Non-symmetric Cauchy kernel, crystals and last passage percolation

Olga Azenhas
based on joint work with
Thomas Gobet, and Cédric Lecouvey, Université Tours
arXiv:2212.06587

CMUC

June 21st, 2023

0. Ulam's problem: random permutations

- S_N the symmetric group on permutations of $\{1, \ldots, N\}$. Let $\pi \in S_N$

$$N = 20, \qquad \pi = 7\ 3\ 1\ 17\ 10\ 18\ 9\ 20\ 6\ 12\ 16\ 13\ 2\ 8\ 19\ 4\ 15\ 11\ 14\ 5$$

  - A longest increasing sequence is 1 6 8 11 14 or 3 6 8 11 14
  - $\ell_N(\pi)$ denotes the maximal length of all increasing subsequences of π.
  - Equip S_N with uniform probability measure (all outcomes are equally likely)

$$\mathbb{P}(\pi) = \frac{1}{N!}.$$

  - How does ℓ_N behave statistically when $N \longrightarrow \infty$?
  - Center and scale $\ell_N(\pi)$,

$$\chi_N(\pi) = \frac{\ell_N(\pi) - 2\sqrt{N}}{N^{1/6}}.$$

  - **Baik-Deift-Johansson 99**. For each $N \geq 1$, let π be a uniformly random permutation of order N. Then, for each $x \in \mathbb{R}$

$$\lim_{N\to\infty} \mathbb{P}(\chi_N \leq x) = F_{GUE}(x) \quad \text{Widom-Tracy distribution function.}$$

Vershik–Kerov, Logan–Shepp, Okounkov.

Connection to random matrix theory

- Consider the Gaussian Unitary Ensemble (GUE) of Hermitian $N \times N$ matrices endowed with a probability measure

  An r.v. x has Gaussian distribution with mean μ and variance σ, if $\mathbb{P}(a \leq x \leq b) = \frac{1}{\sqrt{2\pi\sigma}} \int_a^b e^{-(x-\mu)^2/2\sigma} dx$.

  - Let $\xi_1(M)$ denote the largest eigenvalue of an $N \times N$ GUE matrix M. Then

    $$\lim_{N \to \infty} \mathbb{P}\left(\frac{\xi_1(M) - \sqrt{2N}}{2^{-1/2} N^{-1/6}} \leq x\right) = F_{GUE}(x),$$

    $F_{GUE}(x)$ is the Widom-Tracy distribution function.

- *The length of the longest increasing subsequence of a random permutation behaves statistically like the largest eigenvalue of a GUE matrix, as* $N \to \infty$.

Plancherel measure on the set of partitions: **R-S** correspondence

- The sum of the squares of the dimensions of the irreducible representations S^λ of S_N equals the order of the group S_N

$$N! = \sum_{\lambda \vdash N} (dim\, S^\lambda)^2 = \sum_{\lambda \vdash N} f_\lambda^2$$

$$dim\, S^\lambda = f_\lambda, \quad f_\lambda = \#\text{standard tableaux of shape } \lambda \vdash N$$

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- *Plancherel measure* on the set of partitions of N,

$$\mathbb{P}^P(\lambda) = \frac{f_\lambda^2}{N!}, \quad \lambda \vdash N.$$

- *Robinson-Schensted correspondence* rescues $\ell_N(\pi)$

$$RS : S_N \to \bigsqcup_{\lambda \vdash N} SYT(\lambda, N) \times SYT(\lambda, N)$$
$$\pi \mapsto (P, Q)$$

- For $\pi \in S_N$,
$\ell_N(\pi) = \lambda_1(\pi) =$ the common length of the first rows of P and Q.

$$N = 7, \quad \pi = 4236517, \quad \ell_7(4236517) = 4.$$

Plancherel measure on the set of partitions: R-S correspondence

- $N = 7, \quad \pi = 4236517$

$$4 \to 2 \to 3 \to 6 \to 5 \to 1 \to 7 = 4 \to 2 \to 3 \to 6 \to 5 \to \begin{matrix} 1 & 7 \end{matrix}$$

$$= 4 \to 2 \to 3 \to \begin{matrix} 1 & 7 \\ 5 & \\ 6 & \end{matrix} \quad = 4 \to 2 \to \begin{matrix} 1 & 5 & 7 \\ 3 & & \\ 6 & & \end{matrix} \quad = 4 \to \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & & & \\ 6 & & & \end{matrix}$$

$$= \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{matrix} = P, \qquad Q = \begin{matrix} 1 & 2 & 5 & 6 \\ 3 & 7 & & \\ 4 & & & \end{matrix}$$

- $\ell_7(4236517) = 4 = \lambda_1(\pi)$

- $$\mathbb{P}^P(\lambda_1 \le n) = \mathbb{P}(\ell_N \le n) = \sum_{\ell_N(\sigma) \le n} \mathbb{P}(\sigma) = \frac{1}{N!} \sum_{\lambda \vdash N, \lambda_1 \le n} f_\lambda^2$$

I. Schur measure: a generalization of Plancherel measure on partitions.

- Schur function associated to a partition λ

$$s_\lambda(\mathbf{x}) = \sum_{T \in SSYT(\lambda)} x^T$$

$$\lambda = (2,1) \vdash 3, \qquad T = \begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 2 & \end{array}, \begin{array}{cc} 1 & 1 \\ 3 & \end{array}, \begin{array}{cc} 1 & 3 \\ 3 & \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \cdots$$

$$s_{(2,1)}(\mathbf{x}) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + 2x_1 x_2 x_3 + \cdots + \cdots \qquad f_{(2,1)} = 2$$

$$\lambda \vdash N, \qquad s_\lambda(x) = \cdots + f_\lambda x_1 x_2 \cdots x_N + \cdots$$

- **Cauchy identity**

$$\sum_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \prod_{i,j \geq 1} (1 + x_i y_j + x_i^2 y_j^2 + \cdots).$$

$\mathbf{x} = (x_1, x_2, \dots)$, $\mathbf{y} = (x_1, x_2, \dots)$.

- Taking the coefficient of $x_1 \cdots x_N y_1 \cdots y_N$ on both sides of the Cauchy identity

$$\sum_{\lambda \vdash N} f_\lambda^2 = N!$$

Schur measure

- **Okounkov** 2001. Let $0 \leq x_i, y_i < 1$ and let $0 < \prod_{i,j \geq 1} 1 - x_i y_j < \infty$. Schur measure (with parameters x_i, y_i) is the probability measure on the set of partitions λ (of arbitrary length) given by

$$\mathbb{P}^{Schur}(\lambda) = \prod_{i,j \geq 1} (1 - x_i y_j).s_\lambda(x) s_\lambda(y).$$

Schur polynomials are characters of irreducible representations of the general linear group GL_N and also encode characters of the symmetric group S_N.

RSK and Directed Last Passage Percolation

- **Robinson-Schensted-Knuth** correspondence

$$RSK : \mathcal{M}_{m,n} \to \bigsqcup_{\lambda \in \mathcal{P}_{\min(m,n)}} SSYT(\lambda, m) \times SSYT(\lambda, n)$$

$$A \longmapsto (P(A), Q(A))$$

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \in \mathcal{M}_{4,3}.$$

$$L_A = \boxed{1 \,|\, 1 \,|\, 2 \,|\, 3 \,|\, 3} \otimes \boxed{1 \,|\, 1 \,|\, 3 \,|\, 4} \otimes \boxed{2 \,|\, 3 \,|\, 4} \quad 332114311432$$

$$2 \to 3 \to 4 \to 1 \to 1 \to 3 \to 4 \to \begin{matrix} 1 & 1 & 2 & 3 & 3 \end{matrix} = \begin{matrix} 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 3 & 3 & 4 & & & \\ 4 & & & & & & \end{matrix} = P$$

$$x^P = x_1^4 x_2^2 x_3^4 x_4^2$$

$$Q = \begin{matrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 & & & \\ 3 & & & & & & \end{matrix} \qquad y^Q = y_1^4 y_2^4 y_3^3 \qquad (xy)^A = \prod_{1 \le i \le 4, 1 \le j \le 3} (x_i y_j)^{a_{ij}} = x^P y^Q$$

-

$$\prod_{1 \le i \le m, 1 \le j \le n} \frac{1}{1 - x_i y_j} = \prod_{1 \le i \le m, 1 \le j \le n} \sum_{a_{i,j}=0}^{+\infty} (x_i y_j)^{a_{i,j}} = \sum_{[a_{i,j}] \in \mathcal{M}_{m,n}} \prod_{1 \le i \le m, 1 \le j \le n} (x_i y_j)^{a_{ij}} = \sum_{A \in \mathcal{M}_{m,n}} (xy)^A$$

$$= \sum_{A \in \mathcal{M}_{m,n}} x^{P(A)} y^{Q(A)} = \sum_{\lambda \in \mathcal{P}_{\min(m,n)}} \sum_{P \in SSYT(\lambda,m)} x^P \sum_{Q \in SSYT(\lambda,n)} y^Q = \sum_{\lambda \in \mathcal{P}_{\min(m,n)}} s_\lambda(x) s_\lambda(y).$$

Last Passage Percolation (LPP)

- Let $\Pi_{m,n}$ be the collection of paths p in $\mathbb{N}^2$ with steps $\leftarrow, \downarrow$ starting in $(1, n)$ and ending in $(m, 1)$. Assign a weight $a_{i,j}$ to each coordinate (i, j), $1 \leq i \leq m$, $1 \leq j \leq n$, arranged in the matrix convention, and define $A \in \mathcal{M}_{m,n}$.

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} L_{m,n} &= \max_{p \text{ in } \Pi_{m,n}} \{ \sum_{(i,j)\in p} a_{i,j} \text{ sum of the entries along a path } p \text{ in } \Pi_{m,n} \} \\ &= \text{ common maximal row lenght of } P(A) \text{ and } Q(A) = \lambda_1(A) \\ &= \text{maximal length of a weakly decreasing sequence of the word read off } L_A. \end{aligned}$$

- $L_{m,n}$, **last passage percolation time**, is the largest time it takes to travel from $(1, n)$ to $(m, 1)$ using an admissible path p.

- *Schur measure arises from directed last passage percolation models with an appropriate choice of independent weights* $a_{i,j}$.

Our ensemble of random integer matrices

- Let $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ be two sets of real numbers in the interval $[0, 1[$, and consider a family $w_{i,j}$ of independent r. v. with a *geometric distribution* of parameter $u_i v_j$

$$\mathbb{P}(w_{i,j} = k) = (1 - u_i v_j)(u_i v_j)^k \text{ for any } k \in \mathbb{Z}_{\geq 0}.$$

- $\mathcal{W} = [w_{ij}] \in \mathcal{M}_{m,n}$ is a *random matrix* and since the r. v. $w_{i,j}$ are independent, for any $A \in \mathcal{M}_{m,n}$, we get

$$\mathbb{P}(\mathcal{W} = A) = \prod_{1 \leq i \leq m, 1 \leq j \leq n} \mathbb{P}(w_{i,j} = a_{ij}) = \left(\prod_{1 \leq i \leq m, 1 \leq j \leq n} (1 - u_i v_j) \right) (uv)^A$$

$$= \left(\prod_{1 \leq i \leq m, 1 \leq j \leq n} (1 - u_i v_j) \right) u^{P(A)} v^{Q(A)}$$

where $(uv)^A = \prod_{1 \leq i \leq m, 1 \leq j \leq n} (u_i v_j)^{a_{i,j}}$.

- For $\lambda \in \mathcal{P}_{min(m,n)}$,

$$\mathbb{P}(\lambda) = \sum_{A \in \mathcal{M}_{m,n} : shape(P(A)) = \lambda} \mathbb{P}(\mathcal{W} = A) = \prod_{1 \le i \le m, 1 \le j \le n} (1 - u_i v_j) s_\lambda(u) s_\lambda(v)$$
$$= \mathbb{P}^{Schur}(\lambda).$$

- $\mathrm{perc}(\mathcal{W}) := L_{m,n}$ is a r.v.

$$\mathbb{P}(\mathrm{perc}(\mathcal{W}) = \lambda_1 \le k) = \mathbb{P}^{Schur}(\lambda_1 \le k)$$

- **Johansson** 2000. Directed Last Passage Percolation Model with iid Geometric Weights.
  Fix $q \in (0,1)$. Suppose $m \ge n$. Let $u_i = \sqrt{q},\ i = 1, \ldots, m$ and $v_i = \sqrt{q},\ i = 1, \ldots, n$. Then

$$\mathbb{P}(a_{i,j} = k) = (1-q)q^k,\ k = 0, 1, 2, \ldots$$

  - The Schur measure in this case is closely related to the density function of the eigenvalues in the Gaussian unitary ensemble (GUE).

II. Demazure measures and LPP models in Young shapes

- **Rectangle Cauchy kernel identity**

$$\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{1-x_iy_j}=\sum_{\lambda\in\mathcal{P}_n}s_\lambda(x)s_\lambda(y)$$

LHS rewritten in the basis of Schur polynomials (irreducible characters of GL_n). $\mathcal{P}_n$ the set of partitions with at most n parts.

$$\mathbb{P}^{Schur}(\lambda)=\prod_{1\leq i,j\leq n}(1-x_iy_j).s_\lambda(x)s_\lambda(y).$$

- **Non-symmetric staircase Cauchy kernel identity**, Lascoux 2000

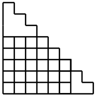
$$\prod_{1\leq j\leq i\leq n}\frac{1}{1-x_iy_j}=\sum_{\mu\in\mathbb{Z}^n_{\geq 0}}\overline{\kappa}^{\mu}(x)\kappa_\mu(y)$$

LHS rewritten in the bases of Demazure and Demazure atom polynomials: $\overline{\kappa}^{\mu}(x_1,\ldots,x_n)$ *opposite Demazure atoms* and $\kappa_\mu(y)$ *Demazure characters*.

- Let $0\leq x_i,y_i<1$, $i=1,\ldots,n$. Demazure measure (with parameters x_i,y_i) is the probability measure on the set $\mathbb{Z}^n_{\geq 0}$ of non negative vectors μ given by

$$\mathbb{P}^{Demazure}(\mu)=\prod_{1\leq j\leq i\leq n}(1-x_iy_j).\overline{\kappa}^{\mu}(x)\kappa_\mu(y).$$

The restriction of RSK correspondence to $\mathcal{M}^{\varrho}_{n,n}$ lower triangular matrices

$\varrho =$ 

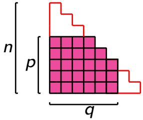
- $$RSK^{\varrho} : \mathcal{M}^{\varrho}_{n,n} \to \bigsqcup_{\mu \in \mathbb{Z}^n_{\geq 0}} \overline{\mathrm{B}}^{\mu} \times \mathrm{B}_{\mu} \quad \text{Lascoux, 2000, A.-Emami,15, Choi-Kwon, 18}$$
  $$A \mapsto (P, Q), \; {\scriptstyle K_+(Q) \leq K_-(P) = K(\mu)}$$

  $\mathrm{B}_{\mu} \subset SSYT(\lambda, n)$ such that $\mu \in S_n\lambda$ for some $\lambda \in \mathcal{P}_n$ is the Demazure crystal consisting of all tableaux $Q \in SSYT(\lambda, n)$ with right key $K_+(Q) \leq K(\mu)$. $\overline{\mathrm{B}}^{\mu} \subset SSYT(\lambda, m)$ opposite Demazure atom crystal consisting of all tableaux $P \in SSYT(\lambda, m)$ with left key $K^-(P) = K(\mu)$.
- The law of the random variable ${\scriptstyle G = perc(\mathcal{W})}$, $\mathcal{W} \in \mathcal{M}^{\varrho}_{n,n}$,

$$\mathbb{P}^{Demazure}(G = k) = \prod_{1 \leq j \leq i \leq n} (1 - u_i v_j) \sum_{\mu \in \mathbb{Z}^n_{\geq 0} | \max(\mu) = k} \overline{\kappa}^{\mu}(u) \kappa_{\mu}(v), \; k \in \mathbb{Z}^+_{\geq} 0.$$

Non-symmetric truncated Cauchy kernel identity

$$q \geq p, \quad \Lambda(p,q) =$$



- **Non-symmetric Cauchy kernel for truncated shapes**

$$\prod_{(i,j)\in\Lambda(p,q)} \frac{1}{1-x_i y_j} = \sum_{\mu\in\mathbb{Z}^p_{\geq 0}} \overline{\kappa}^{\mu}(x_{n-p+1},\ldots,x_n)\kappa_{\widetilde{\mu}}(y_1,\ldots,y_q).$$

- A.-Emami, 14, A.-Gobet-Lecouvey, 22. **Restriction of** RSK **to** $\Lambda(p,q)$

$$RSK^{\Lambda(p,q)} : \mathcal{M}^{\Lambda(p,q)}_{n,n} \to \bigsqcup_{\mu\in\mathbb{Z}^p_{\geq 0}} \overline{\mathrm{B}}^{\mu}_p \times \mathrm{B}_{q,\widetilde{\mu}}$$

$$A \mapsto (P(A), Q(A)) \qquad K_-(P) = K(\mu),\ K_+(Q) \leq K(\tilde{\mu}).$$

- The law of the random variable $G = perc(\mathcal{W})$, $\mathcal{W} \in \mathcal{M}^{\Lambda(p,q)}_{m,n}$. For $k \in \mathbb{Z}^+_{\geq}0$,

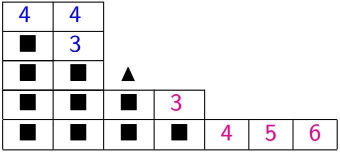
$$\mathbb{P}^{Demazure}(G=k) =$$

$$\prod_{(i,j)\in\Lambda(p,q)} (1-u_i v_j) \sum_{(\mu_1,\ldots,\mu_p)\in\mathbb{Z}^p_{\geq 0}|\max(\mu)=k} \overline{\kappa}_{(\mu_p,\ldots,\mu_1)}(u_n,\ldots,u_{n-p+1})\kappa_{\widetilde{\mu}}(v_1,\ldots,v_q).$$

Non symmetric Cauchy kernel in a Young shape

$n = 8,\ \Lambda = (7,4,2,2,2),\ m = 4,\ \varrho_\Lambda = (4,3,2,1)$

$\Lambda =$



$\sigma(\Lambda, NW) = s_4 s_3 s_4$, $\sigma(\Lambda, SE) = s_3 s_6 s_5 s_4$

- $$\prod_{(i,j)\in\Lambda} \frac{1}{1 - x_i y_j} =$$

  $$\sum_{(\mu_1,\ldots,\mu_m)\in\mathbb{Z}^m} D_{\sigma(\Lambda,NW)}\overline{\kappa}_{(\mu_m,\ldots,\mu_1)}(x_n,\ldots,x_{n-m+1}) D_{\sigma(\Lambda,SE)}\kappa_{(\mu_1,\ldots,\mu_m)}(y_1,\ldots,y_m)$$

  where $D_{\sigma(\Lambda,NW)} = D_{i_1}\cdots D_{i_a}$ and $D_{\sigma(\Lambda,SE)} = D_{j_1}\cdots D_{j_b}$ are Demazure operators.

- **The restriction of RSK to $\mathcal{M}_{n,n}^{D_\Lambda}$**

  $$RSK^{D_\Lambda} : \mathcal{M}_{n,n}^{D_\Lambda} \to \bigsqcup_{(\mu_1,\ldots,\mu_m)\in\mathbb{Z}^m_{\geq 0}} \iota\left(\dot{\Delta}_{\sigma(\Lambda,NW)}\overline{\mathrm{B}}_{(\mu_m,\ldots,\mu_1)}\right) \times \Delta_{\sigma(\Lambda,SE)}\mathrm{B}_{(\mu_1,\ldots,\mu_m)}$$

  where $\Delta_{\sigma(\Lambda,SE)} = \Delta_{j_1}\cdots\Delta_{j_b}$, $\dot{\Delta}_{\sigma(\Lambda,NW)} = \dot{\Delta}_{i_1}\cdots\dot{\Delta}_{i_a}$, ι Schützenberger involution.

$$A_{(7,4,2,2,2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} & 0 \end{pmatrix} \qquad 577 \otimes 45 \otimes 7 \otimes 7 \otimes 8 \otimes \emptyset \otimes 88 \otimes \emptyset$$

$$P = \begin{array}{|c|c|c|c|c|} \hline 4 & 5 & 5 & 7 & 7 \\ \hline 7 & 7 & 8 \\ \cline{1-3} 8 & 8 \\ \cline{1-2} \end{array} \qquad K_-(P) = \begin{array}{|c|c|c|c|c|} \hline 4 & 4 & 4 & 4 & 4 \\ \hline 7 & 7 & 8 \\ \cline{1-3} 8 & 8 \\ \cline{1-2} \end{array} = K(0^3, 5, 0^2, 2, 3)$$

$$Q = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 3 & 4 & 7 \\ \cline{1-3} 5 & 7 \\ \cline{1-2} \end{array} \qquad K_+(Q) = \begin{array}{|c|c|c|c|c|} \hline 2 & 2 & 2 & 2 & 2 \\ \hline 4 & 4 & 7 \\ \cline{1-3} 7 & 7 \\ \cline{1-2} \end{array} = K(0, 5, 0, 2, 0^2, 3, 0)$$

$$\iota\dot{\Delta}_4\dot{\Delta}_3\dot{\Delta}_4\overline{\mathrm{B}}_{(\mu_4,\ldots,\mu_1,0^4)}=$$

$$=\begin{cases}\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,0,0^4)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_2,0^4)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0^2,\mu_2,0^3)} & \text{if } \mu_2>\mu_1=0\\ \iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,0,0^4)} & \text{if } \mu_1=\mu_2=0\\ \iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,\mu_1,0^4)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,0,\mu_1,0^3)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_2,\mu_1,0^3)} & \text{if } \mu_1=\mu_2>0\\ \emptyset, & \text{if } \mu_1>\mu_2\geq 0\\ \iota\overline{\mathrm{B}}_{(\mu_4,\ldots,\mu_1,0^4)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_1,\mu_2,0^4)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,0,\mu_1,0^3)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_2,\mu_1,0^3)}\bigsqcup\\ \bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_1,\mu_2,0^3)}\bigsqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_1,0,\mu_2,0^3)}, & \text{if } \mu_2>\mu_1>0.\end{cases}$$

$$K^-(P)=K(0^3,5,0^2,2,3)\Leftrightarrow P\in\iota\mathrm{B}_{(3,2,0^2,5,0^3)}=\mathrm{B}^{(0^3,5,0^2,2,3)}\Rightarrow\mu=(0,5,2,3)$$

$$K_+(Q)=K(0,5,0,2,0^2,3,0)\leq K(\pi_3\pi_6\pi_5\pi_4(\mu,0^4))=K(0,5,0,2,0^2,3,0).$$

$$(P,Q)\in\iota\mathrm{B}_{(3,2,0^2,5,0^3)}\times\mathrm{B}_{(0,5,0,2,0^2,3,0)}.$$

- The law of the random variable $G = per(\mathcal{W})$, $\mathcal{W} \in \mathcal{M}^{\Lambda}_{m,n}$. For $k \in \mathbb{Z}^{+}_{\geq}0$,

$$\mathbb{P}^{Demazure}(G = k) = \prod_{(i,j)\in D_\Lambda} (1 - u_i v_j)$$

$$\sum_{(\mu_1,\ldots,\mu_m)\in\mathbb{Z}^m|\max(\mu)=k} D_{\sigma(\Lambda,NW)\overline{\kappa}(\mu_m,\ldots,\mu_1)}(u_n,\ldots,u_{n-m+1}) D_{\sigma(\Lambda,SE)\kappa(\mu_1,\ldots,\mu_m)}(v_1,\ldots,v_m).$$