Puzzles
Littlewood-Richardson coefficients
and
Horn inequalities

Olga Azenhas

CMUC, Centre for Mathematics, University of Coimbra

Seminar of the Mathematics PhD Program UCoimbra-UPorto
Porto, 26 October 2009

- Littlewood-Richardson (LR) coefficients $c^{\lambda}_{\mu\,\nu}$ are non-negative integer numbers depending on three non-negative integer vectors μ, ν, λ ordered decreasingly.

- Littlewood-Richardson (LR) coefficients $c_{\mu\,\nu}^{\lambda}$ are non-negative integer numbers depending on three non-negative integer vectors μ, ν, λ ordered decreasingly.

- Who cares?

- Littlewood-Richardson (LR) coefficients $c^{\lambda}_{\mu\,\nu}$ are non-negative integer numbers depending on three non-negative integer vectors μ, ν, λ ordered decreasingly.

- Who cares?

- How can one evaluate them?

- Littlewood-Richardson (LR) coefficients $c_{\mu\,\nu}^{\lambda}$ are non-negative integer numbers depending on three non-negative integer vectors μ, ν, λ ordered decreasingly.

- Who cares?

- How can one evaluate them?
  What do they count?

- Littlewood-Richardson (LR) coefficients $c_{\mu\,\nu}^{\lambda}$ are non-negative integer numbers depending on three non-negative integer vectors μ, ν, λ ordered decreasingly.

- Who cares?

- How can one evaluate them?
  What do they count?

- Are there conditions to see whether or not a given LR-coefficient is non-zero ?

Littlewood-Richardson coefficients: $c_{\mu\,\nu}^{\lambda}$

- Schur functions $\{s_\lambda\}_\lambda$ form a $\mathbb{Z}$-basis for the ring of symmetric functions

$$s_\mu s_\nu = \sum_\lambda c_{\mu\,\nu}^{\lambda} s_\lambda.$$

Littlewood-Richardson coefficients: $c^{\lambda}_{\mu\,\nu}$

- Schur functions $\{s_\lambda\}_\lambda$ form a $\mathbb{Z}$-basis for the ring of symmetric functions
$$s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\,\nu} s_\lambda.$$
  - For which λ does $s_\mu s_\nu$ contain s_λ as a (positive) summand?

Littlewood-Richardson coefficients: $c_{\mu\,\nu}^{\lambda}$

- Schur functions $\{s_\lambda\}_\lambda$ form a $\mathbb{Z}$-basis for the ring of symmetric functions

$$s_\mu s_\nu = \sum_\lambda c_{\mu\,\nu}^{\lambda} s_\lambda.$$

  - For which λ does $s_\mu s_\nu$ contain s_λ as a (positive) summand?
  - Given μ, ν and λ when does one have $c_{\mu\,\nu}^{\lambda} > 0$?

Littlewood-Richardson coefficients: $c_{\mu\,\nu}^{\lambda}$

- Schur functions $\{s_\lambda\}_\lambda$ form a $\mathbb{Z}$-basis for the ring of symmetric functions

$$s_\mu s_\nu = \sum_\lambda c_{\mu\,\nu}^{\lambda} s_\lambda.$$

  - For which λ does $s_\mu s_\nu$ contain s_λ as a (positive) summand?
  - Given μ, ν and λ when does one have $c_{\mu\,\nu}^{\lambda} > 0$?

- The tensor product of two irreducible polynomial representations V_μ and V_ν of the general linear group $GL_d(\mathbb{C})$ decomposes into irreducible representations of $GL_d(\mathbb{C})$

$$V_\mu \otimes V_\nu = \sum_{l(\lambda)\leq d} c_{\mu\,\nu}^{\lambda} V_\lambda.$$

Littlewood-Richardson coefficients: $c_{\mu\,\nu}^{\lambda}$

- Schur functions $\{s_\lambda\}_\lambda$ form a $\mathbb{Z}$-basis for the ring of symmetric functions

$$s_\mu s_\nu = \sum_\lambda c_{\mu\,\nu}^{\lambda} s_\lambda.$$

  - For which λ does $s_\mu s_\nu$ contain s_λ as a (positive) summand?
  - Given μ, ν and λ when does one have $c_{\mu\,\nu}^{\lambda} > 0$?

- The tensor product of two irreducible polynomial representations V_μ and V_ν of the general linear group $GL_d(\mathbb{C})$ decomposes into irreducible representations of $GL_d(\mathbb{C})$

$$V_\mu \otimes V_\nu = \sum_{l(\lambda)\leq d} c_{\mu\,\nu}^{\lambda} V_\lambda.$$

Littlewood-Richardson coefficients: $c_{\mu\,\nu}^{\lambda}$

- Schur functions $\{s_\lambda\}_\lambda$ form a $\mathbb{Z}$-basis for the ring of symmetric functions

$$s_\mu s_\nu = \sum_\lambda c_{\mu\,\nu}^{\lambda} s_\lambda.$$

  - For which λ does $s_\mu s_\nu$ contain s_λ as a (positive) summand?
  - Given μ, ν and λ when does one have $c_{\mu\,\nu}^{\lambda} > 0$?

- The tensor product of two irreducible polynomial representations V_μ and V_ν of the general linear group $GL_d(\mathbb{C})$ decomposes into irreducible representations of $GL_d(\mathbb{C})$

$$V_\mu \otimes V_\nu = \sum_{l(\lambda)\leq d} c_{\mu\,\nu}^{\lambda} V_\lambda.$$

  - Given μ and ν, for which λ does V^λ appear (with positive multiplicity) in $V^\mu \otimes V^\nu$?
  - Given μ, ν and λ when does one have $c_{\mu\,\nu}^{\lambda} > 0$?

Littlewood-Richardson coefficients: $c_{\mu\,\nu}^{\lambda}$

Littlewood-Richardson coefficients: $c_{\mu\,\nu}^{\lambda}$

- Schubert classes σ_λ form a linear basis for $H^*(G(d,n))$, the cohomology ring of the Grassmannian $G(d,n)$ of complex d-dimensional linear subspaces of $\mathbb{C}^n$,

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq d \times (n-d)} c_{\mu\,\nu}^{\lambda} \sigma_\lambda.$$

Littlewood-Richardson coefficients: $c_{\mu\,\nu}^{\lambda}$

- Schubert classes σ_λ form a linear basis for $H^*(G(d,n))$, the cohomology ring of the Grassmannian $G(d,n)$ of complex d-dimensional linear subspaces of $\mathbb{C}^n$,

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq d\times(n-d)} c_{\mu\,\nu}^{\lambda} \sigma_\lambda.$$

- There exist $n \times n$ non singular matrices A, B and C, over a *local principal ideal domain*, with Smith invariants $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ respectively, such that $AB = C$ if and only if $c_{\mu\,\nu}^{\lambda} > 0$.

Littlewood-Richardson coefficients: $c_{\mu\,\nu}^{\lambda}$

- Schubert classes σ_λ form a linear basis for $H^*(G(d,n))$, the cohomology ring of the Grassmannian $G(d,n)$ of complex d-dimensional linear subspaces of $\mathbb{C}^n$,

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq d \times (n-d)} c_{\mu\,\nu}^{\lambda} \sigma_\lambda.$$

- There exist $n \times n$ non singular matrices A, B and C, over a *local principal ideal domain*, with Smith invariants $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ respectively, such that $AB = C$ if and only if $c_{\mu\,\nu}^{\lambda} > 0$.

- There exist $n \times n$ Hermitian matrices A, B and C, with integer eigenvalues arranged in weakly decreasing order $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ respectively, such that $C = A + B$ if and only if $c_{\mu,\nu}^{\lambda} > 0$.

1. Schur functions

Partitions and Young diagrams

- Fix a positive integer $r \geq 1$.

1. Schur functions

Partitions and Young diagrams

- Fix a positive integer $r \geq 1$.
- $\lambda = (\lambda_1, \ldots, \lambda_r)$, with $\lambda_1 \geq \cdots \geq \lambda_r > 0$ positive integers, is a *partition* of length $l(\lambda) = r$.

1. Schur functions

Partitions and Young diagrams

- Fix a positive integer $r \geq 1$.
- $\lambda = (\lambda_1, \ldots, \lambda_r)$, with $\lambda_1 \geq \cdots \geq \lambda_r > 0$ positive integers, is a *partition* of length $l(\lambda) = r$.
- Each *partition* λ is identified with a Young (Ferrer) diagram λ

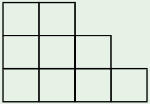
1. Schur functions

Partitions and Young diagrams

- Fix a positive integer $r \geq 1$.
- $\lambda = (\lambda_1, \dots, \lambda_r)$, with $\lambda_1 \geq \cdots \geq \lambda_r > 0$ positive integers, is a *partition* of length $l(\lambda) = r$.
- Each *partition* λ is identified with a Young (Ferrer) diagram λ consisting of $|\lambda| = \lambda_1 + \cdots + \lambda_r$ boxes arranged in r bottom left adjusted rows of lengths $\lambda_1 \geq \cdots \geq \lambda_r > 0$.

Example

$\lambda = (4, 3, 2)$, $|\lambda| = 9$, $l(\lambda) = 3$

$\lambda \quad =$ 

Young Tableaux

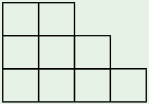
- $n \geq r$, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $l(\lambda) = r$.
- A *semistandard tableau* T of shape λ is a filling of the boxes of the Ferrer diagram λ with elements i in $\{1, \ldots, n\}$ which is
  - weakly increasing across rows from left to right
  - strictly increasing up columns

Young Tableaux

- $n \geq r$, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $l(\lambda) = r$.
- A *semistandard tableau* T of shape λ is a filling of the boxes of the Ferrer diagram λ with elements i in $\{1, \ldots, n\}$ which is
  - weakly increasing across rows from left to right
  - strictly increasing up columns
- T has type $\alpha = (\alpha_1, \ldots, \alpha_n)$ if T has α_i entries equal i.

Example

$\lambda = (4, 3, 2)$, $l(\lambda) = 3$, $n = 6$

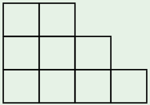


Young Tableaux

- $n \geq r$, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $l(\lambda) = r$.
- A *semistandard tableau* T of shape λ is a filling of the boxes of the Ferrer diagram λ with elements i in $\{1, \ldots, n\}$ which is
  - weakly increasing across rows from left to right
  - strictly increasing up columns
- T has type $\alpha = (\alpha_1, \ldots, \alpha_n)$ if T has α_i entries equal i.

Example

$\lambda = (4, 3, 2)$, $l(\lambda) = 3$, $n = 6$



$T =$

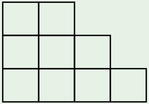
5	6		
4	4	6	
2	3	4	6

Young Tableaux

- $n \geq r$, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $l(\lambda) = r$.
- A *semistandard tableau* T of shape λ is a filling of the boxes of the Ferrer diagram λ with elements i in $\{1, \ldots, n\}$ which is
  - weakly increasing across rows from left to right
  - strictly increasing up columns
- T has type $\alpha = (\alpha_1, \ldots, \alpha_n)$ if T has α_i entries equal i.

Example

$\lambda = (4, 3, 2)$, $l(\lambda) = 3$, $n = 6$



$$T = \begin{array}{|c|c|c|c}\hline 5 & 6 & \multicolumn{2}{c}{} \\ \hline 4 & 4 & \multicolumn{1}{c|}{6} & \\ \hline 2 & 3 & 4 & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

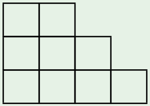
semistandard tableau T of shape $\lambda = (4, 3, 2)$,

Young Tableaux

- $n \geq r$, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $l(\lambda) = r$.
- A *semistandard tableau* T of shape λ is a filling of the boxes of the Ferrer diagram λ with elements i in $\{1, \ldots, n\}$ which is
  - weakly increasing across rows from left to right
  - strictly increasing up columns
- T has type $\alpha = (\alpha_1, \ldots, \alpha_n)$ if T has α_i entries equal i.

Example

$\lambda = (4, 3, 2)$, $l(\lambda) = 3$, $n = 6$



$T =$

5	6		
4	4	6	
2	3	4	6

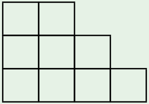
semistandard tableau T of shape $\lambda = (4, 3, 2)$, $\alpha = (0, 1, 1, 3, 1, 3)$.

Young Tableaux

- $n \geq r$, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $l(\lambda) = r$.
- A *semistandard tableau* T of shape λ is a filling of the boxes of the Ferrer diagram λ with elements i in $\{1, \ldots, n\}$ which is
  - weakly increasing across rows from left to right
  - strictly increasing up columns
- T has type $\alpha = (\alpha_1, \ldots, \alpha_n)$ if T has α_i entries equal i.

Example

$\lambda = (4, 3, 2)$, $l(\lambda) = 3$, $n = 6$



$T =$

5	6		
4	4	6	
2	3	4	6

semistandard tableau T of shape $\lambda = (4, 3, 2)$, $\alpha = (0, 1, 1, 3, 1, 3)$.

Schur functions

Example

$n = 7$

$T =$

5	6		
4	4	6	
2	3	4	6

$\alpha(T)$=(0,1,1,3,1,3,0)

$$x^{\alpha(T)} = x_1^0 x_2 x_3 x_4^3 x_5 x_6^3 x_7^0$$

Schur functions continued

- Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of variables.

Schur functions continued

- Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of variables.
- Given the partition λ, the Schur function (polynomial) $s_\lambda(\mathbf{x})$ associated with the partition λ is the homogeneous polynomial of degree $|\lambda|$ on the variables $x_1 \ldots, x_n$

$$s_\lambda(\mathbf{x}) = \sum_T X^{\alpha(T)}$$

where T runs over all semistandard tableaux of shape λ on the alphabet $\{1, \ldots, n\}$.

Example

$\lambda = (2, 1)$, $|\lambda| = 3$

- $n = 3$

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}.$$

Schur functions continued

- Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of variables.
- Given the partition λ, the Schur function (polynomial) $s_\lambda(\mathbf{x})$ associated with the partition λ is the homogeneous polynomial of degree $|\lambda|$ on the variables $x_1 \ldots, x_n$

$$s_\lambda(\mathbf{x}) = \sum_T X^{\alpha(T)}$$

where T runs over all semistandard tableaux of shape λ on the alphabet $\{1, \ldots, n\}$.

Example

$\lambda = (2, 1)$, $|\lambda| = 3$

- $n = 3$

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{|c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{|c|}{3} \\ \hline\end{array}.$$

$$s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

Schur functions continued

- Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of variables.
- Given the partition λ, the Schur function (polynomial) $s_\lambda(\mathbf{x})$ associated with the partition λ is the homogeneous polynomial of degree $|\lambda|$ on the variables $x_1 \ldots, x_n$

$$s_\lambda(\mathbf{x}) = \sum_T X^{\alpha(T)}$$

where T runs over all semistandard tableaux of shape λ on the alphabet $\{1, \ldots, n\}$.

Example

$\lambda = (2, 1)$, $|\lambda| = 3$

- $n = 3$

2	3	2	3	2	3	3	3
1 1	1 1	1 2	1 2	1 3	1 3	2 2	2 3

$$s_\lambda(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

- Kostka number $K_{\lambda,\alpha}$ is the number of semistandard tableaux of shape λ and type α.
- The Schur function on the variables $x_1, \ldots, x_n$

$$s_n(\lambda, \mathbf{x}) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} K_{\lambda,\alpha} x^{\alpha},$$

  with $\alpha_1 + \cdots + \alpha_n = |\lambda|$.

- $K_{\lambda\,\beta} = K_{\lambda\,\alpha}$, with β any permutation of α.

- $K_{\lambda\,\beta} = K_{\lambda\,\alpha}$, with β any permutation of α.

Corollary

- *The Schur function* $s(\lambda, \mathbf{x}) = \sum_{\alpha \text{ weak composition of } |\lambda|} K_{\lambda,\,\alpha} x^{\alpha}$, *is a homogeneous symmetric function in* $x_1, \ldots, x_n$.

Product of Schur functions

- The Schur functions s_λ form an additive basis for the ring of the symmetric functions.
- A product of Schur functions $s_\mu s_\nu$ can be expressed as a non-negative integer linear sum of Schur functions:

$$s_\mu s_\nu = \sum_\lambda c_{\mu\,\nu}^{\lambda} s_\lambda.$$

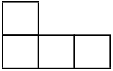
- What does $c_{\mu\nu}^{\lambda}$ count?

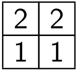
2. Littlewood-Richardson rule

$\mu = (3, 1), \quad \nu = (2, 2)$

2. Littlewood-Richardson rule

$\mu = (3,1), \quad \nu = (2,2)$





2. Littlewood-Richardson rule

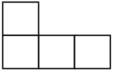
$\mu = (3,1), \quad \nu = (2,2)$

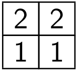
2	2
1	1

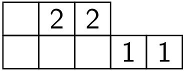
	2	2		
			1	1

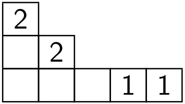
2. Littlewood-Richardson rule

$\mu = (3, 1), \quad \nu = (2, 2)$



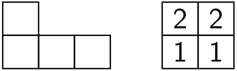


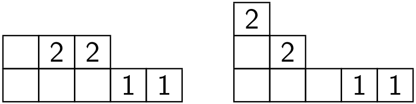
- 

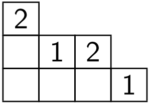


2. Littlewood-Richardson rule

$\mu = (3, 1), \quad \nu = (2, 2)$

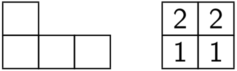


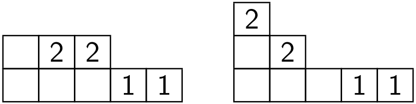
- 

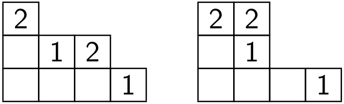
- 

2. Littlewood-Richardson rule

$\mu = (3, 1), \quad \nu = (2, 2)$

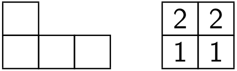


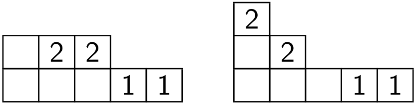
- 

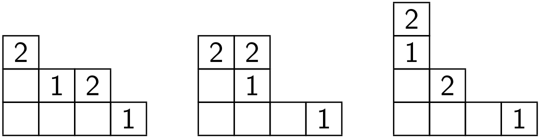
- 

2. Littlewood-Richardson rule

$\mu = (3,1), \quad \nu = (2,2)$

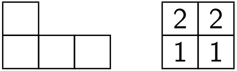


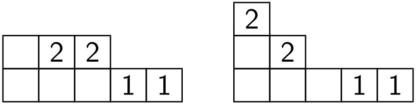
- 

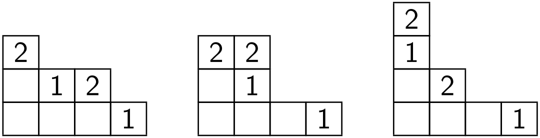
- 

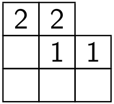
2. Littlewood-Richardson rule

$\mu = (3, 1), \quad \nu = (2, 2)$



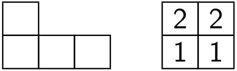
- 

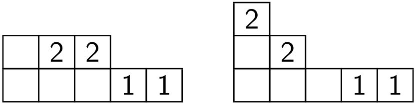
- 

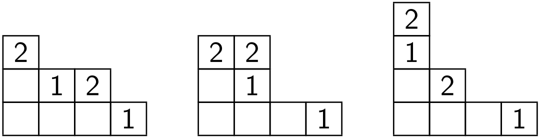
- 

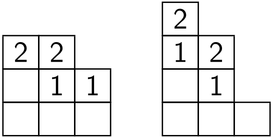
2. Littlewood-Richardson rule

$\mu = (3, 1), \quad \nu = (2, 2)$



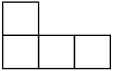
- 

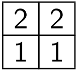
- 

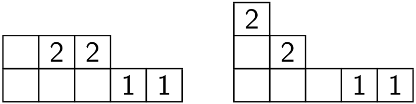
- 

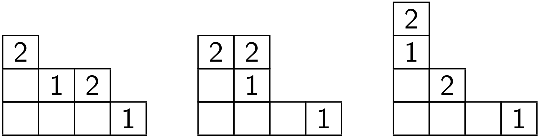
2. Littlewood-Richardson rule

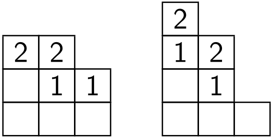
$\mu = (3, 1), \quad \nu = (2, 2)$



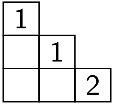


- 

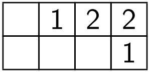
- 

- 

- invalid tableaux



211



1221

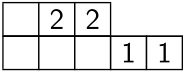
Littlewood-Richardson rule

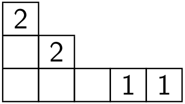
$\mu = (3, 1), \quad \nu = (2, 2)$

	2	2		
			1	1

Littlewood-Richardson rule

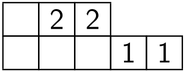
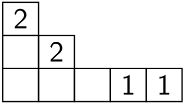
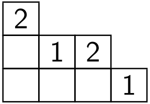
$\mu = (3, 1), \quad \nu = (2, 2)$

- 



Littlewood-Richardson rule

$\mu = (3, 1), \quad \nu = (2, 2)$

-  
- 

Littlewood-Richardson rule

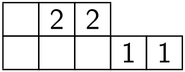
$\mu = (3,1), \quad \nu = (2,2)$

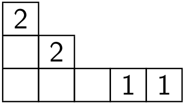
- $\begin{array}{|c|c|c|c|c|} \hline \ & 2 & 2 \\ \hline \ & \ & \ & 1 & 1 \\ \hline \end{array}$ $\quad$ $\begin{array}{|c|c|c|c|c|} \hline 2 \\ \hline \ & 2 \\ \hline \ & \ & \ & 1 & 1 \\ \hline \end{array}$

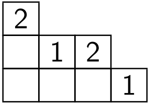
- $\begin{array}{|c|c|c|c|} \hline 2 \\ \hline \ & 1 & 2 \\ \hline \ & \ & \ & 1 \\ \hline \end{array}$ $\quad$ $\begin{array}{|c|c|c|c|} \hline 2 & 2 \\ \hline \ & 1 \\ \hline \ & \ & \ & 1 \\ \hline \end{array}$

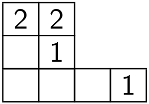
Littlewood-Richardson rule

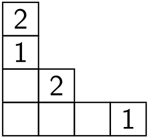
$\mu = (3, 1), \quad \nu = (2, 2)$

- 



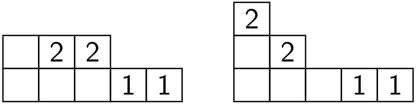
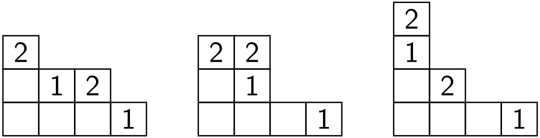
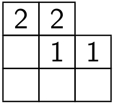
- 





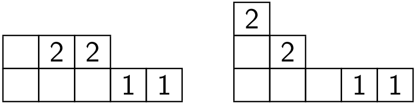
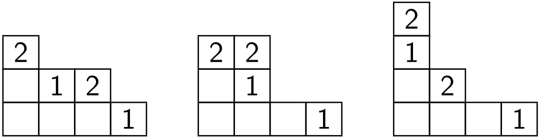
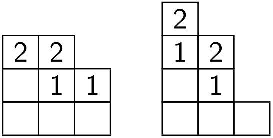
Littlewood-Richardson rule

$\mu = (3,1), \quad \nu = (2,2)$

- 
- 
- 

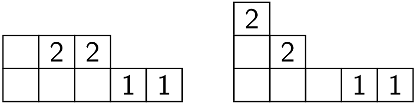
Littlewood-Richardson rule

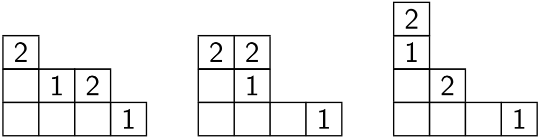
$\mu = (3,1), \quad \nu = (2,2)$

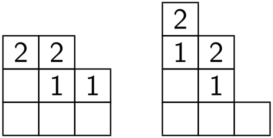
- 
- 
- 

Littlewood-Richardson rule

$\mu = (3,1), \quad \nu = (2,2)$

- 

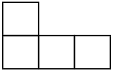
- 

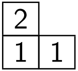
- 

- $s_\mu s_\nu = s_{53} + s_{521} + s_{431} + s_{422} + s_{4211} + s_{332} + s_{3221}$

$\mu = (3, 1), \quad \nu = (2, 1)$

$\mu = (3, 1), \quad \nu = (2, 1)$



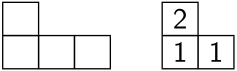


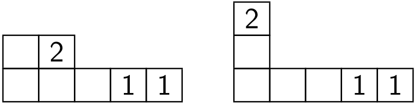
$\mu = (3, 1), \quad \nu = (2, 1)$

2	
1	1

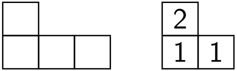
	2			
			1	1

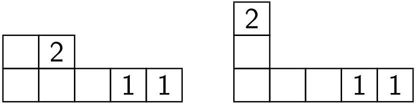
$\mu = (3, 1), \quad \nu = (2, 1)$

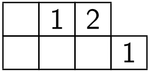


- 

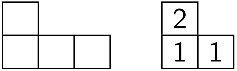
$\mu = (3, 1), \quad \nu = (2, 1)$

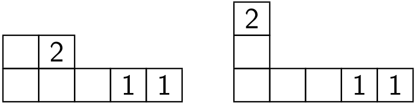


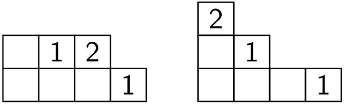
- 

- 

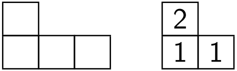
$\mu = (3, 1), \quad \nu = (2, 1)$

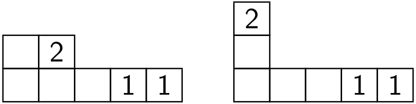


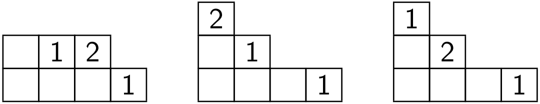
- 

- 

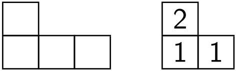
$\mu = (3,1), \quad \nu = (2,1)$

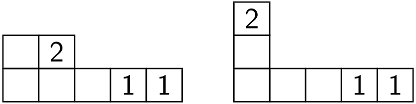


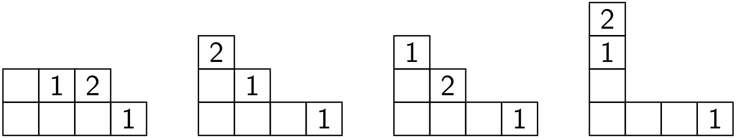
- 

- 

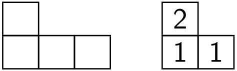
$\mu = (3,1), \quad \nu = (2,1)$

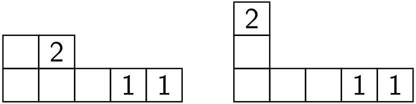


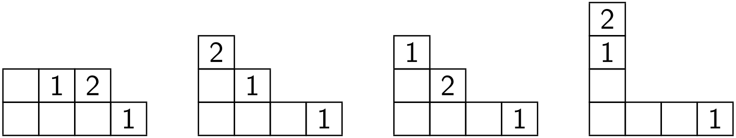
- 

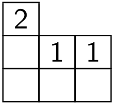
- 

$\mu = (3,1), \quad \nu = (2,1)$

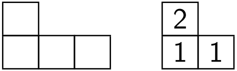


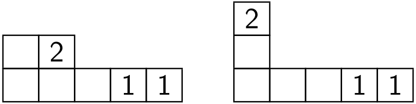
- 

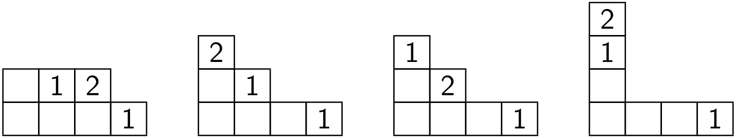
- 

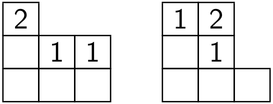
- 

$\mu = (3,1), \quad \nu = (2,1)$

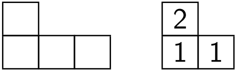


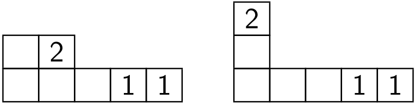
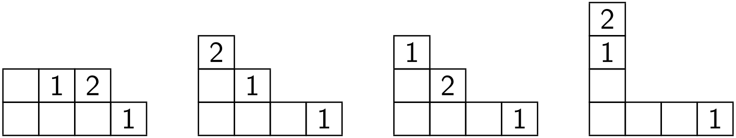
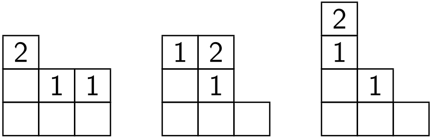
- 

- 

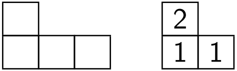
- 

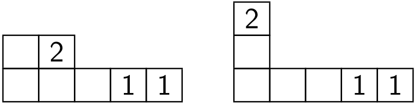
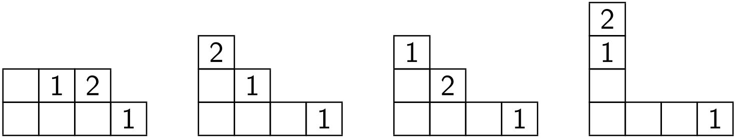
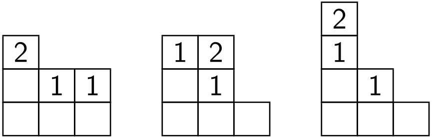
$\mu = (3,1), \quad \nu = (2,1)$



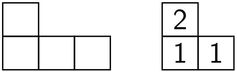
- 
- 
- 

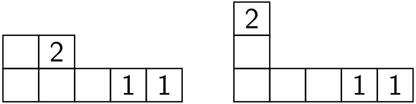
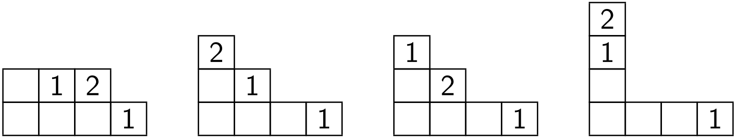
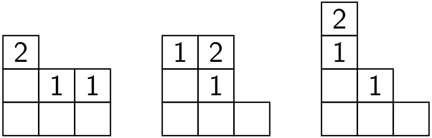
$\mu = (3,1), \quad \nu = (2,1)$



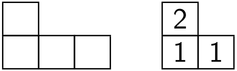
- 
- 
- 
- $s_\mu s_\nu = s_{52} + s_{511} + s_{43} + 2 s_{421} + s_{4111} + s_{331} + s_{322} + s_{3211}$

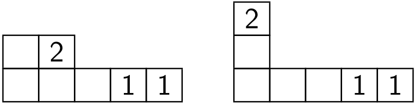
$\mu = (3,1), \quad \nu = (2,1)$

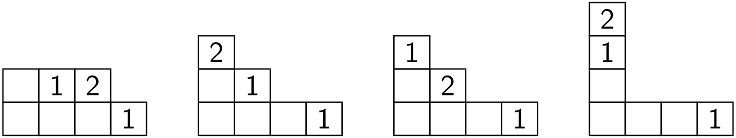


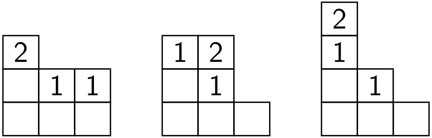
- 
- 
- 
- $s_\mu s_\nu = s_{52} + s_{511} + s_{43} + 2 s_{421} + s_{4111} + s_{331} + s_{322} + s_{3211}$
- $c^{421}_{\mu\,\nu} = 2$

$\mu = (3,1), \quad \nu = (2,1)$



- 

- 

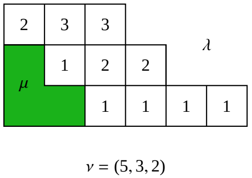
- 

- $s_\mu s_\nu = s_{52} + s_{511} + s_{43} + 2 s_{421} + s_{4111} + s_{331} + s_{322} + s_{3211}$
- $c^{421}_{\mu\,\nu} = 2$

Littlewood-Richardson rule

- $c^{\lambda}_{\mu\nu}$ is the number of tableaux with shape λ/μ and content ν satisfying
  - If one reads the labeled entries in reverse reading order, that is, from right to left across rows taken in turn from bottom to top,

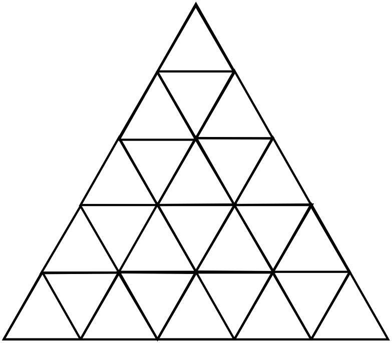
    at any stage, the number of i's encountered is at least as large as the number of $(i+1)$'s encountered, $\#1's \geq \#2's \ldots$.



3.Integer Hives (99)

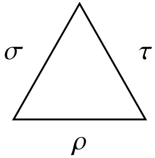
- Knutson-Tao (99)
- An n-integer hive is a triangular graph made of $\binom{n+1}{2} + \binom{n}{2} = n^2$ unitary triangles and $\binom{n+2}{2}$ vertices with non-negative edge labels satisfying a set of conditions given by linear inequalities called *hive conditions*

$n = 5$

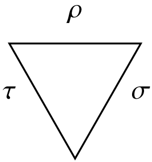


(Edge) Hive conditions

- Two distinct types of elementary triangles with non-negative integer edge labelling



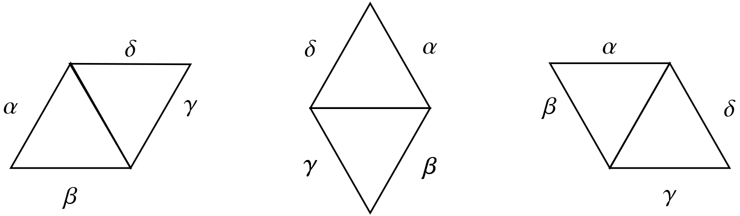
$\sigma + \tau = \rho$



$\sigma + \tau = \rho$

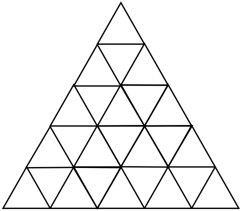
(Edge) Hive conditions continued

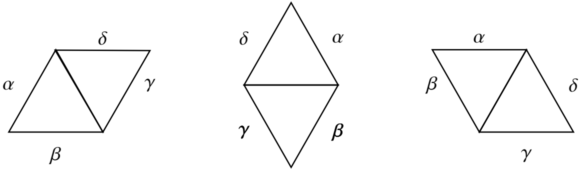
- Three distinct types of rhombi with non-negative integer edge labelling



$$\alpha \geq \gamma$$
$$\beta \geq \delta$$
$$\alpha + \delta = \beta + \gamma$$

Hive conditions continued

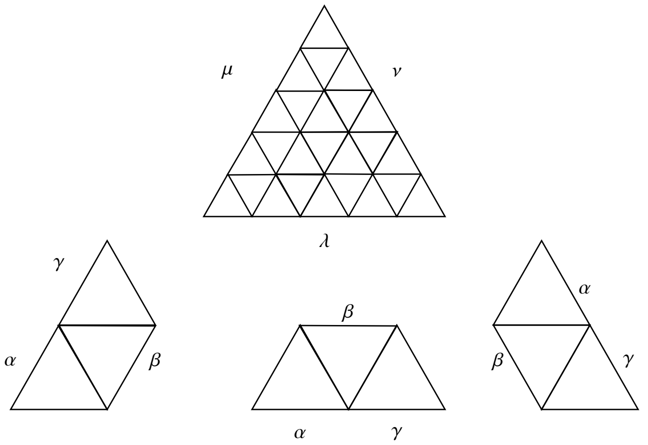




$$\alpha \geq \gamma$$
$$\beta \geq \delta$$
$$\alpha + \delta = \beta + \gamma$$

Hive boundary conditions

- overlapping pairs of rhombi: edge labels along any line parallel to north-west, north-east and southern boundaries are weakly decreasing in the north-east, south-east and easterly directions.



$\alpha \geq \beta \geq \gamma$

Knutson-Tao Hives 99

- The Littlewood-Richardson coefficients $c_{\mu\,\nu}^{\lambda}$ is the number of Hives with boundary μ, ν and λ.

4. Horn conjecture (62)

4. Horn conjecture (62)

- There exist $n \times n$ Hermitian matrices A, B and C, with integer eigenvalues arranged in weakly decreasing order $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ respectively, such that $C = A + B$ if and only if μ, ν and λ satisfy a certain huge system of linear inequalities.

Horn inequalities

- Let $N = \{1, 2, \ldots, n\}$, then for fixed d, with $1 \leq d \leq n$, let $I = \{i_1, i_2, \ldots, i_d\} \subseteq N$.

Horn inequalities

- Let $N = \{1, 2, \ldots, n\}$, then for fixed d, with $1 \leq d \leq n$, let $I = \{i_1, i_2, \ldots, i_d\} \subseteq N$.
- Let $I, J, K \subseteq N$ with $\#I = \#J = \#K = d$ and ordered decreasingly. One defines the partitions

$$\alpha(I) = I - (d, \ldots, 2, 1),$$

$$\beta(J) = J - (d, \ldots, 2, 1),$$

$$\gamma(K) = K - (d, \ldots, 2, 1).$$

Horn inequalities

- Let $N = \{1, 2, \ldots, n\}$, then for fixed d, with $1 \leq d \leq n$, let $I = \{i_1, i_2, \ldots, i_d\} \subseteq N$.

- Let $I, J, K \subseteq N$ with $\#I = \#J = \#K = d$ and ordered decreasingly. One defines the partitions

$$\alpha(I) = I - (d, \ldots, 2, 1),$$

$$\beta(J) = J - (d, \ldots, 2, 1),$$

$$\gamma(K) = K - (d, \ldots, 2, 1).$$

- Let T_d^n be the set of all triples (I, J, K) with $I, J, K \subseteq N$ and $\#I = \#J = \#K = d$ such that $c_{\alpha(I),\beta(J)}^{\gamma(K)} > 0$.

Horn inequalities continued

- μ, ν, λ are said to satisfy the Horn inequalities if

$$\sum_{k=1}^{n} \lambda_k = \sum_{i=1}^{n} \mu_i + \sum_{j=1}^{n} \nu_j$$

$$\sum_{k \in K} \lambda_k \leq \sum_{i \in I} \mu_i + \sum_{j \in J} \nu_j$$

  for all triples $(I, J, K) \in T_d^n$ with $d = 1, \ldots, n-1$.

- Not all of Horn's inequalities are essential. The essential inequalities are those for which (I, J, K) satisfy $c_{\alpha(I),\beta(J)}^{\gamma(K)} = 1$.

Horn inequalities and Littlewood-Richardson coefficients

- $c^{\lambda}_{\mu,\nu} > 0$ if and only if the Horn inequalities are satisfied

$$\sum_{k=1}^{n} \lambda_k = \sum_{i=1}^{n} \mu_i + \sum_{j=1}^{n} \nu_j$$

$$\sum_{k \in K} \lambda_k \leq \sum_{i \in I} \mu_i + \sum_{j \in J} \nu_j$$

for all triples $(I, J, K) \in T^n_d$ with $d = 1, \ldots, n-1$.

Where do Horn inequalities come from?

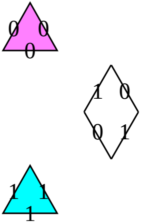
- Impose on a n-hive a puzzle of size n.

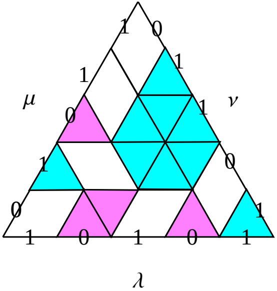
5. Knutson-Tao-Woodward Puzzles (04)

- A puzzle of size n is a tiling of an equilateral triangle of side length n with puzzle pieces each of unit side length.
  - Puzzle pieces may be rotated in any orientation *but not reflected*, and wherever two pieces share an edge, the numbers on the edge must agree.

5. Knutson-Tao-Woodward Puzzles (04)

- A puzzle of size n is a tiling of an equilateral triangle of side length n with puzzle pieces each of unit side length.
  - Puzzle pieces may be rotated in any orientation *but not reflected*, and wherever two pieces share an edge, the numbers on the edge must agree.



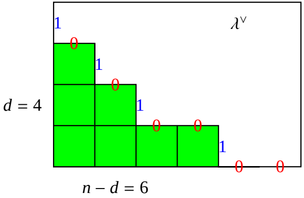


Partitions and 01-strings

- Fix positive integers $0 < d < n$ and consider a $d \times (n-d)$ rectangle.

Partitions and 01-strings

- Fix positive integers $0 < d < n$ and consider a $d \times (n - d)$ rectangle.
- $d = 4 \; n = 10$

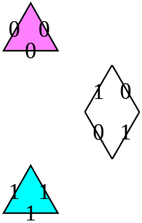


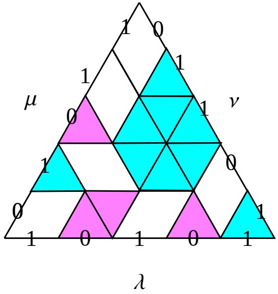
$\lambda = (4, 2, 1, 0) \leftrightarrow 0010010101$

$\lambda^{\vee} = (6, 5, 4, 2) \leftrightarrow 1010100100$

Puzzle rule

- (Knutson-Tao-Woodward) $c_{\mu\,\nu}^{\lambda}$ is the number of puzzles with μ, ν and λ appearing as 01-strings along the boundary.



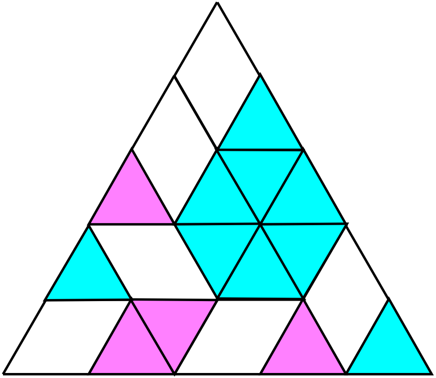


Horn triples and puzzles

- (I, J, K) is Horn triple if it specifies the positions of the 0's on the boundary of any puzzle. It is essential if the puzzle with these boundary 0's edges is unique.

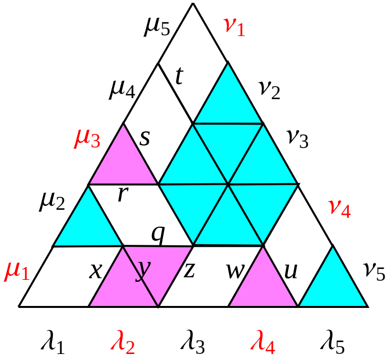
Example

- $I = \{1,3\},\ J = \{1,4\},\ K = \{2,4\}$ is a Horn triple since I, J, and K specify the positions of the 0's on the boundary of the puzzle



Example continued

- $I = \{1, 3\}$, $J = \{1, 4\}$, $K = \{2, 4\}$ is a Horn triple. Superimpose the puzzle, with pink edges specified by those sets I, J, K, on a hive of size 5 and explore the hive conditions



$$\lambda_2 + \lambda_4 = x + y + w + u \leq \mu_1 + y + w + u \leq \mu_1 + z + y + \nu_4$$
$$\leq \mu_1 + z + y + \nu_4 = \mu_1 + q + \nu_4 \leq \mu_1 + r + \nu_4$$
$$\leq \mu_1 + \mu_3 + s + \nu_4 \leq \mu_1 + \mu_3 + \nu_1 + \nu_4$$

Sources

- A. Horn, Pacific J. Math. 12 1962 225–241
- Robert C. Thompson, 401–435, Contemp. Math., 47, Amer. Math. Soc., Providence, RI, 1985.
- O. Azenhas, E. Marques de Sá, Linear and Multilinear Algebra 27 (1990), no. 4, 229–242.
- A. A. Klyachko, Selecta Math. (N.S.) 4 (1998), no. 3, 419-445
- W. Fulton, Séminaire Bourbaki. Vol. 1997/98. Astérisque No. 252 (1998), Exp. No. 845, 5, 255–269.
- A. Knutson, T. Tao, J. Amer. Math. Soc. 12 (1999)
- A. P. Santana, J. F. Queiró, and E. Marques de Sá, Linear and Multilinear Algebra 46 (1999), no. 1-2, 1–23
- W. Fulton, Bull. Amer. Math. Soc. (N.S.) 37 (2000), no. 3, 209–249
- A. S. Buch, Enseign. Math. (2) 46 (2000), no. 1-2, 43–60.
- A. Knutson, T. Tao and C. Woodward, Amer. Math. Soc. 17 (2004) 1948
- Ronald C. King, C. Tollu, F. Toumazet, J. Combin. Theory, Series A,116 (2009), 314-333.