On an involution on the set of Littlewood-Richardson tableaux and the hidden commutativity

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Abstract

We translate to the language of triangles with boundary [9] the involution presented in [3], [4], where the language of skew-tableaux has been used. This involution makes explicit, in a simple way, the commutativity of the Littlewood-Richardson rule.

In [4] the language of skew-tableaux is exclusively used to prove this symmetry. Although the description of the algorithm, using these combinatorial objects, is very simple and makes natural the commutativity of the Littlewood-Richardson rule, the language of triangles suits more the purpose to giving a shorter proof of the involution which exhibits the commutativity of the Littlewood-Richardson rule.

The motivation for this involution is easily understood in the language of the invariant factors of a product of integral matrices when the transposition of this product is considered. This matrix analogue follows from an appropriate decomposition of a product of integral matrices established by R. C. Thompson in [14].

Keywords: Littlewood-Richardson triangles, Littlewood-Richardson rule symmetries. combinatorics of integral matrices.

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1 Introduction

The main feature of this bijection is the decomposition of an LR triangle A, of size $k \geq 1$ and type $[a, b, c]$, into a nested sequence $(A^{(s)})_{s=1}^{k}$, $A = A^{(k)} \supseteq A^{(k-1)} \supseteq \ldots \supseteq A^{(2)} \supseteq A^{(1)}$, of LR triangles of size s and type $[a^{(s)}, (b_1, \ldots, b_s); (c_1, \ldots, c_s)]$, $s = 1, \ldots, k$, where $(a^{(s)})_{s=1}^{k}$ is a sequence of interlacing partitions. On its turn, this sequence of partitions defines a decomposition of an LR triangle B, of type $[b, a, c]$, into a nested sequence $(B^{(s)})_{s=1}^{k}$, $B = B^{(k)} \supseteq B^{(k-1)} \supseteq \ldots \supseteq B^{(2)} \supseteq B^{(1)}$, of LR triangles of size s and type $[(b_1, \ldots, b_s); a^{(s)}; (c_1, \ldots, c_s)]$, $s = 1, \ldots, k$.

In the matrix setting LR triangles, may be interpreted as follows. Let X, Y and Z be k-square nonsingular matrices, over a local principal ideal domain with prime p, such that $XY = Z$. Let $p^{a_1}, \ldots, p^{a_k}$, $p^{b_1}, \ldots, p^{b_k}$, and $p^{c_1}, \ldots, p^{c_k}$ be the invariant factors (Smith invariants) of X, Y and Z respectively, where the exponents of the p-powers are by decreasing order. In [1, 2, 7] is shown that there is one and only one LR triangle A of type $[a, b, c]$ such that $XY = Z$ *realizes* [1, 2, 8], where $a = (a_i)_{i=1}^{n}$, $b = (b_i)_{i=1}^{n}$ and $c = (c_i)_{i=1}^{n}$.

We consider the following algebraic formulation of the LR rule. (For a similar presentation see [6, 15].) There exists an LR triangle of size k and type $[a, b, c]$, iff there exists a sequence of partitions $b^{(s)} = (b_1^{(s)}, \ldots, b_s^{(s)})$, $s = 1, ..., k$, with $b^{(k)} = b$, satisfying the interlacing inequalities

$$b_{i+1}^{(s+1)} \leq b_i^{(s)} \leq b_i^{(s+1)}, \text{ for } 1 \leq i \leq s \leq k-1, \quad (i)$$

schematically

$$\begin{array}{ccccccccccc}
 & & & & b_1^{(1)} & & & & & & \\
 & & & b_1^{(2)} & & b_2^{(2)} & & & & & \\
 & & b_1^{(3)} & & b_2^{(3)} & & b_3^{(3)} & & & & \\
 & & & & \cdots & & & & & & \\
 & b_1^{(k-1)} & & b_2^{(k-1)} & & b_3^{(k-1)} & \cdots & b_{k-1}^{(k-1)} & & & \\
b_1^{(k)} & & b_2^{(k)} & & b_3^{(k)} & \cdots & b_{k-1}^{(k)} & & b_k^{(k)}, & &
\end{array}$$

and the system of linear inequalities

$$a_{s-1} + \sum_{j=1}^{r-1} (b_j^{(s-1)} - b_j^{(s-2)}) \geq a_s + \sum_{j=1}^{r} (b_j^{(s)} - b_j^{(s-1)}),\ r = 1, \ldots, s-1,\ s = 2, \ldots, k, \quad (ii)$$

$$a_s + \sum_{j=1}^{s} (b_j^{(s)} - b_j^{(s-1)}) = c_s,\ s = 1, \ldots, k. \quad (iii)$$

(We put $b_1^{(0)} = 0$.)

Given an LR triangle A of size $k > 1$ and type $[a, b, c]$, for each $s \in \{1, 2, \ldots, k-1\}$, we may associate, by deleting the $k, \ldots, (s+1)$-th rows of A, an LR triangle $T^{(s)}$ of size s and type $[(a_1, \ldots, a_s); (b_1^{(s)}, \ldots, b_s^{(s)}); (c_1, \ldots, c_s)]$, with $T^{(k)} = A$. Indeed, this sequence of LR triangles $T^{(s)}$ of type $[(a_1, \ldots, a_s); (b_1^{(s)}, \ldots, b_s^{(s)}); (c_1, \ldots, c_s)]$, $s = 1, \ldots, k$, is such that the sequence of partitions $b^{(s)} = (b_1^{(s)}, \ldots, b_s^{(s)})$, $s = 1, ..., k$, with $b^{(k)} = b$, satisfy the previous linear inequalities (i), (ii) and (iii). This defines a b-decomposition of A.

The bijection to be exhibited between LR triangles of type $[a, b, c]$ and $[b, a, c]$ is based on a combinatorial deletion operation transforming an LR triangle of size k into one of size

$k-1$, for $k>1$, which defines an a-decomposition of an LR triangle of type $[a,b,c]$. That is, given an LR triangle A of size k and type $[a,b,c]$, this operation defines, uniquely, a nested sequence $A=A^{(k)}\supseteq A^{(k-1)}\supseteq\ldots\supseteq A^{(2)}\supseteq A^{(1)}$ of LR triangles, with $A^{(s)}$ of size s and type $[(a_1^{(s)},\ldots,a_s^{(s)});(b_1,\ldots,b_s);(c_1,\ \ldots,c_s)]$, $s=1,...,k$, and such that the sequence of partitions $a^{(s)}=(a_1^{(s)},\ldots,a_s^{(s)})$, $s=1,...,k$, with $a^{(k)}=a$, satisfies

$$a_{i+1}^{(s+1)}\le a_i^{(s)}\le a_i^{(s+1)},\ \ 1\le i\le s\le k-1. \qquad (iv)$$

Furthermore, since the deletion operation can be performed backwards, we have also

$$b_{s-1}+\sum_{j=1}^{r-1}(a_j^{(s-1)}-a_j^{(s-2)})\ge b_s+\sum_{j=1}^{r}(a_j^{(s)}-a_j^{(s-1)}),\ r=1,...,s-1,\ \ s=2,\ldots,k,\quad (v)$$

$$b_s+\sum_{j=1}^{s}(a_j^{(s)}-a_j^{(s-1)})=c_s,\quad s=1,\ldots,k.\quad (vi)$$

(We put $a_1^{(0)}=0$.)

This combinatorial deletion operation defines, uniquely, an a-decomposition of A. Thus, given an LR triangle A of type $[a,b,c]$, we may associate, uniquely, by means of this deletion operation, an LR triangle B of type $[b,a,c]$, defined by the a-decomposition of A. Moreover, applying to B this combinatorial deletion operation, we recover A.

Considering [14] we may assert:

Let X, Y and Z be k-square non-singular matrices, with entries in a local principal ideal domain, such that $AB=C$. Let $p^{a_1},\ldots,p^{a_k}$, $p^{b_1},\ldots,p^{b_k}$, and $p^{c_1},\ldots,p^{c_k}$ be the invariant factors of X, Y and Z, respectively, where $a_1\ge\ldots\ge a_k$, $b_1\ge\ldots\ge b_k$ and $c_1\ge\ldots\ge c_k$. We may assume that:

(i) X is lower triangular [X diagonal, $X=diag(p^{a_1},\cdots,p^{a_k})$]*;*

(ii) Y is diagonal, $Y=diag(p^{b_1},\ldots,p^{b_k})$ [Y upper triangular]*;*

(iii) $Z=[\gamma_{ij}]$ is lower triangular with $\gamma_{ii}=p^{c_i}$, $p^{c_i}|\gamma_{ij}$ for $i>j$, $1\le i\le k$ [$Z=[\gamma_{ij}]$ is upper triangular with $\gamma_{ii}=p^{c_i}$, $p^{c_i}|\gamma_{ij}$, for $i<j$, $1\le i\le k$] (the symbol "$|$" denotes divisibility).

Denote by A the LR triangle of type $[a,b,c]$ *realized* by $XY=Z$. A is defined by the interlacing decomposition of b as follows. Put X diagonal and Y upper triangular. Let $X^{(k)}:=X$, $Y^{(k)}:=Y$, $Z^{(k)}:=Z$ and $T^{(k)}:=A$, and consider the sequence of product of matrices $X^{(s)}Y^{(s)}=Z^{(s)}$, $s=1,\ldots,k-1$, obtained by deleting the $(s+1)$-th rows and columns of $X^{(s+1)}$, $Y^{(s+1)}$ and $Z^{(s+1)}$. That is, $X^{(s)}$, $Y^{(s)}$ and $Z^{(s)}$ are the s-leading submatrices in the first s rows of $X^{(s+1)}$, $Y^{(s+1)}$ and $Z^{(s+1)}$ respectively, for $s=1,2,\ldots,k-1$. Since X is in the triangular form, by the interlacing property relating the invariant factors of a matrix with those of a submatrix [11, 12, 13, 15], we obtain, for each $s\in\{1,2,\ldots,k\}$, one LR triangle $T^{(s)}$ of type $[(a_1,\ldots,a_s);(b_1^{(s)},\ldots,b_s^{(s)});(c_1,\ \ldots,c_s)]$ *realized* by $X^{(s)}Y^{(s)}=Z^{(s)}$, where the sequence $b^{(s)}=(b_1^{(s)},\ldots b_s^{(s)})$, $s=1,\ldots,k$, satisfies (i).

Now put X lower triangular and Y diagonal and consider the sequence of product of matrices $X^{(s)}Y^{(s)}=Z^{(s)}$, $s=1,\ldots,k-1$, obtained by deleting the $(s+1)$-th rows and columns of $X^{(s+1)}$, $Y^{(s+1)}$ and $Z^{(s+1)}$. That is, $X^{(s)}$, $Y^{(s)}$ and $Z^{(s)}$ are the s-leading submatrices in the first s rows of $X^{(s+1)}$, $Y^{(s+1)}$ and $Z^{(s+1)}$ respectively, for $s=1,2,\ldots,k-1$. Again, since X is in the triangular form, by the interlacing property, we obtain, for each $s\in\{1,2,\ldots,k\}$, one LR triangle of type $[(a_1^{(s)},\ldots,a_s^{(s)});(b_1,\ldots,b_s);(c_1,\ \ldots,c_s)]$ *realized* by $X^{(s)}Y^{(s)}=Z^{(s)}$, where the sequence $a^{(s)} = (a_1^{(s)}, \dots a_s^{(s)})$, $s = 1, \dots, k$, satisfies (iv). By transposition, the s-leading submatrices in the first s rows of $Y^t X^t = Z^t$ (transposing both sides of $XY = Z$), *realize* sequence of LR triangles of type $[(b_1, \dots, b_s); (a_1^{(s)}, \dots, a_s^{(s)}); (c_1, \dots, c_s)]$, $s = 1, \dots, k$, which defines a, unique, LR triangle B of type $[b, a, c]$. Thus the matrix meaning of our combinatorial involution, in the context of the invariant factors, is the transposition of a product of matrices.

Each triple $XY = Z$ of matrices realizes a unique pair (A, B) of LR triangles of types $[a, b, c]$ and $[b, a, c]$ respectively. The triangle A is defined by the sequence of triangles $T^{(s)}$, $s = 1, \cdots, k$, realized by the sequence of matrices $X^{(s)} Y^{(s)} = Z^{(s)}$, $s = 1, \cdots, k$, considering X diagonal and Y upper triangular. On the other hand, considering X lower triangular and Y diagonal, the matrix sequence $X^{(s)} Y^{(s)} = Z^{(s)}$, of s-leading submatrices in the first s rows of $XY = Z$, $s = 1, \dots, k$, *realizes* sequence of LR triangles of type $[(a_1^{(s)}, \dots, a_s^{(s)}); (b_1, \dots, b_s); (c_1, \dots, c_s)]$ where $a^{(s)}$ interlaces with $a^{(s+1)}$, for $s = 1, 2, \dots, k-1$. We point out the analogy between this sequence of LR triangles of type $[(a_1^{(s)}, \dots, a_s^{(s)}); (b_1, \dots, b_s); (c_1, \dots, c_s)]$ and the sequence of LR triangles $A^{(s)}$, $s = 1, \dots, k$, with $A^{(k)} = A$ *realized* by $XY = Z$, achieved by means of a combinatorial deletion operation, which decomposes a into a sequence of interlacing partitions.

Therefore, if Pak-Vallejo Conjecture 1 in [10] is true, this means that the bijections ρ_1, ρ_2, ρ_2' and and the one presented here (denoted by ρ_3 in [10]) have a matrix analogue. In [5] is shown that symmetries ρ_1, ρ_2, and ρ_2' coincide.

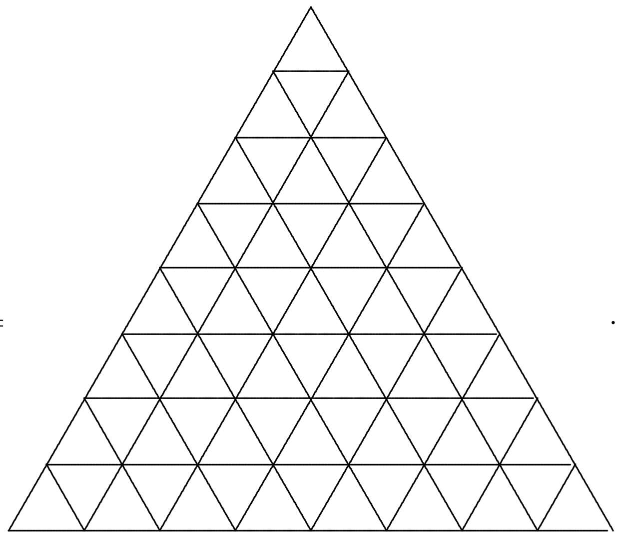
2 The space of triangles and the hive graph

Let k be a positive integer. Let T_k be the space of triangles of size k [7] consisting of all sequences

$$A = (V^{(0)}, V^{(1)}, \dots, V^{(k)}),$$

where $V^{(j)} = (a_{jj}, \dots, a_{kj}) \in \mathbb{R}^{k-j+1}$, $0 \le j \le k$, and $a_{00} = 0$. As a vector space $T_k \simeq \mathbb{R}^{\frac{(k+1)(k+2)}{2}-1}$.

The hive graph Δ_k of size k is a graph in the plane with $\left(\begin{array}{c} k+2 \\ 2 \end{array}\right)$ vertices arranged in a triangular grid, consisting of k^2 small equilateral triangles.

$\Delta_8 =$  . (1)

We identify T_k with the vector space of all labelling $A = (a_{ij})_{0 \le j \le i \le k}$ of Δ_k by real numbers such that $a_{00} = 0$.

We write $A \in T_k$ as a triangular array of real numbers

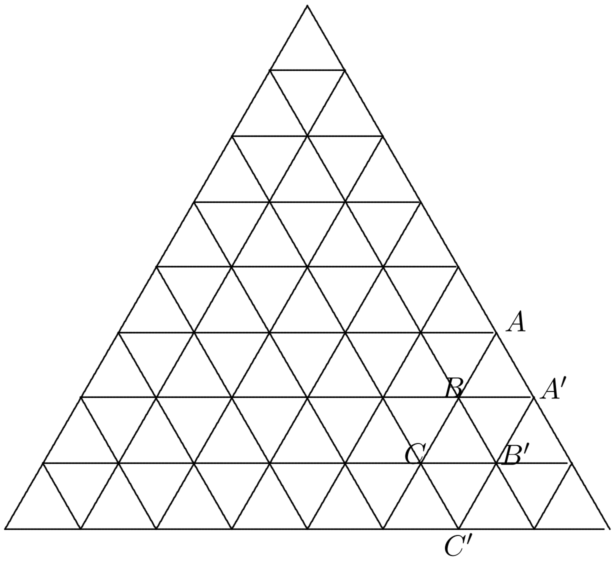
$$A = \begin{array}{ccccccc} & & & a_{00} & & & \\ & & a_{10} & & a_{11} & & \\ & a_{20} & & a_{21} & & a_{22} & \\ a_{30} & & a_{31} & & a_{32} & & a_{33} \end{array} \in T_3.$$

A Littlewood-Richardson (**LR**) triangle of size k [9] is an element $A = (a_{ij})_{0 \le j \le i \le k}$ of T_k that satisfies the following inequalities

$$\begin{array}{lll} (P) & a_{ij} \ge 0, & 0 \le i, j \le k, \\ (I) & \sum_{q=j}^{i} a_{qj} \ge \sum_{q=j+1}^{i+1} a_{q,j+1}, & 1 \le j \le i < k, \\ (S) & \sum_{p=0}^{j-1} a_{ip} \ge \sum_{p=0}^{j} a_{i+1,p}, & 1 \le j \le i < k. \end{array}$$

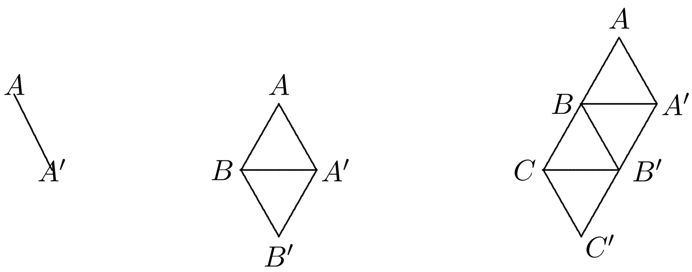
For each $j = 1, \dots, k-1$, we consider, in Δ_k (2),the labelled parallelogram $\mathbf{p}_j = [a_{jj}, \dots, a_{k-1,j}, V^{(j+1)}]$. (We convention $\mathbf{p}_{k-1} = [a_{k-1,k-1}, a_{kk}]$ as a degenerated parallelogram.) The labels of parallelograms $\mathbf{p}_j$, $1 \le j \le k-1$, satisfy inequalities (I).

For $k = 8$, we have the parallelogram $\mathbf{p}_5 = [ABC; A'B'C']$ in Δ_8



(2)

where the labels of



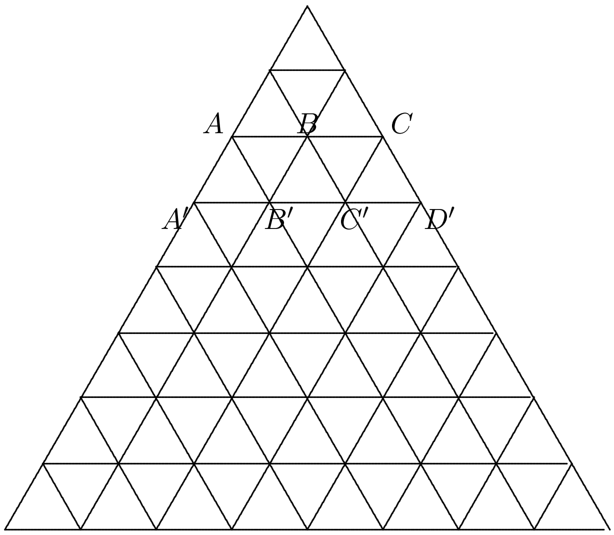
(3)

satisfy the inequalities

$$\begin{aligned} &A \geq A' \\ &A + B \geq A' + B' \\ &A + B + C \geq A' + B' + C'. \end{aligned}$$

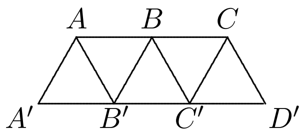
For $i = 1, \ldots, k-1$, we consider the labelled trapezoids $\mathbf{t}_i = [a_{i0}, a_{i1}, \ldots, a_{ii}; a_{i+10}, a_{i+11}, \ldots, a_{i+1,i+1}]$ in Δ_k (4). The labels of $\mathbf{t}_j$, $1 \leq j < k$, satisfy inequalities (S).

For $k = 8$, we have the trapezoid $\mathbf{t}_2 = [ABC; A'B'C'D']$ in Δ_8



(4)

The labels of



(5)

satisfy
$A \geq A' + B'$,
$A + B \geq A' + B' + C'$,
$A + B + C \geq A' + B' + C' + D'$.

We denote by LR_k the cone of all Littlewood-Richardson triangles in T_k, and call it the Littlewood-Richardson cone of order k.

To each triangle $A = (a_{ij})_{0 \leq j \leq i \leq k} \in T_k$ we associate the real vectors $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k)$ and $c = (c_1, \ldots, c_k)$, where

$$\begin{aligned} a_i &= a_{i0}, \;\; 1 \leq i \leq k \\ b_j &= \Sigma_{q=j}^{k} a_{qj}, \;\; 1 \leq j \leq k \\ c_i &= \Sigma_{q=0}^{i} a_{iq}, \;\; 1 \leq i \leq k. \end{aligned}$$

We call (a, b, c) the *type* of A, b the *weight* of A, and a the *boundary* of T_k. Note that a is the label of the right edge of the hive graph Δ_k.

Let x be a real vector, and denote by $|x|$ the sum of its entries. If $A \in LR_k$, it follows from (P), (S) and (I) that the vectors a, b, and c satisfy $a_1 \geq \ldots \geq a_k \geq 0$, $b_1 \geq \ldots \geq b_k \geq 0$, $c_1 \geq \ldots \geq c_k \geq 0$, and $|a| + |b| = |c|$, $a \leq c$.

We denote by $LR_k(a,b,c)$ the set of all **LR** triangles in T_k of type (a,b,c).

For example, the triangle below is in LR_4 with type given by $a = (5,3,2,0)$, $b = (5,4,2,1)$ and $c = (8,6,5,3)$

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 5 & & 3 & & & \\
 & & 3 & & 1 & & 2 & & \\
 & 2 & & 0 & & 1 & & 2 & \\
0 & & 1 & & 1 & & 0 & & 1
\end{array}.$$

Let $LR_k(\mathbb{Z}) := LR_k \cap \mathbb{Z}^{\frac{(k+1)(k+2)}{2}-1}$ be the set of all integral **LR** triangles of size k, that is, the set of integer points of LR_k. Since LR_k is a rational polyhedral cone, $LR_k(\mathbb{Z})$ is a finitely generated semigroup and the cone generated by $LR_k(\mathbb{Z})$ is LR_k.

Let P_k denote the set of all k-tuples $x = (x_1, \ldots, x_k)$ of nonnegative integers such that $x_1 \geq \ldots \geq x_k \geq 0$.

Let a, b, c partitions in P_k such that $a \leq c$ and $|a| + |b| = |c|$. To each Littlewood-Richardson triangle $\mathcal{T}$ of type (a,b,c), we associate an integral Littlewood-Richardson triangle $A = (a_{ij})_{0 \leq j \leq i \leq k} \in T_k$ defined by

$$\begin{array}{l}
a_{00} = 0, \;\; a_{i0} = a_i, \; 1 \leq i \leq k, \\
a_{ij} \;\; \text{the number of } j\text{'s in row } i \text{ of } \mathcal{T}, \; 0 < j \leq i \leq k.
\end{array}$$

For example,

$$\mathcal{T} = \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 & 2 & & \\
0 & 0 & 2 & 3 & 3 & & & \\
1 & 2 & 4 & & & & &
\end{array} \qquad \longleftrightarrow \quad A_{\mathcal{T}} = \begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 5 & & 3 & & & \\
 & & 3 & & 1 & & 2 & & \\
 & 2 & & 0 & & 1 & & 2 & \\
0 & & 1 & & 1 & & 0 & & 1
\end{array}.$$

Proposition 1 *[9] Let a, b, c partitions in P_k such that $|c| = |a| + |b|$ and $a \leq c$. Then the correspondence $\mathcal{T} \longleftrightarrow A_{\mathcal{T}}$ is a bijection between $LR_k(a,b,c) \cap \mathbb{Z}^{\frac{(k+1)(k+2)}{2}-1}$ and the set of all LR-skew tableaux of type (a,b,c). In particular, $|LR_k(a,b,c) \cap \mathbb{Z}^{\frac{(k+1)(k+2)}{2}-1}| = N_{ab}^c$.*

3 Triangles with 0-boundary by excavation of a partition

We write $A = (0; a_{ij})_{1 \leq j \leq i \leq k} \in T_k$ when the right edge of Δ_k is labelled by the null vector, that is, $a_{i0} = 0$, $1 \leq i \leq k$.

Definition 1 *Let $b = (b_1, \ldots, b_k) \in P_k$. A nonnegative integral triangle $A = (0; a_{ij}) \in T_k$, of weight b, is an excavation of b if*

$$\begin{array}{l}
b_i - a_{ki} \geq b_{i+1}, \; 1 \leq i < k \\
b_i - \Sigma_{j=k}^{s} a_{ji} \geq b_{i+1} - \Sigma_{j=k}^{s+1} a_{ji+1}, \; 1 \leq i \leq s < k.
\end{array}$$

An excavation of $b=(b_1,b_2,b_3,b_4)$ may be seen as a decomposition of b. Put $b:=b^{(4)}$ and split b into a sequence of partitions

$$\begin{array}{l} b^{(4)}:=(b_1,b_2,b_3,b_4=a_{44}) \\ b^{(3)}=(b_1-a_{41},b_2-a_{42},b_3-a_{43}=a_{33}) \\ b^{(2)}=(b_1^{(3)}-a_{31},b_2^{(3)}-a_{32}=a_{22}) \\ b^{(1)}=(b_1^{(2)}-a_{21}=a_{11}), \end{array}$$

as shown in the following sequence of arrays

$$\begin{array}{ccccccccc} &&&&0&&&& \\ &&&0&&b_1^{(4)}&&& \\ &&0&&&&b_2^{(4)}&& \\ &0&&&&&&b_3^{(4)}& \\ 0&&&&&&&&b_4^{(4)} \end{array} \longrightarrow \begin{array}{ccccccccc} &&&&0&&&& \\ &&&0&&b_1^{(3)}&&& \\ &&0&&&&b_2^{(3)}&& \\ &0&&&&&&b_3^{(3)}& \\ 0&&a_{41}&&a_{42}&&a_{43}&&a_{44} \end{array} \longrightarrow$$

$$\begin{array}{ccccccccc} &&&&0&&&& \\ &&&0&&b_1^{(2)}&&& \\ &&0&&&&b_2^{(2)}&& \\ &0&&a_{31}&&a_{32}&&a_{33}& \\ 0&&a_{41}&&a_{42}&&a_{43}&&a_{44} \end{array} \longrightarrow \begin{array}{ccccccccc} &&&&0&&&& \\ &&&0&&a_{11}&&& \\ &&0&&a_{21}&&a_{22}&& \\ &0&&a_{31}&&a_{32}&&a_{33}& \\ 0&&a_{41}&&a_{42}&&a_{43}&&a_{44}. \end{array} \qquad (6)$$

That is,

$$\begin{array}{l} b_{i+1}^{(4)}\le b_i^{(3)}\le b_i^{(4)},\ \ 1\le i\le 3, \\ b_{i+1}^{(3)}\le b_i^{(2)}\le b_i^{(3)},\ \ 1\le i\le 2, \\ b_2^{(2)}\le b_1^{(1)}\le b_1^{(2)}, \end{array} \qquad \text{schematically} \qquad \begin{array}{ccccccc} &&&b_1^{(1)}&&& \\ &&b_1^{(2)}&&b_2^{(2)}&& \\ &b_1^{(3)}&&b_2^{(3)}&&b_3^{(3)}& \\ b_1^{(4)}&&b_2^{(4)}&&b_3^{(4)}&&b_4^{(4)}, \end{array}$$

or equivalently

$$\begin{array}{l} a_{11}+a_{21}+a_{31}\ge a_{22}+a_{32}+a_{42}\ge 0 \\ a_{11}+a_{21}\ge a_{22}+a_{32}\ge a_{33}+a_{43}\ge 0 \\ a_{11}\ge a_{22}\ge a_{33}\ge a_{44}\ge 0. \end{array}$$

Proposition 2 *Let $b=(b_1,\ldots,b_k)\in P_k$. Let $A=(0;a_{ij})_{1\le i\le j\le k}$ be a nonnegative integral triangle in T_k of boundary 0 and weight b. The following conditions are equivalent:*
(a) A is an excavation of b.
(b) A satisfy the interlacing inequalities (I), that is,

$$\Sigma_{q=j}^{i}a_{qj}\ge\Sigma_{q=j+1}^{i+1}a_{q,j+1},\ 1\le j\le i\le k.$$

(c) b has a decomposition into a sequence of interlacing partitions $b^{(j)}\in P_j$, $1\le j\le k$, with $b^{(k)}=b$,

$$b_{i+1}^{(j+1)}\le b_i^{(j)}\le b_i^{(j+1)},\ \ 1\le i\le j\le k-1.$$

Proof: A is an excavation of b iff b has a decomposition into a sequence of interlacing partitions $b^{(j)} \in P_j$, $1 \leq j \leq k$, with $b^{(k)} = b$, $b_{i+1}^{(j+1)} \leq b_i^{(j)} \leq b_i^{(j+1)}$, $1 \leq i \leq j \leq k-1$, where

$$b^{(j)} = (b_1^{(j+1)} - a_{j+1,1}, b_2^{(j+1)} - a_{j+1,2}, \ldots, a_{j+1,j+1}),\ 1 \leq j \leq k.$$

This is equivalent to

$$b_s^{(j)} - a_{js} = \Sigma_{q=s}^{j-1} a_{qs} \geq b_{s+1}^{(j+1)} - a_{j+1,s+1} = \Sigma_{q=s+1}^{j} a_{q,s+1},\ \ 1 \leq j \leq s \leq k.$$

■

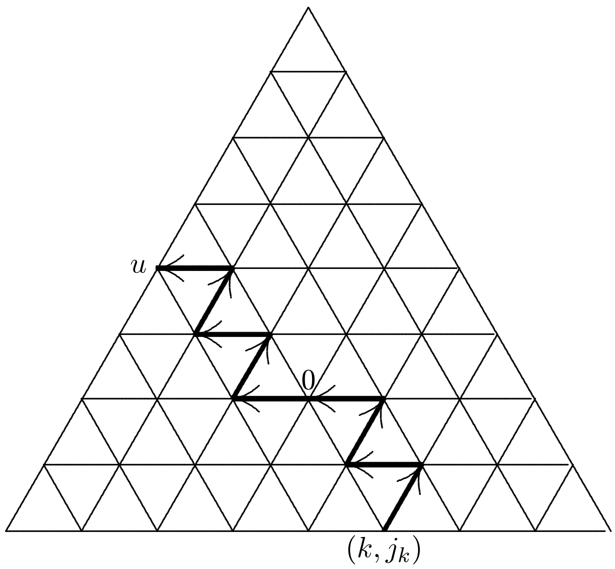
Given a partition b in P_k, this proposition shows that the triangles of size k obtained by excavation of b are exactly the triangles of weight b with 0-boundary satisfying inequalities (I); and may be identified with a decomposition of b into a sequence of interlacing partitions.

4 Deletion and insertion routes of LR triangles

Let $A \in LR_k$.

• A walk $\pi_\uparrow$ in the hive graph Δ_k

$$(k, j_k) \to (k-1, j_k) \to (k-1, j_{k-1}) \to (k-2, j_{k-1}) \to \ldots \to (u, j_{u+1}) \to (u, 0),$$



(7)

with $k > j_k > j_{k-1} > \ldots > j_{u+1} > j_u = 0$, such that

$$a_{sj_s} > 0,\ s = k, \ldots, u,$$
$$a_{sx} = 0, j_s < x < j_{s+1},\ s \in \{u, \ldots, k-1\},$$

is called an *deletion route* of A. The vertices (k, j_k) and $(u, 0)$ are called the *initial and final vertices of the route*, respectively. Clearly, and $k - 1 \geq u \geq k - j_k > 0$.

Do there exist always a deletion route of $A \in LR_k$ for each bottom vertex (k, j) in Δ_k, $1 \le j \le k-1$, with non zero label?

There is one and only one deletion route of A with initial vertex (k, j). The uniqueness is clear from the definition of deletion route. To prove the existence suppose that for some bottom vertex (k, j_k) in Δ_k, with $1 \le j_k \le k-1$, with non zero label, we had, at a certain point of the route, $a_{qj} = 0$, $0 \le j < j_{q+1}$, with $a_{k,j_k}, \ldots, a_{q+1,j_{q+1}} > 0$. Then, as $a_{q+1,j_{q+1}} > 0$, the trapezoid $\mathbf{p}_q$ wouldn't satisfy the (S) inequalities.

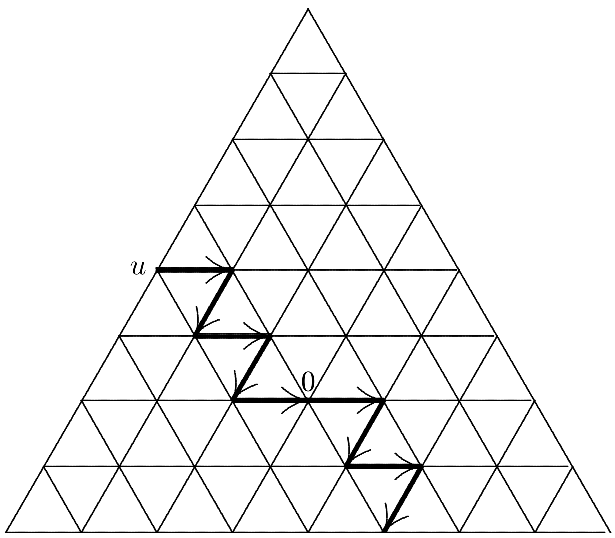
Given a bottom vertex (k, r), there exists a unique deletion route $\pi_\uparrow$ of $A \in LR_k$, thus we associate the deletion route triangle $\Pi_\uparrow = (p_{ij}) \in T_k$ such that

$$\begin{array}{l} p_{sj_{s+1}} = 1, \ s \in \{u, \ldots, k-1\} \\ p_{sj_s} = -1, \ s \in \{u, \ldots, k\} \\ 0, \ \text{otherwise.} \end{array}$$

The map $(A, \Pi_\uparrow) \to A + \Pi_\uparrow = (a_{ij} + p_{ij})$ defines a *deletion operation* on A.

• A walk $\pi_\downarrow$ in the hive graph Δ_k

$$(k, j_k) \leftarrow (k-1, j_k) \leftarrow (k-1, j_{k-1}) \leftarrow \ldots \leftarrow (u+1, j_{u+1}) \leftarrow (u, j_{u+1}) \leftarrow (u, 0),$$



(8)

with $k > j_k > j_{k-1} > \ldots > j_{u+1} > j_u = 0$, such that

$$\begin{array}{r} a_{u-1} \ge a_u + 1, \quad u > 1, \\ a_{sj_{s+1}} > 0, \ s = k-1, \ldots, u, \\ a_{sx} = 0, j_s < x < j_{s+1}, \ s \in \{u, \ldots, k-1\}, \end{array}$$

is called an *insertion route* of A. The vertices $(u, 0)$ and (k, j_k) are called the *initial and final vertices of the route*, respectively. Clearly, $k-1 \ge u \ge k - j_k > 0$.

Given $1 \le u \le k-1$, unlike the deletion route, we haven't always an insertion route with initial vertex $(u, 0)$ such that $a_{u-1} \ge a_u + 1, \ \ u > 1$. Clearly if there exists an insertion route of A with initial vertex $(u, 0)$, that route is unique and the final vertex is uniquely determined by u and A. For example, the LR_3 triangle

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 2 & & 1 & & \\ & 2 & & 0 & & 0 & \\ 0 & & 1 & & 1 & & 0 \end{array}$$

has no insertion route with initial vertex $(1, 0)$, but has a deletion route with initial vertex $(3, 2)$ and final vertex $(1, 0)$.

In fact, there exists an insertion route of $A \in LR_k$ with initial vertex $(u, 0)$ iff $a_{u-1} \geq a_u + 1$, $u > 1$, and there exists a sequence of vertices (s, j_{s+1}) in Δ_k, with $j_s < j_{s+1} \leq s$, such that $a_{s,j_{s+1}} > 0$, $s = u, \ldots, k-1$. Clearly, if $a_{k-1,k-1} > 0$ and $a_{u-1} \geq a_u + 1$, $u > 1$, there exists always an insertion route of $A \in LR_k$ with initial vertex $(u, 0)$.

Given the insertion route $\pi_\downarrow$, we associate the insertion route triangle $\Pi_\downarrow = (p_{ij}) \in T_k$ such that

$$\begin{array}{l} p_{sj_{s+1}} = -1, \ s \in \{u, \ldots, k-1\} \\ p_{sj_s} = 1, \ s \in \{u, \ldots, k\} \\ 0, \ \text{otherwise.} \end{array}$$

The map $(A, \Pi_\downarrow) \to A + \Pi_\downarrow = (a_{ij} + p_{ij})$ defines an *insertion operation* on A.

Note that the triangles $\Pi_\uparrow$ and $\Pi_\downarrow$ have weight 0 and, respectively, types $(-e_u, 0, -e_k)$ and $(e_u, 0, e_k)$ (e_i denotes, as usual, the elementary vector with 1 in entry i and zero elsewhere).

When we are referring indistinctly to a walk corresponding either to a deletion or insertion route, we drop the arrows.

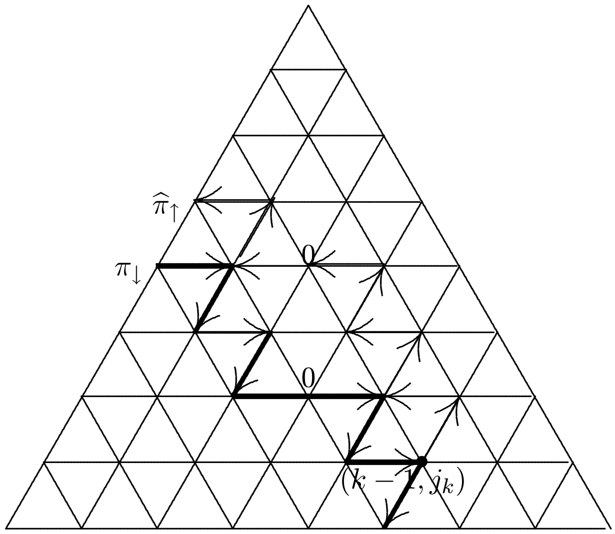
Definition 2 *Given two walks π and $\hat{\pi}$ in Δ_k with, respectively, bottom and top vertices (k, r), $(\hat{k}, \hat{r})$, and $(u, 0)$, $(\hat{u}, 0)$. We say that $\hat{\pi}$ is to the right of π if*

(i) either $k = \hat{k}$ and $\hat{r} > r$ or $k > \hat{k}$ and the vertices $(\hat{k}, j)$, $(\hat{k}, j + x)$, $x > 1$, of π satisfy $\hat{r} \geq j + x$;

(ii) (h, j), $(h, j + x)$, $x \geq 1$ are vertices of π, with $h > \hat{k}$, then $(h, \hat{j})$, $(h, \hat{j} + y)$, $y \geq 1$, are vertices of $\hat{\pi}$, such that $\hat{j} \geq j + x$.

By symmetry, we say that π is to the left of $\hat{\pi}$.

Clearly, if $\hat{\pi}$ is to the right of π then $\hat{u} < u$.



(9)

The walk $\hat{\pi}_{\downarrow}$, restricted to Δ_7, is to the right of $\hat{\pi}_{\uparrow}$.

Proposition 3 *Let $A \in LR_k$, with $k \geq 3$, and $\pi_{\downarrow}$ an insertion route of A with initial vertex $(u, 0)$ and final vertex (k, j_k), where $1 \leq j_k \leq k-2$. Then there exists a deletion route $\hat{\pi}_{\uparrow}$ in the restriction of A to Δ_{k-1}, with initial vertex $(k-1, j_k)$ and final vertex $(v, 0)$ such that $u > v \geq 1$.*

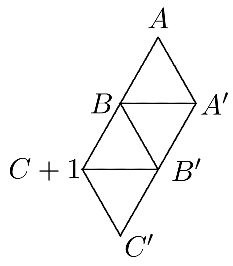
Proof: The vertices $(k-1, j_k)$, $(k-2, j_{k-1})$,..., $(u+1, j_{u+2})$, (u, j_{u+1}) of $\pi_{\downarrow}$ have positive labels. Therefore there exists a deletion route $\hat{\pi}_{\uparrow}$ with initial vertex $(k-1, j_k)$ to the right of $\pi_{\downarrow}$ and henceforth with final vertex at $(v, 0)$, with $u > v$.■

We have showed that for each insertion route of $A \in LR_k$, $k \geq 3$, with final vertex $1 \leq j_k \leq k-2$, there exists to its right a deletion route in the restriction of A to Δ_{k-1}.

By symmetry for each deletion route in the restriction of A to Δ_{k-1}, there exists to its left an insertion route of A, with final vertex $1 \leq j_k \leq k-2$.

Proposition 4 *Let $A \in LR_k$ of type (a, b, c). If $\pi_{\uparrow}$ is a deletion route of A with final vertex $(u, 0)$, and $\Pi_{\uparrow}$ its triangle, then $A + \Pi_{\uparrow} \in LR_k$ and is of type $(a - e_u, b, c - e_k)$.*

Proof: As the deletion route (7) traverses Δ_k, the labels of the parallelograms $\mathbf{p}$ of A might change according the following situations



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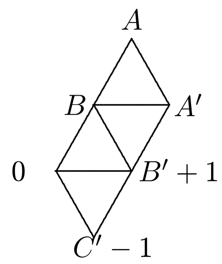

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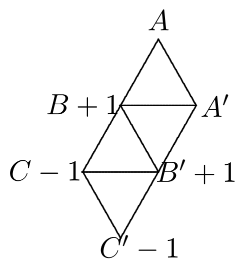

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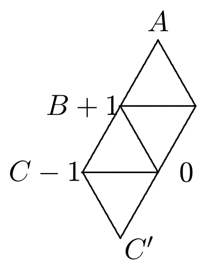
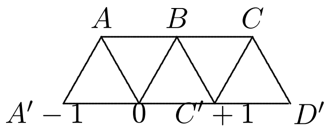


(13)

Clearly the labels of these parallelograms still satisfy inequalities (I), and therefore $A + \Pi_{\uparrow}$ satisfy inequalities (I).

Similarly, as the deletion route (7) traverses Δ_k, the labels of the trapezoids $\mathbf{t}$ of A might change according to the following situations



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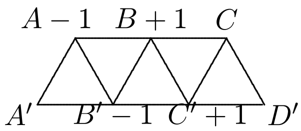

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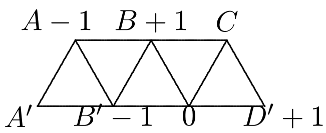

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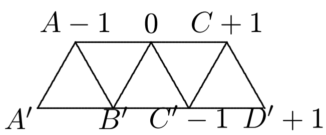


(17)

$$A \geq A' + B' + C', \;\; C' > 0 \;\Rightarrow A > A' + B' \;\Rightarrow A - 1 \geq A' + B'.$$

Clearly, the (S) inequalities are still satisfied by these trapezoids.■

This proposition shows that deletion operation preserve parallelogram and trapezoid inequalities.

As usual, let $e_j \in \mathbb{Z}^n$, $1 \leq j \leq n$, be the integral vector with j-entry equals 1 and 0 elsewhere.

Lemma 1 *Let $A \in LR_k$ of type (a, b, c). Let $0 < s \leq r < k$ such that $a_{ks}, a_{kr} > 0$ (if $s = r$ put $a_{kr} > 1$). Let $\Pi_{\uparrow}^{(r)}$ be the deletion route triangle of A with initial vertex (k, r), and $\Pi_{\uparrow}^{(s)}$ the deletion route triangle of $A + \Pi_{\uparrow}^{(r)}$ with initial vertex (k, s).*

Then $A+\Pi_{\uparrow}^{(r)} \in LR_k$ is of type $(x^{(1)}, b, c-e_k)$, $A+\Pi_{\uparrow}^{(r)}+\Pi_{\uparrow}^{(s)} \in LR_k$ is of type $(x^{(2)}, b, c-2e_k)$, where $x^{(i)}$ interlaces with a, for $1 \leq i \leq 2$.

Proof: If $(u, 0)$ is the final vertex of the deletion route of A with initial vertex (k, r), then, as $s \leq r$, if $(v, 0)$ is the final vertex of the deletion route of $A + \Pi_{\uparrow}^{(r)}$ with initial vertex (k, s), we have $v \geq u$. Therefore $x^{(1)} = a - e_u$ with $a_u - 1 \geq a_{u+1}$, and $x^{(2)} = a - e_u - e_v$, either with $a_u - 2 \geq a_{u+1}$, if $v = u$, or with $a_u - 1 \geq a_{u+1} \geq a_v \geq a_v - 1 \geq a_{v+1}$, otherwise. Therefore, $a - e_u$ and $a - e_u - e_v$ interlace with a. ■

Let $A \in LR_k$ of type (a, b, c). Let $\alpha_k := c_k - a_{k0} - a_{kk}$. Put $r_0 = 0$ and define recursively, for $s = 1, \ldots, \alpha_k$, $\Pi_{\uparrow}^{(r_{s-1})}$ the triangle of the deletion route of $A + \Sigma_{i=1}^{r_s} \Pi_{\uparrow}^{(r_i)} \in LR_k$ with initial vertex (k, r_s) such that a_{k,r_s} is the rightmost non zero label, with $r_s < k$, By induction on α_k, we conclude that the triangle $A + \Sigma_{i=1}^{r_s} \Pi_{\uparrow}^{(r_i)}$ in LR_k of type $(x^{(s)}, b, c - se_k)$ is such that $x^{(s)}$ interlaces with a, for $1 \leq s \leq \alpha_k$.

Let $A_{\uparrow}^{(k-1)} \in LR_{k-1}$ of type $(a^{(k-1)}; (b_1, \ldots, b_{k-1}); (c_1, \ldots, c_{k-1}))$ with $a^{(k-1)} := x^{(\alpha_k)}$, be the triangle obtained from $X^{(\alpha_k)}$ deleting the $(k-1)$-th row. Repeating the previous process with $A_{\uparrow}^{(k-1)}$ we obtain $A_{\uparrow}^{(k-2)} \in LR_{k-2}$ of type $(a^{(k-2)}; (b_1, \ldots, b_{k-2}); (c_1, \ldots, c_{k-2}))$, eventually we obtain $A_{\uparrow}^{(1)} \in LR_1$ of type $(a^{(0)}, b_1, c_1)$. Therefore, as shown in (6), the sequence $(a^{(s)})_{s=1}^{k}$ define a triangle $(0, Y) \in T_k$ by excavation of a.

Note that since the final vertex $(u, 0)$ of a deletion route of A with initial vertex (k, r) satisfies $k > u \geq k - r$, the triangles A and $(0, Y)$ are such that $Y = (y_{ij})_{1 \leq i \leq j \leq k}$ satisfy

$$(y_{s1}, \ldots, y_{s,s-1}) \preceq (a_{s,s-1}, \ldots, a_{s1}), \quad s = 2, \ldots, k.$$

($\preceq$ denotes majorization). Let $k \geq 2$, for $s = 2, \ldots, k$, we call $(y_{s1}, \ldots, y_{s,s-1})$ the *s-deletion sequence* of A.

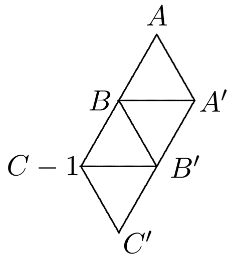
Next we shall prove that $T = (b, Y) \in LR_k$ of type (b, a, c).

Lemma 2 *Let $A \in LR_{k+1}$ of type (a, b, c) such that $a_{kr} > 0$, with $1 \leq r \leq k$. If $\pi_{\downarrow}$ is an insertion route of A with initial vertex $(u, 0)$ and final vertex $(k+1, r)$ such that $a_{u-1} > a_u + 1$, and $\Pi_{\downarrow}$ its triangle, then $A + \Pi_{\downarrow} \in LR_{k+1}$ and is of type $(a + e_u, b, c + e_k)$, if and only if*

$$\Sigma_{i=r}^{k} a_{ir} \geq \Sigma_{i=r+1}^{k+1} a_{ir+1} + 1.$$

In particular, if $r = k$, we have $a_{kk} \geq a_{k+1,k+1} + 1$, and if $a_{k+1,r+1} = 0$, we have always $\Sigma_{i=r}^{k} a_{ir} \geq \Sigma_{i=r+1}^{k} a_{ir+1} + 1$.

Proof: As the insertion route (8) traverses Δ_k, the labels of the parallelograms $\mathbf{t}$ of A might change according to the following situations



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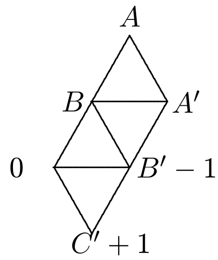

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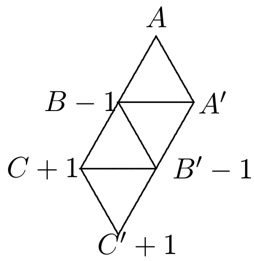

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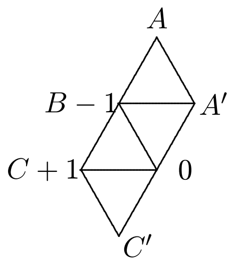

(21)

Situation (18) implies

$$\Sigma_{i=r}^{k} a_{ir} \geq \Sigma_{i=r+1}^{k+1} a_{ir+1} + 1.$$

But if $C' = 0$, then$A + B \geq A' + B'$ implies trivially $A + B + C \geq A' + B' + C' + 1$.

In the case of the degenerated parallelogram $[a_{kk}\ ;\ a_{k+1,k+1}]$ we have $a_{kk} \geq a_{k+1,k+1} + 1$.

The trapezoid inequalities can be easily checked, reversing the signs $+1$ and -1 in situations (14, 15, 16, 17).■

This lemma shows that trapezoid inequalities are always preserved by insertion operations, but the preservation of parallelogram inequalities requires additional conditions.

Proposition 5 *Let $A \in LR_{k+1}$ such that $a_{k+1,j} = 0$, $1 \leq j \leq k$, and $(y_{k1}, \ldots, y_{kk-1})$ its k-deletion sequence. Let $1 \leq u \leq k$ and $m > 0$. Then, for $i = 1, \ldots, m$, $\pi_{\downarrow}^{(u;r_i)}$ is an insertion route of $A + \Sigma_{s=1}^{i-1}\Pi_{\downarrow}^{(u;r_s)} \in LR_{k+1}$, with initial vertex $(u, 0)$, and final vertex $(k+1, r_i)$, such that $A + \Sigma_{i=1}^{m}\Pi_{\downarrow}^{(u;r_i)} \in LR_{k+1}$ iff*

$$a_{kk} + \Sigma_{j=1}^{u-1} y_{kj} \geq a_{k+1,k+1} + m. \tag{22}$$

In particular, if $u = 1$, $a_{k,k} \geq a_{k+1,k+1} + m$. We call m an u-insertion number of $A \in LR_{k+1}$.

Proof: By induction on m. If $m = 1$, by lemma 1, with $a_{k+1,j} = 0$, $1 \leq j \leq k$, we have two cases:

Case 1: $r_1 = k$. Then $A + \Pi_{\downarrow}^{(u;r_1)} \in LR_{k+1}$ iff $a_{kk} \geq a_{k+1,k+1} + 1$. From proposition 3, $\Sigma_{j=1}^{u-1} y_{kj} = 0$, we have $a_{kk} \geq a_{k+1,k+1} + 1$ equivalent to $\Sigma_{j=1}^{u-1} y_{kj} + a_{kk} \geq a_{k+1,k+1} + 1$.

Case 2: $k + 1 - u \leq r_1 < k$. We have always $A + \Pi_{\downarrow}^{(u;r_1)} \in LR_{k+1}$. By proposition 3, this situation is equivalent to $\Sigma_{j=1}^{u-1} y_{kj} > 0$. As $a_{kk} \geq a_{k+1,k+1}$, this situation is also equivalent to $\Sigma_{j=1}^{u-1} y_{kj} + a_{kk} \geq a_{k+1,k+1} + 1$.

Let $m > 1$ and $A_{\downarrow}^{(s)} := A + \Sigma_{i=1}^{s}\Pi_{\downarrow}^{(u;r_i)}$, $s = 1, \ldots, m-1$. Suppose that $A_{\downarrow}^{(s)} \in LR_{k+1}$, $s = 1, \ldots, m-1$. As $\pi_{\downarrow}^{(u;r_m)}$ is an insertion route of $A_{\downarrow}^{(m-1)}$, we consider again two cases:

Case 1: $r_{m-1} \leq r_m < k$. We have $a_{k+1,j}^{(m-1)} = a_{k+1,j} = 0$, $r_{m-1} < j \leq k$, then, by lemma 1, we have always $A^{(m-1)} + \Pi_{\downarrow}^{(u;r_m)} = A + \Sigma_{i=1}^{m}\Pi_{\downarrow}^{(u;r_i)} \in LR_{k+1}$.

On the other hand, by proposition 3, $\Sigma_{j=1}^{u-1} y_{kj} \geq m$. As $a_{kk} \geq a_{k+1,k+1}$, $\Sigma_{j=1}^{u-1} y_{kj} \geq m$ is equivalent to $\Sigma_{j=1}^{u-1} y_{kj} + a_{kk} \geq a_{k+1,k+1} + m$.

Case 2: $r_m = k$. By lemma 1, $A_{\downarrow}^{(m-1)} + \Pi_{\downarrow}^{(u;r_m)} = A + \Sigma_{i=1}^{m}\Pi_{\downarrow}^{(u;r_i)} \in LR_{k+1}$ iff $a_{kk}^{(m-1)} \geq a_{k+1,k+1} + 1$. On the other hand, by proposition 3, $\Sigma_{j=1}^{u-1} y_{kj} + a_{kk} - m + 1 \geq a_{kk}^{(m-1)}$. Therefore, $\Sigma_{j=1}^{u-1} y_{kj} + a_{kk} - m + 1 \geq a_{k+1,k+1} + 1$. ■

We say that $(m_1, \ldots, m_k)$ is an *insertion sequence* of A if m_s is an s-insertion number of $A_{\downarrow}^{(m_1+\ldots+m_{s-1})}$, for $s = 1, \ldots, k$.

Example 1 In the triangle below, the 3-deletion sequence is $(y_{31}, y_{32}) = (1, 1)$. According to (22), 1 is an 2-insertion number, but 2 is not, since

$$a_{33} + y_{31} = 1 + 1 = a_{44} + 1 = 1 + 1,$$

and

$$a_{33} + y_{31} = 1 + 1 < a_{44} + 2 = 1 + 2.$$

$$
[1 \rightarrow] \quad 1 \rightarrow \quad
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 4 & & 1 & & & \\
 & & 2 & & 1 & & 1 & & \\
 & 0 & & 1 & & 1 & & 1 & \\
0 & & 0 & & 0 & & 0 & & 1
\end{array}
\quad \longrightarrow
$$

$$
\longrightarrow \quad [1 \rightarrow] \quad
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 4 & & 1 & & & \\
 & & 2+1 & & 1-1 & & 1 & & \\
 & 0 & & 1+1 & & 1-1 & & 1 & \\
0 & & 0 & & 0+1 & & 0 & & 1
\end{array}
\quad \in LR_4
$$

But

$$
1 \rightarrow \quad
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 4 & & 1 & & & \\
 & & 3 & & 0 & & 1 & & \\
 & 0 & & 2 & & 0 & & 1 & \\
0 & & 0 & & 1 & & 0 & & 1
\end{array}
\quad \longrightarrow
$$

$$
\longrightarrow \quad
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 4 & & 1 & & & \\
 & & 3+1 & & 0 & & 1-1 & & \\
 & 0 & & 2 & & 0+1 & & 1-1 & \\
0 & & 0 & & 0 & & 1+0 & & 1
\end{array}
$$

$$
= \quad
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 4 & & 1 & & & \\
 & & 4 & & 0 & & 0 & & \\
 & 0 & & 2 & & 1 & & 0 & \\
0 & & 0 & & 0 & & 1 & & 1
\end{array}
\quad \notin LR_4.
$$

Proposition 6 *Let $A \in LR_{k+1}$ such that $a_{k+1,j} = 0$, $1 \leq j \leq k$, and $(y_{k1}, \ldots, y_{kk-1})$ its k-deletion sequence . Then $(0, \ldots, 0, m_v, 0, \ldots, 0, m_u, 0, \ldots, 0)$, $u > v$ is an insertion sequence of A iff*

$$
\begin{array}{l}
a_{kk} + \Sigma_{j=1}^{v-1} y_{kj} \geq a_{k+1,k+1} + m_v, \\
a_{kk} + \Sigma_{j=1}^{u-1} y_{kj} \geq a_{k+1,k+1} + m_u + m_v.
\end{array}
$$

■

By an inductive argument, we conclude that $(m_1, \ldots, m_k)$ is an insertion sequence of A iff

$$\begin{aligned} a_{kk} &\geq a_{k+1,k+1} + m_1, \\ a_{kk} + y_{k1} &\geq a_{k+1,k+1} + m_1 + m_2 \\ &\ldots \\ a_{kk} + \Sigma_{j=1}^{k-1} y_{kj} &\geq a_{k+1,k+1} + m_1 + \ldots + m_k. \end{aligned} \tag{23}$$

Example 2 In the last example, we have seen that $(0, 1, 0)$ is an insertion sequence of

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 4 & & 1 & & & \\ & & 2 & & 1 & & 1 & & \\ & 0 & & 1 & & 1 & & 1 & \\ 0 & & 0 & & 0 & & 0 & & 1 \end{array} \quad \in LR_4.$$

Notice

$$a_{33} + y_{31} + y_{32} = 1 + 1 + 1 \geq a_{44} + 1 + 1 = 1 + 1 + 1.$$

So $(0, 1, 1)$ is an insertion sequence as well. On the other hand,considering the triangle

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 4 & & 1 & & & \\ & & 2 & & 1 & & 1 & & \\ & 0 & & 0 & & 1 & & 1 & \\ 0 & & 0 & & 0 & & 0 & & 1 \end{array} \quad \in LR_4.$$

$(0, 1, 0)$ is an insertion sequence, but $(0, 1, 1)$ is not. Notice that the 3-deletion sequence is $(1, 0)$ and thus

$$a_{33} + y_{31} + y_{32} = 1 + 0 + 1 < a_{44} + 1 + 1 = 1 + 1 + 1.$$

Example 3 The 3-deletion sequence of the triangle below is $(2, 1)$. The sequences $(1, 1, 0)$ and $(1, 0, 0)$ are not insertion sequences, although we have $a_{33} + y_{31} = 1 + 2 \geq a_{44} + 1 + 1 = 1 + 2$. That is, conditions (23) must be fulfilled

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 4 & & 1 & & & \\ & & 2 & & 1 & & 1 & & \\ & 0 & & 1 & & 2 & & 1 & \\ 0 & & 0 & & 0 & & 0 & & 1 \end{array} \quad \in LR_4 \quad .$$

Theorem 1 *(Symmetry by boundary excavation) Let $A \in LR_k$ of type (a, b, c) and $B = (b, Y)$ the triangle of type (b, a, c) obtained by excavation of the boundary of A. Then $B \in LR_k$.*

Proof: It remains to prove that $B = (b, Y)$ satisfy the trapezoid inequalities. The LR triangle A may be obtained by insertion of Y into A. Put $A(\Delta_0) := 0$. Using proposition 6, let, for $r = 1, \ldots, k$, $A(\Delta_r)$ be obtained from $A(\Delta_{r-1})$ by insertion of $(y_{r1}, \ldots, y_{r,r-1})$. Clearly, $A(\Delta_k) = A$. On the other hand, the $r-1$-deletion sequence of $A(\Delta_{r-1})$ is $(y_{r-1,1}, \ldots, y_{r-1,r-2})$. Thus, by proposition 6, the triangle $B = (b, Y)$ satisfy the trapezoid inequalities as well. Hence $B \in LR_k$.■

We write $excav(A) = B$.

Theorem 2 *(Symmetry by boundary insertion) Let $A \in LR_k$ of type (a, b, c) and $\Gamma = (b, Z)$ the triangle of type (b, a, c) obtained by insertion of A. Then $\Gamma \in LR_k$.*

Proof: The triangle A satisfy the trapezoid inequalities. Start with the triangle $\Gamma_1 = (a_{11}; a_1; c_1) \in LR_1$, using proposition 6, we get $\Gamma_2 = (a_{11} + a_{21}; (a_1, a_2); (c_1, c_2)) \in LR_2$ by insertion of a_{21} in Γ_1. By an inductive argument we get Γ_r from Γ_{r-1} in LR_{r-1} by insertion of $(a_{r1}, \ldots, a_{r,r-1})$, $r > 2$. Notice that the deletion sequence of Γ_{r-1} is $(a_{r-1,1}, \ldots, a_{r-1,r-2})$ as the insertion operation can be reversed.■

We write $insert(A) = \Gamma$.

Example 4 Symmetry by excavation of the boundary

$$\begin{array}{l}
1 \quad 2 \quad 4 \quad 5 \quad 0 \\
\qquad\qquad\qquad 5 \quad 3 \\
\qquad\qquad 3 \quad 1 \quad 2 \\
\qquad 2 \quad 0 \quad 1 \quad 2 \\
0 \quad 1 \quad 1 \quad 0 \quad 1
\end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{l}
1 \quad 2 \quad 4 \quad 5 \quad 0 \\
0 \quad 5 \quad 3 \\
0 \quad 3 \quad 1 \quad 2 \\
1 \quad 2-1 \quad 0 \quad 1+1 \quad 2 \\
0 \quad 1 \quad 1-1 \quad 0 \quad 1
\end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{l}
1 \quad 2 \quad 4 \quad 5 \quad 0 \\
0 \quad 5 \quad 3 \\
0 \quad 3 \quad 1 \quad 2 \\
2 \quad 1-1 \quad 0+1 \quad 2 \quad 2 \\
0 \quad 1-1 \quad 0 \quad 0 \quad 1
\end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{l}
1 \quad 2 \quad 4 \quad 5 \quad 0 \\
0 \quad 1 \quad 5-1 \quad 3+1 \\
0 \quad 3 \quad 1-1 \quad 2+1 \\
2 \quad 0 \quad 1 \quad 2-1 \quad 2 \\
0 \quad 1
\end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{l}
1 \quad 2 \quad 4 \quad 5 \quad 0 \\
0 \quad 1 \quad 4 \quad 4 \\
0 \quad 1 \quad 3-1 \quad 0 \quad 3+1 \\
2 \quad 0 \quad 1 \quad 1-1 \quad 2 \\
0 \quad 1
\end{array} \longrightarrow$$

$$\longrightarrow \quad \begin{array}{ccccccccccccc}
1 & & 2 & & 4 & & 5 & & 0 \\
& 0 & & 1 & & & & 4 & & 4 \\
& & 0 & & 1+1 & & 2-1 & & 0+1 & & 4 \\
& & & 2 & & 0 & & 1-1 & & 0 & & 2 \\
& & & & 0 & & & & & & & & 1
\end{array} \quad \longrightarrow$$

$$\longrightarrow \quad \begin{array}{ccccccccccccc}
1 & & 2 & & 4 & & 5 & & 0 \\
& 0 & & 1 & & 1 & & 4-1 & & 4+1 \\
& & 0 & & 2 & & 1 & & 1-1 & & 4 \\
& & & 2 & & 0 & & & & & & 2 \\
& & & & 0 & & & & & & & & 1
\end{array} \quad \longrightarrow$$

$$\longrightarrow \quad \begin{array}{ccccccccccccc}
1 & & 2 & & 4 & & 5 & & 0 \\
& 0 & & 1 & & 1 & & 3 & & 5 \\
& & 0 & & 2 & & 1 & & & & 4 \\
& & & 2 & & 0 & & & & & & 2 \\
& & & & 0 & & & & & & & & 1
\end{array} \tag{24}$$

Example 5 Symmetry by boundary insertion

$$\begin{array}{ccccccccccccc}
& & & & 0 & & 5 & & 3 & & 2 & & 0 \\
& & & 5 & & 3 \\
& & 3 & & \underline{1} & & 2 \\
& 2 & & 0 & & 1 & & 2 \\
0 & & 1 & & 1 & & 0 & & 1
\end{array} \quad \longrightarrow$$

$$\begin{array}{ccccccccccccc}
& & & & 0 & & 5-1 & & 3 & & 2 & & 0 \\
& & & 5 & & 3+1 & & 1 \\
& & 3 & & & & 2 \\
& 2 & & 0 & & \underline{1} & & 2 \\
0 & & 1 & & 1 & & 0 & & 1
\end{array} \quad \longrightarrow$$

$$\begin{array}{ccccccccccccc}
& & & & 0 & & 4 & & 3 & & 2 & & 0 \\
& & & 5 & & 4 & & 1-1 & & 0 \\
& & 3 & & & & 2+1 & & 1 \\
& 2 & & & & & & 2 \\
0 & & 1 & & \underline{1} & & 0 & & 1
\end{array} \quad \longrightarrow$$

$$
\begin{array}{ccccccccccccc}
 & & & & 0 & & 4 & & 3-1 & & 2-1 & & 0 \\
 & & & 5 & & 4 & & 0 & & 1 & & 1 & \\
 & & 3 & & & & 3+1 & & 1 & & & & \\
 & 2 & & & & & & 2 & & & & & \\
0 & & \underline{1} & & & & & & 1 & & & &
\end{array}
\quad \longrightarrow
$$

$$
\begin{array}{ccccccccccccc}
 & & & & 0 & & 4-1 & & 2-1 & & 1-1 & & 0 \\
 & & & 5 & & 5 & & 0+1 & & 1+1 & & 1+1 & \\
 & & 3 & & & & 4 & & 1 & & 0 & & \\
 & 2 & & & & & & 2 & & 0 & & & \\
0 & & & & & & & & 1 & & & &
\end{array}
\quad =
$$

$$
\begin{array}{ccccccccccccc}
 & & & & 0 & & 3 & & 1 & & 0 & & 0 \\
 & & & 5 & & 5 & & 1 & & 2 & & 2 & \\
 & & 3 & & & & 4 & & 1 & & 0 & & \\
 & 2 & & & & & & 2 & & 0 & & & \\
0 & & & & & & & & 1 & & & &
\end{array}
\tag{25}
$$

Notice that triangles (24) and (25) are the same.

5 Deletion and insertion are identical bijections

Clearly, deletion and insertion operations are the backwards of each other. That is, $insert(excav(A)) = A = excav(insert(A))$. But we have even more,

Theorem 3 *Let $A \in LR_k$, then*

$$excav(A) = insert(A). \tag{26}$$

This equality follows by induction, on the size of the triangle, and from the following interesting property

Theorem 4 *Let $A \in LR_{k+1}$ of type (a, b, c) with bottom row $(a_{k+1,0}, e_r, a_{k+1,k+1})$. Let $\Pi_{\uparrow}^{(r)}$ be the deletion triangle of A with respect to vertex $(k+1, r)$. Then*

$$insert\left[(A + \Pi_{\uparrow}^{(r)})_{|\Delta_k}\right] = [insert(A)]_{|\,\Delta_k}.$$

Proof: By induction on k. Easy for $k = 2, 3, 4$.■

Corollary 1 *Let $A \in LR_{k+1}$ of type (a, b, c). Let $\Pi_{\uparrow} = \sum_{r=1}^{k} \Pi_{\uparrow}^{(r)}$ Then*

$$insert\left[(A + \Pi_{\uparrow})_{|\Delta_k}\right] = [insert(A)]_{|\,\Delta_k}.$$

In plain language, these theorem and corollary say that a deletion operation on A, with initial vertex $(k+1, r)$, means an r-insertion of the label $a_{k+1,r}$ on $insert(A_{|\Delta_k})$.

Proof of Theorem 3. For $k = 2, 3$, it is easy to check. By definition of excavation, we have

$$[excav(A)]_{|\Delta_k} = excav\left[(A + \Pi_{\uparrow})_{|\Delta_k}\right]. \tag{27}$$

By induction on k and previous theorem, we get

$$[excav(A)]_{|\Delta_k} = excav\left[(A + \Pi_{\uparrow})_{|\Delta_k}\right] = insert\left[(A + \Pi_{\uparrow})_{|\Delta_k}\right] = [insert(A)]_{|\Delta_k}. \blacksquare \tag{28}$$

Example 6

$$A \quad = \quad \begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 5 & & 3 & & & & \\
 & & & 3 & & 1 & & 2 & & & \\
 & & 2 & & 0 & & 1 & & 2 & & \\
 & 0 & & 1 & & 1 & & 0 & & 1 & \\
0 & & 0 & & 0 & & (\mathbf{1}) & & 0 & & 0
\end{array} \qquad \longrightarrow$$

$$[A + \Pi^{(3)} \uparrow]_{|\Delta_4} \quad = \quad \begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 5 & & 3 & & & \\
 & & 3 & & 1 & & 2 & & \\
 & 2-(\mathbf{1}) & & 0 & & 1+(\mathbf{1}) & & 2 & \\
0 & & 1 & & 1-(\mathbf{1}) & & 0+(\mathbf{1}) & & 1
\end{array} \qquad \longrightarrow$$

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 5 & & 3 & 2-(\mathbf{1}) & 0 \\
 & & & 5 & & 3 & & & & & \\
 & & 3 & & 1 & & 2 & & & & \\
 & 2-(\mathbf{1}) & & 0 & & 1+(\mathbf{1}) & & 2 & & & \\
0 & & 1 & & 1-(\mathbf{1}) & & 0+(\mathbf{1}) & & 1 & &
\end{array} \qquad \longrightarrow$$

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 5-1 & & 3 & 2-(\mathbf{1}) & 0 \\
 & & & 5 & & 3+1 & & 1 & & & \\
 & & 3 & & & & 2 & & & & \\
 & 2-(\mathbf{1}) & & 0 & & 1+(\mathbf{1}) & & 2 & & & \\
0 & & 1 & & 1-(\mathbf{1}) & & 0+(\mathbf{1}) & & 1 & &
\end{array} \qquad \longrightarrow$$

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 5-1 & & 3 & 2-(\mathbf{1}) & 0 \\
 & & & 5 & & 4 & & 1-1 & & & \\
 & & 3 & & & & 2+1 & & 1 & & \\
 & 2-(\mathbf{1}) & & 0 & & (\mathbf{1}) & & 2 & & & \\
0 & & 1 & & 1-(\mathbf{1}) & & 0+(\mathbf{1}) & & 1 & &
\end{array} \qquad \longrightarrow$$

$$
\begin{array}{cccccccccccccc}
 & & & & 0 & & 4 & & 3-(1) & & 2-(\mathbf{1}) & & 0 & \\
 & & & 5 & & 4 & & 0 & & (1) & & 0 & & \\
 & & 3 & & & & 2+1+(1) & & 1 & & 0 & & & \longrightarrow \\
 & 2-(\mathbf{1}) & & & & & & 2 & & 0 & & & & \\
0 & & 1 & & 1-(\mathbf{1}) & & 0+(\mathbf{1}) & & 1 & & & & &
\end{array}
$$

Denote by $\hat{Q}$ the right triangle.

$$
\begin{array}{cccccccccccccc}
 & & & & 0 & & 4 & & 2 & & 1 & & 0 & \\
 & & & 5 & & 4 & & 0 & & 1 & & 0 & & \\
 & & 3 & & & & 4 & & 1-(1) & & & & & \longrightarrow \\
 & 2-(\mathbf{1}) & & & & & & 2+(1) & & (1) & & & & \\
0 & & 1 & & & & & & 1 & & & & &
\end{array}
$$

Denote by $\hat{R}$ the right triangle.

$$
\begin{array}{cccccccccccccc}
 & & & & 0 & & 4-1 & & 2-1 & & 1-1 & & 0 & \\
 & & & 5 & & 4+1 & & 0+1 & & 1+1 & & 1 & & \\
 & & 3 & & & & 4 & & 0 & & 0 & & & = \\
 & 2-\mathbf{1} & & & & & & 3 & & 1 & & & & \\
0 & & & & & & & & 1 & & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & 0 & & 3 & & 1 & & 0 & & 0 \\
 & & & 5 & & 5 & & 1 & & 2 & & 1 & \\
 & & 3 & & & & 4 & & 0 & & 0 & & \\
 & 2-\mathbf{1} & & & & & & 3 & & 1 & & & \\
0 & & & & & & & & 1 & & & &
\end{array}
$$

Denote the triangle on the right by $\hat{T}$.

On the other hand

$$
\begin{array}{ccccccccccccccc}
 & & & & & 0 & & 4 & & 3 & & 2 & 0 & 0 & \\
 & & & & 5 & & 4 & & 0 & & 0 & & & & \\
 & & & 3 & & & & 3 & & 1 & & & & & \\
 & & 2 & & & & & & 2 & & & & & & \longrightarrow \\
 & 0 & & 1 & & 1 & & 0 & & 1 & & & & & \\
0 & & 0 & & 0 & & (1) & & 0 & & 0 & & & &
\end{array}
$$

Denote by Q the right triangle, and notice that

$$
\hat{Q} = \quad
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 4 & & 4 & & \\
 & 3+(1) & & 0 & & 3-(1) & \\
2 & & 1 & & (1) & & 2-(1)
\end{array}
$$

is obtained from

$$Q = \qquad \begin{array}{ccccccc} & & & 0 & & & \\ & & 4 & & 4 & & \\ & 3 & & 0 & & 3 & \\ 2 & & 1 & & 0 & & 2 \end{array}$$

by 2-insertion of 1 and then restricted to Δ_3.

$$\begin{array}{ccccccccccccccc} & & & & & 0 & & 4 & & 2 & & 1 & & 0 & 0 \\ & & & & 5 & & 4 & & 0 & & 1 & & 1 & & \\ & & & 3 & & & & 4 & & 1 & & 0 & & & \\ & & 2 & & & & & & 2 & & 0 & & & & \\ & 0 & & 1 & & 0 & & 0 & & 1 & & & & & \\ 0 & & 0 & & 0 & & (1) & & 0 & & 0 & & & & \end{array} \qquad \longrightarrow$$

Denote the right triangle by Q'.
Notice that $\hat{Q} = Q'_{|\Delta_3}$, and

$$R' = \qquad \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 4 & & 4 & & & \\ & & 4 & & 0 & & 2 & & \\ & 3 & & 0 & & 1 & & 1 & \\ 1 & 1 & & 0 & & 0 & & 0 & 0 \end{array}$$

is obtained from Q' by 3-insertion of 1 and then restricted to Δ_4.

$$\begin{array}{ccccccccccccccc} & & & & & 0 & & 3 & & 1 & & 0 & & 0 & 0 \\ & & & & 5 & & 5 & & 1 & & 2 & & 2 & & \\ & & & 3 & & & & 4 & & 1 & & 0 & & & \\ & & 2 & & & & & & 2 & & 0 & & & & \\ & 0 & & & & & & & & 1 & & & & & \\ 0 & & 0 & & 0 & & (1) & & 0 & & 0 & & & & \end{array} \qquad \longrightarrow$$

$$\begin{array}{cccccccccccccccc} & & & & & 0 & & 3 & & 1 & & 0 & & 0 & & 0 \\ & & & & 5 & & 5 & & 1 & & 2 & & 2-(1) & & 0 & \\ & & & 3 & & & & 4 & & 1-(1) & & 0 & & (1) & & \\ & & 2 & & & & & & 2+(1) & & 0+(1) & & 0 & & & \\ & 0 & & & & & & & & 1 & & 0 & & & & \\ 0 & & & & & & & & & & 0 & & & & & \end{array} \qquad =$$

$$
\begin{array}{cccccc}
0 & 3 & 1 & 0 & 0 & 0 \\
5 & 5 & 1 & 2 & 1 & 0 \\
3 & & 4 & 0 & 0 & 1 \\
2 & & & 3 & 1 & 0 \\
0 & & & & 1 & 0 \\
0 & & & & & 0
\end{array} \quad .
$$

The triangle T' on the right is such that $T'_{|\Delta_4}$ is the triangle $\hat{T}$ above.

Example 7

$$
A \quad = \quad
\begin{array}{c}
0 \\
5 \quad 4 \\
4 \quad 1 \quad 4 \\
4 \quad 0 \quad 0 \quad 2 \\
3 \quad 1 \quad 0 \quad 0 \quad 1 \\
0 \quad 0 \quad (1) \quad \bar{1} \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad [\mathbf{1}] \quad 0 \quad 0 \quad 0
\end{array}
\qquad \longrightarrow
$$

$$
[A+\Pi_{\uparrow}^{(3)}]_{|\Delta_5} \quad = \quad
\begin{array}{c}
0 \\
5 \quad 4 \\
4 \quad 1 \quad 4 \\
4-(\mathbf{1}) \quad 0+(\mathbf{1}) \quad 0 \quad 2 \\
3 \quad 0 \quad 0+(\mathbf{1}) \quad 0 \quad 1 \\
0 \quad 0 \quad 0 \quad \bar{1}+[\mathbf{1}] \quad 0 \quad 0
\end{array}
$$

By induction

$$
\begin{array}{c}
0 \quad 3 \quad 3 \quad 3 \quad 3 \quad 0 \\
5 \quad 6 \quad 2-(\mathbf{1}) \quad 1-(\mathbf{1}) \quad 1-(\mathbf{1}) \quad 0 \\
4 \quad 4+(\mathbf{1}) \quad 0+(\mathbf{1}) \quad 0+(\mathbf{1}) \quad 0 \\
4-(1) \quad 2 \quad 0 \quad 0 \\
3 \quad 1 \quad 0 \\
0 \quad 0 \quad 0 \quad \bar{\mathbf{1}}+[1] \quad 0 \quad 0
\end{array}
$$

$$
\begin{array}{c}
0 \quad 3 \quad 3 \quad 3 \quad 3 \quad 0 \\
5 \quad 6 \quad 2-(\mathbf{1}) \quad 1-(\mathbf{1}) \quad 1-(\mathbf{1}) \quad 0 \\
4-(1) \quad 4+(\mathbf{1}) \quad (\mathbf{1})-\bar{1} \quad (\mathbf{1})-\bar{1} \quad 0 \\
4 \quad 2+\bar{1} \quad \bar{1} \quad \bar{1} \\
3 \quad 1 \quad 0 \\
0 \quad 0 \quad 0 \quad [1] \quad 0 \quad 0
\end{array}
$$

$$
\begin{array}{c}
0 \quad 3 \quad 3 \quad 3-[1] \quad 3-[1] \quad 0 \\
5 \quad 6 \quad 2-(\mathbf{1}) \quad 1-(\mathbf{1}) \quad 1-(\mathbf{1})+[1] \quad [1] \\
4-(1) \quad 4+(\mathbf{1}) \quad 0+(\mathbf{1})-\bar{\mathbf{1}} \quad 0+(\mathbf{1})-\bar{1} \quad 0 \\
4 \quad 2+\bar{\mathbf{1}}+[1] \quad 0+\bar{\mathbf{1}} \quad \bar{\mathbf{1}} \\
3 \quad 1 \quad 0 \\
0 \quad 0
\end{array}
$$

On the other hand,

$$\begin{array}{ccccccc}
 & & & 0 \quad 3 \quad 3 \quad 3 \quad 3 \quad 0 \quad 0 \\
 & & 5 \quad 6 \quad 2 \quad 1 \quad 1 \\
 & 4 \quad 4 \quad 0 \quad 0 \\
 4 \quad 2 \quad 0 \\
 3 \quad 1 \\
 0 \quad 0 \quad (1) \quad \bar{1} \quad 0 \quad 0 \\
 0 \quad 0 \quad 0 \quad [1] \quad 0 \quad 0 \quad 0
\end{array}$$

$$\begin{array}{l}
0 \quad 3 \quad 3 \quad 3 \quad 3 \quad 0 \quad 0 \\
5 \quad 6 \quad 2 \quad 1-\bar{1} \quad 1-\bar{1} \quad 0 \quad 0 \\
4 \quad 4 \quad 0 \quad 0+\bar{1} \quad \bar{1} \quad 0 \\
4 \quad 2+\bar{1} \quad 0 \quad 0 \quad 0 \\
3 \quad 1 \quad 0 \quad 0 \\
0 \quad 0 \quad (1) \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad [1] \quad 0 \quad 0 \quad 0
\end{array}$$

$$\begin{array}{l}
0 \quad 3 \quad 3 \quad 3-(1) \quad 3-\bar{1}-(1) \quad 0 \quad 0 \\
5 \quad 6 \quad 2-(1) \quad 1-\bar{1} \quad 1-\bar{1}+(1) \quad (1) \quad 0 \\
4 \quad 4+(1) \quad (1) \quad \bar{1} \quad \bar{1} \quad 0 \\
4 \quad 2+\bar{1} \quad 0 \quad 0 \quad 0 \\
3 \quad 1 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad [1] \quad 0 \quad 0 \quad 0
\end{array}$$

$$\begin{array}{l}
0 \quad 3 \quad 3 \quad 3-(1) \quad 3-(1) \quad 0 \quad 0 \\
6 \quad 2-(1) \quad 1-\bar{1} \quad 1-\bar{1}+(1) \quad (1) \quad 0 \\
4+(1) \quad (1)-[1] \quad \bar{1}-[1] \quad \bar{1}-[1] \quad 0 \\
2+\bar{1}+[1] \quad [1] \quad [1] \quad [1] \\
1 \quad 0 \quad 0 \\
0 \quad 0 \\
0
\end{array}$$

References

[1] G. D. Appleby, A simple approach to matrix realizations for Littlewood-Richardson sequences, *Linear Algebra and its Applications*, 291:1-14, 1999.

[2] O. Azenhas and E. Marques de Sá, Matrix realizations of Littlewood-Richardson sequences, *Linear and Multilinear Algebra*, 27:229-242, 1990.

[3] O. Azenhas, Littlewood-Richardson fillings and their symmetries, *Matrices and Group Representations, Coimbra, 6-8 May 1998*, Textos de Matemática, Série B, Departamento de Matemática, Universidade de Coimbra, 1999.

[4] O. Azenhas, On an involution on the set of Littlewood-Richardson tableaux and the hidden commutativity, Pré-Publicações do Departamento de Matemática, Universidade de Coimbra, preprint number 00-27(2000), available from http://www.mat.uc.pt/preprints/2000.html

[5] V. I. Danilov, G. A. Koshevoy, Arrays and the octahedron recurrence, arXiv:math.CO/0504299, April 2005.

[6] S. Johnson, *The Schubert Calculus and Eigenvalue Inequalities for Sums of Hermitian Matrices*, Dissertation, Santa Barbara, 1979 (unpublished).

[7] A. N. Kirillov, A. D. Berenstein, Groups generated by involutions, Gelfand-Tsetlin patterns, and combinatorics of Young tableaux, *Algebra i Analiz* 7, 1995.

[8] T. Klein, The multiplication of Schur functions and extensions of p-modules, *Journal of the London Mathematical Society*, 43:280-284, 1968.

[9] I. Pak, Ernesto Vallejo, Combinatorics and geometry of Littlewood-Richardson cones, preprint (2003) 15 pp., to appear in Europ. J. Combinatorics, available from http://www-math.mit.edu/ pak/research.html

[10] I. Pak, E. Vallejo, Reductions of Young tableaux bijections, preprint (2004), 42 pp, available from http://www-math.mit.edu/ pak/research.html

[11] J. F. Queiró, E. M. de Sá, A. P. Santana. O. Azenhas, Interlacing of eigenvalues and invariant factors, *Mathematical Inequalities and Applications*, 2:149-154, 2000.

[12] E. Marques de Sá, Imbedding conditions for λ-matrices, *Linear Algebra and its Applications* 24, 33-50, 1979.

[13] R. C. Thompson, Interlacing inequalities for invariant factors, *Linear Algebra and its Applications* 24, 1-32, 1979.

[14] R. C. Thompson, An inequality for invariant factors, *Proceedings of the American Mathematical Society*, 86, 1982.

[15] R. C. Thompson, Smith invariants of a product of integral matrices, in Linear Algebra and its Role in Systems Theory, *Contemporary Mathematics*, 47:401-435, AMS, 1985.