theta, \quad \tilde\Sigma = \begin{pmatrix}1&2&33&4&5\\2&2&21&1&2\end{pmatrix} \longleftrightarrow \tilde\Sigma' = \begin{pmatrix}5321&43\\2222&11\end{pmatrix}.$$

$\theta: (21)(21)11 \to (21)(21)22 \qquad \tilde\theta: (21)(21)1)1 \to (22(21)1)2$

If $(\emptyset \leftarrow w) = (P,Q)$ then $(\emptyset \leftarrow \tilde\theta w) = (\theta P, Q')$, $\theta w \equiv \tilde\theta w$.
$Q = Q'$ if and only if $\tilde\theta = \theta$.

- **Under certain conditions, operators $\tilde\Theta$, ($\tilde\theta$,) can be extended to t-column frank words (words on a t-letter alphabet, $t \ge 2$)**

Corollary 0.1. Generalized recursive Lascoux-Schützenberger criterion. *Let $J = J_k \cdots J_1$ be a frank word and J_{k+1} a column. Then, $J_{k+1}J$ is frank if and only if $J_{k+1}J_k$ and $\tilde J_k J_{k-1} \cdots J_1$ are frank words, where $\tilde J_{k+1} \tilde J_k = \tilde\Theta_k(J_{k+1}J_k)$ for some operator $\tilde\Theta_k$.*

- **Why *non-standard* reflection crystal operators and variants of the *jeu de taquin*?**

They describe the skew-tableaux on a two-letter alphabet defined by $\Delta_\alpha UK(m)$ and $\Delta_\alpha UK(s_1m)$

Theorem 0.1. *Let $m = (m_1, m_2)$. Let T and T' be respectively the tableaux defined by the sequences $\Delta_\alpha UK(m)$ and $\Delta_\alpha UK(s_1m)$, with indexing-set words J_2J_1, $J'_2J'_1$, and column-reading words w, w'. Then,*

- *J_2J_1, $J'_2J'_1$ are frank words such that $\tilde\Theta_1 J_2J_1 = J'_2J'_1$.*
- *$w \equiv K(m)$ and $w' = \tilde\theta_1 w \equiv K(s_1m)$.*

Conversely, if T and T' are respectively tableaux of inner-shape α with indexing-set frank words J_2J_1 and $J'_2J'_1$ satisfying $J'_2J'_1 = \tilde\Theta_1 J_2J_1$, then $\exists\, U, U' \sim I$ such that $\Delta_\alpha UK(m)$ and $\Delta_\alpha U'K(s_1m)$ define the tableaux T and T' respectively.

- **Example**

$$\Delta_{2,1}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&p\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}K(3,2) \to T = \begin{matrix}2\\\bullet\ 1\ 2\\\bullet\ \bullet\ 1\ 1\end{matrix} \qquad w = 21211, \qquad J = \begin{matrix}3&4\\1&3\\&2\end{matrix}$$

$$\Delta_\alpha\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&p\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}K(2,3) \to \tilde T = \begin{matrix}2\\\bullet\ 2\ 2\\\bullet\ \bullet\ 1\ 1\end{matrix} \qquad \tilde w = 22211, \qquad \tilde J = \begin{matrix}3\\2&4.\\1&3\end{matrix}$$

$$\tilde\theta: w = (21(21)1) \longleftrightarrow \tilde w = (22(21)1) \qquad \tilde\Theta: \begin{matrix}3&4\\1&3\\\blacksquare&2\end{matrix} \longleftrightarrow \begin{matrix}3&4\\1&3\\2&\blacksquare\end{matrix} \longleftrightarrow \begin{matrix}3&\blacksquare\\2&4\\1&3\end{matrix}.$$

$$\Delta_{2,1}\begin{pmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&p\\0&0&1&0\\0&0&0&1\end{pmatrix}K(32) \to T = \begin{matrix}2\\\bullet\ 1\ 2\\\bullet\ \bullet\ 1\ 1\end{matrix} \qquad w = 21211, \qquad J = \begin{matrix}3&4\\1&3\\&2\end{matrix}$$

$$\Delta_{2,1}\begin{pmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&p\\0&0&1&0\\0&0&0&1\end{pmatrix}K(23) \to \hat T = \begin{matrix}2\\\bullet\ 1\ 2\\\bullet\ \bullet\ 1\ 2\end{matrix} \qquad \hat w = 21212, \qquad \hat J = \begin{matrix}4\\3&3.\\&1&2\end{matrix}$$

$$\theta: w = (21)(21)1 \longleftrightarrow \hat w = (21)(21)2 \qquad \Theta: \begin{matrix}\blacksquare&4\\3&3\\1&2\end{matrix} \longleftrightarrow \begin{matrix}4&\blacksquare\\3&3\\1&2\end{matrix}$$

- **Shuffling columns ordered by inclusion is compatible with Knuth operations**

By Greene's theorem the shuffles of a finite set of columns ordered by inclusion are contained in the Knuth class of the key with those columns. However under certain conditions we have equality.

Corollary 0.2. *The following statements are equivalent:*

- *The Knuth class of a key-tableau over a three-letter alphabet is the set of all shuffles of its columns.*
- *J is a three-column frank word if and only if J has one of the following forms*

$$\begin{matrix}A^1_1&&\\A^2_1&A^2_2&\\A^3_1&A^3_2&A^3_3\end{matrix},\quad \begin{matrix}A^1_1&&\\A^2_1&A^2_2&\\A^3_1&A^3_2&A^3_3\end{matrix},\quad \begin{matrix}&A^1_1&\\A^2_1&A^2_2&\\A^3_1&A^3_2&A^3_3\end{matrix},\quad \begin{matrix}&&A^1_1\\&A^2_2&A^2_3\\A^3_1&A^3_2&A^3_3\end{matrix},\quad \begin{matrix}&A^1_1&\\A^2_1&&A^2_3\\A^3_1&A^3_2&A^3_3\end{matrix},\quad \begin{matrix}A^1_1&&\\A^2_1&&A^2_3\\A^3_1&A^3_2&A^3_3\end{matrix},\quad \begin{matrix}&&A^1_1\\&A^2_2&A^2_3\\A^3_1&A^3_2&A^3_3\end{matrix}$$

where $A^3_1 \le A^3_2 \le A^3_3$, with $|A^3_1| = |A^3_2| = |A^3_3|$; $A^r_i \cap A^s_i = \emptyset$, for $r \ne s$, $i = 1,2,3$, and $A^2_1 \le A^2_2$, $A^2_1 \le A^2_3$, $A^2_2 \le A^2_3$, with $|A^2_1| = |A^2_2| = |A^2_3|$.

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3 An action of the symmetric group on Young tableaux of skew-shape

$U \sim I$, $\qquad \beta = (\beta_1, \beta_2, \beta_3)$ a partition, $\qquad (\beta_{i_1}, \beta_{i_2}, \beta_{i_3}) \in S_3(\beta)$.

$$\begin{matrix} & \overset{s_1}{\nearrow} & \Delta_\alpha UK(\beta_2,\beta_1,\beta_3) \overset{s_2}{\text{—}} \Delta_\alpha UK(\beta_2,\beta_3,\beta_1) & \overset{s_1}{\searrow} & \\ \Delta_\alpha UK(\beta_1,\beta_2,\beta_3) & & & & \Delta_\alpha UK(\beta_3,\beta_2,\beta_1). \\ & \underset{s_2}{\searrow} & \Delta_\alpha UK(\beta_1,\beta_3,\beta_2) \underset{s_1}{\text{—}} \Delta_\alpha UK(\beta_3,\beta_1,\beta_2) & \underset{s_2}{\nearrow} & \end{matrix} \tag{0.2}$$

Theorem 0.2. *Let $\sigma \in <s_1, s_2>$, $\theta \in <\theta_1, \theta_2>$ and $\Theta \in <\Theta_1, \Theta_2>$ with the same reduced word. Let $T(\sigma)$ be the tableau defined by $\Delta_\alpha UK(\sigma\beta)$, with word σw and indexing frank word σJ of shape the reverse of $\sigma\beta$. Then $\{T(\sigma) : \sigma \in <s_1, s_2>\}$ are the vertices of an hexagon such that*

- *there exist $\tilde\theta_1$ and $\tilde\theta_2$ satisfying the Moore-Coxeter relations of the symmetric group S_3,*

$$\begin{matrix} & \nearrow & \tilde\theta_1 w \text{ — } \tilde\theta_2\tilde\theta_1 w & \searrow & \\ w & & & & \tilde\theta_1\tilde\theta_2\tilde\theta_1 w\\ & \searrow & \tilde\theta_2 w \text{ — } \tilde\theta_1\tilde\theta_2 w & \nearrow & \end{matrix} \tag{0.3}$$

where $\tilde\theta \in <\tilde\theta_1, \tilde\theta_2>$, with the same reduced word as θ, verifies $\sigma w = \tilde\theta w \equiv \theta K(\beta) = K(\sigma\beta)$.

- *there exist $\tilde\Theta_1$ and $\tilde\Theta_2$ satisfying the Moore-Coxeter relations of the symmetric group S_3,*

$$\begin{matrix} & \nearrow & \tilde\Theta_1 J = J_3G_2G_1 \text{ — } \tilde\Theta_2\tilde\Theta_1 J = F_3F_2G_1 & \searrow & \\ J = J_3J_2J_1 & & & & \tilde\Theta_1\tilde\Theta_2\tilde\Theta_1 J = F_3XH_1.\\ & \searrow & \tilde\Theta_2 J = L_3L_2J_1 \text{ — } \tilde\Theta_1\tilde\Theta_2 J = L_3H_2H_1 & \nearrow & \end{matrix} \tag{0.4}$$

where $\tilde\Theta \in <\tilde\Theta_1, \tilde\Theta_2>$, with the same reduced word as Θ, verifies $\sigma J = \tilde\Theta J$.

- **The operators $\tilde\theta_i$ and $\tilde\Theta_i$ closing the hexagons** (0.2), (0.3) **and** (0.4)

Lemma 0.3. *The operators closing hexagons (0.2) and (0.4) obey the following conditions*

- *If L_3L_2 and F_3F_2 are, respectively, the indexing frank words of $\Delta_\alpha UK(\beta_1, \beta_3)$ and $\Delta_\alpha UK(\beta_2, \beta_3)$, it holds $F_2 \le L_2$.*
- *If L_3H_2 and J_3G_2 are, respectively, the indexing frank words of $\Delta_\alpha UK(\beta_3, \beta_1)$ and $\Delta_\alpha UK(\beta_2, \beta_1)$, it holds $G_2 \le H_2$.*
- *The operations $\tilde\Theta_1$ and $\tilde\Theta_2$ defining the hexagon (0.4) are such that $\tilde\Theta_2[\tilde\Theta_1 J] = F_3F_2G_1$ with $F_2 \le L_2$, and $\tilde\Theta_1[\tilde\Theta_2 J] = L_3H_2H_1$ with $G_2 \le H_2$.*

4 An action of the symmetric group on frank words not always preserving the Knuth class

Theorem 0.4. *Let $J = J_3J_2J_1$ be a contretableau. The following assertions are equivalent.*

- *There exist $\tilde\Theta_1$ and $\tilde\Theta_2$ defining the hexagon (0.4) such that $\tilde\Theta_2[\tilde\Theta_1 J] = F_3F_2G_1$ with $F_2 \le L_2$, and $\tilde\Theta_1[\tilde\Theta_2 J] = L_3H_2H_1$ with $G_2 \le H_2$.*
- *The contretableau J has a decomposition giving rise to an hexagon*

$$J = \begin{matrix}&A^5_1\ A^5_2\ A^5_3\\&A^4_1\ A^4_2\\&A^3_3\ A^3_2\\A^2_3&A^2_1\\&A^1_1\end{matrix} \quad \begin{matrix}\overset{\tilde\Theta_1}{\nearrow}\ \begin{matrix}A^5_2\ A^5_1\\A^4_2\ A^4_1\\A^3_2\ A^3_3\\A^2_3\ A^2_1\\A^1_1\end{matrix} \overset{\tilde\Theta_2}{\text{—}} \begin{matrix}A^5_2\ A^5_3\ A^5_1\\A^4_2\ A^4_1\\A^3_2\\A^2_3\ A^2_1\\A^1_1\end{matrix}\ \overset{\tilde\Theta_1}{\searrow}\\ \underset{\tilde\Theta_2}{\searrow}\ \begin{matrix}A^5_3\ A^5_1\\A^4_2\ A^4_1\\A^3_2\ A^3_3\\A^2_3\\A^1_1\end{matrix} \underset{\tilde\Theta_1}{\text{—}} \begin{matrix}A^5_3\ A^5_2\ A^5_1\\A^4_2\ A^4_1\\A^3_2\ A^3_3\\A^2_3\ A^2_1\\A^1_1\end{matrix}\ \underset{\tilde\Theta_2}{\nearrow}\end{matrix} \quad \begin{matrix}A^5_3\ A^5_2\ A^5_1\\A^4_1\\A^3_2\ A^3_3\\A^2_3\ A^2_1\\A^1_1\end{matrix} = T, \tag{0.5}$$

where the sets A^j_i are pairwise disjoint in each column J_i, $A^j_{i+1} \le A^j_i$, with $|A^j_{i+1}| = |A^j_i|$,

$$A^3_2 \le A^2_1 < A^4_2 \le A^4_1,$$

$|A^2_3| = |A^2_1| = |A^4_2| = |A^4_1|$, *and* $J_1 \cap A^5_2 \subseteq A^5_1$, $(J_1 \setminus A^5_1) \cap A^4_2 \subseteq A^4_1$, $[J_1 \setminus (A^5_1 \cup A^4_1)] \cap A^3_2 \subseteq A^3_1$, $[J_2 \cup (A^2_1 \cup A^4_1)] \cap A^3_2 \subseteq A^2_1$, *and* $[J_2 \cup (A^2_1 \cup A^4_1)] \cap A^5_3 \subseteq A^5_2$.

$$\tilde K_+(T) = \begin{matrix}A^5_1\ A^5_1\ A^5_1\\A^4_1\ A^4_1\ A^4_1\\A^3_1\ A^3_1\\A^2_1\\A^1_1\end{matrix} \quad \ge \quad \tilde K_-(T) = \begin{matrix}A^5_3\ A^5_3\ A^5_3\\A^4_3\\A^3_3\ A^3_3\\A^2_3\ A^2_3\ A^2_3\\A^1_1\end{matrix}$$

- **Example**

Two decompositions of the tableau $T = \begin{matrix}5\\4\\3\ 3\\2\ 2\ 5\end{matrix}$ leading to different left and right keys. The second hexagon gives the frank words in the Knuth class of T.

$$(I)\quad \begin{matrix}3\ 3\\5\ 5\\2\ \ 2\\4\end{matrix} \ \begin{matrix}\overset{\tilde\Theta_1}{\nearrow} \begin{matrix}3\ 3\\5\ 5\\2\ 2\\4\end{matrix} \overset{\tilde\Theta_2}{\text{—}} \begin{matrix}3\ \ 3\\5\ \ 5\\2\ 2\\4\end{matrix} \overset{\tilde\Theta_1}{\searrow}\\ \underset{\tilde\Theta_2}{\searrow} \begin{matrix}3\ \ 3\\5\ 5\\2\ \ 2\\4\end{matrix} \underset{\Theta_1}{\text{—}} \begin{matrix}3\ 3\\5\ 5\\2\ 2\\4\end{matrix} \underset{\tilde\Theta_2}{\nearrow}\end{matrix} \begin{matrix}3\ 3\\5\ \ 5\\2\ 2\\4\end{matrix} = T, \qquad (II)\quad \begin{matrix}3\ 3\\5\ 5\\2\ \ 2\\4\end{matrix} \ \begin{matrix}\overset{\Theta_1}{\nearrow} \begin{matrix}3\ 3\\5\ 5\\2\ 2\\4\end{matrix} \overset{\tilde\Theta_2}{\text{—}} \begin{matrix}3\ \ 3\\5\ \ 5\\2\ 2\\4\end{matrix} \overset{\Theta_1}{\searrow}\\ \underset{\tilde\Theta_2}{\searrow} \begin{matrix}3\ 3\\5\ \ 5\\2\ \ 2\\4\end{matrix} \underset{\Theta_1}{\text{—}} \begin{matrix}3\ 3\\5\ 5\\2\ 2\\4\end{matrix} \underset{\tilde\Theta_2}{\nearrow}\end{matrix} \begin{matrix}3\ \ 3\\5\ 5\\2\ 2\\4\end{matrix} = T \tag{0.6}$$

$$\tilde K_+(T) = \begin{matrix}5\\4\\3\ 5\\2\ 3\ 5\end{matrix} \qquad K_+(T) = \begin{matrix}5\\4\\3\ 5\\2\ 3\ 3\end{matrix}$$

5 The dual action of the symmetric group on words congruent with keys not always preserving the Q-symbol

- **A Yamanouchi word on a 3-letter alphabet is a shuffle of words** 3121, 321, 21, 1

Split the contretableau $J = J_3J_2J_1$ into row-words $X_1 = c^2a^2b^4a^4$, $X_2 = c^5b^5a^5$, $X_3 = b^3a^3$, and $X_4 = a^1$

$$\Sigma' = \begin{pmatrix}J_3 & J_2 & J_1\\ 3^2 & 2^3 & 1^5\end{pmatrix} \longleftrightarrow \Pi = \begin{pmatrix}c^2a^2b^4a^4 & c^5b^5a^5 & b^3a^3 & a^1\\ 3\ 1\ 2\ 1 & 3\ 2\ 1 & 2\ 1 & 1\end{pmatrix} \longleftrightarrow \Sigma = \begin{pmatrix}(J_3J_2J_1)\uparrow\\ w\end{pmatrix}, \qquad w \in 3121 ⧢ 321 ⧢ 21 ⧢ 1 \tag{0.7}$$

- **The dual hexagon (0.3) is a "*shuffle*" of four elementary hexagons**

$$\begin{pmatrix}c^2a^2b^4a^4\\3\ 1\ 2\ 1\end{pmatrix} \begin{matrix}\overset{\theta_1}{\nearrow}\begin{pmatrix}c^2a^2b^4a^4\\3\ 2\ 2\ 1\end{pmatrix}\overset{\theta_2}{\text{—}}\begin{pmatrix}c^2a^2b^4a^4\\3\ 2\ 3\ 1\end{pmatrix}\overset{\theta_1}{\searrow}\\ \underset{\theta_2}{\searrow}\begin{pmatrix}c^2a^2b^4a^4\\3\ 1\ 2\ 1\end{pmatrix}\underset{\theta_1}{\text{—}}\begin{pmatrix}c^2a^2b^4a^4\\3\ 2\ 2\ 1\end{pmatrix}\underset{\theta_2}{\nearrow}\end{matrix} \begin{pmatrix}c^2a^2b^4a^4\\3\ 2\ 3\ 1\end{pmatrix} \qquad \begin{pmatrix}c^5a^5b^5\\3\ 2\ 1\end{pmatrix} \begin{matrix}\overset{\theta_1}{\nearrow}\begin{pmatrix}c^5a^5b^5\\3\ 2\ 1\end{pmatrix}\overset{\theta_2}{\text{—}}\begin{pmatrix}c^5a^5b^5\\3\ 2\ 1\end{pmatrix}\overset{\theta_1}{\searrow}\\ \underset{\theta_2}{\searrow}\begin{pmatrix}c^5a^5b^5\\3\ 2\ 1\end{pmatrix}\underset{\theta_1}{\text{—}}\begin{pmatrix}c^5a^5b^5\\3\ 2\ 1\end{pmatrix}\underset{\theta_2}{\nearrow}\end{matrix} \begin{pmatrix}c^5a^5b^5\\3\ 2\ 1\end{pmatrix}$$

$$\begin{pmatrix}b^3a^3\\2\ 1\end{pmatrix} \begin{matrix}\overset{\theta_1}{\nearrow}\begin{pmatrix}b^3a^3\\2\ 1\end{pmatrix}\overset{\theta_2}{\text{—}}\begin{pmatrix}b^3a^3\\3\ 1\end{pmatrix}\overset{\theta_1}{\searrow}\\ \underset{\theta_2}{\searrow}\begin{pmatrix}b^3a^3\\3\ 1\end{pmatrix}\underset{\theta_1}{\text{—}}\begin{pmatrix}b^3a^3\\3\ 2\end{pmatrix}\underset{\theta_2}{\nearrow}\end{matrix} \begin{pmatrix}b^3a^3\\3\ 2\end{pmatrix} \qquad \begin{pmatrix}a^1\\1\end{pmatrix} \begin{matrix}\overset{\theta_1}{\nearrow}\begin{pmatrix}a^1\\2\end{pmatrix}\overset{\theta_2}{\text{—}}\begin{pmatrix}a^1\\3\end{pmatrix}\overset{\theta_1}{\searrow}\\ \underset{\theta_2}{\searrow}\begin{pmatrix}a^1\\1\end{pmatrix}\underset{\theta_1}{\text{—}}\begin{pmatrix}a^1\\2\end{pmatrix}\underset{\theta_2}{\nearrow}\end{matrix} \begin{pmatrix}a^1\\3\end{pmatrix}. \tag{0.8}$$

Theorem 0.5. *The hexagon (0.3) is a shuffle of the bottoms rows of the four hexagons (0.8) with the appropriate multiplicities.*

- **The dual hexagon of** (0.6)**(I)**

$$w = 3121121 \begin{matrix}\overset{\theta_1}{\nearrow} 3221221 \overset{\theta_2}{\text{—}} 3231331 \overset{\tilde\theta_1}{\searrow}\\ \underset{\tilde\theta_2}{\searrow} 3131121 \underset{\theta_1}{\text{—}} 3232221 \underset{\theta_2}{\nearrow}\end{matrix} 3232331 \tag{0.9}$$

6 Lascoux-Schützenberger actions of the symmetric group are obtained by a particular row decomposition of a 3-column tableau /shuffle decomposition of a 3-letter Yamanouchi word

Algorithm 0.6. $w = 33121121 \equiv K(4,2,2)$.

- Bracket all the successive factors 21 of $w_{|\{1,2\}}$: 331(21)1(21). We get the decomposition

$$w = w|\{\{4,5\},\{7,8\},\{1\},\{2\},\{3\},\{6\}\}, \qquad w|\{4,5\} = w|\{7,8\} = 21 \qquad w|\{1\} = w|\{2\} = 3 \qquad w|\{3\} = w|\{6\} = 1.$$

- Remove the letters 1 in $w|\{4,5\}$ and $w|\{7,8\}$: $w' = 3312 - 12-$. Bracket first all the successive factors 31 and then all successive factors 32 of w': $w' = (3(31)2) - 12 -$. We get the decomposition

$$w = w|\{\{1,4,5\},\{7,8\},\{2,3\},\{6\}\} \qquad \text{with} \qquad w|\{1,4,5\} = 321, \qquad w|\{7,8\} = 21 \qquad w|\{2,3\} = 31 \qquad w|\{6\} = 1.$$

- Remove the subwords $w|\{1,4,5\} = 321$ and $w|\{6\} = 1$ of w: $-31 - - - 21$. We get the shuffle decomposition

$$w = w|\{\{2,3,7,8\},\{1,4,5\},\{6\}\} = 33121121.$$

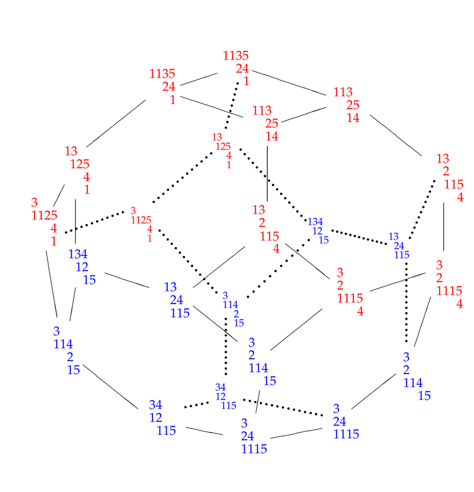
- The Lascoux-Schützenberger actions of the symmetric group

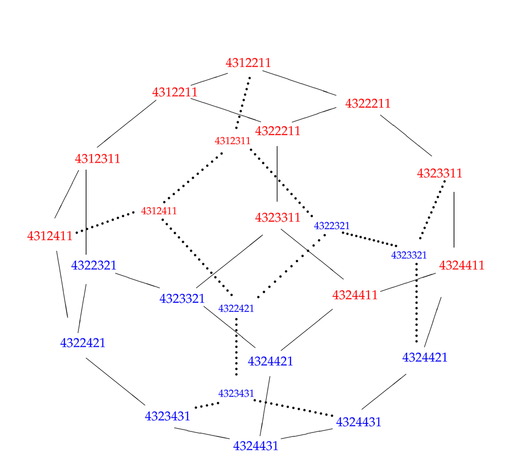
$$33121121 \begin{matrix}\overset{\theta_1}{\nearrow} 33221221 \overset{\theta_2}{\text{—}} 33221331 \overset{\theta_1}{\searrow}\\ \underset{\theta_2}{\searrow} 33121121 \underset{\theta_1}{\text{—}} 33221221 \underset{\theta_2}{\nearrow}\end{matrix} 33221331, \qquad \begin{matrix}1\ 3\ 3\\5\ 5\\2\ \ 2\\4\end{matrix} \begin{matrix}\overset{\Theta_1}{\nearrow} \begin{matrix}1\ 3\ 3\\5\ 5\\2\ 2\\4\end{matrix} \overset{\Theta_2}{\text{—}} \begin{matrix}1\ 3\ 3\\5\ \ 5\\2\ 2\\4\end{matrix} \overset{\Theta_c}{\searrow}\\ \underset{\Theta_2}{\searrow} \begin{matrix}1\ 3\ 3\\5\ 5\\2\ \ 2\\4\end{matrix} \underset{\Theta_1}{\text{—}} \begin{matrix}1\ 3\ 3\\5\ 5\\2\ 2\\4\end{matrix} \underset{\Theta_2}{\nearrow}\end{matrix} \begin{matrix}1\ 3\ 3\\5\ \ 5\\2\ 2\\4\end{matrix}$$

Theorem 0.7. *To each hexagon (0.5) corresponds an hexagon (0.2). That is , given an hexagon (0.5), $\exists\, U \sim I$ such that, for some partition α, $\{\Delta_\alpha UK(\sigma\beta) : \sigma \in S_3\}$ is an hexagon whose indexing frank words are those of (0.5).*

7 Dual permutohedra in S_4 generated by variants of *jeu de taquin* and non-standard reflection crystal operators

$$P = \begin{matrix}3\\2\ 5\\1\ 1\ 1\ 4\end{matrix} \qquad P = \begin{matrix}3\\2\ 4\\1\ 1\ 1\ 5\end{matrix}$$





$$Q = \begin{matrix}6\\3\\2\ 7\\1\ 4\ 5\end{matrix} \qquad Q = \begin{matrix}7\\3\\2\ 6\\1\ 4\ 5\end{matrix}$$