Semi-skyline augmented fillings and non-symmetric Cauchy kernels for stair-type shapes

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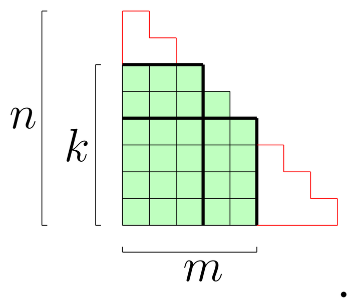
Abstract

Using an analogue of the Robinson-Schensted-Knuth (RSK) algorithm for semi-skyline augmented fillings (SSAF), due to Sarah Mason, we exhibit expansions of non-symmetric Cauchy kernels $\prod_{(i,j)\in\eta}(1-x_iy_j)^{-1}$, where the product is over all cell-coordinates (i,j) of the truncated staircase partition η in green,



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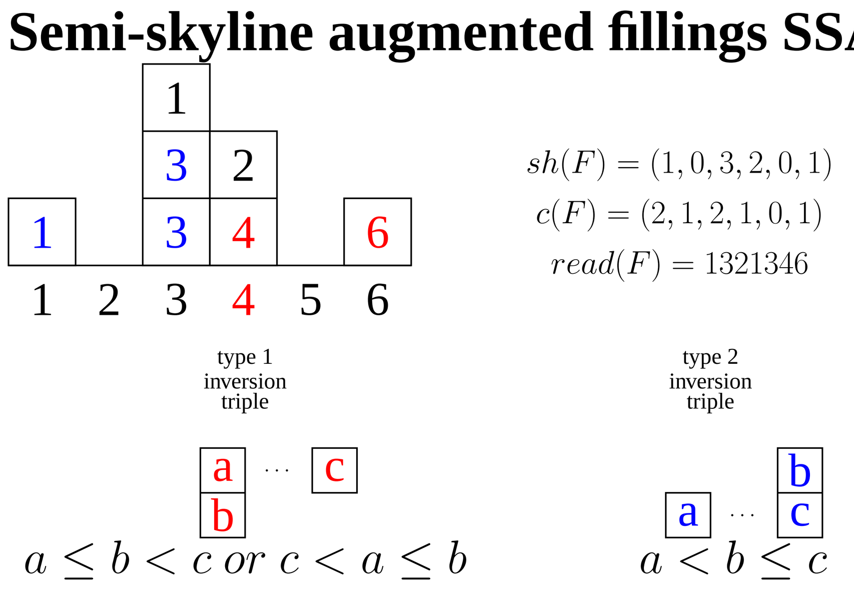
SSYTs are in bijection with SSAFs whose shapes determine the right keys of the first. Since the RSK analogue refines the classical RSK for SSYTs by exhibiting their right keys, it provides an interpretation of the Cauchy kernel over truncated staircases as a family of pairs of semi-skyline augmented fillings, where the pairs of key tableaux, determined by their shapes, lead to expansions as a sum of products of two families of key polynomials, the basis of Demazure characters of type A, and the basis of Demazure atoms. Our expansions, explicit in the pairs of tableaux, are

$$\prod_{\substack{(i,j)\in\eta\\ m\le k}}(1-x_iy_j)^{-1}=\sum_{\nu=(\nu_1,\dots,\nu_m,0^{n-m})}\widehat{\kappa}_\nu(y)\pi^{-1}_{>k}\kappa_{\omega\nu}(x),$$

$$\prod_{\substack{(i,j)\in\eta\\ k\le m}}(1-x_iy_j)^{-1}=\sum_{\nu=(\nu_1,\dots,\nu_k,0^{n-k})}\widehat{\kappa}_\nu(x)\pi^{-1}_{>m}\kappa_{\omega\nu}(y),$$

where the key polynomial $\pi^{-1}_{>l}\kappa_{\omega\nu}(z)$, for $l=k,m$ and $z=x,y$, is combinatorially defined by the SSYTs with entries $\le k$ in the Demazure crystal graph $\mathfrak{B}_{\omega\nu}$, with ω the longest element in $\mathfrak{S}_n$. A previous expansion of the Cauchy kernel in type A, for the staircase, was given by Alain Lascoux, based on the structure of double crystal graphs, which was then used to recover expansions for Ferrers shapes.

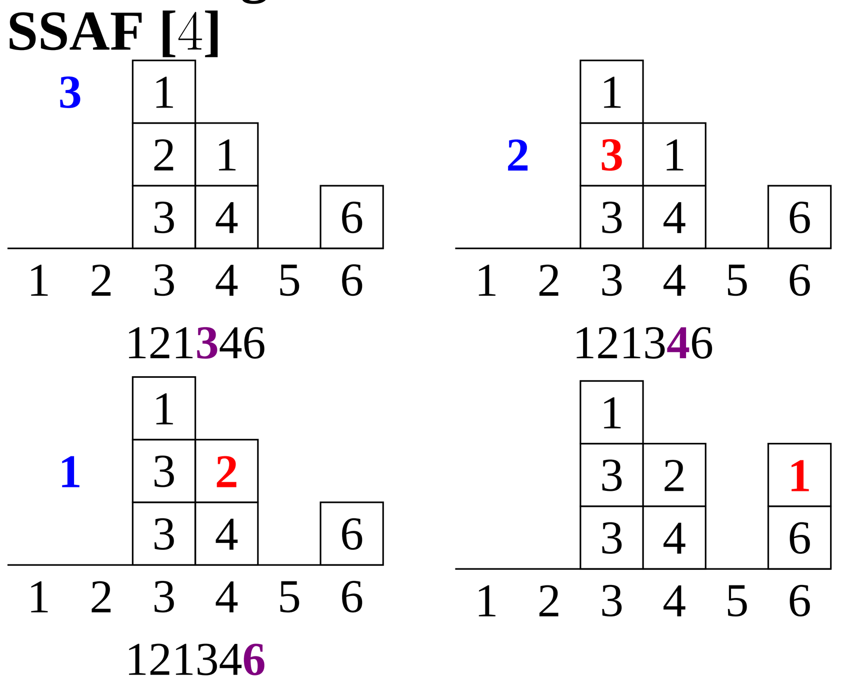
Semi-skyline augmented fillings SSAF [1]



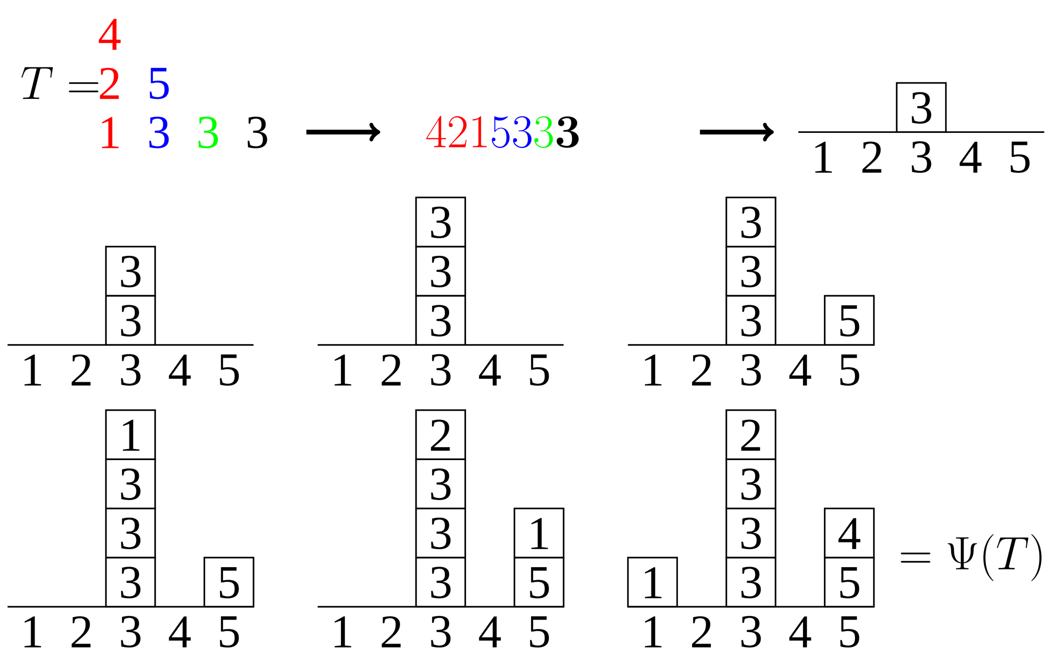
$sh(F)=(1,0,3,2,0,1)$
$c(F)=(2,1,2,1,0,1)$
$read(F)=1321346$

$a\le b<c$ or $c<a\le b$ $\qquad a<b\le c$

An analogue of Schensted row insertion in SSAF [4]



A weight preserving, shape rearranging bijection Ψ between SSYT and SSAF [4]



Right key of tableau [2, 3]

$$K_+(T)=key(sh(\Psi(T)))=key((1,0,4,0,2))=\begin{matrix}5\\3\ 5\\1\ 3\ 3\ 3\end{matrix}$$

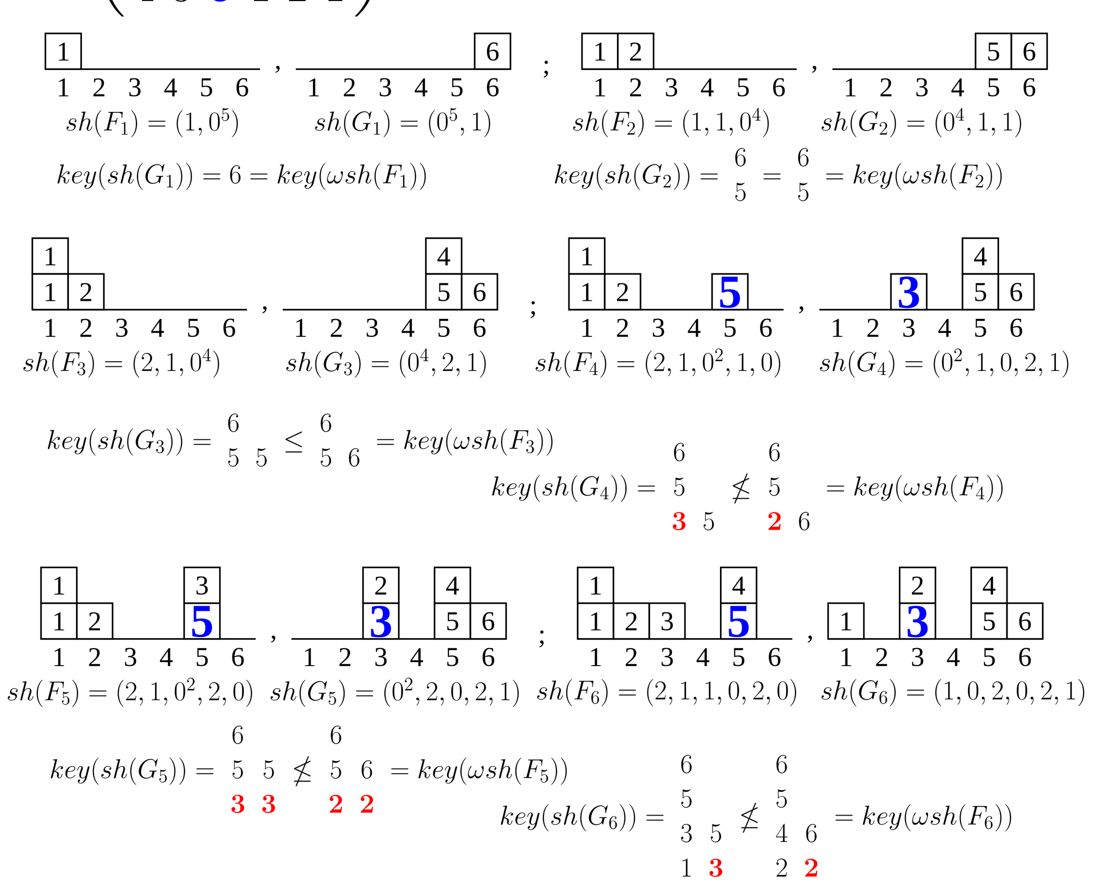
Bruhat order in type A

$\alpha,\beta\in\mathbb{N}^n$, $\alpha^+=\beta^+$, $\alpha\le\beta\iff key(\alpha)\le key(\beta)$

$$(3,1,2,1)\le(1,1,3,2)\iff\begin{matrix}4\\3\\2\ 3\\1\ 1\ 1\end{matrix}\le\begin{matrix}4\\3\\2\ 4\\1\ 3\ 3\end{matrix}$$

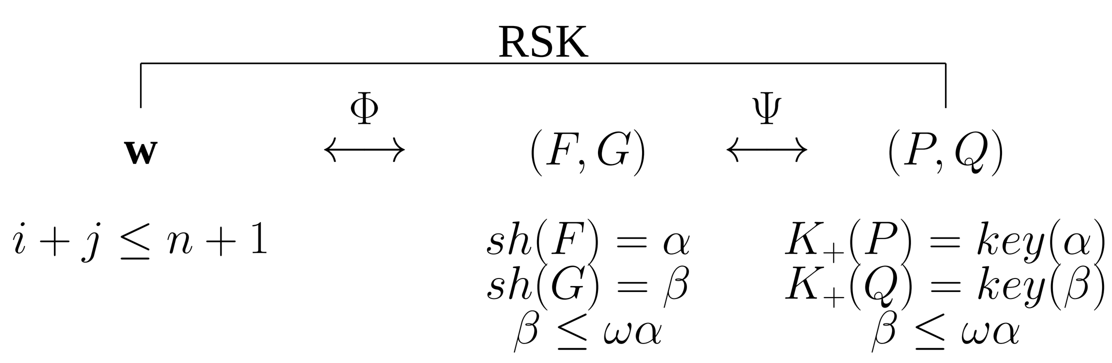
Φ: RSK analogue for SSAF [4]

$$\mathrm{w}=\begin{pmatrix}1&2&3&4&5&6\\4&3&5&1&2&1\end{pmatrix}\longrightarrow$$



Main Theorem

Fix $n\in\mathbb{N}\setminus\{0\}$. Let w be a biword in lexicographic order in the alphabet $[n]$, and let $\Phi(\mathrm{w})=(F,G)$. For each biletter $\binom{i}{j}$ in w one has $i+j\le n+1$ if and only if $key(sh(G))\le key(\omega sh(F))$, where ω is the longest permutation of $\mathfrak{S}_n$. Moreover, if the first [respectively the second] row of w is a word in the alphabet $[m]$, with $1\le m\le n$, the shape of G [respectively F] has the last $n-m$ entries equal to zero.



Demazure operator, Demazure crystal and Key polynomial

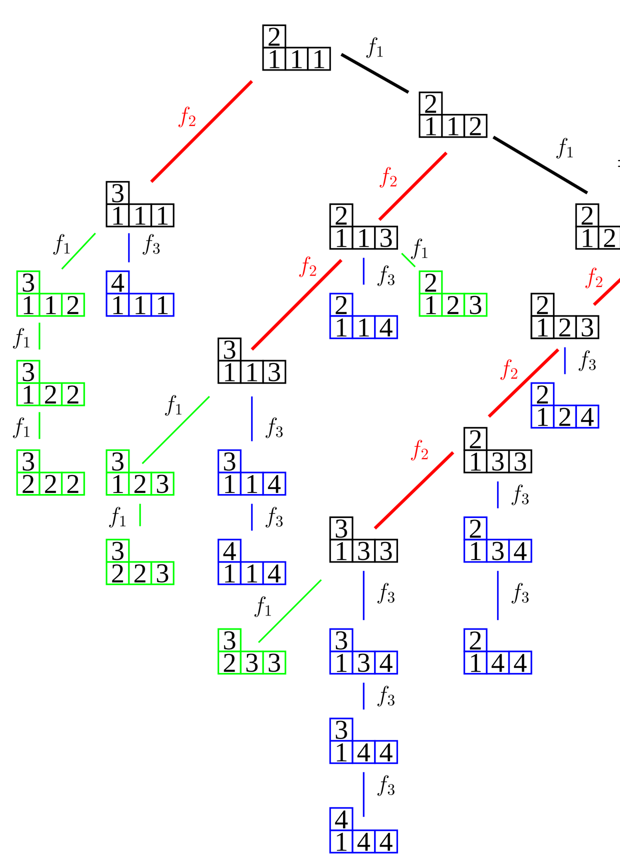
Demazure operators:

$$\pi_i,\widehat{\pi}_i:\mathbb{Z}[x_1,\dots,x_n]\to\mathbb{Z}[x_1,\dots,x_n]$$

$$\pi_i:f\mapsto\pi_i(f):=\frac{x_if-x_{i+1}s_i(f)}{x_i-x_{i+1}},\ 1\le i<n,\ \widehat{\pi}_i:=\pi_i-1.$$

Key polynomial and Demazure atom: Given the partition λ in $\mathbb{N}^n$, let σ be a minimal length representative in $\mathfrak{S}_n/stab_\lambda$, such that $\sigma\lambda=\alpha$. Then $\kappa_\alpha(x)=\pi_\sigma(x^\lambda)$ and $\widehat{\kappa}_\alpha(x)=\widehat{\pi}_\sigma(x^\lambda)$.

Demazure crystal: The Demazure crystal $\mathfrak{B}_{s_2s_1(\lambda)}$ with $\lambda=(3,1,0,0)$.(Black tableaux)



$$\kappa_{s_2s_1(3100)}(x_1,x_2,x_3,x_4)$$
$$=\kappa_{1030}(x_1,x_2,x_3,x_4)=\pi_2\pi_1(x^{3100})$$
$$=\pi_2(x^{2200}+x^{1300}+x^{3100})$$
$$=x^{2200}+x^{1300}+x^{3100}+x^{3010}+x^{2110}$$
$$+x^{2020}+x^{1210}+x^{1120}+x^{1030}.$$

$$\kappa_{s_2s_1(\lambda)}=\sum_{T\in\mathfrak{B}_{s_2s_1(\lambda)}}x^T=\sum_{K_+(T)\le key(1030)}x^T$$

$$\widehat{\kappa}_{s_2s_1(\lambda)}=\sum_{T\in\mathfrak{B}_{s_2s_1(\lambda)}\setminus\mathfrak{B}_{s_1(\lambda)}\cup\mathfrak{B}_{s_2(\lambda)}}x^T=\sum_{K_+(T)=key(1030)}x^T$$

$$\pi^{-1}_{>3}\kappa_{1003}(x)=\sum_{T\in\mathfrak{B}_{013}\cap\mathfrak{B}_{1003}}x^T$$

Expansions of Cauchy kernels on truncated stair shapes

Truncated stair shapes.

$$\prod_{\substack{(i,j)\in\eta\\ m\le k}}(1-x_iy_j)^{-1}=\sum_{\nu=(\nu_1,\dots,\nu_m,0^{n-m})}\ \sum_{sh(F)=\nu}y^F\sum_{\substack{\beta=(\beta_1,\dots,\beta_k,0^{n-k})\\ \beta\le\omega\nu}}\ \sum_{sh(G)=\beta}x^G=\sum_{\nu=(\nu_1,\dots,\nu_m,0^{n-m})}\widehat{\kappa}_\nu(y)\pi^{-1}_{>k}\kappa_{\omega\nu}(x),$$

$$\prod_{\substack{(i,j)\in\eta\\ m\ge k}}(1-x_iy_j)^{-1}=\prod_{\substack{(j,i)\in\overline{\eta}\\ m\ge k}}(1-y_jx_i)^{-1}=\sum_{\nu=(\nu_1,\dots,\nu_k,0^{n-k})}\widehat{\kappa}_\nu(x)\pi^{-1}_{>m}\kappa_{\omega\nu}(y),$$

Stair shape partitions. $\eta=(n,n-1,\dots,1)=\overline{\eta}$

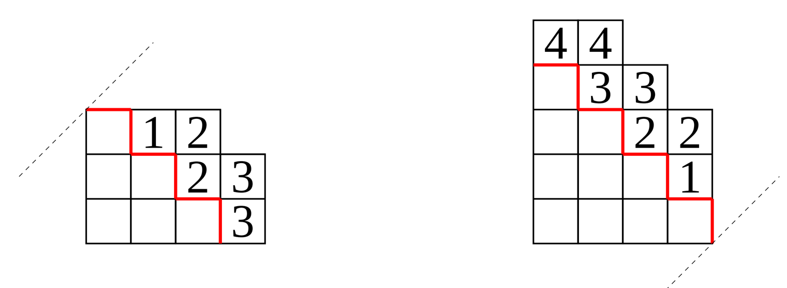
$$\prod_{i+j\le n+1}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(y)\kappa_{\omega\nu}(x)=\sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y),$$

Rectangle partitions.

$$\prod_{(i,j)\in(m^{n-m+1})}(1-x_iy_j)^{-1}=\sum_\lambda s_\lambda(x)s_\lambda(y),$$

Lascoux's expansion for the Cauchy kernel over an arbitrary Ferrers shape λ [2]

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^r}(\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\nu(x))(\pi_{\sigma(\lambda,SE)}\kappa_{\omega\nu}(y)),$$



$\sigma(\lambda_1,NW)=id$
$\sigma(\lambda_1,SE)=s_2s_1s_3s_2s_3$

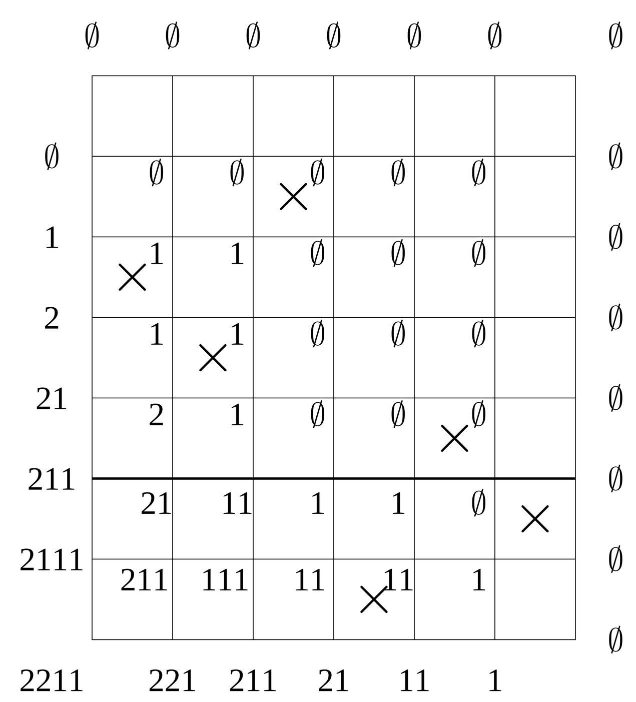
$\sigma(\lambda_2,NW)=s_2s_1s_3s_2s_4s_3s_4$
$\sigma(\lambda_2,SE)=id$

Comparison between Lascoux's formula in the case λ = η and our formula

$$\prod_{\substack{(i,j)\in\eta\\ m\ge k}}(1-x_iy_j)^{-1}=\sum_{\mu\in\mathbb{N}^k}\pi_{\sigma(\eta,SE)}\kappa_{\omega\mu}(y)\widehat{\kappa}_\mu(x)=\sum_{\nu=(\nu_1,\dots,\nu_k,0^{n-k})}\widehat{\kappa}_\nu(x)\pi^{-1}_{>m}\kappa_{\omega\nu}(y),$$

$$\prod_{\substack{(i,j)\in\eta\\ m\le k}}(1-x_iy_j)^{-1}=\sum_{\mu\in\mathbb{N}^m}\kappa_{\omega\mu}(y)\pi_{\sigma(\eta,NW)}\widehat{\kappa}_\mu(x)=\sum_{\nu=(\nu_1,\dots,\nu_m,0^{n-m})}\widehat{\kappa}_\nu(y)\pi^{-1}_{>k}\kappa_{\omega\nu}(x).$$

Fomin's growth diagram for RSK analogue and reverse RSK



References

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4. S. Mason, A decomposition of Schur functions and an analogue of the Robinson-Schensted- Knuth algorithm. Sém. Lothar. Combin. 57: B57e,24,2006/08.